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Proper links, algebraically split links and Arf invariant

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Abstract. In this paper we study certain kinds of links; proper links, algebraically split links and $\boldsymbol{Z}_2$-algebraically split links. These links have 'algebraic' definitions. In fact these are defined in terms of the linking number. We shall give these links certain 'geometric' definitions. By using the geometric definitions, we study the Arf invariants of these links.

Introduction.

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular all links are oriented. For an oriented manifold M, $-M$ denotes M with the opposite orientation. For a surface F in a 4-manifold M, $[F]$ denotes a second homology class represented by F.

A link $L = K_1 \cup K_2 \cup \cdots \cup K_n$ is *proper* if the linking number $\mathrm{lk}(K_i, L - K_i)$ is even for any $i (= 1, 2, \dots, n)$. A link $L = K_1 \cup K_2 \cup \cdots \cup K_n$ is *algebraically split* (resp. $\boldsymbol{Z}_2$-*algebraically split*) if $\mathrm{lk}(K_i, K_j) = 0$ (resp. $\mathrm{lk}(K_i, K_j)$ is even) for any i, j $(1 \leq i < j \leq n)$.

In section 1, we give a necessary and sufficient condition for links to be proper by using 2-spheres in 4-manifolds representing characteristic second homology class. Here, for a compact, connected, orientable 4-manifold M whose boundary is either empty or a disjoint union of 3-spheres, a homology class $\xi \in H_2(M, \partial M; \boldsymbol{Z})$ is *characteristic* if its mod 2 reduction ξ' is dual to the second Stiefel-Whitney class. An equivalent condition is that the mod 2 intersection number $\xi' \cdot x$ is equal to the mod 2 self intersection number $x \cdot x$ for every $x \in H_2(M, \partial M; \boldsymbol{Z}_2)$. In [**18**], R. A. Robertello defined the Arf invariants for proper links. We give an alternative definition of the Arf invariants for proper links by using planar surfaces in 4-manifolds representing characteristic homology classes. Our definition is similar to but different from Robertello's definition.

Let F_i $(i = 1, 2, \dots, n)$ be compact (not necessarily orientable) surfaces in S^3 with $\partial F_i \cong S^1$. The union $F_1 \cup F_2 \cup \cdots \cup F_n$ is an *R-complex* if the following conditions hold [**5**];

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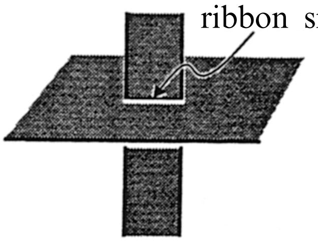

Figure 0.

(1) $F_1, F_2, \ldots, F_n$ are in general position,

(2) $F_1 \cup F_2 \cup \cdots \cup F_n$ has no triple singularities, and

(3) the set of singularities of $F_1 \cup F_2 \cup \cdots \cup F_n$ are all ribbon, see Figure 0.

An R-complex $F_1 \cup F_2 \cup \cdots \cup F_n$ is *orientable* if F_i $(i = 1, 2, \ldots, n)$ are orientable.

In section 2, we show that a link is algebraically split if and only if it bounds an orientable R-complex. For an orientable R-complex $R = F_1 \cup F_2 \cup \cdots \cup F_n$, we define a certain quadratic function on each $H_1(F_i; \boldsymbol{Z}_2)$, and denote by $\mathrm{Arf}(F_i, R)$ the Arf invariant [**1**] of this quadratic function. From our definition, we note that $\mathrm{Arf}(F_i, R)$ is an invariant of R but not an invariant of a knot ∂F_i. However, we find that the Arf invariant of a link $L = \partial F_1 \cup \partial F_2 \cup \cdots \cup \partial F_n$ is mod 2 congruent to the sum of $\mathrm{Arf}(F_i, R)$ $(i = 1, 2, \ldots, n)$, that is, $\sum_{i=1}^{n} \mathrm{Arf}(F_i, R) \pmod 2$ is an invariant of L.

For a proper link L, suppose that a sublink L' of L is proper and that L' bounds an orientable surface F in S^3 with $F \cap L = \partial F = L'$, then we can also define $\mathrm{Arf}(F, L)$ to be the Arf invariant of a certain *proper* quadratic function on $H_1(F; \boldsymbol{Z}_2)$. We show that $\mathrm{Arf}(F, L)$ is an invariant of L. In general, $\mathrm{Arf}(F, L)$ is not equal to the Arf invariant of $L' (= \partial F)$. When L is a 2-component, algebraically split link, each component K of L bounds an orientable surface F in S^3 with $F \cap L = \partial F = K$ and the difference between $\mathrm{Arf}(F, L)$ and the Arf invariant of K is mod 2 congruent to the Sato-Levine invariant [**20**] of L. When $L = K_1 \cup K_2 \cup K_3$ is a 3-component, algebraically split link, each component K_i of L bounds an orientable surface F_i in S^3 with $F_i \cap L = \partial F_i = K_i$ $(i = 1, 2, 3)$ and each 2-component sublink $K_i \cup K_j$ of L bounds an orientable surface F_{ij} in S^3 with $F_{ij} \cap L = \partial F_{ij} = K_i \cup K_j$ $(1 \le i < j \le 3)$. Then we have that both $\sum_{i=1}^{3} (\mathrm{Arf}(F_i, L) - \mathrm{Arf}(K_i, L))$ and $\sum_{i<j} (\mathrm{Arf}(F_{ij}, L) - \mathrm{Arf}(K_i \cup K_j))$ are mod 2 congruent to the Sato-Levine invariant of L defined by T. Cochran [**4**].

In section 3, we show that a link is $\boldsymbol{Z}_2$-algebraically split if and only if it bounds an unoriented R-complex. For an unoriented R-complex $R = F_1 \cup F_2 \cup \cdots \cup F_n$, we define a certain $\boldsymbol{Z}_4$-quadratic function on each $H_1(F_i; \boldsymbol{Z}_2)$ and denote by $\mathrm{B}(F_i, R)$ the Brown invariant [**3**] of this quadratic function. The Arf invariant of a link $L = \partial F_1 \cup \partial F_2 \cup \cdots \cup \partial F_n$ is represented by $\mathrm{B}(F_i, R)$, the linking numbers $\mathrm{lk}(\partial F_i, \widehat{\partial F_i})$ $(i = 1, 2, \ldots,$

n) and the total linking number $\sum_{i<j} \mathrm{lk}(\partial F_i, \partial F_j)$ of L, where $\widehat{\partial F_i}$ is a parallel copy of ∂F_i on F_i oriented in the same direction as K_i.

For a proper link L, suppose that a sublink L' of L bounds an unoriented surface F in S^3 with $F \cap L = \partial F = L'$, then we can also define $\mathrm{B}(F, L)$ to be the Brown invariant [**10**] of a certain *proper* $\boldsymbol{Z}_4$-quadratic function on $H_1(F; \boldsymbol{Z}_2)$. We find that the difference between $\mathrm{B}(F, L)$ and $\mathrm{lk}(L', \hat{L}')/2$ is an invariant of L. When L is a 2-component link, each component K bounds an unoriented surface F with $F \cap L = \partial F = K$ and the unoriented Sato-Levine invariant [**19**] of L is represented by $\mathrm{B}(F, L)$, $\mathrm{lk}(K, \hat{K})$, the linking number of L and the Arf invariant of $K (= \partial F)$.

In the last section, we study connections between the Arf invariant of proper links and certain local moves on links, and show some useful results to computing the Arf invariants of links.

1. Proper links.

The following theorem gives us a geometric characterization of a proper link.

