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Nonexistence results of positive solutions for semilinear elliptic equations in $\boldsymbol{R}^n$

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Abstract. We consider the global properties of nonnegative solutions of the semilinear elliptic equations in the entire space. By employing Pohozaev identity in the entire space and the results concerning the asymptotic behavior of nonnegative solutions, we establish some theorems of Liouville type.

1. Introduction and statement of the results.

In this paper we consider the elliptic equations of the form

$$\Delta u + K(x)u^{\sigma} = 0 \quad \text{in } \boldsymbol{R}^n, \tag{1.1}$$

where $n \geq 3$ and $\sigma > 1$ is a positive constant. In (1.1) we assume that $K \in C^1(\boldsymbol{R}^n)$ and $K(x) \geq 0$ in $\boldsymbol{R}^n$. We are concerned with the global properties of classical nonnegative solutions of (1.1), and establish some results of Liouville-type.

The properties of radial solutions of (1.1) in the case $K = K(|x|)$ have been investigated in full detail by [**3**], [**8**], [**10**], [**7**], [**15**], [**16**]. For a radial solution $u = u(r)$, $r = |x|$, equation (1.1) is rewritten in the form

$$(r^{n-1}u_r)_r + r^{n-1}K(r)u^{\sigma} = 0, \quad r > 0, \tag{1.2}$$

with the condition $u_r(0) = 0$. Let u_λ be a solution of (1.2) satisfying the initial condition

$$u(0) = \lambda \quad \text{and} \quad u_r(0) = 0, \tag{1.3}$$

where λ is a positive parameter. Define $K_{n,\sigma}(r)$ as

$$K_{n,\sigma}(r) = \frac{n+2-\sigma(n-2)}{2}K(r) + rK_r(r), \quad r > 0.$$

Ding and Ni [**3**], and Kusano and Naito [**8**] proved the following: if $K_{n,\sigma}(r) \leq 0$, $\not\equiv 0$, for

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$r > 0$ then u_λ remains positive on $[0, \infty)$ and satisfies $\liminf_{r\to\infty} r^{(n-2)/2}u_\lambda(r) > 0$ for all $\lambda > 0$, and if $K_{n,\sigma}(r) \geq 0, \not\equiv 0$, for $r > 0$ then u_λ has a zero in $(0, \infty)$ for all $\lambda > 0$. For more precise properties concerning the structure of positive solutions of the problem (1.2)–(1.3), see Kawano et al. [**7**] and Yanagida and Yotsutani [**15**], [**16**].

Our main results are the following.

Theorem 1. *Assume that*

$$\frac{n+2-\sigma(n-2)}{2}K(x) + x \cdot \nabla K(x) \leq 0, \not\equiv 0, \quad x \in \boldsymbol{R}^n. \tag{1.4}$$

Let $u \in C^2(\boldsymbol{R}^n)$ *be a nonnegative solution of* (1.1) *satisfying*

$$u(x) = O(|x|^{-\alpha}) \quad \text{as } |x| \to \infty \quad \text{for some } \alpha > \frac{n-2}{2}. \tag{1.5}$$

Then $u \equiv 0$ *in* $\boldsymbol{R}^n$.

We consider the case where K satisfies the condition:

$$K(x) = O(|x|^{\ell}) \quad \text{as } |x| \to \infty \quad \text{for some } \ell \in \boldsymbol{R}. \tag{1.6}$$

Combining Theorem 1 and the asymptotic properties by Li and Ni [**11**, Theorem 3.2], we obtain the following.

Theorem 2. *Assume that* (1.4) *and* (1.6) *hold. Let* $u \in C^2(\boldsymbol{R}^n)$ *be a bounded nonnegative solution of* (1.1) *satisfying*

$$\begin{cases} \liminf_{|x|\to\infty} u(x) = 0 & \text{if } \ell < -2 \\ u(x) = o((\log|x|)^{-1/(\sigma-1)}) \quad \text{as } |x| \to \infty & \text{if } \ell = -2 \\ u(x) = o(|x|^{-(\ell+2)/(\sigma-1)}) \quad \text{as } |x| \to \infty & \text{if } \ell > -2. \end{cases} \tag{1.7}$$

Then $u \equiv 0$ *in* $\boldsymbol{R}^n$.

Remark. For the case $\ell < -2$, the result is proved by Li and Ni [**10**, Theorem 1.4].

