J. Math. Soc. Japan
Vol. 53, No. 2, 2001

Folding maps and the surgery theory on manifolds

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(Received Nov. 27, 1997)
(Revised Nov. 8, 1999)

Abstract. Let $f : N \to P$ be a smooth map between n-dimensional oriented manifolds which has only folding singularities. Such a map is called a folding map. We prove that a folding map $f : N \to P$ canonically determines the homotopy class of a bundle map of $TN \oplus \theta_N$ to $TP \oplus \theta_P$, where θ_N and θ_P are the trivial line bundles over N and P respectively. When P is a closed manifold in addition, we define the set $\Omega_{\text{fold}}(P)$ of all cobordism classes of folding maps of closed manifolds into P of degree 1 under a certain cobordism equivalence. Let SG denote the space $\lim_{k\to\infty} SG_k$, where SG_k denotes the space of all homotopy equivalences of S^{k-1} of degree 1. We prove that there exists an important map of $\Omega_{\text{fold}}(P)$ to the set of homotopy classes $[P, SG]$. We relate $\Omega_{\text{fold}}(P)$ with the set of smooth structures on P by applying the surgery theory.

Introduction.

Let N and P denote oriented smooth manifolds of dimension n in this paper. We shall say that a smooth map germ of (N, x) into (P, y) has a singularity of folding type at x if it is written as $(x_1, x_2, \ldots, x_n) \mapsto (x_1^2, x_2, \ldots, x_n)$ under suitable local coordinate systems of N and P near x and y respectively. A smooth map $f : N \to P$ is called a *folding map* if it has only folding singularities.

It follows from Eliashberg [**E**] that given a smooth map $f : N \to P$, in many cases there exists a folding map of N into P homotopic to f if and only if the vector bundles TN and $f^*(TP)$ are stably equivalent. In this paper we shall clarify the reason why this phenomenon occurs by using the results of [**An**] and it will lead us to prove by the surgery theory due to [**K-M**], [**B2**] and [**Su**] that if $n \geq 5$, P is a closed oriented and simply connected manifold and if the surgery obstruction of Kervaire invariant vanishes for P, then a given folding map of degree 1 determines, up to a certain equivalence, a smooth manifold P' and a homotopy equivalence of P' into P of degree 1.

2000 *Mathematics Subject Classification.* Primary 58K15; Secondary 57R45, 57R67, 57R55, 55Q10.

Key Words and Phrases. Folding singularity, jet space, manifold, surgery theory, homotopy class.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 09640114), Ministry of Education, Culture, Sports, Science and Technology, Japan.

In the 2-jet space $J^2(n,n)$ we shall consider the subspace Ω^{10} consisting of all jets of either regular germs or germs with folding singularities at the origin. In [**An**] we have proved that there exists a topological embedding i_n of $SO(n+1)$ into Ω^{10} giving a homotopy equivalence. The rotation group $SO(n)$ acts on $J^2(n,n)$ through the source space $\boldsymbol{R}^n$ and the target space $\boldsymbol{R}^n$ and also does on $SO(n+1)$ from the left-hand and the right-hand sides through $SO(n) \times SO(1)$. We shall show in §2 that i_n is equivariant with respect to these actions of $SO(n) \times SO(n)$.

In the 2-jet bundle $J^2(N,P)$, let $\Omega^{10}(N,P)$ be its subbundle associated with Ω^{10}. If we provide N and P with Riemannian metrics, then we can reduce the structure groups of $J^2(N,P)$ and $\Omega^{10}(N,P)$ to $SO(n) \times SO(n)$. Let $SO(TN \oplus \theta_N, TP \oplus \theta_P)$ be the subbundle of $\mathrm{Hom}(TN \oplus \theta_N, TP \oplus \theta_P)$ associated with $SO(n+1)$, where θ_N and θ_P are the trivial line bundles over N and P respectively. Then we obtain a topological embedding $i(N,P): SO(TN \oplus \theta_N, TP \oplus \theta_P) \to \Omega^{10}(N,P)$, which is a fibre map over $N \times P$ associated with i_n.

Theorem 1. *Let N and P be oriented smooth manifolds with Riemannian metrics of dimension n. Then the embedding $i(N,P): SO(TN \oplus \theta_N, TP \oplus \theta_P) \to \Omega^{10}(N,P)$ covering the identity of $N \times P$ gives a homotopy equivalence of fibre bundles.*

This theorem together with Proposition 3.1 will yield the following, where bundle maps are fiberwise linear.

Corollary 2. *The homotopy classes of orientation preserving bundle maps of $TN \oplus \theta_N$ into $TP \oplus \theta_P$ correspond bijectively to the homotopy classes of continuous sections of $\Omega^{10}(N,P)$ over N. This correspondence does not depend on the choice of Riemannian metrics of N and P. In particular, the jet-extension $j^2 f$ of a folding map $f: N \to P$ determines the homotopy class of an orientation preserving bundle map of $TN \oplus \theta_N$ into $TP \oplus \theta_P$.*

The proofs will be given in §3. These results should be compared with [**E**, 3.9 and Theorem 3.10] and [**Sa**, Lemma 3.1].

Let $f_i: N_i \to P$ $(i = 0,1)$ be two folding maps of degree 1. We shall say that they are *fold-cobordant* when there exists a folding map F of $(W, \partial W)$ into $(P \times [0,1], P \times 0 \cup P \times 1)$ of degree 1 such that

(i) W is oriented with $\partial W = N_0 \cup (-N_1)$ and the collar of ∂W is identified with $N_0 \times [0,\varepsilon) \cup N_1 \times (1-\varepsilon, 1]$,

(ii) $F|N_0 \times [0,\varepsilon) = f_0 \times id_{[0,\varepsilon)}$ and $F|N_1 \times (1-\varepsilon,1] = f_1 \times id_{(1-\varepsilon,1]}$,

where ε is a sufficiently small positive real number. Let $\Omega_{\mathrm{fold}}(P)$ denote the set of all fold-cobordism classes of folding maps to P of degree 1.

Let G_k (resp. SG_k) denote the space of all homotopy equivalences (resp. of degree 1) of S^{k-1} with compact-open topology. The suspension of a homotopy equivalence yields the inclusion of G_k into G_{k+1} (resp. SG_k into SG_{k+1}). We set $G=\lim_{k\to\infty} G_k$ and $SG=\lim_{k\to\infty} SG_k$ respectively. Similarly set $O=\lim_{k\to\infty} O(k)$. By considering the quotient space $G_k/O(k)$ by the acton of $O(k)$ on G_k, set $G/O = \lim_{k\to\infty} G_k/O(k)$.

In §4 we shall prove by using Corollary 2 and the results about spherical fibre spaces ([**At1**], [**B2**], [**W1**] and [**W2**]) that there exists an important map ω of $\Omega_{\text{fold}}(P)$ to $[P, SG]$ (Theorem 4.2 and 5.5). This enables us to define new invariants of $\Omega_{\text{fold}}(P)$ (Corollary 4.7).

A homotopy equivalence $f : N \to P$ of degree 1 is called a smooth structure on P. We will say that two smooth structures on P, $f_i : N_i \to P$ $(i = 0, 1)$, are equivalent if there is a diffeomorphism $d : N_0 \to N_1$ such that f_0 is homotopic to $f_1 \circ d$. Let $\mathscr{S}(P)$ denote the set of all equivalence classes of smooth structures on P. Let $\mathscr{S}^{\text{tang}}(P)$ denote its subset of all equivalence classes of smooth structures $f : P' \to P$ such that TP' and $f^*(TP)$ are stably equivalent. Then the group bP_{n+1} of homotopy n-spheres bounding parallelizable manifolds acts on $\mathscr{S}^{\text{tang}}(P)$ as usual. Let $i_\Sigma : \Sigma \to S^n$ be a map of degree 1 for a homotopy n-sphere $\Sigma \in bP_{n+1}$. The action of Σ on f is defined by the connected sum of maps, $f \sharp i_\Sigma : P' \sharp \Sigma \to P \sharp S^n = P$. The quotient set of $\mathscr{S}^{\text{tang}}(P)$ by this action of bP_{n+1} is denoted by $\mathscr{S}^{\text{tang}}(P)/bP_{n+1}$.

In §5 we shall prove the following theorem by applying Sullivan's exact sequence in the surgery theory ([**Su**]) and [**E**, Theorem 3.10] to the map ω projected to $[P, G/O]$.

THEOREM 3. *Let P be a closed oriented and simply connected smooth manifold of dimension $n \geq 5$. We assume that if $n \equiv 2 \pmod 4$, then the surgery obstruction of Kervaire invariant vanishes for P. Then there exists a surjection of $\Omega_{\text{fold}}(P)$ onto $\mathscr{S}^{\text{tang}}(P)/bP_{n+1}$ such that a smooth structure $f : P' \to P$ of class C^∞ with only folding singularities in $\Omega_{\text{fold}}(P)$ is mapped to the equivalence class of f modulo bP_{n+1}.*

All manifolds are of class C^∞. Maps are basically continuous, but may be smooth (of class C^∞) if so stated.

The author would like to thank the refree for his careful reading of the manuscript of the paper and kind comments.

§1. Notations.

The space of all homomorphisms of a vector space V into a vector space W will be denoted by $\mathrm{Hom}(V, W)$. Let $J^2(N, P)$ be the 2-jet space of manifolds N and P. Let π_N and π_P be the projections mapping a jet to its source and target respectively. Let $L^2(n)$ be the group of 2-jets of all diffeomorphisms of $(\boldsymbol{R}^n, 0)$. $J^2(N,P)$ has the structure group $L^2(n) \times L^2(n)$. Let $\mathrm{Hom}(TN \oplus S^2(TN), TP)$ be the vector bundle over $N \times P$ with structure group $GL(n) \times GL(n) \subset L^2(n) \times L^2(n)$, which is the union of all spaces $\mathrm{Hom}(T_xN \oplus S^2(T_xN), T_yP)$ for (x, y) of $N \times P$, where $S^2(T_xN)$ denotes the 2-fold symmetric product of T_xN. If we provide N and P with Riemannian metrics and Riemannian connections, then we have the exponential maps (see, for example, [**N**]) defined on neighbourhoods of the zero vectors of T_xN and T_yP,

$$\exp_{N,x} : (T_xN, \mathbf{0}) \to (N, x) \quad \text{and}$$
$$\exp_{P,y} : (T_yP, \mathbf{0}) \to (P, y)$$

respectively. An orthonormal basis of T_xN (resp. T_yP) gives its local coordinate system of (N, x) (resp. (P, y)) compatible with the differentiable structure of N (resp. P). By using them we can define the map $J : J^2(N,P) \to \mathrm{Hom}(TN \oplus S^2(TN), TP)$ as follows.

Let $z = j^2_x f$ with $y = f(x)$ be a 2-jet in $J^2_{x,y}(N,P)$, which is the subset of $J^2(N,P)$ consisting of all 2-jets of smooth map germs of (N, x) into (P, y). Then define $J(z)$ to be the 2-jet of $(\exp_{P,y})^{-1} \circ f \circ \exp_{N,x}$ at $\mathbf{0}$, which becomes a linear map of $T_xN \oplus S^2(T_xN)$ into T_yP. It is shown by the properties of exponential maps that J is a bundle map between bundles with structure group $L^2(n) \times L^2(n)$. The Riemannian metrics of TN and TP reduce the structure group of $\mathrm{Hom}(TN \oplus S^2(TN), TP)$ to $SO(n) \times SO(n)$. We note that J depends on metrics and Riemannian connections of N and P. However, any two Riemannian metrics are homotopic ([**Ste**, 12.12]) and so all J's are homotopic. This kind of observation can be found in [**P**]. Another homotopy theoretic approach to this fact can be found in [**D**].