THEOREM 1.1. *The following conditions are mutually equivalent.*

(1) *L is a proper link.*

(2) *There exist a closed, simply connected* 4*-manifold M, a* 2*-sphere Σ^2 in M and a* 3*-sphere Σ^3 in M such that Σ^2 represents a characteristic homology class in $H_2(M; \boldsymbol{Z})$ and $(\Sigma^3, \Sigma^2 \cap \Sigma^3) \cong (S^3, L)$.*

(3) *There exist a compact, simply connected* 4*-manifold M with $\partial M \cong S^3$, and a disjoint union Δ of* 2*-disks in M such that Δ represents a characteristic homology class in $H_2(M, \partial M; \boldsymbol{Z})$ and $(\partial M, \partial \Delta) \cong (S^3, L)$.*

PROOF. In the theorem above '(3) $\Rightarrow$ (2)' is clear, and '(1) $\Rightarrow$ (3)' follows from [**18**, proof of Theorem 2]. We shall prove '(2) $\Rightarrow$ (1)'.

Let M_1 and M_2 be the closures of the components $M - \Sigma^3$. Suppose $(\partial M_1, \partial(\Sigma^2 \cap M_1)) \cong (S^3, L)$. Then we note that $(\partial M_2, \partial(\Sigma^2 \cap M_2)) = (-S^3, -L)$. For a component K of L, let D and D' be the closures of the components of $\Sigma^2 - K$. Let $F_i = D \cap M_i$ and $F_i' = D' \cap M_i$ $(i = 1, 2)$. We may assume that F_1 contains K. Then F_2' contains K. Since $D = F_1 \cup F_2$ and $D' = F_1' \cup F_2'$, $\partial F_1 - K = \partial F_2$ and $\partial F_1' = \partial F_2' - K$. Set $(S^3, \partial F_1 - K) = (S^3, L_1)$ and $(S^3, \partial F_1') = (S^3, L_2)$. Note that since $\partial M_i \cong S^3$, by using the isomorphism $H_2(M_i, \partial M_i; \boldsymbol{Z}) \cong H_2(M_i; \boldsymbol{Z})$, we have a well-defined intersection pairing on $H_2(M_i, \partial M_i; \boldsymbol{Z})$. Then we see $\mathrm{lk}(K \cup L_1, L_2) = -[F_1] \cdot [F_1']$ and $\mathrm{lk}(K \cup L_2, L_1) = [F_2'] \cdot [F_2]$ since $F_1 \cap F_1' = F_2 \cap F_2' = \varnothing$. Hence we have

$$\mathrm{lk}(K, L-K) = \mathrm{lk}(K, L_1 \cup L_2) = -[F_1]\cdot[F_1'] + [F_2']\cdot[F_2] - 2\mathrm{lk}(L_1, L_2).$$

The fact that $[\Sigma^2]$ is characteristic implies $[F_i] + [F_i']$ $(i = 1, 2)$ are characteristic in $H_2(M_i, \partial M_i; \boldsymbol{Z})$. Thus we have

$$([F_i] + [F_i'])\cdot[F_i] = [F_i]\cdot[F_i] + [F_i']\cdot[F_i] \equiv [F_i]\cdot[F_i] \pmod 2.$$

Hence we have $[F_i']\cdot[F_i] \equiv 0 \pmod 2$. This completes the proof. □

For any proper link L, by Theorem 1.1, there exist a simply connected 4-manifold M with $\partial M \cong S^3$, and a planar surface F in M such that $(\partial M, \partial F) \cong (S^3, L)$ and $[F]$ is a characteristic homology class in $H_2(M, \partial M; \boldsymbol{Z})$. Thus, the following proposition gives us an alternate definition of Arf invariants for proper links (cf. [**18**]).

PROPOSITION 1.2. *Let M be a compact, simply connected 4-manifold with $\partial M \cong S^3$ and L a proper link in ∂M. If L bounds a planar surface F in M that represents characteristic homology class in $H_2(M, \partial M; \boldsymbol{Z})$, then*

$$\mathrm{Arf}(L) \equiv \frac{[F]\cdot[F] - \sigma(M)}{8} \pmod 2.$$

PROOF. Since L is a proper link, by Theorem 1.1, there exist a compact, simply connected 4-manifold M' with $\partial M' \cong S^3$, and a disjoint union Δ of 2-disks in M' such that $(\partial M', \partial\Delta) \cong (S^3, L)$ and $[\Delta]$ is a characteristic homology class in $H_2(M', \partial M'; \boldsymbol{Z})$ with $[\Delta]\cdot[\Delta] = l$. Set $M'' = M \cup_f (-M')$ and $\Sigma = F \cup_f (-\Delta)$, where f is an orientation reversing diffeomorphism from $(-\partial M', -\partial\Delta)$ to $(\partial M, \partial F)$. Since Σ is a 2-sphere and Σ represents a characteristic homology class in $H_2(M''; \boldsymbol{Z})$, by [**9**, Theorem 1], $[\Sigma]\cdot[\Sigma] \equiv \sigma(M'') \pmod{16}$. Hence we have

$$[F]\cdot[F] - l \equiv \sigma(M) - \sigma(M') \pmod{16}.$$

From [**18**, proof of Theorem 2], we have

$$\mathrm{Arf}(L) \equiv \frac{l - \sigma(M')}{8} \pmod 2.$$

It follows that

$$\mathrm{Arf}(L) \equiv \frac{l - \sigma(M')}{8} \equiv \frac{[F]\cdot[F] - \sigma(M)}{8} \pmod 2.$$ □

From Proposition 1.2, we have the following well known result [**18**].

Corollary 1.3 ([**18**]). *Let L be a proper link in the boundary of a* 4*-ball. If L bounds a planar surface in the* 4*-ball, then* $\mathrm{Arf}(L) = 0$. □

Combining Theorem 1.1 and Proposition 1.2, we have the following two propositions.

Proposition 1.4. *Let L_1 and L_2 be proper links (possibly the numbers of the components of L_1 and L_2 are not same). Let M be a compact, simply connected* 4*-manifold with $\partial M \cong (-S^3) \cup S^3$. If there is a planar surface F in M such that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and F represents a characteristic homology class in $H_2(M, \partial M; \boldsymbol{Z})$, then*

$$\mathrm{Arf}(L_2) - \mathrm{Arf}(L_1) \equiv \frac{[F] \cdot [F] - \sigma(M)}{8} \pmod 2.$$

Proof. Since L_1 is a proper link, by Theorem 1.1, there exist a compact, simply connected 4-manifold M' with $\partial M' \cong S^3$, and a disjoint union Δ of 2-disks in M' such that Δ represents a characteristic homology class in $H_2(M', \partial M'; \boldsymbol{Z})$ and $(\partial M', \partial \Delta) \cong (S^3, L_1)$. Set $M'' = M \cup_f M'$ and $F' = F \cup_f \Delta$, where f is an orientation reversing diffeomorphism from $(\partial M', \partial \Delta)$ to $(-S^3, -L_1)$. By Proposition 1.2,

$$\mathrm{Arf}(L_2) \equiv \frac{[F'] \cdot [F'] - \sigma(M'')}{8} \pmod 2,$$

and

$$\mathrm{Arf}(L_1) \equiv \frac{[\Delta] \cdot [\Delta] - \sigma(M')}{8} \pmod 2.$$

Since $\sigma(M'') = \sigma(M) + \sigma(M')$ and $[F'] \cdot [F'] = [F] \cdot [F] + [\Delta] \cdot [\Delta]$, we obtain the desired formula. □

The following proposition extends a result given by E. Ogasa [**17**] and S. Satoh [**21**] in the case when M is a 4-sphere.