Theorem 3. *Assume that*

$$\frac{n+2-\sigma(n-2)}{2}K(x) + x \cdot \nabla K(x) \geq 0, \not\equiv 0, \quad x \in \boldsymbol{R}^n, \tag{1.8}$$

and that there exist constants $C_1, C_2, C_3 > 0$, *and* $\ell \in \boldsymbol{R}$ *such that*

$$C_1|x|^{\ell} \leq K(x) \leq C_2|x|^{\ell} \quad \text{and} \quad |\nabla K(x)| \leq C_3|x|^{\ell-1} \quad \text{for all large } |x|. \tag{1.9}$$

Assume in addition that

$$1 < \sigma < \min\left\{\frac{n+2+2\ell}{n-2}, \frac{n+2}{n-2}\right\}.$$

Let $u \in C^2(\boldsymbol{R}^n)$ *be a nonnegative solution of* (1.1), *then* $u \equiv 0$ *in* $\boldsymbol{R}^n$.

Remark. For the case $K = K(|x|)$, Bianchi in [**2**] showed the radial symmetry of positive solutions by employing the method of moving planes, and then obtained some nonexistence results.

Example 1. Consider the equation (1.1) with

$$K(x) = 1 + \eta(|x|) \quad \text{and} \quad \sigma = \frac{n+2}{n-2}, \tag{1.10}$$

where $\eta \in C^1[0, \infty)$ is a nonincreasing function satisfying $\eta \geq 0$, $\not\equiv 0$. By Theorem 1, if there is a nonnegative solution u satisfying (1.5) then $u \equiv 0$ in $\boldsymbol{R}^n$. On the other hand, it has been shown by [**3**], [**8**] that (1.1) with (1.10) has a positive radial solution u satisfying

$$0 < \liminf_{r\to\infty} r^{(n-2)/2} u(r) \leq \limsup_{r\to\infty} r^{(n-2)/2} u(r) < \infty.$$

Then the condition (1.5) in Theorem 1 is optimal.

Let us consider the Matukuma-type equation

$$\Delta u + \frac{1}{1+|x|^\tau} u^\sigma = 0 \quad \text{in } \boldsymbol{R}^n, \quad \tau \geq 0. \tag{1.11}$$

Example 2. Consider the equation (1.11) with $\sigma > (n+2)/(n-2)$. By Theorem 2, if there is a bounded nonnegative solution u satisfying (1.7) with $\ell = -\tau$, then $u \equiv 0$ in $\boldsymbol{R}^n$. On the other hand, it has been shown by [**7**], [**10**] that (1.11) has a positive radial solution u satisfying

$$\begin{cases} \lim_{r\to\infty} u(r) = \text{const} > 0 & \text{if } \tau > 2 \\ \lim_{r\to\infty} (\log r)^{1/(\sigma-1)} u(r) = \text{const} > 0 & \text{if } \tau = 2 \\ \lim_{r\to\infty} r^{(\ell+2)/(\sigma-1)} u(r) = \text{const} > 0 & \text{if } \tau < 2. \end{cases} \tag{1.12}$$

Therefore, the condition (1.7) in Theorem 2 is optimal.

Example 3. Consider the equation (1.11) with $1 < \sigma < (n+2-2\tau)/(n-2)$. By Theorem 3, if there is a nonnegative solution u then $u \equiv 0$ in $\boldsymbol{R}^n$. On the other hand, it has been shown by [**7**] that if $\sigma > (n+2-2\tau)/(n-2)$ and $\sigma > 1$, then (1.11) has positive radial solutions.

In our proofs the key ingredient of the method is the Pohozaev identity in the entire space $\boldsymbol{R}^n$. In the work of Pohozaev [**12**] and in recent works [**1**], [**4**], [**13**], [**14**], the Pohozaev identities are discussed which are useful for solving various questions about elliptic differential equations. In this paper we consider the Pohozaev identity for the equation $\Delta u + f(x,u) = 0$ in $\boldsymbol{R}^n$, which plays an important role in the proofs of Theorems 1–3.

2. Proofs of Theorems.

First we consider the Pohozaev identity for the equation

$$\Delta u + f(x,u) = 0 \quad \text{in } \boldsymbol{R}^n. \tag{2.1}$$

We assume that $f(x,u) \in C(\boldsymbol{R}^n \times \boldsymbol{R})$ satisfies

$$uf(x,u) \geq 0 \quad \text{and} \quad f(x,-u) = -f(x,u) \quad \text{for } (x,u) \in \boldsymbol{R}^n \times \boldsymbol{R}.$$

Let

$$F(x,u) = \int_0^u f(x,t)\,dt.$$

We recall the following Pohozaev identity. For the proof, see, e.g., [**3**, Lemma 3.7].