Set $\boldsymbol{D} = \pi_N^*(TN)$ and $\boldsymbol{P} = \pi_P^*(TP)$. Then there is a homomorphism $d_1 : \boldsymbol{D} \to \boldsymbol{P}$ defined as follows. Let $z = j^2_x f$ with $y = f(x)$ be a jet of $J^2_{x,y}(N,P)$. Let $\boldsymbol{D}_z$ and $\boldsymbol{P}_z$ be the fibres of $\boldsymbol{D}$ and $\boldsymbol{P}$ over z respectively. Then $d_{1,z} : \boldsymbol{D}_z \to \boldsymbol{P}_z$ refers to $df : TN_x \to TP_y$. We define $\Sigma^i(N,P)$ (resp. $\Sigma^i_{x,y}(N,P)$) to be the set of all jets z in $J^2(N,P)$ (resp. $J^2_{x,y}(N,P)$) with $\dim(\mathrm{Ker}(d_{1,z})) = i$. Then we have the subbundle $\boldsymbol{K} = \mathrm{Ker}(d_1)$ and the cokernel bundle $\boldsymbol{Q} = \mathrm{Cok}(d_1)$ over $\Sigma^i(N,P)$. In [**L**] the second intrinsic derivative $d_2 : \boldsymbol{K} \to \mathrm{Hom}(\boldsymbol{K}, \boldsymbol{Q})$ is defined using the second derivatives of z. We define $\Sigma^{10}(N,P)$ (resp. $\Sigma^{10}_{x,y}(N,P)$) to be the set of all jets z in $J^2(N,P)$ (resp. $J^2_{x,y}(N,P)$) such that $\dim(\mathrm{Ker}(d_{1,z})) = 1$ and $d_{2,z} : \boldsymbol{K}_z \to \mathrm{Hom}(\boldsymbol{K}_z, \boldsymbol{Q}_z)$ is an isomorphism. Let $\Omega^{10}(N,P)$ denote the union of all regular jets and $\Sigma^{10}(N,P)$, which becomes an open subbundle of $J^2(N,P)$.

Let $J^k(n,n)$ refer to $J^k_{0,0}(\boldsymbol{R}^n, \boldsymbol{R}^n)$. We shall identify $J^1(n,n)$ with $\mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ as usual. Let Σ^i be its subspace consisting of all homomorphisms $\alpha : \boldsymbol{R}^n \to \boldsymbol{R}^n$ with $\dim(\mathrm{Ker}(\alpha)) = i$. Next $J^2(n,n)$ is identified with

$\mathrm{Hom}(\boldsymbol{R}^n \oplus S^2\boldsymbol{R}^n, \boldsymbol{R}^n)$. We usually denote its element as (α, β) with $\alpha : \boldsymbol{R}^n \to \boldsymbol{R}^n$ and $\beta : S^2\boldsymbol{R}^n \to \boldsymbol{R}^n$. Consider the composition of the restriction $\beta|S^2(\mathrm{Ker}(\alpha))$ and the natural projection of $\boldsymbol{R}^n$ onto $\mathrm{Cok}(\alpha)$. It induces a new homomorphism of $\mathrm{Ker}(\alpha)$ into $\mathrm{Hom}(\mathrm{Ker}(\alpha), \mathrm{Cok}(\alpha))$ denoted by $\bar{\beta}$. Let Σ^{10} be the subspace of $J^2(n,n)$ consisting of all elements (α, β) such that $\dim(\mathrm{Ker}(\alpha)) = 1$ and $\dim(\mathrm{Ker}(\bar{\beta})) = 0$. The notation Σ^i is often used for $\Sigma^i \times \mathrm{Hom}(S^2\boldsymbol{R}^n, \boldsymbol{R}^n)$ if there is no confusion. The space $\Omega^{10}(n,n)$ means $\Sigma^0 \cup \Sigma^{10}$ in $J^2(n,n)$. It is an open subspace. We say that a 2-jet of Σ^{10} and also its singularity at the origin are of folding type.

The two constructions above of $\Sigma^i_{0,0}(\boldsymbol{R}^n, \boldsymbol{R}^n)$, $\Sigma^{10}_{0,0}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ and Σ^i, Σ^{10} correspond to each other by J. Let $\Omega^{10}(N,P)'$ denote the subbundle of $\mathrm{Hom}(TN \oplus S^2(TN), TP)$ associated with Ω^{10}. It is clear that J gives a bundle map of $\Omega^{10}(N,P)$ to $\Omega^{10}(N,P)'$.

For two square matrices A and B, let $A \dotplus B$ denote the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Let $\boldsymbol{e}_j$ be the j-th unit vector. Let I_j be the unit matrix of rank j. Let I_- be the matrix $I_{n-1} \dotplus (-1)$. Let $\Delta(\boldsymbol{d})$ with $\boldsymbol{d} = (d_1, \ldots, d_n)$ be the diagonal matrix with diagonal components $\boldsymbol{d}$. Let $SO(n) \times SO(1)$ be the set consisting of all matrices $T \dotplus (1)$ with $T \in SO(n)$.

Let $p_i : \xi_i \to X_i$ $(i = 0, 1)$ be fibre bundles or spherical fibre spaces. In this paper a continuous map $\tilde{c} : \xi_0 \to \xi_1$ is called a fibre map over $c : X_0 \to X_1$ if $p_1 \circ \tilde{c} = c \circ p_0$.

An equivalence class with representative x in a set will be denoted by $[x]$ and it is often abbreviated as x if there is no confusion.

§2. Homotopy type of $\Omega^{10}(n,n)$.

First we briefly review the result of [**An**]. There has been constructed an embedding i_n of $SO(n+1)$ into $\Omega^{10}(n,n)$, which has its image as a deformation retract. For $M \in SO(n+1)$, let $M\boldsymbol{e}_{n+1}$ be written as $\boldsymbol{x} = {}^t(x_1, \ldots, x_n, b)$. If b is not equal to -1 (resp. 1), let $r(\boldsymbol{x})$ (resp. $\bar{r}(\boldsymbol{x})$) be the matrix of the rotation which is the identity on the subspace orthogonal to $\boldsymbol{x}$ and $\boldsymbol{e}_{n+1}$ and rotates the great circle through $\boldsymbol{x}$ and $\boldsymbol{e}_{n+1}$ so as to carry $\boldsymbol{e}_{n+1}$ (resp. $-\boldsymbol{e}_{n+1}$) to $\boldsymbol{x}$. Note that $r(\boldsymbol{e}_{n+1}) = \bar{r}(-\boldsymbol{e}_{n+1}) = I_{n+1}$. Then the matrices $r(\boldsymbol{x})^{-1}M$ and $\bar{r}(\boldsymbol{x})^{-1}M$ have the forms $U(M) \dotplus (1)$ and $I_-U(M)' \dotplus (-1)$ with some n-matrices $U(M)$ and $U(M)'$ in $SO(n)$ respectively.

Let $\boldsymbol{s}(M)$ be the vector ${}^t(s_1, \ldots, s_n)$ such that $s_i = x_i/(1-b^2)^{1/2}$ for $b \neq \pm 1$ and that $\boldsymbol{s}(M)$ can be any vector of length 1 for $b = \pm 1$.

For $n \geq 2$, let $\boldsymbol{d}_{ab}$ denote the n-vector $(a/\sqrt{n-1}, \ldots, a/\sqrt{n-1}, b/\sqrt{n})$, where a and b are real numbers with $a^2 + (b^2/n) = 1$ and $a \geq 0$. Let S be any matrix of $SO(n)$ with $S\boldsymbol{e}_n = s(M)$. An elementary observation in linear algebra shows the following relations in which $G(\boldsymbol{s}(M)) = SI_{-}{}^tS$ and $b \neq \pm 1$:

$$SI_{-}{}^tS = (\delta_{ij} - 2s_is_j), \quad \bar{r}(\boldsymbol{x})^{-1} \circ r(\boldsymbol{x}) = G(\boldsymbol{s}(M)) \dotplus (-1) \quad \text{and}$$
$$U(M)' = I_{-}G(\boldsymbol{s}(M))U(M).$$

Consider the homomorphism $\alpha(M) : \boldsymbol{R}^n \to \boldsymbol{R}^n$ defined as follows:

$$\text{for } n \geq 2, \quad \alpha(M) = \begin{cases} S\Delta(\boldsymbol{d}_{ab})^tSU(M) & \text{for } b \geq 0 \\ S\Delta(\boldsymbol{d}_{ab})I_{-}{}^tSI_{-}U(M)' & \text{for } b \leq 0 \end{cases} \quad \text{and}$$
$$\text{for } n = 1, \quad \alpha(M) = b.$$

This definition does not depend on the choice of S for $\boldsymbol{s}(M)$. In particular, if $b = 1$, then $\alpha(M) = (1/\sqrt{n})U(M)$ and if $b = -1$, then $\alpha(M) = (1/\sqrt{n})I_{-}U(M)'$. Note that if $n = 1$, then $U(M) = U(M)' = (1)$.

For a vector $\boldsymbol{s} \in \boldsymbol{R}^n$ of length 1, $U \in SO(n)$ and u with $0 \leq u \leq 1$, we let $q_u(\boldsymbol{s}, U)(\boldsymbol{x}, \boldsymbol{y})$ denote the quadratic form ${}^t\boldsymbol{x}{}^tUS\Delta(0, \dots, 0, u){}^tSU\boldsymbol{y}$ and define the homomorphism $\beta_u(\boldsymbol{s}, U) : S^2\boldsymbol{R}^n \to \boldsymbol{R}^n$ by $\beta_u(\boldsymbol{s}, U)(\boldsymbol{x}, \boldsymbol{y}) = q_u(\boldsymbol{s}, U)(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{s}$. Now we define the homomorphism $\beta(M) : S^2\boldsymbol{R}^n \to \boldsymbol{R}^n$ as follows:

$$\text{for } n \geq 2, \quad \beta(M) = \begin{cases} \beta_{1-b}(\boldsymbol{s}(M), U(M)) & \text{for } b \geq 0 \\ \beta_{1+b}(\boldsymbol{s}(M), G(\boldsymbol{s}(M))I_{-}U(M)') & \text{for } b \leq 0 \end{cases} \quad \text{and}$$
$$\text{for } n = 1, \quad \beta(M) = \begin{cases} (1-b)\boldsymbol{s}(M) & \text{for } b \geq 0 \\ (1+b)\boldsymbol{s}(M) & \text{for } b \leq 0. \end{cases}$$

In particular, if $b = \pm 1$, then $\beta(M)$ is the null homomorphism.

Then the topological embedding giving a homotopy equivalence

$$i_n : SO(n+1) \to \Omega^{10}(n, n)$$

is defined by $i_n(M) = (\alpha(M), \beta(M))$ (this is denoted by $h \circ h'$ in [**An**, §5]).

Remark 2.1. We give two remarks concerning $i_n(M)$.

(i) Let us explain how M is constructed from a folding map germ associated to $i_n(M)$. The Jacobian matrix of this germ is $\alpha(M)$, from which we obtain the number b and the matrices $U(M)$ and $U(M)'$ by [**An**, §3]. The vector $\boldsymbol{s}(M)$ is constructed to be a vector of length 1 on $\mathrm{Cok}(\alpha(M))$ and to have the inward direction with respect to the image of this folding map germ. By definition, M is determined from b, $\boldsymbol{s}(M), U(M)$ and $U(M)'$.

(ii) In the definition of $\beta(M)$ it does not matter to replace $1+b$ and $1-b$ by $(1-b^2)^{1/2}$.

Here we consider the actions of $SO(n) \times SO(n)$ on $SO(n+1)$ and on $J^2(n, n)$ as follows. An element $(O', {}^tO)$ of $SO(n) \times SO(n)$ acts on each element M of

$SO(n+1)$ and (α, β) of $J^2(n,n)$ as

$$(O', {}^tO) \cdot M = (O' \dotplus (1))M(O \dotplus (1)) \quad \text{and}$$
$$(O', {}^tO) \cdot (\alpha, \beta) = (\alpha', \beta'),$$

where $\alpha'(\boldsymbol{x}) = O'\alpha(O\boldsymbol{x})$ and $\beta'(\boldsymbol{x}, \boldsymbol{y}) = O'\beta(O\boldsymbol{x}, O\boldsymbol{y})$ respectively. Note that $\Omega^{10}(n,n)$ is invariant with respect to this action.