Proposition 1.5. *Let M be a closed, simply connected* 4*-manifold, Σ^2 a* 2*-sphere in M and Σ^3 a* 3*-sphere in M such that Σ^2 represents a characteristic homology class in $H_2(M; \boldsymbol{Z})$. Let M_1 and M_2 be the closures of the components of $M - \Sigma^3$, and $F_i = \Sigma^2 \cap M_i$ $(i = 1, 2)$. If $(\Sigma^3, \Sigma^2 \cap \Sigma^3) \cong (S^3, L)$, then L is a proper link and*

$$\mathrm{Arf}(L) \equiv \frac{[F_i] \cdot [F_i] - \sigma(M_i)}{8} \pmod 2 \quad (i = 1, 2).$$

Furthermore, if M is prime, then $\mathrm{Arf}(L) = 0$. □

2. Algebraically split links.

The following proposition gives us a geometric characterization of an algebraically split link.

Proposition 2.1. *The following conditions are mutually equivalent.*

(1) $L = K_1 \cup K_2 \cup \cdots \cup K_n$ *is an algebraically split link.*

(2) *There is an orientable R-complex* $F_1 \cup F_2 \cup \cdots \cup F_n$ *such that* $\partial F_i = K_i$ $(i = 1, 2, \dots, n)$.

(3) *There is a disjoint union* Λ *of once punctured, orientable surfaces in a* 4*-ball* B^4 *such that* $(\partial B^4, \partial \Lambda) \cong (S^3, L)$.

Proof. Since '(3) ⇒ (1)' is clear, we shall prove '(1) ⇒ (2)' and '(2) ⇒ (3)'.

(1) ⇒ (2). Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an algebraically split link. Since $\mathrm{lk}(K_1, K_i) = 0$ $(i = 2, 3, \dots, n)$, there is an orientable surface F_1 of K_1 without intersecting to $L - K_1$. By deforming F_1 into a small neighborhood of a spine of F_1, we can choose an orientable surface F_2 of K_2 so that $F_2 \cap (L - K_1 \cup K_2) = \emptyset$ and that $F_1 \cap F_2$ has only ribbon singularities. Repeating the process above, we have a desired orientable R-complex.

(2) ⇒ (3). Let $R = F_1 \cup F_2 \cup \cdots \cup F_n$ be an orientable R-complex in the boundary of a 4-ball B^4, and $s_1, s_2, \dots, s_m$ the ribbon singularities of R. For each s_j $(j = 1, 2, \dots, m)$, we may assume that a small neighborhood of s_j in R is a union of 2-disks D_{j1} and D_{j2} such that $D_{j1} \cap D_{j2} = s_j$ and $D_{j1} \subset \mathrm{int}(F_1) \cup \mathrm{int}(F_2) \cup \cdots \cup \mathrm{int}(F_n)$. By pushing each D_{j1} into B^4, we obtain from R a desired orientable surface in B^4. □

Let $R = F_1 \cup F_2 \cup \cdots \cup F_n$ be an orientable R-complex. For each F_i $(i = 1, 2, \dots, n)$, we can define a quadratic function $q : H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ as follows. Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be a link such that $K_i = \partial F_i$ $(i = 1, 2, \dots, n)$. Suppose $\alpha \in H_1(F_i; \boldsymbol{Z}_2)$ is represented by a simple closed curve a in F_i without intersecting the singularities contained in $\mathrm{int}(F_i)$. Define $q(\alpha) \in \boldsymbol{Z}_2$ by

$$q(\alpha) \equiv \mathrm{lk}(a, a^*) + \mathrm{lk}(a, L - K_i) \pmod 2,$$

where a^* denotes the result of pushing a a very small amount into $S^3 - F_i$ along the positive normal direction to F_i. This gives a well-defined function $q : H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ that is a quadratic function with respect to the intersection pairing $\cdot : H_1(F_i; \boldsymbol{Z}_2) \otimes H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$. Choose a symplectic basis $\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \beta_2, \dots, \beta_g$ of $H_1(F_i; \boldsymbol{Z}_2)$ satisfying $\alpha_k \cdot \alpha_l = \beta_k \cdot \beta_l = 0$ and $\alpha_k \cdot \beta_l = \delta_{kl}$ (Kronecker's delta). We define the *Arf invariant* $\mathrm{Arf}(F_i, R)$ of F_i to be $\sum_{k=1}^{g} q(\alpha_k) q(\beta_k) \pmod 2$. Note that $\mathrm{Arf}(F_i, R)$ is an invariant of R in S^3. Though $\mathrm{Arf}(F_i, R)$ is not an invariant of a knot $K_i = \partial F_i$, the following theorem holds.

THEOREM 2.2. *Let* $R = F_1 \cup F_2 \cup \cdots \cup F_n$ *be an orientable* R*-complex and* $L = \partial F_1 \cup \partial F_2 \cup \cdots \cup \partial F_n$ *a link. Then the following formula holds.*

$$\mathrm{Arf}(L) \equiv \sum_{i=1}^{n} \mathrm{Arf}(F_i, R) \pmod 2.$$

Hence $\sum_{i=1}^{n} \mathrm{Arf}(F_i, R) \pmod 2$ *is an invariant of* L.

Let L be a proper link and L' a sublink of L. Suppose that there is an orientable, possibly disconnected surface F such that $\partial F = L'$ and $F \cap (L - L') = \emptyset$. (Note that $L - L'$ is also a proper link.) For this surface F, if L' is a proper link, then we can define a quadratic function $q : H_1(F; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$ as follows. Suppose $\alpha \in H_1(F; \boldsymbol{Z}_2)$ is represented by a simple closed curve a in F. Define $q(\alpha) \in \boldsymbol{Z}_2$ by

$$q(\alpha) \equiv \mathrm{lk}(a, a^*) + \mathrm{lk}(a, L - L') \pmod 2.$$

Let $V = \{\alpha \in H_1(F; \boldsymbol{Z}_2) \mid \alpha \cdot x = 0 \text{ for any } x \in H_1(F; \boldsymbol{Z}_2)\}$. Then by Claim 3.4, q vanishes on V. In this case, $\cdot$ and q induce well-defined nonsingular bilinear and quadratic forms on $H_1(F; \boldsymbol{Z}_2)/V$. Choose a symplectic basis $\alpha_1, \alpha_2, \ldots, \alpha_g, \beta_1, \beta_2, \ldots, \beta_g$ of $H_1(F; \boldsymbol{Z}_2)/V$. We define the *Arf invariant* $\mathrm{Arf}(F, L)$ of F to be $\sum_{k=1}^{g} q(\alpha_k) q(\beta_k) \pmod 2$.

THEOREM 2.3. *Let* L *be a proper link and* L' *a sublink of* L. *Suppose that* L' *is proper and it bounds an orientable, possibly disconnected surface* F *with* $F \cap (L - L') = \emptyset$. *Then we have* $\mathrm{Arf}(L) \equiv \mathrm{Arf}(L - L') + \mathrm{Arf}(F, L) \pmod 2$. *Hence* $\mathrm{Arf}(F, L)$ *is an invariant of* L.

Note that if L is an algebraically split link, then any sublink L' of L is proper and L' bounds an orientable surface F with $F \cap (L - L') = \emptyset$.

Theorems 2.2 and 2.3 will be proved in the last section. Theorem 2.3 implies the following.