LEMMA 1. *Let B_r be the ball of radius r centered at the origin and let $S_r = \partial B_r$. Let u be a solution of* (2.1). *Then we have*

$$\int_{B_r} \left[nF(x,u) - \frac{n-2}{2} uf(x,u) + x \cdot \nabla_x F(x,u) \right] dx \tag{2.2}$$

$$= \int_{S_r} \left[(x \cdot \nabla u) \frac{\partial u}{\partial \nu} - \frac{r}{2} |\nabla u|^2 + rF(x,u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right] ds,$$

where ds is the volume element of S_r and ν is the unit outer normal of S_r.

Define $P(x,u)$ as

$$P(x,u) = nF(x,u) - \frac{n-2}{2} uf(x,u) + x \cdot \nabla_x F(x,u) \quad \text{for } (x,u) \in \boldsymbol{R}^n \times \boldsymbol{R}.$$

For the case $f(x,u) = K(x)|u|^{\sigma-1}u$, we find that

$$P(x,u) = \frac{1}{\sigma+1} \left[\frac{n+2-\sigma(n-2)}{2} K(x) + x \cdot \nabla K(x) \right] |u|^{\sigma+1}.$$

We obtain the following.

Proposition 1. *Let u be a solution of* (2.1) *satisfying*

$$F(x,u),uf(x,u) \in L^1(\boldsymbol{R}^n) \quad \text{and} \quad |u(x)| = O\left(|x|^{-(n-2)/2}\right) \quad \text{as } |x| \to \infty. \tag{2.3}$$

Assume one of the following properties:

(i) $x \cdot \nabla_x F(x,u) \in L^1(\boldsymbol{R}^n)$;

(ii) $P(x,u) \geq 0$ *or* $P(x,u) \leq 0$ *in* $\boldsymbol{R}^n$.

Then

$$\int_{R^n} P(x,u)\,dx = 0. \tag{2.4}$$

Proof. First we show that

$$\int_{R^n} |\nabla u|^2\,dx < \infty. \tag{2.5}$$

Assume to the contrary that

$$\int_{R^n} |\nabla u|^2\,dx = \infty.$$

Multiplying (2.1) by u and applying the divergence theorem in B_r, we have

$$\int_{S_r} u\frac{\partial u}{\partial v}\,ds = \int_{B_r} |\nabla u|^2\,dx - \int_{B_r} uf(x,u)\,dx, \quad r > 0.$$

By $uf(x,u) \in L^1(\boldsymbol{R}^n)$, we observe that there exists a constant $R > 0$ such that

$$\int_{S_r} u\frac{\partial u}{\partial v}\,ds \geq 0, \quad r \geq R.$$

Set

$$U(r) = \int_{S_r} u^2\,ds.$$

Then, we find that

$$U_r - \frac{n-1}{r}U = 2\int_{S_r} u\frac{\partial u}{\partial v}\,ds \geq 0, \quad r \geq R.$$

It follows that $(r^{1-n}U(r))_r \geq 0$ for $r \geq R$, and

$$U(r) \geq R^{1-n}U(R)r^{n-1}, \quad r \geq R. \tag{2.6}$$

On the other hand, by the second of (2.3), we have

$$U(r) \le Cr, \quad r \ge R,$$

for some $C > 0$, which contradicts (2.6). Thus we conclude that (2.5) holds.

From (2.2) we have

$$\left| \int_{B_r} P(x,u)\,dx \right| \le \frac{3}{2} r \int_{S_r} |\nabla u|^2\,ds + r \int_{S_r} F(x,u)\,ds + \frac{n-2}{2} \int_{S_r} u|\nabla u|\,ds \tag{2.7}$$

for $r > 0$. Using Hölder's inequality, we drive

$$\begin{aligned} \int_{S_r} u|\nabla u|\,ds &\le \left(\int_{S_r} u^2\,ds \right)^{1/2} \left(\int_{S_r} |\nabla u|^2\,ds \right)^{1/2} \\ &= \left(r^{-1} \int_{S_r} u^2\,ds \right)^{1/2} \left(r \int_{S_r} |\nabla u|^2\,ds \right)^{1/2}. \end{aligned} \tag{2.8}$$