We shall prove that i_n is equivariant with respect to the actions of $SO(n) \times SO(n)$. Its proof needs a complicated observation about the embedding i_n. First we shall prepare two lemmas. According to [**Ste**, (23.3)], the matrix representations of $r(\boldsymbol{x})$ and $\bar{r}(\boldsymbol{x})$ are given by

$$r(\boldsymbol{x}) = \left(\begin{array}{c:c} \delta_{ij} - \dfrac{x_i x_j}{1+b} & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ \hdashline -x_1 \ldots\ldots -x_n & b \end{array} \right) \quad \text{and}$$

$$\bar{r}(\boldsymbol{x}) = \left(\begin{array}{c:c} \delta_{ij} - \dfrac{x_i x_j}{1-b} & \begin{matrix} -x_1 \\ \vdots \\ -x_n \end{matrix} \\ \hdashline x_1 \ldots\ldots x_n & -b \end{array} \right) \quad \text{respectively.}$$

Lemma 2.2. *Let $\boldsymbol{x} = {}^t(x_1, \ldots, x_n, b)$ be $M\boldsymbol{e}_{n+1}$ as above. Let $\boldsymbol{y} = {}^t(y_1, \ldots, y_n, b)$ be $(O' \dotplus (1))\boldsymbol{x}$ for an O' in $SO(n)$. Then we have*

$$r(\boldsymbol{y})^{-1}(O' \dotplus (1)) = (O' \dotplus (1))r(\boldsymbol{x})^{-1} \quad \textit{and}$$
$$\bar{r}(\boldsymbol{y})^{-1}(O' \dotplus (1)) = (O' \dotplus (1))\bar{r}(\boldsymbol{x})^{-1}.$$

Proof. Since $r(\boldsymbol{x})$ is equal to the matrix

$$(I_n \dotplus (-1)) - (1/(1+b))\,{}^t(x_1, \ldots, x_n, 1+b)(x_1, \ldots, x_n, -(1+b))$$

and ${}^t(y_1, \ldots, y_n) = O'\,{}^t(x_1, \ldots, x_n)$, it follows from a direct calculation that $r(\boldsymbol{y}) = (O' \dotplus (1))r(\boldsymbol{x})\,{}^t(O' \dotplus (1))$. The second formula follows similarly. $\square$

Lemma 2.3. *Let $M' = (O' \dotplus (1))M(O \dotplus (1))$ for O and O' in $SO(n)$. Then we have*

$$\text{(i)}\quad U(M') = O'U(M)O \qquad \textit{for } b \geq 0$$

$$\text{(ii)}\quad U(M')' = I_{-}O'I_{-}U(M)'O \quad \textit{for } b \leq 0 \quad \textit{and}$$

$$\text{(iii)}\quad G(\boldsymbol{s}(M'))I_{-}U(M')' = O'G(\boldsymbol{s}(M))I_{-}U(M)'O.$$

Proof. It follows from Lemma 2.2 that

$$\begin{aligned} r(\boldsymbol{y})^{-1}M' &= r(\boldsymbol{y})^{-1}(O' \dotplus (1))M(O \dotplus (1)) \\ &= (O' \dotplus (1))r(\boldsymbol{x})^{-1}M(O \dotplus (1)) \\ &= (O' \dotplus (1))(U(M) \dotplus (1))(O \dotplus (1)) \\ &= O'U(M)O \dotplus (1) \quad \text{and} \\ \bar{r}(\boldsymbol{y})^{-1}M' &= \bar{r}(\boldsymbol{y})^{-1}(O' \dotplus (1))M(O \dotplus (1)) \\ &= (O' \dotplus (1))\bar{r}(\boldsymbol{x})^{-1}M(O \dotplus (1)) \\ &= (O' \dotplus (1))(I_{-}U(M)' \dotplus (-1))(O \dotplus (1)) \\ &= O'I_{-}U(M)'O \dotplus (-1) \\ &= I_{-}(I_{-}O'I_{-}U(M)'O) \dotplus (-1). \end{aligned}$$

Thus (i) and (ii) follow from the definition of $U(M')$ and $U(M')'$. From (ii), it follows that

$$\begin{aligned} G(\boldsymbol{s}(M'))I_{-}U(M')' &= G(O'\boldsymbol{s}(M))I_{-}U(M')' \\ &= O'SI_{-}{}^{t}S^{t}O'I_{-}(I_{-}O'I_{-}U(M)'O) \\ &= O'SI_{-}{}^{t}SI_{-}U(M)'O \\ &= O'G(\boldsymbol{s}(M))I_{-}U(M)'O. \end{aligned}$$ □

We are ready to prove the following.

Proposition 2.4. *The embedding i_n is equivariant with respect to the actions of $SO(n) \times SO(n)$ on $SO(n+1)$ and on $J^2(n,n)$.*

Proof. The assertion for $n = 1$ is trivial, since the actions are trivial. Hence we assume $n \geq 2$ in the following. Let M, O', O and M' be as in Lemma 2.3 and set $\boldsymbol{x}' = M'\boldsymbol{e}_{n+1} = (O' \dotplus (1))\boldsymbol{x}$. We use the notations given in the definition of i_n. We have that $\boldsymbol{s}(M') = O'\boldsymbol{s}(M)$. Using Lemma 2.3 we obtain the following.

If $b \geq 0$, then

$$\begin{aligned} \alpha(M') &= O'S\Delta(\boldsymbol{d}_{ab})^{t}S^{t}O'U(M') \\ &= O'S\Delta(\boldsymbol{d}_{ab})^{t}S^{t}O'O'U(M)O \\ &= O'S\Delta(\boldsymbol{d}_{ab})^{t}SU(M)O \\ &= O'\alpha(M)O \quad \text{and} \end{aligned}$$

$$\begin{aligned}
\beta(M')(\boldsymbol{x},\boldsymbol{y}) &= q_{1-b}(\boldsymbol{s}(M'),\, U(M'))(\boldsymbol{x},\boldsymbol{y})\boldsymbol{s}(M') \\
&= {}^t\boldsymbol{x}^t U(M')O'S\varDelta(0,\dots,0,1-b)^t S^t O' U(M')\boldsymbol{y}\boldsymbol{s}(M') \\
&= {}^t\boldsymbol{x}^t (O'U(M)O)O'S\varDelta(0,\dots,0,1-b)^t S^t O' \\
&\quad \times (O'U(M)O)\boldsymbol{y}O'\boldsymbol{s}(M) \\
&= {}^t\boldsymbol{x}^t O^t U(M)S\varDelta(0,\dots,0,1-b)^t SU(M)O\boldsymbol{y}O'\boldsymbol{s}(M) \\
&= O'\beta(M)(O\boldsymbol{x},\, O\boldsymbol{y}).
\end{aligned}$$

If $b \leq 0$, then

$$\begin{aligned}
\alpha(M') &= O'S\varDelta(\boldsymbol{d}_{ab})I_-{}^tS^tO'I_-U(M')' \\
&= O'S\varDelta(\boldsymbol{d}_{ab})I_-{}^tS^tO'I_-(I_-O'I_-U(M)'O) \\
&= O'S\varDelta(\boldsymbol{d}_{ab})I_-{}^tSI_-U(M)'O \\
&= O'\alpha(M)O \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\beta(M')(\boldsymbol{x},\boldsymbol{y}) &= q_{1+b}(\boldsymbol{s}(M'),\, G(\boldsymbol{s}(M'))I_-U(M')')(\boldsymbol{x},\boldsymbol{y})\boldsymbol{s}(M') \\
&= {}^t\boldsymbol{x}^t(O'G(\boldsymbol{s}(M))I_-U(M)'O)O'S\varDelta(0,\dots,0,1+b) \\
&\quad \times {}^tS^tO'(O'G(\boldsymbol{s}(M))I_-U(M)'O)\boldsymbol{y}O'\boldsymbol{s}(M) \\
&= {}^t\boldsymbol{x}^tO^t(G(\boldsymbol{s}(M))I_-U(M)')S\varDelta(0,\dots,0,1+b) \\
&\quad \times {}^tS(G(\boldsymbol{s}(M))I_-U(M)')O\boldsymbol{y}O'\boldsymbol{s}(M) \\
&= q_{1+b}(\boldsymbol{s}(M), G(\boldsymbol{s}(M))I_-U(M)')(O\boldsymbol{x},\, O\boldsymbol{y})O'\boldsymbol{s}(M) \\
&= O'\beta(M)(O\boldsymbol{x},\, O\boldsymbol{y}).
\end{aligned}$$

This proves that i_n is equivariant with respect to the actions of $SO(n) \times SO(n)$. □

§3. Associated fibre bundles.

First we shall give the precise definition of the embedding $i(N,P)$ and prove Theorem 1 and Corollary 2.

By providing N and P with Riemannian metrics, we have the bundle map $J : J^2(N,P) \to \mathrm{Hom}(TN \oplus S^2(TN), TP)$ over $N \times P$ as defined in §1. The last bundle has the structure group $SO(n) \times SO(n)$ through TN and TP. The map J induces a diffeomorphism between fibers $J^2_{x,y}(N,P)$ and $\mathrm{Hom}(T_xN \oplus S^2(T_xN), T_yP)$. We shall apply the embedding $i_n : SO(n+1) \to$

$\Omega^{10}(n,n)(\subset \mathrm{Hom}(\boldsymbol{R}^n \oplus S^2\boldsymbol{R}^n, \boldsymbol{R}^n))$ to $\mathrm{Hom}(TN \oplus S^2(TN), TP)$. Then we obtain the subspace homeomorphic to $SO(n+1)$ denoted by $SO_{x,y}(N,P)$ in $\mathrm{Hom}(T_xN \oplus S^2(T_xN), T_yP)$. This space is well defined by Proposition 2.4. The space $SO(N,P)$ is defined to be the union of all spaces $SO_{x,y}(N,P)$ in $\Omega^{10}(N,P)'$, where (x,y) varies all over $N \times P$. It becomes a subbundle with structure group $SO(n) \times SO(n)$ coming from those of TN and TP.

By the Riemannian metrics of N and P we have the subbundle $SO(TN \oplus \theta_N, TP \oplus \theta_P)$ of $\mathrm{Hom}(TN \oplus \theta_N, TP \oplus \theta_P)$ associated with $SO(n+1)$. Let $i(N,P)'$ be the map of $SO(TN \oplus \theta_N, TP \oplus \theta_P)$ to $\Omega^{10}(N,P)'$ associated with i_n. It is clear that its image coincides with $SO(N,P)$ and is homotopy equivalent to $\Omega^{10}(N,P)'$ by [**An**, §5] and Proposition 2.4. Then $i(N,P)$ in Introduction is defined to be $(J^{-1}) \circ i(N,P)'$.

Proof of Theorem 1. The assertion follows from the fact that $i(N,P)'$ is a homotopy equivalence of fibre bundles. □

Let $GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$ be the subbundle of $\mathrm{Hom}(TN \oplus \theta_N, TP \oplus \theta_P)$ associated with $GL^+(n+1)$. Let $i_{SO} : SO(TN \oplus \theta_N, TP \oplus \theta_P) \to GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$ be the inclusion, which becomes a homotopy equivalence of fibre bundles covering $id_{N\times P}$. Let $(i(N,P)')^{-1} : \Omega^{10}(N,P)' \to SO(TN \oplus \theta_N, TP \oplus \theta_P)$ be the homotopy inverse of $i(N,P)'$. Then we consider the fibre map

$$\begin{aligned} & i_{SO} \circ (i(N,P)')^{-1} \circ J|\Omega^{10}(N,P) : \\ & \qquad \Omega^{10}(N,P) \to \Omega^{10}(N,P)' \to SO(TN \oplus \theta_N, TP \oplus \theta_P) \\ & \qquad\qquad \to GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P) \end{aligned}$$

giving a homotopy equivalence of fibre bundles. Then the following proposition follows from the fact that all of the maps J's are homotopic to each other as explained in §1.

Proposition 3.1. *The homotopy class of the fibre map* $i_{SO} \circ (i(N,P)')^{-1} \circ J|\Omega^{10}(N,P)$ *covering* $id_{N\times P}$ *does not depend on the choice of Riemannian metrics of* N *and* P.