COROLLARY 2.4. *Let* L *be a proper link and* K *a component of* L. *Suppose that* $\mathrm{lk}(K, K') = 0$ *for any component* $K' (\neq K)$. *Then for any orientable surface* F *with* $F \cap L = \partial F = K$, $\mathrm{Arf}(L) \equiv \mathrm{Arf}(L - K) + \mathrm{Arf}(F, L) \pmod 2$. □

REMARK 2.5. For a non-proper link L, if there is a component K of L with $\mathrm{lk}(K, K') = 0$ for any $K' (\neq K)$, then there is an orientable surface F with $F \cap L = \partial F =$

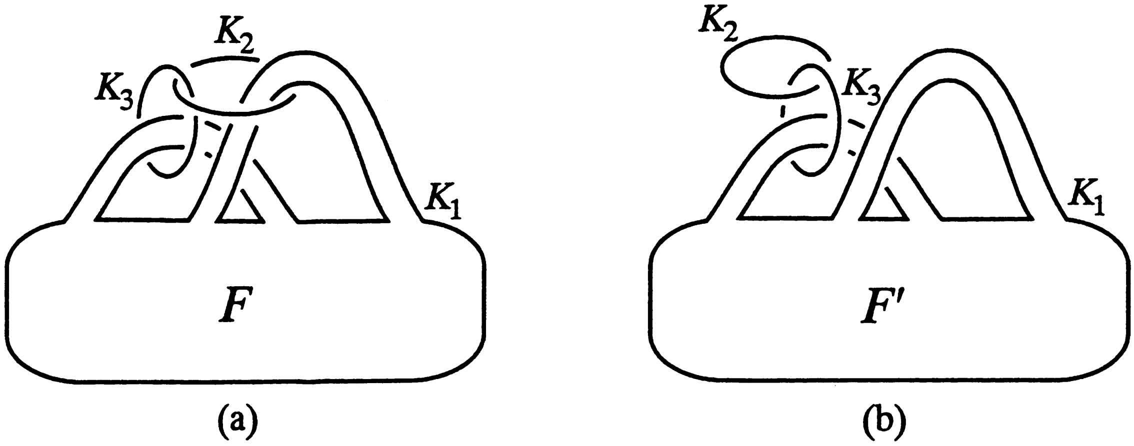

Figure 1.

K and then we can define $\mathrm{Arf}(F, L)$. Though $\mathrm{Arf}(F, L)$ is an invariant of $F \cup (L - K)$ in S^3, it is not always an invariant of L. For example, the links as in Figure 1, (a) and (b) are ambient isotopic but $\mathrm{Arf}(F, L) \neq \mathrm{Arf}(F', L)$. Thus in Corollary 2.4 (Theorem 2.3), the condition that L is proper is essential.

R. S. Beiss [**2**] has shown that the Sato-Levine invariant [**20**] $\beta(L) \in \boldsymbol{Z}$ of a 2-component algebraically split link $L = K \cup K'$ is mod 2 congruent to the sum of $\mathrm{Arf}(L)$, $\mathrm{Arf}(K)$ and $\mathrm{Arf}(K')$. Combining this and Corollary 2.4, we have

PROPOSITION 2.6. *Let $L = K \cup K'$ be a 2-component algebraically split link. For any orientable surface F with $F \cap L = \partial F = K$,* $\mathrm{Arf}(F, L) - \mathrm{Arf}(K) \equiv \beta(L) \pmod 2$. □

T. D. Cochran [**4**] defined the Sato-Levine invariant $\beta(L) \in \boldsymbol{Z}$ for a 3-component, algebraically split link $L = K_1 \cup K_2 \cup K_3$ and showed that $a_4(L) = (\beta(L))^2 = (\bar{\mu}_L(123))^2$, where $a_4(L)$ is the fourth coefficient of the Conway polynomial of L and $\bar{\mu}_L(123)$ is Milnor's $\bar{\mu}$-invariant [**11**] of L. H. Murakami [**13**] and J. Hoste [**8**] showed that the sum $\mathrm{Arf}(L) + \sum_{i=1}^{3}(\mathrm{Arf}(L - K_i) + \mathrm{Arf}(K_i))$ is mod 2 congruent to $a_4(L)$. Thus $\sum_{i=1}^{3}(\mathrm{Arf}(L) + \mathrm{Arf}(L - K_i) + \mathrm{Arf}(K_i))$ is mod 2 congruent to $\beta(L) (= \bar{\mu}_L(123))$. Combining this and Theorem 2.3, we get

PROPOSITION 2.7. *Let $L = K_1 \cup K_2 \cup K_3$ be a 3-component, algebraically split link. For any orientable surfaces F_i $(i = 1, 2, 3)$ with $F_i \cap L = \partial F_i = K_i$, and for any orientable surfaces F_{ij} $(1 \leq i < j \leq 3)$ with $F_{ij} \cap L = \partial F_{ij} = K_i \cup K_j$,*

$$\sum_{i=1}^{3}(\mathrm{Arf}(F_i, L) - \mathrm{Arf}(K_i)) \equiv \sum_{i<j}(\mathrm{Arf}(F_{ij}, L) - \mathrm{Arf}(K_i \cup K_j))$$

$$\equiv \beta(L) \equiv \bar{\mu}_L(123) \pmod 2.$$ □

3. $\boldsymbol{Z}_2$-algebraically split links.

By the arguments similar to that in the proof of Proposition 2.1, we have the following proposition. This proposition gives us a geometric definition of a $\boldsymbol{Z}_2$-algebraically split link.

Proposition 3.1. *The following conditions are mutually equivalent.*

(1) $L = K_1 \cup K_2 \cup \cdots \cup K_n$ *is a* $\boldsymbol{Z}_2$*-algebraically split link.*

(2) *There is an unoriented, possibly non-orientable R-complex* $F_1 \cup F_2 \cup \cdots \cup F_n$ *such that* $\partial F_i = K_i$ $(i = 1, 2, \dots, n)$.

(3) *There is a disjoint union* Λ *of once punctured, unoriented, possibly non-orientable surfaces in a* 4*-ball* B^4 *such that* $(\partial B^4, \partial \Lambda) \cong (S^3, L)$. □

Let $R = F_1 \cup F_2 \cup \cdots \cup F_n$ be an unoriented, possibly non-orientable R-complex. For each F_i $(i = 1, 2, \dots, n)$, we can define a $\boldsymbol{Z}_4$-quadratic function $\varphi : H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_4$ as follows. Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be a link such that $K_i = \partial F_i$ $(i = 1, 2, \dots, n)$. Suppose $\alpha \in H_1(F_i; \boldsymbol{Z}_2)$ is represented by a simple closed curve a in F_i without intersecting the singularities contained in $\mathrm{int}(F_i)$. Define $\varphi(\alpha) \in \boldsymbol{Z}_4$ by

$$\varphi(\alpha) \equiv \mathrm{lk}(a, \tau a) + 2\,\mathrm{lk}(a, L - K_i) \pmod 4,$$

where τa denotes the result of pushing $2a$ a very small amount into $S^3 - F_i$. This gives a well-defined function $\varphi : H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_4$ that is a $\boldsymbol{Z}_4$-quadratic function with respect to the intersection pairing $\cdot : H_1(F_i; \boldsymbol{Z}_2) \otimes H_1(F_i; \boldsymbol{Z}_2) \to \boldsymbol{Z}_2$. That is, $\varphi(x + y) \equiv \varphi(x) + \varphi(y) + 2(x \cdot y) \pmod 4$ for all $x, y \in H_1(F_i; \boldsymbol{Z}_2)$. We define the *Brown invariant* $\mathrm{B}(F_i, R) \in \boldsymbol{Z}_8$ to be the Brown invariant [**3**] of the $\boldsymbol{Z}_4$-quadratic function φ. That is, $\mathrm{B}(F_i, R)$ is defined by

$$\sqrt{2}^{\dim H_1(F_i; \boldsymbol{Z}_2)} \exp(\pi\sqrt{-1}\mathrm{B}(F_i, R)/4) = \sum_{x \in H_1(F_i; \boldsymbol{Z}_2)} \sqrt{-1}^{\varphi(x)}.$$