We observe from the second of (2.3) that

$$r^{-1} \int_{S_r} u^2\,ds = O(1) \quad \text{as } r \to \infty. \tag{2.9}$$

From (2.5) and $F(x,u) \in L^1(\boldsymbol{R}^n)$, we have

$$\liminf_{r \to \infty} r \int_{S_r} (|\nabla u|^2 + F(x,u))\,ds = 0.$$

Then there exists a sequence $r_j \to \infty$ such that

$$\lim_{j \to \infty} r_j \int_{S_{r_j}} |\nabla u|^2\,ds = 0 \quad \text{and} \quad \lim_{j \to \infty} r_j \int_{S_{r_j}} F(x,u)\,ds = 0. \tag{2.10}$$

We now put $r = r_j$ in (2.7) and let $j \to \infty$. Then, from (2.8), (2.9), and (2.10) we have

$$\lim_{j \to \infty} \int_{B_{r_j}} P(x,u)\,dx = 0. \tag{2.11}$$

If $x \cdot \nabla_x F(x,u) \in L^1(\boldsymbol{R}^n)$, then $P(x,u) \in L^1(\boldsymbol{R}^n)$. By (2.11) we have (2.4). If $P(x,u) \ge 0$ or $P(x,u) \le 0$ in $\boldsymbol{R}^n$, then $\int_{B_r} P(x,u)\,dx$ is monotone in $r > 0$. By (2.11) we have (2.4). This completes the proof of Proposition 1. □

PROOF OF THEOREM 1. First we claim that

$$\int_{R^n} K(x) u^{\sigma+1}\,dx < \infty. \tag{2.12}$$

Set

$$H(r) = r^{(4-n-\sigma(n-2))/2} \int_{S_r} K(x)\, ds, \quad r > 0.$$

We find that $H(R) > 0$ for some $R > 0$ and

$$\frac{d}{dr} H(r) = r^{(2-n-\sigma(n-2))/2} \int_{S_r} \left[\frac{n+2-\sigma(n-2)}{2} K(x) + x \cdot \nabla K(x) \right] ds \le 0, \quad r > 0.$$

It follows that

$$\int_{S_r} K(x)\, ds \le H(R) r^{-(4-n-\sigma(n-2))/2}, \quad r \ge R. \tag{2.13}$$

From (1.5) and (2.13), we have

$$\begin{aligned} \int_{R^n} K(x) u^{\sigma+1}\, dx &= \int_0^\infty \left(\int_{S_r} K(x) u^{\sigma+1}\, ds \right) dr \\ &\le C_1 + C_2 \int_R^\infty r^{-\alpha(\sigma+1)} \left(\int_{S_r} K(x)\, ds \right) dr \\ &\le C_1 + C_2 H(R) \int_R^\infty r^{-1-(\sigma+1)(\alpha-(n-2)/2)}\, dr < \infty, \end{aligned}$$

where C_1 and C_2 are positive constants.

Then, by Proposition 1, we have

$$\int_{R^n} \left[\frac{n+2-\sigma(n-2)}{2} K(x) + x \cdot \nabla K(x) \right] u^{\sigma+1} dx = 0. \tag{2.14}$$

From (1.4) we obtain $u(x_0) = 0$ for some $x_0 \in \boldsymbol{R}^n$. By the strong maximum principle, see e.g., [**6**], we have $u \equiv 0$ in $\boldsymbol{R}^n$. □

Proof of Theorem 2. By the proof of Theorem 3.2 in [**11**], we have $u(x) = O(|x|^{2-n})$ as $|x| \to \infty$. By Theorem 1, we conclude that $u \equiv 0$ in $\boldsymbol{R}^n$. □

Proof of Theorem 3. From (1.9) we have

$$|\nabla(\log K(x))| = \frac{|\nabla K(x)|}{K(x)} \le \frac{C}{|x|} \quad \text{for all large } |x|,$$

where $C = C_3/C_1$. Then by Theorem 3.6 in [**5**], we obtain

$$u(x) = O(|x|^{-(\ell+2)/(\sigma-1)}) \quad \text{as } |x| \to \infty.$$

Since $K(x) = O(|x|^{\ell})$ as $|x| \to \infty$ and $\sigma < (n+2+2\ell)/(n-2)$, we have (2.12) and $(\ell+2)/(\sigma-1) > (n-2)/2$. By Proposition 1 we obtain (2.14). Then we conclude that $u \equiv 0$ in $\boldsymbol{R}^n$. □

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