Proof of Corollary 2. The set of all continuous sections of $GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$ over N corresponds bijectively to that of all bundle maps of $TN \oplus \theta_N$ to $TP \oplus \theta_P$ preserving orientations. For a folding map $f : N \to P$, the section j^2f determines the homotopy class of a section of $GL^+_{n+1}(TN \oplus \theta_N, TP \oplus \theta_P)$ by Proposition 3.1. It gives a bundle map $\bar{f} : TN \oplus \theta_N \to TP \oplus \theta_P$ determined up to homotopy. □

The notation $\bar{f}$ always refers to the bundle map in the proof above.

Let ξ and η (resp. ξ' and η') be vector bundles over a topological space X (resp. Y). For bundle maps $b : \xi \to \xi'$ and $b' : \eta \to \eta'$ covering the same map of X into Y, define the sum $b \oplus b' : \xi \oplus \eta \to \xi' \oplus \eta'$ by $(b \oplus b')(\boldsymbol{v} \oplus \boldsymbol{w}) = b(\boldsymbol{v}) \oplus b'(\boldsymbol{w})$. In the rest of the paper, k denotes an integer greater than $n+2$. Let ξ_X and η_X be vector bundles of dimension k over the same manifold X of dimension n. Let θ_X^ℓ denote the trivial bundle $X \times \boldsymbol{R}^\ell$ $(\ell \geq 1)$. Then the set of homotopy classes of bundle maps of ξ_X to η_X corresponds bijectively to that of bundle maps of $\xi_X \oplus \theta_X^\ell$ to $\eta_X \oplus \theta_X^\ell$ by the correspondence defined by mapping h to $h \oplus id_{(\theta_X^\ell)}$. In fact, this can be proved by using the fact that $\pi_i(O(k+\ell), O(k)) \cong \{0\}$ for $i < k$. When two bundle maps h_0 and h_1 are homotopic, we write $h_0 \simeq h_1$. All vector bundles of dimension not less than k over X will be called stable. For the tangent bundle TX of X, we will denote $TX \oplus \theta_X^\ell$ $(\ell > 2)$ by the symbol τ_X without specifying the number ℓ, which is called the stable tangent bundle of X. When N and P are embedded in $\boldsymbol{R}^{n+k}$, v_N and v_P refer to the stable normal bundles of N and P of dimension k respectively. Then we have the following by virtue of Proposition 3.3 below.

Proposition 3.2. *Let N and P be oriented manifolds of dimension n with fixed trivializations $t_N : \tau_N \oplus v_N \to \theta_N^{2k}$ and $t_P : \tau_P \oplus v_P \to \theta_P^{2k}$. Then a folding map $f : N \to P$ determines the homotopy class of a bundle map $v(\bar{f}) : v_N \to v_P$ over f such that $\bar{f} \oplus v(\bar{f})$ satisfies the property described in Proposition* 3.3.

Let ξ_N and η_N (resp. ξ_P and η_P) be vector bundles over N (resp. P) of dimension $k > n+2$. Suppose that we have the trivializations $t_N : \xi_N \oplus \eta_N \to \theta_N^{2k}$ and $t_P : \xi_P \oplus \eta_P \to \theta_P^{2k}$. A bundle map $b : \theta_N^{2k} \to \theta_P^{2k}$ over a map $f : N \to P$ is canonically identified with the pair (f, α), where the continuous map $\alpha : N \to GL(2k)$ satisfies $b(x, \boldsymbol{v}) = (f(x), \alpha(x)(\boldsymbol{v}))$ for any $\boldsymbol{v}$ in $\boldsymbol{R}^{2k}$ and x in N. Hence b is often denoted as α by neglecting f when there is no confusion. Let $\alpha^{-1} : N \to GL(2k)$ be the map defined by $\alpha^{-1}(x) = \alpha(x)^{-1}$. Let $i_m : N \to GL(m)$ be the map such that $i_m(x)$ is always the unit matrix I_m. Let $\beta : N \to GL(2k)$ be another map. Then the following is easy to prove (see, for example, [**At2**, p. 76]):

(i) $\alpha \simeq \beta$ if and only if $\alpha \oplus i_m \simeq \beta \oplus i_m$ for $m \geq 1$,

(ii) $\alpha \oplus \alpha^{-1}$ is homotopic to i_{4k},

(iii) $\alpha \oplus \beta$ is homotopic to $\beta \oplus \alpha$.

Then we have the following proposition.

Proposition 3.3. *Under the notations above, a homotopy class of a bundle map $\tilde{f} : \xi_N \to \xi_P$ over f determines the unique homotopy class of a bundle map $\tilde{g} : \eta_N \to \eta_P$ over f such that $t_P \circ (\tilde{f} \oplus \tilde{g}) \circ (t_N)^{-1}$ is homotopic to the bundle map* (f, i_{2k}).

Proof. By considering the induced bundles $f^*\xi_P$ and $f^*\eta_P$, it is enough to consider the case where $N = P$ and f is the identity of N. But we use the given notations except for f. Since ξ_N and ξ_P are equivalent, there is an isomorphism of η_N into η_P, say b. So we have the isomorphism $t_P \circ (\tilde{f} \oplus b) \circ t_N^{-1}$, which we identify with $\alpha : N \to GL(2k)$ as above. Then we consider the isomorphism

$$b \oplus (\alpha^{-1}) : \eta_N \oplus \theta_N^{2k} \to \eta_P \oplus \theta_P^{2k}.$$

Let us denote by i_{2k} the identity of θ_N^{2k} or θ_P^{2k}. Then by the dimensional reason, there exists a unique homotopy class of a bundle map $\tilde{g} : \eta_N \to \eta_P$ such that $b \oplus (\alpha^{-1}) \simeq \tilde{g} \oplus i_{2k}$.

We shall show that $\tilde{g}$ is the required bundle map. We have

$$\begin{aligned}
(t_P \circ (\tilde{f} \oplus \tilde{g}) \circ t_N^{-1}) \oplus i_{2k} &= (t_P \oplus i_{2k}) \circ (\tilde{f} \oplus \tilde{g} \oplus i_{2k}) \circ (t_N^{-1} \oplus i_{2k}) \\
&\simeq (t_P \oplus i_{2k}) \circ (\tilde{f} \oplus b \oplus (\alpha^{-1})) \circ (t_N^{-1} \oplus i_{2k}) \\
&= (t_P \circ (\tilde{f} \oplus b) \circ t_N^{-1}) \oplus \alpha^{-1} \\
&= \alpha \oplus \alpha^{-1} \\
&\simeq i_{4k}.
\end{aligned}$$

Therefore, it follows from (i) that $\tilde{g}$ satisfies the first required property.

Next we show the uniqueness of $\tilde{g}$. Let $t_N' : \eta_N \oplus \xi_N \to \theta_N^{2k}$ (resp. $t_P' : \eta_P \oplus \xi_P \to \theta_P^{2k}$) denote the trivialization defined by $t_N'(\boldsymbol{v}_1 \oplus \boldsymbol{v}_0) = t_N(\boldsymbol{v}_0 \oplus \boldsymbol{v}_1)$ (resp. $t_P'(\boldsymbol{w}_1 \oplus \boldsymbol{w}_0) = t_P(\boldsymbol{w}_0 \oplus \boldsymbol{w}_1)$), where $\boldsymbol{v}_0 \in \xi_N$ and $\boldsymbol{v}_1 \in \eta_N$ (resp. $\boldsymbol{w}_0 \in \xi_P$ and $\boldsymbol{w}_1 \in \eta_P$). Then it follows that $t_P' \circ (\tilde{g} \oplus \tilde{f}) \circ t_N'^{-1} = t_P \circ (\tilde{f} \oplus \tilde{g}) \circ t_N^{-1}$. In fact, if $t_N(\boldsymbol{v}_0 \oplus \boldsymbol{v}_1) = t_N'(\boldsymbol{v}_1 \oplus \boldsymbol{v}_0) = \boldsymbol{x} \oplus \boldsymbol{y}$ with $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^k$, then we have

$$\begin{aligned}
t_P' \circ (\tilde{g} \oplus \tilde{f}) \circ t_N'^{-1}(\boldsymbol{x} \oplus \boldsymbol{y}) &= t_P' \circ (\tilde{g} \oplus \tilde{f})(\boldsymbol{v}_1 \oplus \boldsymbol{v}_0) \\
&= t_P' \circ (\tilde{g}(\boldsymbol{v}_1) \oplus \tilde{f}(\boldsymbol{v}_0)) \\
&= t_P(\tilde{f}(\boldsymbol{v}_0) \oplus \tilde{g}(\boldsymbol{v}_1)) \\
&= t_P \circ (\tilde{f} \oplus \tilde{g})(\boldsymbol{v}_0 \oplus \boldsymbol{v}_1) \\
&= t_P \circ (\tilde{f} \oplus \tilde{g}) \circ t_N^{-1}(\boldsymbol{x} \oplus \boldsymbol{y}).
\end{aligned}$$

Suppose that there exists another bundle map $\tilde{g}' : \eta_N \to \eta_P$ such that $t_P \circ (\tilde{f} \oplus \tilde{g}') \circ t_N^{-1} \simeq i_{2k}$. Since $\tilde{f} \oplus \tilde{g} \simeq \tilde{f} \oplus \tilde{g}'$, we have $\tilde{g} \oplus \tilde{f} \oplus \tilde{g} \simeq \tilde{g} \oplus \tilde{f} \oplus \tilde{g}'$.

On the other hand, we have

$$\begin{aligned}(t'_P \oplus i_k) \circ (\tilde{g} \oplus \tilde{f} \oplus \tilde{g}) \circ (t'^{-1}_N \oplus i_k) &= (t'_P \circ (\tilde{g} \oplus \tilde{f}) \circ t'^{-1}_N) \oplus \tilde{g} \\ &= (t_P \circ (\tilde{f} \oplus \tilde{g}) \circ t^{-1}_N) \oplus \tilde{g} \\ &\simeq i_{2k} \oplus \tilde{g} \quad \text{and} \\ (t'_P \oplus i_k) \circ (\tilde{g} \oplus \tilde{f} \oplus \tilde{g}') \circ (t'^{-1}_N \oplus i_k) &= (t'_P \circ (\tilde{g} \oplus \tilde{f}) \circ t'^{-1}_N) \oplus \tilde{g}' \\ &= (t_P \circ (\tilde{f} \oplus \tilde{g}) \circ t^{-1}_N) \oplus \tilde{g}' \\ &\simeq i_{2k} \oplus \tilde{g}'.\end{aligned}$$

Consequently, we have $i_{2k} \oplus \tilde{g} \simeq i_{2k} \oplus \tilde{g}'$. Therefore we obtain $\tilde{g} \simeq \tilde{g}'$. $\square$

EXAMPLE 3.4. If $TN \oplus \theta_N$ and $TP \oplus \theta_P$ are trivial bundles with fixed trivializations, then the bundle map $\bar{f} : TN \oplus \theta_N \to TP \oplus \theta_P$ in the proof of Corollary 2 induces a map $M(\bar{f}) : N \to SO(n+1)$. However, in order for the map $M(\bar{f})$ to inherit the intuitive geometric properties of f, we must select the trivializations very naturally, even though they do not exist on the whole spaces of N and P. Otherwise $M(\bar{f})$ can often be homotopic to a constant map. This actually occurs in the case where $P = \boldsymbol{R}^n$, N is the unit sphere of $\boldsymbol{R}^{n+1}$ and $TS^n \oplus \theta_{S^n} \cong T\boldsymbol{R}^{n+1}|_{S^n}$ with canonical trivialization and f is the canonical projection of S^n into $\boldsymbol{R}^n \times 0 = \boldsymbol{R}^n$.

We shall give two examples of $M(\bar{f})$. Let $R(x)$ denote the matrix $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$.

(1) Let S^1 be parametrized by $x \mapsto e^{ix}$ $(0 \le x \le 2\pi)$ inducing the trivialization of $T(S^1)$. Then the folding map $f : S^1 \to \boldsymbol{R}^1$ defined by $f(x) = \cos x$ induces the map $M(\bar{f}) = R(x + \pi/2)$.