Let $\hat{K}_i$ be a parallel copy of K_i on F_i oriented in the same direction as K_i, and set

$$\mathrm{A}(F_i, R) \equiv \mathrm{B}(F_i, R) - \frac{1}{2}\,\mathrm{lk}(K_i, \hat{K}_i) \pmod 8.$$

Since $\mathrm{B}(F_i, R)$ is an invariant, $\mathrm{A}(F_i, R)$ is also an invariant of R in S^3. Though $\mathrm{A}(F_i, R)$ is not an invariant of a knot $K_i = \partial F_i$, we have

Theorem 3.2. *Let* $R = F_1 \cup F_2 \cup \cdots \cup F_n$ *be an unoriented, possibly non-orientable R-complex and* $L = \partial F_1 \cup \partial F_2 \cup \cdots \cup \partial F_n$ *a link. Then the following formula holds.*

$$4\,\mathrm{Arf}(L) \equiv \sum_{i=1}^{n} \mathrm{A}(F_i, R) - \sum_{i<j} \mathrm{lk}(\partial F_i, \partial F_j) \pmod 8.$$

Hence $\sum_{i=1}^{n} \mathrm{A}(F_i, R)$ *is an invariant of* L.

Remark 3.3. For an R-complex $R = F_1 \cup F_2 \cup \cdots \cup F_n$, if $F_1, F_2, \ldots, F_n$ are mutually disjoint, then $\varphi(\alpha) \equiv \mathrm{lk}(a, \tau a) \pmod 4$ for any α. This implies that $\mathrm{A}(F_i, R)$ is the same as an invariant of a knot ∂F_i, defined by P. Gilmer [**6**], [**7**]. Hence we have $\mathrm{A}(F_i, R) = 4\,\mathrm{Arf}(\partial F_i)$ [**6**].

Let L be a proper link and L' a sublink of L. Suppose that L' is proper and it bounds an unoriented, possibly disconnected surface F with $F \cap (L - L') = \emptyset$. (Note that $L - L'$ is also a proper link.) For this surface F, we can define a $\boldsymbol{Z}_4$-quadratic function $\varphi : H_1(F; \boldsymbol{Z}_2) \to \boldsymbol{Z}_4$ as follows. Suppose $\alpha \in H_1(F; \boldsymbol{Z}_2)$ is represented by a simple closed curve a in F. Define $\varphi(\alpha) \in \boldsymbol{Z}_4$ by

$$\varphi(\alpha) \equiv \mathrm{lk}(a, \tau a) + 2\,\mathrm{lk}(a, L - L') \pmod 4.$$

Claim 3.4. *Let* $V = \{\alpha \in H_1(F; \boldsymbol{Z}_2) \mid \alpha \cdot x = 0$ *for any* $x \in H_1(F; \boldsymbol{Z}_2)\}$. *The* $\boldsymbol{Z}_4$-*quadratic function* $\varphi : H_1(F; \boldsymbol{Z}_2) \to \boldsymbol{Z}_4$ *above vanishes on* V.

We call a $\boldsymbol{Z}_4$-quadratic function on $H_1(F; \boldsymbol{Z}_2)$ *proper* [**6**] (or *informative* [**10**]) if it vanishes on V.

Proof. Set $\partial F = L' = K_1 \cup K_2 \cup \cdots \cup K_m$ and let $\alpha_i \in H_1(F; \boldsymbol{Z}_2)$ be 1-cycle that represented by K_i $(i = 1, 2, \ldots, m)$. Let $\alpha \in V$ and a a simple closed curve in F representing α. Since $\alpha \cdot x = 0$ for any $x \in H_1(F; \boldsymbol{Z}_2)$, we may assume that a separates F. This implies that V is generated by $\alpha_1, \alpha_2, \ldots, \alpha_m$. Hence it is sufficient to show that $\varphi(\alpha_i) = 0 \in \boldsymbol{Z}_4$ for any $i (= 1, 2, \ldots, m)$. By the assumption that L' is proper and the fact that K_i and $L' - K_i$ cobound F, we have $\mathrm{lk}(K_i, \tau K_i) = 2\mathrm{lk}(K_i, \hat{K}_i) \equiv 2\mathrm{lk}(K_i, L' - K_i) \equiv 0 \pmod 4$, for any K_i. Thus we have

$$\varphi(\alpha_i) \equiv \mathrm{lk}(K_i, \tau K_i) + 2\mathrm{lk}(K_i, L - L') \equiv 2\mathrm{lk}(K_i, L - K_i) \pmod 4.$$

Since L is a proper link, we have $2\mathrm{lk}(K_i, L - K_i) \equiv 0 \pmod 4$. □

We define the *Brown invariant* $B(F) \in \boldsymbol{Z}_8$ to be the Brown invariant of the proper $\boldsymbol{Z}_4$-quadratic function φ. That is, $B(F)$ is defined by

$$\sqrt{2}^{\dim H_1(F; \boldsymbol{Z}_2) + \dim V} \exp(\pi\sqrt{-1}\mathrm{B}(F, L)/4) = \sum_{x \in H_1(F; \boldsymbol{Z}_2)} \sqrt{-1}^{\varphi(x)}.$$

This formula is due to E. H. Brown [**3**] in the case that $V = \{0\}$. Its extension to proper forms is due to V. M. Kharlamov and O. Ya. Viro [**10**]. Let $\hat{L}'$ be a parallel copy of L' on F oriented in the same direction as L', and set

$$\mathrm{A}(F, L) \equiv \mathrm{B}(F, L) - \frac{1}{2}\,\mathrm{lk}(L', \hat{L}') \pmod 8.$$

THEOREM 3.5. *Let L be a proper link and L' a sublink of L. Suppose that L' is proper and it bounds an unoriented, possibly disconnected surface F with $F \cap (L - L') = \emptyset$. Then we have*

$$4\,\mathrm{Arf}(L) \equiv 4\,\mathrm{Arf}(L - L') + \mathrm{A}(F, L) - \mathrm{lk}(L', L - L') \pmod 8.$$

Hence $\mathrm{A}(F, L)$ is an invariant of L.

REMARK 3.6. Let L, L' and F be as in the theorem above.

(1) Since $\mathrm{lk}(L', L - L')(= \mathrm{lk}(\partial F, L - \partial F))$ is even, by Theorem 3.5, $\mathrm{A}(F, L)/2$ is a $\boldsymbol{Z}_4$-valued link invariant. Let L_0, L_1, L_{-1}, and L_2 be the 2-component trivial link, $(2, 4)$-torus link, its mirror image, and the Whitehead link, respectively. Let F_i be an unoriented surface bounding one component of L_i without intersecting the other component ($i = 0, \pm 1, 2$). It follows from Theorem 3.5 that $\mathrm{A}(F_i, L) \equiv i \pmod 4$. This implies $\mathrm{A}(*)/2$ can take any value in $\boldsymbol{Z}_4$.

(2) If $L - L'$ and F are separated by a 2-sphere, then by $\varphi(\alpha) \equiv \mathrm{lk}(a, \tau a) \pmod 4$ for any α. This implies that $\mathrm{A}(F, L)$ is the same as an invariant of a link $\partial F = L'$, defined by P. Gilmer [**6**], [**7**]. Hence we have $\mathrm{A}(F, L) = 4\,\mathrm{Arf}(L')$ [**6**].

We shall prove Theorems 3.2 and 3.5 in the last section. Theorem 3.5 implies the following.