(2) Let $S^1 \times S^1$ be parametrized by $(x, y) \mapsto (e^{ix}, e^{iy})$ $(0 \le x, y \le 2\pi)$ inducing the trivialization of $T(S^1 \times S^1)$. Consider the folding map $f : S^1 \times S^1 \to \boldsymbol{R}^2$ defined by $(f_1(x,y), f_2(x,y)) = ((3 + \cos y)\cos x, (3 + \cos y)\sin x)$. Then $M(\bar{f})$ is homotopic to the map $\Pi : S^1 \times S^1 \to SO(3)$ defined by $\Pi(x,y) = ((1) \dotplus R(y))(R(x) \dotplus (1))$. We give a sketch of the proof.

The Jacobian matrix $J(x,y)$ of f is equal to

$$\begin{aligned}J(x,y) &= \begin{pmatrix} -(3+\cos y)\sin x & -\sin y \cos x \\ (3+\cos y)\cos x & -\sin y \sin x \end{pmatrix} \\ &= R(x + \pi/2)\Delta(3 + \cos y, \sin y) \\ &= R(x + \pi/2)\Delta(3 + \cos y, \sin y)\,{}^tR(x + \pi/2)R(x + \pi/2).\end{aligned}$$

Here recall the definition of $i_n(M)$ in §2 and Remark 2.1. Set $\boldsymbol{s}_x = {}^t(-\cos x, -\sin x)$, $a = (1-\sin^2 y/2)^{1/2}$ and $b = \sin y$ for $\Delta(\boldsymbol{d}_{ab})$ and define $J' : S^1 \times S^1 \to \mathrm{Hom}(\boldsymbol{R}^2, \boldsymbol{R}^2)$ by $J'(x,y) = R(x+\pi/2)\Delta(\boldsymbol{d}_{ab}){}^tR(x+\pi/2)R(x+\pi/2)$. The Hessians $H(f_1)$ and $H(f_2)$ are equal to

$$H(f_1)(x,y) = \begin{pmatrix} -(3+\cos y)\cos x & \sin y \sin x \\ \sin y \sin x & -\cos y \cos x \end{pmatrix},$$

$$H(f_2)(x,y) = \begin{pmatrix} -(3+\cos y)\sin x & -\sin y \cos x \\ -\sin y \cos x & -\cos y \sin x \end{pmatrix}$$

respectively. We define $H : S^1 \times S^1 \to \mathrm{Hom}(S^2\boldsymbol{R}^2, \boldsymbol{R}^2)$ by $H(x,y) = (H_1(f)(x,y), H_2(f)(x,y))$. We obtain the two maps (J, H) and (J', H) of $S^1 \times S^1$ to $\Omega^{10}(2,2)(\subset \mathrm{Hom}(\boldsymbol{R}^2, \boldsymbol{R}^2) \oplus \mathrm{Hom}(S^2\boldsymbol{R}^2, \boldsymbol{R}^2))$, which are homotopic to each other.

We note that $(J'(x,y), H(x,y))$ is a fold jet when $\sin y = 0$ and $\cos y = \pm 1$. At these folding singularities, $\mathrm{Ker}(J'(x,y))$ is generated by $\boldsymbol{e}_2$ and $\mathrm{Cok}(J'(x,y))$ is generated by $\boldsymbol{s}_x$. Furthermore, $H(f_1)(x,y)(\boldsymbol{e}_2, \boldsymbol{e}_2) = -\cos y \cos x$ and $H(f_2)(x,y)(\boldsymbol{e}_2, \boldsymbol{e}_2) = -\cos y \sin x$. Hence the quadratic form q of $S^2\boldsymbol{K}$ to $\boldsymbol{Q}$ induced from $H(x,y)$ has the property $q(\boldsymbol{e}_2, \boldsymbol{e}_2) = \cos y\, \boldsymbol{s}_x$. We shall construct $M(\bar{f})$ from (J', H) by using these observations and the defintion of i_n with the notation $i_n(M) = (\alpha(M), \beta(M))$ given in §2. First we take $J'(x,y)$ as $\alpha(M)$. Then we have $R(x+\pi/2)$ as $U(M)$ for $\sin y \geq 0$ and $I_- G(\boldsymbol{s}_x)R(x+\pi/2)$ as $U(M)'$ for $\sin y \leq 0$. Then by Remark 2.1, we need to consider the vector $\boldsymbol{x}(x,y) = {}^t(-\cos x \cos y, -\sin x \cos y, \sin y)$ and obtain the matrices $M_+(x,y) = r(\boldsymbol{x}(x,y))(R(x+\pi/2) \dotplus (1))$ for $\sin y \geq 0$ and $M_-(x,y) = \bar{r}(\boldsymbol{x}(x,y))(G(\boldsymbol{s}_x)R(x+\pi/2) \dotplus (-1))$ for $\sin y \leq 0$, where by the formulas in §2, $r(\boldsymbol{x}(x,y))$ or $\bar{r}(\boldsymbol{x}(x,y))$ is equal to the matrix

$$\begin{pmatrix} 1-\cos^2 x(1-\delta \sin y) & -\sin x \cos x(1-\delta \sin y) & -\delta \cos x \cos y \\ -\sin x \cos x(1-\delta \sin y) & 1-\sin^2 x(1-\delta \sin y) & -\delta \sin x \cos y \\ \delta \cos x \cos y & \delta \sin x \cos y & \delta \sin y \end{pmatrix}.$$

Here δ is equal to 1 for $M_+(x,y)$ and equal to -1 for $M_-(x,y)$ respectively. Then it is elementary to prove that we can take the map $\tilde{M} : S^1 \times S^1 \to SO(3)$ defined by $\tilde{M}(x,y) = M_+(x,y)$ for $\sin y \geq 0$ and $\tilde{M}(x,y) = M_-(x,y)$ for $\sin y \leq 0$ for $M(\bar{f})$. We note that $\tilde{M}(3\pi/2, y) = (1) \dotplus R(y - \pi/2)$ and $\tilde{M}(x, \pi/2) = R(x+\pi/2) \dotplus (1)$. Define the new map $\Pi' : S^1 \times S^1 \to SO(3)$ by $\Pi'(x,y) = \tilde{M}(3\pi/2, y)\tilde{M}(x, \pi/2)$. Since $\Pi'(3\pi/2, y) = \tilde{M}(3\pi/2, y)$ and $\Pi'(x, \pi/2) = \tilde{M}(x, \pi/2)$, $\tilde{M}$ coincides with Π' on $(3\pi/2) \times S^1 \cup S^1 \times (\pi/2)$. Since $\pi_2(SO(3)) \cong \{0\}$, it follows that $\tilde{M}$ is homotopic to Π'. Thus the assertion follows from the fact that Π is homotopic to Π'.

§4. Map of $\Omega_{\mathrm{fold}}(P)$ to $[P, SG]$.

First we shall recall the results about spherical fibre spaces (see, for example, [**B1**], [**B2**], [**W1**] and [**W2**]). In the rest of the paper sphere bundles associated with oriented vector bundles ξ of dimension $k > n+2$ over manifolds X of dimension n will be denoted by $S(\xi)$. A fibre map $h : S(\xi) \to S(\xi)$ is called an *automorphism* if h is a homotopy equivalence, that is, if it gives a homotopy equivalence on each fibre. In this paper an automorphism of an oriented spherical fibre space is always assumed to be an orientation preserving one.

Let $\mathrm{End}(\xi)$ denote the group of the homotopy classes of automorphisms of $S(\xi)$. Note that h is extended to a self-fibre map of ξ (denoted by the same letter h) by fibrewise cone construction (this is not necessarily fibrewise linear). Let $h' : S(\eta) \to S(\eta)$ be an automorphism of another vector bundle η over X. Then we can define the *Whitney sum* $h + h' : \xi \oplus \eta \to \xi \oplus \eta$ of the fibre maps h and h' similarly as in the case of bundle maps between vector bundles and it yields an automorphism denoted by $h + h' : S(\xi \oplus \eta) \to S(\xi \oplus \eta)$. There is an isomorphism of $\mathrm{End}(\xi)$ to $\mathrm{End}(\xi \oplus \theta^{\ell})$ ($\theta^{\ell} = \theta_X^{\ell}, \ell \geq 1$), which maps h to $h + id_{\theta^{\ell}}$. This is proved by using the following fact. Let G_k and G be the spaces given in Introduction. Then it is known that $\pi_i(G_{k+1}, G_k) = \{0\}$ for $i \leq k-2$ (see [**B2**, I.4.10 Proposition]). Set $\mathscr{E}(\xi) = \lim_{\ell \to \infty} \mathrm{End}(\xi \oplus \theta^{\ell})$. Then it follows that $\mathscr{E}(\xi) \cong \mathscr{E}(\xi \oplus \theta^{\ell})$. Suppose that $\xi \oplus \eta$ is trivial and has its trivialization $t : \xi \oplus \eta \to \theta^{2k}$. Let a homomorphism $E(t) : \mathrm{End}(\xi) \to \mathrm{End}(\theta^{2k})$ be defined by $E(t)(h) = [t \circ (h + id_{\eta}) \circ t^{-1}]$, where the bracket is often abbreviated. Then it induces an isomorphism (see [**B2**, p. 22] and [**W1**, Proof of Theorem 3.5])

$$\mathscr{E}(t) : \mathscr{E}(\xi) \to \mathscr{E}(\theta^{2k}).$$

Lemma 4.1. *The map $\mathscr{E}(t)$ does not depend on the choice of a trivialization t.*

Proof. If $t' : \xi \oplus \eta \to \theta^{2k}$ is another trivialization, then $t' \circ t^{-1} : \theta^{2k} \to \theta^{2k}$ gives a continuous map of X into $GL(2k)$. Then the following is proved by using the properties (i), (ii) and (iii) described just before Proposition 3.3.

$$\begin{aligned}
E(t)(h) + id_{\theta^{2k}} &= (t \circ (t')^{-1} \circ E(t')(h) \circ t' \circ t^{-1}) + id_{\theta^{2k}} \\
&= (t \circ (t')^{-1} + id_{\theta^{2k}}) \circ (E(t')(h) + id_{\theta^{2k}}) \circ (t' \circ t^{-1} + id_{\theta^{2k}}) \\
&\simeq (id_{\theta^{2k}} + t \circ (t')^{-1}) \circ (E(t')(h) + id_{\theta^{2k}}) \circ (t' \circ t^{-1} + id_{\theta^{2k}}) \\
&= (E(t')(h) + t \circ (t')^{-1}) \circ (t' \circ t^{-1} + id_{\theta^{2k}}) \\
&\simeq (E(t')(h) + t \circ (t')^{-1}) \circ (id_{\theta^{2k}} + t' \circ t^{-1}) \\
&= E(t')(h) + id_{\theta^{2k}}.
\end{aligned}$$

Hence we have $\mathscr{E}(t) = \mathscr{E}(t')$. $\square$

In the following $\mathscr{E}(t)$ will be denoted simply by $\mathscr{E}$. Since SG is a homotopy commutative H-space (Remark 4.6), the set of homotopy classes $[X, SG]$ has a structure of an abelian group. We obtain a canonical isomorphism of $\mathrm{End}(\theta^k)$ onto $[X, SG_k]$ by mapping $h : S(\theta^k) \to S(\theta^k)$ to the continuous map $\phi : X \to SG_k$ defined by $\phi(x) = h \,|\, x \times S^{k-1}$. It also induces an isomorphism of $\mathscr{E}(\theta^k)$ onto $[X, SG]$. By composing it with $\mathscr{E}$, we obtain the isomorphism

$$\boldsymbol{c}_{SG} : \mathscr{E}(\xi) \to [X, SG].$$

Now we take and fix an embedding e of P into $\boldsymbol{R}^{n+k}$ for a while. Consider $\nu_P' = T(\boldsymbol{R}^{n+k})|_{e(P)}/T(e(P))$ and $\nu_P = e^*(\nu_P')$ with bundle map $\bar{e} : \nu_P \to \nu_P'$. Then the usual metric of $\boldsymbol{R}^{n+k}$ induces a splitting of the sequence $0 \to T(P) \to \theta_P^{n+k} \to \nu_P \to 0$ by orthogonality, which yields a trivialization $t_P : \tau_P \oplus \nu_P \to \theta^{2k}$ with dimension of τ_P being equal to k. Take an embedding of N into $\boldsymbol{R}^{n+k}$, which yields a trivialization $t_N : \tau_N \oplus \nu_N \to \theta^{2k}$ similarly. Given a folding map $f : N \to P$, there is a bundle map $\bar{f} : \tau_N \to \tau_P$ determined up to homotopy by Corollary 2. By t_P and t_N, we obtain a bundle map $\nu(\bar{f}) : \nu_N \to \nu_P$ determined up to homotopy by Proposition 3.2. Let $T(\nu(\bar{f})) : T(\nu_N) \to T(\nu_P)$ be the Thom map associated with $\nu(\bar{f})$. Let $\phi : S^{n+k} \to T(\nu_P')$ be the Pontrjagin-Thom construction for the fixed embedding e of P. Then we have a homotopy class $\alpha_P = [T(\bar{e}^{-1}) \circ \phi]$ in $\pi_{n+k}(T(\nu_P))$, where $[*]$ refers to the homotopy class. In the rest of the paper we also call α_P the homotopy class obtained by the Pontrjagin-Thom construction for the embedding e of P into $\boldsymbol{R}^{n+k}$. Similarly we obtain the class $\alpha_N \in \pi_{n+k}(T(\nu_N))$.

Consider two homotopy classes α_P and $T(\nu(\bar{f}))_*(\alpha_N)$ of $\pi_{n+k}(T(\nu_P))$. Then it follows from [**B2**, I.4.19 Theorem] that they determine an automorphism $h(f) : S(\nu_P) \to S(\nu_P)$ up to homotopy such that $T(h(f))_*(\alpha_P) = T(\nu(\bar{f}))_*(\alpha_N)$. Thus we obtain an element $\boldsymbol{c}_{SG}(h(f))$ of $[P, SG]$ by applying $\boldsymbol{c}_{SG}$ for $\xi = \nu_P$.

Theorem 4.2. *There exists a well defined map*

$$\omega : \Omega_{\mathrm{fold}}(P) \to [P, SG]$$

such that a fold-cobordism class of f is mapped to $\boldsymbol{c}_{SG}(h(f))$.

For the proof we need the following two lemmas.

Lemma 4.3. *Fix the embedding e of P into $\boldsymbol{R}^{n+k}$. Then the element $\boldsymbol{c}_{SG}(h(f))$ defined for a folding map f of degree 1 is a fold-cobordism invariant.*

Proof. Set $I = [0, 1]$. Let $f_i : N_i \to P$ $(i = 0, 1)$ be folding maps with fold-cobordism $F : W \to P \times I$ as described in Introduction. Take an embedding E of W into $\boldsymbol{R}^{n+k} \times I$ by the Whitney embedding theorem (see [**G-G**, II, §5]) such that for a sufficiently small positive real number ε,

(i) $E(N_0 \times u) \subset \boldsymbol{R}^{n+k} \times u$ and $E(N_1 \times u) \subset \boldsymbol{R}^{n+k} \times u$

for $u \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$,

(ii) $E|N_0 \times [0, \varepsilon) = (E|N_0 \times 0) \times id_{[0,\varepsilon)}$ and

(iii) $E|N_1 \times (1 - \varepsilon, 1] = (E|N_1 \times 1) \times id_{(1-\varepsilon, 1]}$.

We note that $\tau_W|N_0 = \tau_{N_0} \oplus \theta^1_{N_0}$, $\tau_W|N_1 = \tau_{N_1} \oplus \theta^1_{N_1}$. Then we obtain a trivialization $t_W : \tau_W \oplus \nu_W \to \theta^{2k+1}_W$ such that $t_W|_{N_i} \simeq t_{N_i} \oplus id$ with id denoting the identity of $\theta^1_{N_i}$ $(i = 0, 1)$. Since F is a folding map, we have the bundle map $\bar{F} : \tau_W \to \tau_{P\times I}$ by Corollary 2. Using t_W, we obtain a bundle map $\nu(\bar{F}) : \nu_W \to \nu_{P\times I} = \nu_P \times I$ from Propositon 3.2 such that $\nu(\bar{F})|_{N_i} = \nu(\bar{f}_i) \times i$. By the Pontrjagin-Thom construction for the embedding E and $T(\nu(\bar{F})) : T(\nu_W) \to T(\nu_P \times I)$, we obtain a homotopy to prove $T(\nu(\bar{f}_0))_*(\alpha_{N_0}) = T(\nu(\bar{f}_1))_*(\alpha_{N_1})$. This yields $\mathscr{E}(h(f_0)) = \mathscr{E}(h(f_1))$. □

By this lemma we can define the map ω if the embedding e is fixed. Theorem 4.2 follows from the following.

Lemma 4.4. *The map ω does not depend on the choice of an embedding e.*

Proof. Let e_0 and e_1 be two embeddings of P into $\boldsymbol{R}^{n+k}$. Then there exists an embedding $E : P \times I \to \boldsymbol{R}^{n+k} \times I$ such that $E(P \times u) \subset \boldsymbol{R}^{n+k} \times u$ for all $u \in I$ and that $E(x, i) = (e_i(x), i)$ $(i = 0, 1)$. In fact, if $e_0(P) \cap e_1(P)$ is not empty, then there is a smooth isotopy e_u $(0 \le u \le 1/2)$ of parallel translations such that $e_{1/2}(P) \cap e_1(P) = \phi$. Next we can take an embedding $E' : P \times [1/2, 1] \to \boldsymbol{R}^{n+k}$ such that $E'|P \times u = e_u$ for $u = 1/2$ and 1 by the Whitney embedding theorem. Then we can construct a level preserving embedding of $P \times I$ into $\boldsymbol{R}^{n+k} \times I$ by

$$E(x, u) = \begin{cases} (e_u(x), u), & 0 \le u \le 1/2 \\ (E'(x, u), u), & 1/2 \le u \le 1. \end{cases}$$

Set $e_u(x) = E(x, u)$ and let ν'_u be the normal bundle of the embedded manifold $e_u(P)$. Set $\nu_u = e^*_u(\nu'_u)$ with bundle map $\bar{e}_u : \nu_u \to \nu'_u$ over e_u. We have a trivialization $t_u : \tau_P \oplus \nu_u \to \theta^{2k}_P$. Let ν be the normal bundle over $P \times I$ to the embedding E with the property $\nu|_{P\times u} = \nu_u$. Then there is a bundle map $B : \nu_0 \times I \to \nu$ covering $id_{P\times I}$ with $B|_{\nu_0 \times 0} = id_{\nu_0}$. Set $b_u = B|_{\nu_0 \times u}$. Then $t_u \circ (id_{\tau_P} \oplus b_u) \simeq t_0$. The composition map

$$S^{n+k} \xrightarrow{\phi_u} T(\nu'_u) \xrightarrow{T(\bar{e}_u^{-1})} T(\nu_u) \xrightarrow{T(b_u^{-1})} T(\nu_0)$$

gives a homotopy between $T(\bar{e}_0^{-1}) \circ \phi_0$ and $T(b_1^{-1}) \circ T(\bar{e}_1^{-1}) \circ \phi_1$, where ϕ_u denotes the Pontrjagin-Thom construction for e_u. Set $\alpha_u = T(\bar{e}_u^{-1})_*(\alpha_{e_u(P)}) =$

$[T(\bar{e}_u^{-1}) \circ \phi_u] \in \pi_{n+k}(T(\nu_u))$. Then we have

$$\begin{aligned}
\alpha_0 &= T(\bar{e}_0^{-1})_*(\alpha_{e_0(P)}) \\
&= T(b_1^{-1})_* \circ T(\bar{e}_1^{-1})_*(\alpha_{e_1(P)}) \\
&= T(b_1^{-1})_*(\alpha_1).
\end{aligned}$$

Recall the definition of $\mathscr{E}(h(f))$. For the embedding e_u, the automorphism $h(f)_u : S(\nu_u) \to S(\nu_u)$ is defined to satisfy $T(h(f)_u)_*(\alpha_u) = T(\nu(\bar{f})_u)_*(\alpha_N)$, where $\nu(\bar{f})_u : \nu_N \to \nu_u$ is the bundle map associated with $\bar{f} : \tau_N \to \tau_P$ and t_u. Since $t_u \circ (id_{\tau_P} \oplus b_u) \simeq t_0$, we have $t_0 \circ (\bar{f} \oplus \nu(\bar{f})_0) \circ t_N^{-1} \simeq t_u \circ (\bar{f} \oplus b_u \circ \nu(\bar{f})_0) \circ t_N^{-1}$, which is homotopic to (f, id_{2k}) by Proposition 3.2. Hence by the definition of $\nu(\bar{f})_u$ we have $b_u \circ \nu(\bar{f})_0 \simeq \nu(\bar{f})_u$. In particular, $b_1 \circ \nu(\bar{f})_0 \simeq \nu(\bar{f})_1$. Since $T(b_1)_*(\alpha_0) = \alpha_1$, we have

$$\begin{aligned}
T(h(f)_1)_*(T(b_1)_*(\alpha_0)) &= T(h(f)_1)_*(\alpha_1) \\
&= T(\nu(\bar{f})_1)_*(\alpha_N) \\
&= T(b_1)_*(T(\nu(\bar{f})_0)_*(\alpha_N)) \\
&= T(b_1)_*(T(h(f)_0)_*(\alpha_0)).
\end{aligned}$$

Thus by [**B2**, I.4.19 Theorem] the following diagram is commutative up to homotopy.

$$\begin{array}{ccccc}
S(\nu_0) & \xrightarrow{h(f)_0} & S(\nu_0) & \xleftarrow{\nu(\bar{f})_0} & S(\nu_N) \\
\downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle b_1} & & \| \\
S(\nu_1) & \xrightarrow{h(f)_1} & S(\nu_1) & \xleftarrow{\nu(\bar{f})_1} & S(\nu_N)
\end{array}$$

Now $\mathscr{E}(h(f)_u)$ is defined to be the automorphism $t_u \circ (id_{\tau_P} + h(f)_u) \circ t_u^{-1}$. Hence

$$\begin{aligned}
&t_0 \circ (id_{\tau_P} + h(f)_0) \circ t_0^{-1} \\
&\quad \simeq t_1 \circ (id_{\tau_P} + b_1) \circ (id_{\tau_P} + (b_1^{-1} \circ h(f)_1 \circ b_1)) \circ (id_{\tau_P} + b_1)^{-1} \circ t_1^{-1} \\
&\quad = t_1 \circ (id_{\tau_P} + h(f)_1) \circ t_1^{-1}.
\end{aligned}$$

This shows that $\boldsymbol{c}_{SG}(h(f)_0) = \boldsymbol{c}_{SG}(h(f)_1)$. Therefore $\boldsymbol{c}_{SG}(h(f))$ does not depend on the choice of an embedding of P into $\boldsymbol{R}^{n+k}$. □

The inclusion $i : SO \to SG$ induces the map $i_* : [P, SO] \to [P, SG]$ (see [**Ad**], [**Q**] and [**T**] concerning the results about the image of i_*).

Proposition 4.5. *Suppose that a diffeomorphism $d : P \to N$ of degree 1 is given. If $f : N \to P$ is a folding map homotopic to d^{-1}, then $\omega(f)$ lies in the image $i_*([P, SO])$.*

Proof. Consider the diffeomorphism $H : P \times (1/3, 2/3) \to N \times (1/3, 2/3)$ defined by $H(x, t) = (d(x), t)$ and the manifold W constructed by pasting $P \times [0, 2/3)$ and $N \times (1/3, 1]$ by H. It is easy to see that $f \circ d$ and f are fold-cobordant and so $\omega(f \circ d) = \omega(f)$.