COROLLARY 3.7. *Let L be a proper link and K a component of L. Suppose that $\mathrm{lk}(K, K')$ is even for any component $K'(\neq K)$. Then for any unoriented surface F with $F \cap L = \partial F = K$,*

$$4\,\mathrm{Arf}(L) \equiv 4\,\mathrm{Arf}(L - K) + \mathrm{A}(F, L) - \mathrm{lk}(K, L - K) \pmod 8. \qquad \square$$

In [**19**], M. Saito defined the unoriented Sato-Levine invariant $\beta^*(L) \in \boldsymbol{Z}_4$ for a 2-component proper link L. Note that when L is a 2-component link, L is proper if and only if L is $\boldsymbol{Z}_2$-algebraically split. By using Corollary 3.7, we have the following proposition.

PROPOSITION 3.8. *Let $L = K \cup K'$ be a 2-component proper link. For any unoriented surface F with $F \cap L = \partial F = K$, $\mathrm{A}(F, L) - 4\,\mathrm{Arf}(K) - 2\,\mathrm{lk}(K, K') \equiv 2\beta^*(L) \pmod 8$.*

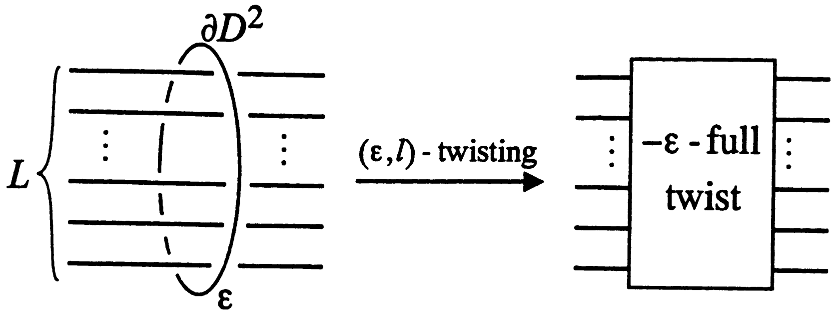

Figure 2.

Since $\beta^*(L)$ is in $\boldsymbol{Z}_4$, by the proposition above, we can identify $\mathrm{A}(F,L)/2 - 2\,\mathrm{Arf}(K) - \mathrm{lk}(K,K') \in \boldsymbol{Z}_4$ with $\beta^*(L)$.

Proof. It follows from [**19**, Theorem 4.1] that $2\beta^*(L) \equiv 4a_3(L) - \mathrm{lk}(K,K')$ $(\mathrm{mod}\,8)$, where $a_3(L)$ is the third coefficient of the Conway polynomial of L. K. Murasugi [**16**], H. Murakami [**13**] and J. Hoste [**8**] have shown that the sum of $\mathrm{Arf}(L)$, $\mathrm{Arf}(K)$ and $\mathrm{Arf}(K')$ is mod 2 congruent to $a_3(L)$. Combining these and Corollary 3.7, we get

$$4a_3(L) \equiv 4\,\mathrm{Arf}(L) + 4\,\mathrm{Arf}(K) + 4\,\mathrm{Arf}(K') \equiv \mathrm{A}(F,L) - 4\,\mathrm{Arf}(K) - \mathrm{lk}(K,K') \pmod 8.$$

This completes the proof. □

Remark 3.9. If F is an orientable surface, then $\mathrm{A}(F,L) = \mathrm{B}(F,L)$ and then by [**3**, Theorem 1.20, (vii)], $\mathrm{B}(F,L) = 4\,\mathrm{Arf}(F,L)$. By [**19**, Remark 2.3], $2\beta(L) \equiv \beta^*(L)$ $(\mathrm{mod}\,4)$. It follows from Proposition 3.8 that $4\mathrm{Arf}(F,L) - 4\,\mathrm{Arf}(K) \equiv 4\beta(L) \pmod 8$. This gives us an alternate proof of Proposition 2.6.

4. Local moves.

Let L be a link in S^3, and D^2 a disk intersecting L in its interior. Let $l = |\mathrm{lk}(\partial D^2, L)|$, and $\varepsilon = 1$ or $= -1$. An ε-Dehn surgery along ∂D^2 changes L into a new link L' in S^3 (Figure 2). We say that L' is obtain from L by (ε, l)-*twisting*.

Proposition 4.1. *Let L_1 and L_2 be links such that L_2 is obtained from L_1 by a single (ε, l)-twisting. If L_1 is a proper link and l is odd, then L_2 is proper and*

$$\mathrm{Arf}(L_2) - \mathrm{Arf}(L_1) \equiv \frac{l^2-1}{8} \pmod 2.$$

A local move on a link diagram as shown in Figure 3 is called a $\sharp(l,m)$-move. If both l and m are multiples of a prime p, then a $\sharp(l,m)$-move is called a $\sharp^p$-move [**12**]. A

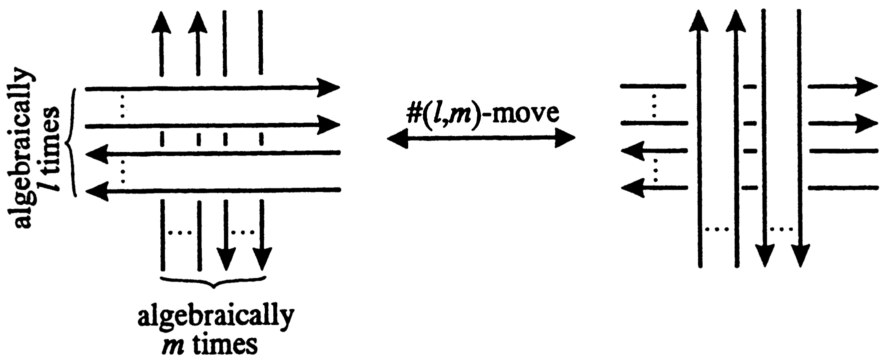

Figure 3.

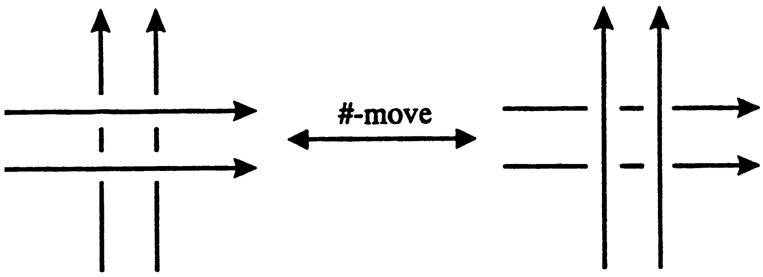

Figure 4.

$\sharp^p$-move is a generalized $\sharp$-move, where the $\sharp$-move is a local move, defined by H. Murakami [**14**], as in Figure 4.

Proposition 4.2. *Let L_1 and L_2 be links such that L_2 is obtained from L_1 by a single $\sharp(l,m)$-move. If L_1 is a proper link and l, m are even, then L_2 is proper and*

$$\mathrm{Arf}(L_2) - \mathrm{Arf}(L_1) \equiv \frac{lm}{4} \pmod 2.$$

If a link L is proper, then by [**15**, Theorem A.2], L is deformed into a trivial link by $\sharp$-moves. Since the $\sharp$-move is an example of $\sharp(2,2)$-move and Arf invariant of a trivial link is equal to 0 (Corollary 1.3), Proposition 4.2 gives the following proposition.

Proposition 4.3 ([**14**, Theorem 3.5]). *Let L be a proper link and m a number of $\sharp$-moves needed to deform L into a trivial link. Then* $\mathrm{Arf}(L) \equiv m \pmod 2$. □

To prove Proposition 4.1, we need the following lemma.