For the map $f \circ d$, we obtain that by definition, $h(f \circ d)$ is equal to the spherical fibre map induced from $v(\overline{f \circ d}) : v_P \to v_P$ up to homotopy. Since $v(\overline{f \circ d})$ is a bundle map, $\boldsymbol{c}_{SG}(h(f \circ d))$ lies in $i_*([P, SO])$. □

Remark 4.6. (i) Since SG is a homotopy commutative H-space, which is weakly homotopy equivalent to the identity component of the loop space $\Omega^\infty S^\infty$ (see, [**Sta**, Definition 4.1 and Page 65] and [**M-M**, Corollary 3.8]), $[P, SG]$ has the structure of an abelian group. Let F_k^m denote the space of all self-maps of S^k preserving the base point of degree m. It is known that there is an isomorphism $\mu : [S^n, SG] \to \lim_{k\to\infty} \pi_{n+k}(S^k)$, which is induced from the following (see [**At1**, Lemma 1.3 and (i), (ii) on page 295]).

$$[S^n, SG_k] \cong \pi_n(SG_k) \cong \pi_n(F_k^1) \cong \pi_n(F_k^0) \cong \pi_{n+k-1}(S^{k-1}) \quad (k > n+2)$$

(ii) Many authors have contributed to the study of the very difficult structure of the algebras $H_*(SG; \boldsymbol{Z}/p\boldsymbol{Z})$ and $H^*(SG; \boldsymbol{Z}/p\boldsymbol{Z})$, where p is a prime number (consult [**M-M**, Chapter 6] and [**M**, Theorem 6.1 and Conjecture 6.2]).

Corollary 4.7. *For an element $[f]$ of $\Omega_{\mathrm{fold}}(P)$, we have the homomorphism $\omega(f)^* : H^*(SG; \boldsymbol{Z}/p\boldsymbol{Z}) \to H^*(P; \boldsymbol{Z}/p\boldsymbol{Z})$. Then for any element a of $H^*(SG; \boldsymbol{Z}/p\boldsymbol{Z})$, $\omega(f)^*(a)$ of $H^*(P; \boldsymbol{Z}/p\boldsymbol{Z})$ is a fold-cobordism invariant.*

It may be reasonable to call all the elements of the form $\omega(f)^*(a)$ the characteristic classes of $[f]$ associated with ω. It is natural to ask how $\omega(f)^*(a)$ is related to the topological structure of $S(f)$ in N and $f(S(f))$ in P, where $S(f)$ is the set of folding singularities of f.

Example 4.8. (1) If $P = S^1$, then it is not difficult to prove that $\omega : \Omega_{\mathrm{fold}}(S^1) \to [S^1, SG] \cong \boldsymbol{Z}/2\boldsymbol{Z}$ is bijective (note that $i_* : [S^1, SO] \to [S^1, SG]$ is bijective). Cosider the folding maps f_A and f_B described in figures (A) and (B) respectively, which are constructed by projecting the outer circles to the inner circles along the half-lines with end-point O. By arguments similar to those given in Example 3.4, we see that $\omega(f_A)$ is a nontrivial element and $\omega(f_B)$ is a trivial one.

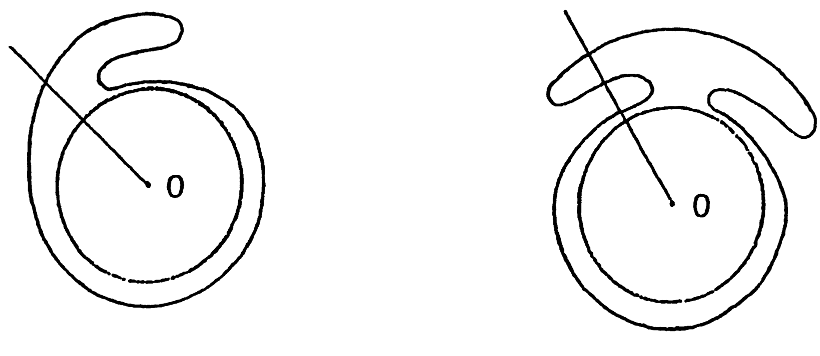

Figure 1

(2) We shall give a folding map $g : S^1 \times S^1 \to S^2$ of degree 1 such that $\omega(g)$ represents a nontrivial element of $[S^2, SG] \cong \boldsymbol{Z}/2\boldsymbol{Z}$ (note that $\pi_2(SO) = \{0\}$). Let $f_T : S^1 \times S^1 \to S^2$ the folding map defined by $f_T(x, y) = f(2x, 2y)$, where f is the folding map given in Example 3.4 (2), by identifying $S^2 \backslash \{\text{a point}\}$ with $\boldsymbol{R}^2$. Delete a very small open disk $\operatorname{Int} D_\varepsilon^2$ from $S^1 \times S^1$ which is mapped diffeomorphically into S^2 by f_T and paste $S^2 \backslash \operatorname{Int}(f_T(D_\varepsilon))$ instead. Then this construction defines a folding map g through $f_T|(S^1 \times S^1 \backslash \operatorname{Int} D_\varepsilon^2)$ and $id_{(S^2 \backslash \operatorname{Int}(f_T(D_\varepsilon^2)))}$ by a slight modification near ∂D_ε^2, which folds on $S(f_T)$ and ∂D_ε^2. We can prove that $\omega(g)$ is the unique nontrivial element by using an explicit description of the isomorphism μ of Remark 4.6 and [**T**, Propositions 3.1 and 5.3]. The details will appear elsewhere.

§5. Surgery theory.

In this section we need to recall the surgery theory due to Kervaire-Milnor ([**K-M**]), Browder ([**B2**]), Sullivan ([**Su**]) and Wall ([**W2**]). First we give a fold-cobordism invariant related to surgery obstructions. Let $Ln(\pi_1(P))$ be the Wall group for an oriented manifold P of dimension n. If $n \geq 5$, then we have the surgery obstruction $\Theta : [P, G/O] \to Ln(\pi_1(P))$. In particular, if $n \equiv 2 \pmod 4$ and P is simply connected, then the surgery obstruction is the Kervaire invariant defined in $\boldsymbol{Z}/2\boldsymbol{Z}$. Let $p_{SG} : SG \to SG/SO = G/O$ be the canonical projection and $(p_{SG})_* : [P, SG] \to [P, G/O]$ the induced map.

PROPOSITION 5.1. *Let P be an oriented manifold of dimension $n \geq 5$. Then the map $\Theta \circ (p_{SG})_* \circ \omega : \Omega_{\text{fold}}(P) \to Ln(\pi_1(P))$ gives a fold-cobordism invariant in $Ln(\pi_1(P))$.*

The author does not know to what extent this invariant does not vanish. If $n \equiv 2 \pmod 4$ and $P = S^n$, then the surgery obstruction of Kervaire invariant vanishes except for the dimensions $n \neq 2^\ell - 2$ and it is defined also in the dimension $n = 2$. It is known that it is nontrivial in dimensions 2, 6, 14 and 30 (see [**B1**, Corollary 1]). So the Kervaire invariant seems to be a non-vanishing fold-cobordism invariant (see Example 4.8 (2)).

Let ν_X be the normal bundle of an embedded manifold X in S^{n+k} as defined in §4 inducing a trivialization of $\tau_X \oplus \nu_X$. Now we shall recall the results developped in the surgery theory along [**B2**] and [**M-M**].

Let η be a vector bundle over P of dimension k. A *normal map* of degree 1 is defined to be a pair (f, b), where $f : N \to P$ is a map of degree 1 and b is a bundle map (this is not necessarily orientation preserving) of ν_N into η covering f ([**B2**, page 31]). Two normal maps $f_i : N_i \to P$ and $b_i : \nu_{N_i} \to \eta_i$ $(i = 0, 1)$ are normally cobordant when there exists an oriented manifold W of dimension $n + 1$ with bundle map $B : \nu_W \to \eta_0 \times I$ over a map $F : W \to P \times I$ such that

(i) $\partial W = N_0 \cup (-N_1)$ and $\nu_W|_{N_i} = \nu_{N_i}$,

(ii) $F|_{N_i} = f_i$ and

(iii) $B|_{N_0} = b_0$ and $B|_{N_1} = a \circ b_1$,

where $a : \eta_1 \to \eta_0$ is a bundle isomorphism (see [**M-M**, Definition 2.13]). Let $NM_O(P)$ denote the set of all normal cobordism classes of normal maps (f, b) of degree 1 into P. A *G/O-bundle structure* on $S(\xi)$ is a pair (η, h), where ξ is a vector bundle of dimension k over P and $h : S(\xi) \to S(\eta)$ is a fibre homotopy equivalence, which refers to a fibre map giving a homotopy equivalence. Two G/O-bundle structures $(\eta_i, h_i)(i = 0, 1)$ on $S(\xi)$ are equivalent if there is a bundle map $a : \eta_1 \to \eta_0$ such that $h_0 \simeq a \circ h_1$. Let $\mathscr{S}_{G/O}(\xi)$ denote the set of equivalence classes of G/O-bundle structures on $S(\xi)$.

Theorem 5.2. (i) ([**B2**, *Proof of* II.4.8 *Lemma*] *and* [**M-M**, *p.* 35]) *There exists a bijection* $\mathfrak{n} : NM_O(P) \to \mathscr{S}_{G/O}(\nu_P)$.

(ii) ([**Su**] *and* [**B2**, II.4.4 *Theorem and* II.4.7 *Proposition*]) *There exists a bijection* $\boldsymbol{c}_{G/O} : \mathscr{S}_{G/O}(\nu_P) \to [P, G/O]$.

Here we shall give a sketch of the definition of these maps. Let $\alpha_N \in \pi_{n+k}(T(\nu_N))$ and $\alpha_P \in \pi_{n+k}(T(\nu_P))$ be the elements as in §4. Let (f, b) be an element of $NM_O(P)$. Then α_P and $T(b)_*(\alpha_N)$ determine the homotopy class of a fibre homotopy equivalence $h : S(\nu_P) \to S(\eta)$ such that $T(h)_*(\alpha_P) = T(b)_*(\alpha_N)$ by [**B2**, I.4.19 Theorem]. Since h can be considered as a G/O-bundle structure on $S(\nu_P)$, the map $\mathfrak{n}$ is defined by $\mathfrak{n}(f, b) = [\eta, h]$. We see the surjectivity of $\mathfrak{n}$ as follows. Let $h : S(\nu_P) \to S(\eta)$ be a G/O-bundle structure. Then we have the composition of α_P and $T(h) : T(\nu_P) \to T(\eta)$. By deforming $T(h) \circ \alpha_P$ so that it is transverse to the zero section P of η, we obtain a map f of $N = (T(h) \circ \alpha_P)^{-1}(P)$ into P with $b : \nu_N \to \eta$. Then $\mathfrak{n}(f, b) = [\eta, h]$.

The map $\boldsymbol{c}_{G/O}$ is a little complicated. Consider the following:

$$\mathscr{S}_{G/O}(\nu_P) \xrightarrow{\mathscr{j}} \mathscr{S}_{G/O}(\tau_P \oplus \nu_P) \xrightarrow{\mathscr{S}(t_P^{-1})} \mathscr{S}_{G/O}(\theta_P^{2k}),$$

where $\mathscr{j}([\eta, h]) = [\tau_P \oplus \eta, id_{\tau_P} + h]$ and $\mathscr{S}(t_P^{-1})([\eta', h']) = [\eta', h' \circ t_P^{-1}]$. Let us prove that there exists a bijection $\boldsymbol{c}'_{G/O} : \mathscr{S}_{G/O}(\theta_P^{2k}) \to [P, G/O]$. Let $\bar{\gamma}$ (resp. γ) be a universal vector bundle over $BO(2k)$ (resp. a universal spherical fibre space over BG_{2k}). Let $\bar{\rho} : S(\bar{\gamma}) \to \gamma$ be a classifying fibre map covering $\rho : BO(2k) \to BG_{2k}$. Let $\bar{c} : S(\theta_P^{2k}) \to \gamma$ be a fixed classifying fibre map covering a constant map $c : P \to BG_{2k}$. Let $h : S(\theta_P^{2k}) \to S(\eta)$ be a G/O-bundle structure and $\bar{c}_\eta : \eta \to \bar{\gamma}$ be a classifying bundle map of η covering $c_\eta : P \to BO(2k)$. Then there exists a homotopy between $\bar{\rho} \circ \bar{c}_\eta \circ h : S(\theta_P^{2k}) \to S(\eta) \to S(\bar{\gamma}) \to \gamma$ and $\bar{c}$ by the universality of γ. Hence it covers a homotopy c_u of $P \times I$ to BG_{2k} such that $c_0 = \rho \circ c_\eta$ and $c_1 = c$. By applying the homotopy lifting property of ρ for c_u and c_η, there exists a homotopy $(c_\eta)_u$ of $P \times I$ to $BO(2k)$ with $(c_\eta)_0 = c_\eta$ and $\rho \circ (c_\eta)_1 = c$. Hence $(c_\eta)_1$ gives a map of P into $\rho^{-1}(\text{point}) = G_{2k}/O(2k) \to G/O$, which is what we want for $[\eta, h]$. The inverse of $\boldsymbol{c}'_{G/O}$ is given as follows. Given a map $(c_\eta)_1 : P \to \rho^{-1}(\text{point}) = G_{2k}/O(2k)$, set $\eta = (c_\eta)_1^*(\bar{\gamma})$. By the universality of γ, $(c_\eta)_1$ determines the unique homotopy class of a fibre homotopy equivalence $h : S(\theta_P^{2k}) \to S(\eta)$ which is what we want. The map $\boldsymbol{c}_{G/O}$ is the composition $\boldsymbol{c}'_{G/O} \circ \mathscr{S}(t_p^{-1}) \circ \mathscr{j}$.

Next we have a canonical map for any vector bundle ξ

$$\mathscr{I} : \mathscr{E}(\xi) \to \mathscr{S}_{G/O}(\xi)$$

by mapping an automorphism h of $\mathscr{E}(\xi)$ to the G/O-bundle structure $[\xi, h]$ on $S(\xi)$.

Propositon 5.3. *We have the following commutative diagram with* $(p_{SG})_* \circ \boldsymbol{c}_{SG} = \boldsymbol{c}_{G/O} \circ \mathscr{I}$.

$$\begin{array}{ccc} \mathscr{E}(\nu_P) & \xrightarrow{\boldsymbol{c}_{SG}} & [P, SG] \\ {\scriptstyle \mathscr{I}}\big\downarrow & & \big\downarrow{\scriptstyle (p_{SG})_*} \\ \mathscr{S}_{G/O}(\nu_P) & \xrightarrow{\boldsymbol{c}_{G/O}} & [P, G/O] \end{array}$$

Proof. We define the map $\mathscr{j}_{\mathscr{E}} : \mathscr{E}(\nu_P) \to \mathscr{E}(\tau_P \oplus \nu_P)$ by $\mathscr{j}_{\mathscr{E}}(h) = id_{\tau_P} + h$. Then we have the following commutative diagram, where $\mathscr{E}'(t_P)(h) = [t_P \circ h \circ t_P^{-1}]$ and $\mathscr{I}'(h) = [\tau_P \oplus \nu_P, t_P^{-1} \circ h]$.

$$\begin{array}{ccccccc}
\mathscr{E}(\nu_P) & \xrightarrow{\mathfrak{s}_{\mathscr{E}}} & \mathscr{E}(\tau_P \oplus \nu_P) & \xrightarrow{\mathscr{E}'(t_P)} & \mathscr{E}(\theta_P^{2k}) & \xrightarrow{\boldsymbol{c}'_{SG}} & [P, SG] \\
\Big\downarrow \mathscr{I} & & \Big\downarrow \mathscr{I} & & \Big\downarrow \mathscr{I}' & & \\
\mathscr{S}_{G/O}(\nu_P) & \xrightarrow{\mathfrak{s}} & \mathscr{S}_{G/O}(\tau_P \oplus \nu_P) & \xrightarrow{\mathscr{S}(t_P^{-1})} & \mathscr{S}_{G/O}(\theta_P^{2k}) & \xrightarrow{\boldsymbol{c}'_{G/O}} & [P, G/O]
\end{array}$$

Here $\boldsymbol{c}'_{SG}$ is the isomorphism which appeared in the definition of $\boldsymbol{c}_{SG}$ in §4. The fact that $\boldsymbol{c}'_{SG}$ in the diagram is bijective can also be explained similarly as in the case of $\boldsymbol{c}'_{G/O}$ above by considering the universal fibre space $ESG \to BSG$ with fibre SG. Then the equality $(p_{SG})_* \circ \boldsymbol{c}'_{SG} = \boldsymbol{c}'_{G/O} \circ \mathscr{I}'$ follows from the homotopy commutative diagram

$$\begin{array}{ccccc}
SG & \longrightarrow & ESG & \longrightarrow & BSG \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
G/O & \longrightarrow & BO & \longrightarrow & BG.
\end{array} \qquad \square$$

If $n \geq 5$ and P is simply connected, then we have the following exact sequence due to D. Sullivan [**Su**] (see also [**B2**, II.4.10] and [**M-M**]):

$$bP_{n+1} \cdots \to \mathscr{S}(P) \xrightarrow{\P} [P, G/O] \cong NM_O(P) \xrightarrow{\Theta} P_n,$$

where

$$P_n = \begin{cases} 0 & (n\text{: odd}) \\ \boldsymbol{Z} & (n \equiv 0 \pmod 4) \\ \boldsymbol{Z}/2\boldsymbol{Z} & (n \equiv 2 \pmod 4) \end{cases}$$

and Θ is the surgery obstruction of normal maps. $\Theta([f, b])$ in dimensions $n \equiv 0 \pmod 4$ is, by definition, $(I(N) - I(P))/8$ with I being the index and it is represented by the Pontrjagin classes of tangent bundles ([**B2**, III.3.11]). Hence if τ_N and τ_P are equivalent, then it vanishes. The dotted arrow of bP_{n+1} to $\mathscr{S}(P)$ does not refer to a map but refers to the action of bP_{n+1} on $\mathscr{S}(P)$ described in Introduction.

Here we shall explain the definition of ¶. Let $f : N \to P$ be a smooth structure on P. Let f^{-1} be its homotopy inverse and $\eta = (f^{-1})^*(\nu_N)$. Then there exists a bundle map $b : \nu_N \to \eta$ over f. Hence (f, b) becomes a normal map of degree 1. As before, $T(b)_*(\alpha_N)$ and α_P give a G/O-bundle structure $h : S(\nu_P) \to S(\eta)$ with $T(h)_*(\alpha_P) = T(b)_*(\alpha_N)$. Then $\P(f)$ is defined to be $\boldsymbol{c}_{G/O}([\eta, h])$ in $[P, G/O]$.

Let $[P, G/O]^{\mathrm{tang}}$ refer to the image $(p_{SG})_*([P, SG])$ in this paper. Then we have the following.

Corollary 5.4. *Let $n \geq 5$ and P be simply connected. Then the image $\P(\mathscr{S}^{\mathrm{tang}}(P))$ is contained in $[P, G/O]^{\mathrm{tang}}$.*

Proof. For a smooth structure $f : N \to P$ in $\mathscr{S}^{\mathrm{tang}}(P)$, we have $(f^{-1})^*(\nu_N) \cong \nu_P$. So there exists an automorphism $h : S(\nu_P) \to S(\nu_P)$ such that $\P(f) = \boldsymbol{c}_{G/O} \circ \mathscr{I}(h)$, which is equal to $(p_{SG})_* \circ \boldsymbol{c}_{SG}(h)$ by Proposition 5.3. This proves the assertion. $\square$

The following theorem is another formulation of Theorem 3 in Introduction.

Theorem 5.5. *Let $n \geq 5$ and P be simply connected. Consider the map $(p_{SG})_* \circ \omega : \Omega_{\mathrm{fold}}(P) \to [P, G/O]$. Then we have the following.*

(1) *The image of $(p_{SG})_* \circ \omega$ contains $\P(\mathscr{S}^{\mathrm{tang}}(P))$.*

(2) *If either* (i) $n \equiv 0, 1$ *or* 3 $(\mathrm{mod}\, 4)$ *or* (ii) $n \equiv 2$ $(\mathrm{mod}\, 4)$ *and Θ vanishes for P, then $\P(\mathscr{S}^{\mathrm{tang}}(P))$ and the image of $(p_{SG})_* \circ \omega$ are equal to $[P, G/O]^{\mathrm{tang}}$.*

Proof. Let x be any element of $[P, G/O]^{\mathrm{tang}}$. Let (f', b) be the associated normal map $b : \nu_M \to \nu_P$ covering a map $f' : M \to P$ of degree 1 and $h : S(\nu_P) \to S(\eta)$ the associated G/O-bundle structure as in Theorem 5.2. Then $[\eta, h]$ is contained in $\mathscr{I}(\mathscr{E}(\nu_P))$ by Proposition 5.3. Hence ν_M must be equivalent to $f^*(\nu_P)$. For dimensions $n \equiv 0$ $(\mathrm{mod}\, 4)$, we have $\Theta(x) = 0$, since $I(M) = I(P)$. Therefore we obtain the assertion in (2) that $[P, G/O]^{\mathrm{tang}}$ is equal to $\P(\mathscr{S}^{\mathrm{tang}}(P))$. Thus it is enough for (1) and (2) to consider an element x of $[P, G/O]^{\mathrm{tang}}$ such that there is a smooth structure $f : N \to P$ with $x = \P(f)$.

Next we prove, using [**E**, Theorem 3.10], that the map f above is homotopic to a folding map g. In fact, if $n = 7$, then it is a direct consequence. If n is even or n is odd and TN and $f^*(TP)$ are equivalent, then take two small disks D_ε and $D_{2\varepsilon}$ of radii ε and 2ε respectively in N. Let N_1 be $(N \backslash \mathrm{Int}\, D_{2\varepsilon}) \cup D_\varepsilon$ and N_2 be $D_{2\varepsilon} \backslash \mathrm{Int}\, D_\varepsilon$. Then they satisfy the condition (b) or (c) of [**E**, Theorem 3.10] so that we obtain g. If n is odd and TN and $f^*(TP)$ are not equivalent, then by taking a small sphere bounding a disk of N and splitting N into two manifolds, we can prove the condition (d) of [**E**, Theorem 3.10] again to obtain g.

Thus $\P(f) = \P(g)$. By definition, $\P(g)$ is equal to $\boldsymbol{c}_{G/O} \circ \mathscr{I}(h(g))$, which is equal to $(p_{SG})_* \circ \boldsymbol{c}_{SG}(h(g))$ by Proposition 5.3. By the definition of ω we have $\omega(g) = \boldsymbol{c}_{SG}(h(g))$ and so $\P(g) = (p_{SG})_* \circ \omega(g)$. This is what we want. $\square$

Proof of Theorem 3. The assertion follows from Theorem 5.5 and the fact that if f is a homotopy equivalence and is a folding map of degree 1, then we have $\P(f) = (p_{SG})_* \circ \omega(f)$. $\square$

Example 5.6. Consider the case of $P = S^n$ $(n \geq 5)$. Let θ_n denote the group of the h-cobordism classes of homotopy n-spheres, which is identified with $\mathscr{S}^{\mathrm{tang}}(S^n) = \mathscr{S}(S^n)$ in this case. Hence, $\P(\mathscr{S}^{\mathrm{tang}}(S^n))$ is identified with the well kown group θ_n/bP_{n+1}, which has been discussed in [**K-M**, §4 and p. 581]. Therefore, the image $(p_{SG})_* \circ \omega(\Omega_{\mathrm{fold}}(S^n))$ contains the group θ_n/bP_{n+1} by (1) of Theorem 5.5.

Remark 5.7. The results of the present paper suggest that folding maps are closely related to differentiable structures of manifolds. Another kind of phenomena in low dimensions can be found in [**B-D**], [**S-S1**] and [**S-S2**] and their references due to Saeki and Sakuma.

We propose a problem: Find a method to construct an explicit folding map $h : P' \to P$ between homotopy equivalent manifolds.

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