Lemma 4.4. *Let L_1 and L_2 be links. Let M be a twice punctured εCP^2. If L_2 is obtained from L_1 by a single (ε, l)-twisting, then there exists a disjoint union F of annuli in M such that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and F represents a homology class $l\gamma$, where γ is a standard generator of $H_2(M, \partial M; \boldsymbol{Z})$ with $\gamma \cdot \gamma = \varepsilon$.*

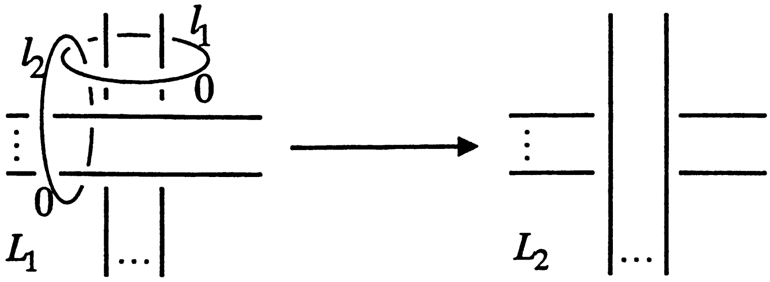

Figure 5.

Proof. Set $(M_0, F) = (S^3, L_1) \times I$. We may assume that $(\partial M_0, \partial F) = (-S^3 \times \{0\}, -L_1) \cup (S^3 \times \{1\}, L_1)$. Let D^2 be a disk in $S^3 \times \{1\}$ such that $|\mathrm{lk}(\partial D^2, L_1)| = l$ and an ε-Dehn surgery along ∂D^2 changes L_1 into L_2. Attach 2-handle H to M_0 along ∂D^2 with framing ε. The resulting 4-manifold $M = M_0 \cup H$ is a twice punctured εCP^2. It is not hard to see that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and F represents a homology class $l\gamma$. □

Proof of Proposition 4.1. By Lemma 4.4, there exists a disjoint union F of annuli in a twice punctured εCP^2, say M, such that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and $[F] = l\gamma$. Note that $l\gamma$ is characteristic because l is odd. If L_2 is a proper link, then the proposition follows from Proposition 1.4. We shall prove that L_2 is proper.

Since L_1 is a proper link, by Theorem 1.1, there exist a compact, simply connected 4-manifold M' with $\partial M' \cong S^3$, and a disjoint union Δ of 2-disks in M' such that Δ represents a characteristic homology class in $H_2(M', \partial M'; \boldsymbol{Z})$ and $(\partial M', \partial \Delta) \cong (S^3, L_1)$. Set $M'' = M \cup_f M'$ and $\Delta' = F \cup_f \Delta$, where f is an orientation reversing diffeomorphism from $(\partial M', \partial \Delta)$ to $(-S^3, -L_1)$. Then we note that $(\partial M'', \partial \Delta') \cong (S^3, L_2)$, Δ' is a disjoint union of 2-disks in M'' and $[\Delta']$ is characteristic. It follows from Theorem 1.1 that L_2 is proper. We have completed the proof. □

To prove Proposition 4.2, we use the following lemma.

Lemma 4.5. *Let L_1 and L_2 be links. Let M be a twice punctured $S^2 \times S^2$. If L_2 is obtained from L_1 by a single $\sharp(l, m)$-move, then there exists a disjoint union F of annuli in M such that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and F represents a homology class either $l\alpha + m\beta$ or $l\alpha - m\beta$, where α, β are standard generators of $H_2(M, \partial M; \boldsymbol{Z})$ with $\alpha \cdot \alpha = \beta \cdot \beta = 0$ and $\alpha \cdot \beta = 1$.*

Proof. Set $(M_0, F) = (S^3, L_1) \times I$. We may assume that $(\partial M_0, \partial F) = (-S^3 \times \{0\}, -L_1) \cup (S^3 \times \{1\}, L_1)$. Figure 5 shows that doing 0-surgeries along l_1 and l_2 have the same effect on L_1 as $\sharp(l, m)$-move. Attach 2-handles H_1 and H_2 to M_0 with framing 0 along l_1 and l_2 respectively. The resulting 4-manifold $M = M_0 \cup H_1 \cup H_2$ is a twice

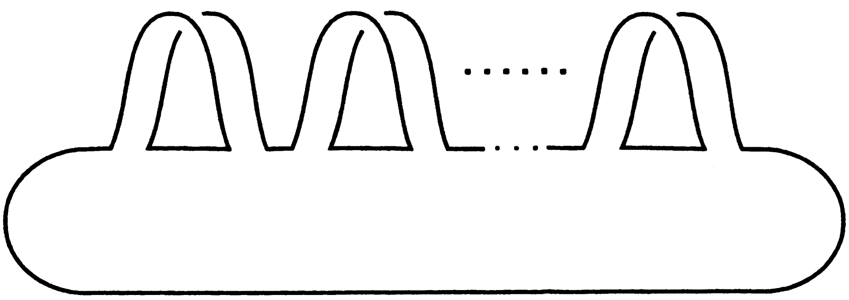

Figure 6.

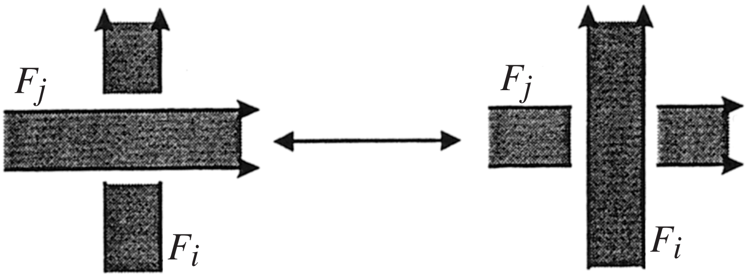

Figure 7.

punctured $S^2 \times S^2$. It is not hard to see that $(\partial M, \partial F) \cong (-S^3, -L_1) \cup (S^3, L_2)$ and F represents a homology class either $l\alpha + m\beta$ or $l\alpha - m\beta$. □

Proof of Proposition 4.2. By Lemma 4.5 and the arguments similar to that in Proof of Proposition 4.1, we obtain the proposition. □

A link is a *$\boldsymbol{Z}_2$-boundary link* if the components bounds mutually disjoint (not necessarily orientable) surface in S^3.

T. Shibuya [**22**] and M. Saito [**19**] proved the following proposition. T. Shibuya showed that the proposition below holds in more general situation. Here we give a proof by using Proposition 4.3.

Proposition 4.6. ([**22**, Theorem], [**19**, Proposition 5.3]). *If $L = K_1 \cup K_2 \cup \cdots \cup K_n$ is a $\boldsymbol{Z}_2$-boundary link, then the following formula holds.*

$$\mathrm{Arf}(L) \equiv \sum_{i=1}^{n} \mathrm{Arf}(K_i) - \frac{1}{4}\sum_{i<j} \mathrm{lk}(K_i, K_j) \pmod 2.$$

Proof. Let $F_1, F_2, \ldots, F_n$ be mutually disjoint surfaces with $\partial F_i = K_i$ $(i = 1, 2, \ldots, n)$. We may assume that each F_i is non-orientable and that each F_i is an image of embedding of a surface as in Figure 6. (When a surface F_i is orientable, attach a 1-handle H^1 to F_i so that $F_i \cup H^1$ is non-orientable and $H^1 \cap (F_1 \cup F_2 \cup \cdots \cup F_n - F_i) = \varnothing$.) By $\sharp$-move as in Figure 7, we can deform $F_1 \cup F_2 \cup \cdots \cup F_n$ into a split sum without

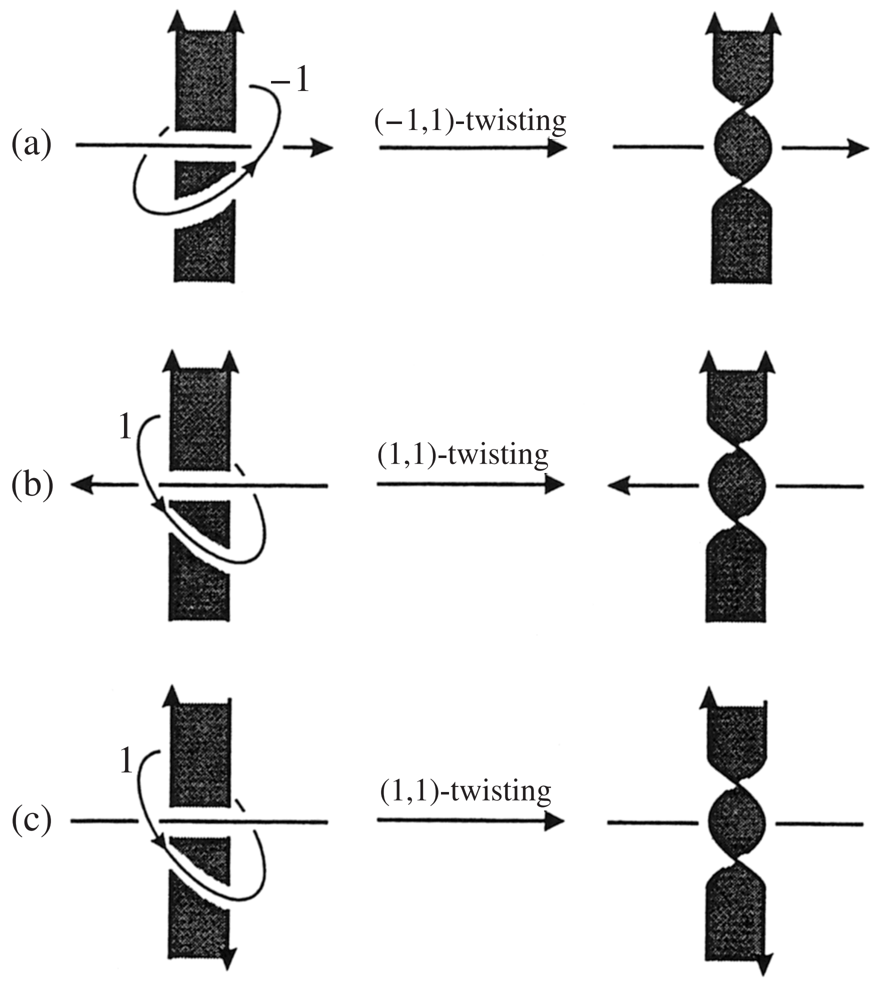

Figure 8.

changing each F_i. Let p be a number of $\sharp$-moves needed in this deformations. Note that

$$p \equiv \frac{1}{4}\sum_{i<j} \mathrm{lk}(K_i, K_j) \pmod 2.$$

Let q_i be a number of $\sharp$-moves needed to deform K_i into a trivial knot $(i = 1, 2, \dots, n)$. Hence, a number of $\sharp$-moves needed to deform L into a trivial link is equal to $\sum_{i=1}^{n} q_i + p$. By Proposition 4.3, we have

$$\mathrm{Arf}(L) \equiv \sum_{i=1}^{n} q_i - p \equiv \sum_{i=1}^{n} \mathrm{Arf}(K_i) - \frac{1}{4}\sum_{i<j} \mathrm{lk}(K_i, K_j) \pmod 2. \qquad \square$$

Proof of Theorem 3.2. Deform R-complex $R = F_1 \cup F_2 \cup \cdots \cup F_n$ into a disjoint union $R' = F_1' \cup F_2' \cup \cdots \cup F_n'$ of surfaces by $(\pm 1, 1)$-twistings as in Figure 8. Set $\partial F_i' = K_i'$ $(i = 1, 2, \dots, n)$. It is not hard to see that neither the $\boldsymbol{Z}_4$-quadratic functions on $H_1(F_i; \boldsymbol{Z}_2)$ $(i = 1, 2, \dots, n)$ nor the value of $\sum_{i=1}^{n} \mathrm{lk}(K_i, \hat{K}_i)/2 + \sum_{i<j} \mathrm{lk}(K_i, K_j)$ changes

under $(\pm 1, 1)$-twistings. Hence we have

$$\sum_{i=1}^{n} \mathrm{A}(F_i, R) - \sum_{i<j} \mathrm{lk}(K_i, K_j) \equiv \sum_{i=1}^{n} \mathrm{A}(F_i', R') - \sum_{i<j} \mathrm{lk}(K_i', K_j') \pmod 8.$$

By Proposition 4.1, $\mathrm{Arf}(L) = \mathrm{Arf}(\partial R')$. Since $\partial R'$ is a $\boldsymbol{Z}_2$-boundary link, by Remark 3.3, $\mathrm{A}(F_i', R') = 4\,\mathrm{Arf}(\partial F_i')$, and by Proposition 4.6,

$$\mathrm{Arf}(\partial R') \equiv \sum_{i=1}^{n} \mathrm{Arf}(\partial F_i') - \frac{1}{4}\sum_{i<j} \mathrm{lk}(K_i', K_j') \pmod 2.$$

Thus we have the desired formula. □

Proof of Theorem 3.5. By $(\pm 1, 1)$-twistings as in Figure 8, deform F into F' so that $L - L'$ and F' are separated by a 2-sphere. Since $(\pm 1, 1)$-twistings preserve both the $\boldsymbol{Z}_4$-quadratic function on $H_1(F; \boldsymbol{Z}_2)$ and the value of $\mathrm{lk}(L', \hat{L}')/2 + \mathrm{lk}(L', L - L')$, we have

$$\mathrm{A}(F, L) - \mathrm{lk}(L', L - L') \equiv \mathrm{A}(F', (L - L') \cup \partial F') \pmod 8.$$

By Proposition 4.1, $\mathrm{Arf}(L) = \mathrm{Arf}((L - L') \cup \partial F')$. Since $L - L'$ and F' are separated, by Remark 3.6, (2), $\mathrm{A}(F', (L - L') \cup \partial F') = 4\,\mathrm{Arf}(\partial F')$, and by Proposition 4.3,

$$\mathrm{Arf}((L - L') \cup \partial F') \equiv p + q \equiv \mathrm{Arf}(L - L') + \mathrm{Arf}(\partial F') \pmod 2,$$

where p (resp. q) is a number of $\sharp$-moves needed to deform $L - L'$ (resp. $\partial F'$) to be trivial. Hence we have the desired formula. □

Proof of Theorem 2.2. If an R-complex $R = F_1 \cup F_2 \cup \cdots \cup F_n$ is orientable, then $\mathrm{lk}(K_i, \hat{K}_i) = 0$, and then by [**3**, Theorem 1.20, (vii)], $\mathrm{B}(F_i, R) = 4\,\mathrm{Arf}(F_i, R)$ $(i = 1, 2, \dots, n)$. Hence we have $\mathrm{A}(F_i, R) = 4\,\mathrm{Arf}(F_i, R)$. Theorem 2.2 follows from Theorem 3.2. □

Proof of Theorem 2.3. If F is orientable, then $\mathrm{lk}(L', \hat{L}') = 0$, and then by [**3**, Theorem 1.20, (vii)], $\mathrm{B}(F, L) = 4\,\mathrm{Arf}(F, L)$. Hence we have $\mathrm{A}(F, L) = 4\,\mathrm{Arf}(F, L)$. Theorem 2.3 follows from Theorem 3.5. □

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