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A Kummer type construction of self-dual metrics on the connected sum of four complex projective planes

By Nobuhiro HONDA† and Mitsuhiro ITOH‡

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Abstract. We show that there exist on $4\boldsymbol{CP}^2$, the connected sum of four complex projective planes, self-dual metrics with the following properties: (i) the sign of the scalar curvature is positive, (ii) the identity component of the isometry group is $U(1)$, (iii) the metrics are not conformally isometric to the self-dual metrics constructed by LeBrun [**LB1**]. These are the first examples of self-dual metrics with non semi-free $U(1)$-isometries on simply connected manifolds. Our proof is based on the twistor theory: we use an equivariant orbifold version of the construction of Donaldson and Friedman [**DF**]. We also give a rough description of the structure of the algebraic reduction of the corresponding twistor spaces.

1. Introduction.

Let $S^3 \times S^1$ be the Riemannian product of the standard spheres, which is a conformally flat 4-manifold. We denote by R_p and R_q the reflections with respect to points $p \in S^3$ and $q \in S^1$ respectively. Let $\phi_p := R_p \cdot (-1)$ and $\phi_q := R_q \cdot (-1)$ denote the composition of the reflections with the anti-podal maps (-1), whose fixed points are $\{\pm p\}$ and $\{\pm q\}$ respectively. Then it is easy to see that $\tau := (\phi_p, \phi_q)$ is an orientation preserving isometric involution on $S^3 \times S^1$ and that the fixed points set consists of four points $(p,q), (-p,q), (p,-q)$ and $(-p,-q)$. Let $M_0 := S^3 \times S^1/\langle\tau\rangle$ denote the conformally flat Riemannian orbifold with four orbifold points obtained as a quotient of $S^3 \times S^1$. It is readily seen that M_0 is simply connected and $b_2 = 0$.

On the other hand, let M_{EH} be the (compactified) Eguchi–Hanson space, which has a unique orbifold point whose isotropy group is $\boldsymbol{Z}_2$. M_{EH} is also simply connected and $b_2 = 1$.

Let $M_0 \#_{\boldsymbol{Z}_2} 4M_{EH}$ be the (smooth) 4-manifold obtained by connecting M_0 with four copies of M_{EH} at the orbifold points. $M_0 \#_{\boldsymbol{Z}_2} 4M_{EH}$ is diffeomorphic to $4\boldsymbol{CP}^2$, the connected sum of four complex projective planes. Then the results of LeBrun–Singer ([**LS**, Theorem A]) and Pontecorvo ([**Pont**, Proposition 2.4]) imply that $4\boldsymbol{CP}^2$ admits a self-dual metric. In brief, $4\boldsymbol{CP}^2$ has a self-dual metric originated from so called a *Kummer type construction* ([**LS**]).

In this paper, we shall investigate this example, which seems to be basic one, in detail and show that among the self-dual metrics on $4\boldsymbol{CP}^2$ obtained in this way, there exists a

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family of self-dual metrics of positive scalar curvature with $U(1)$-symmetry which are not LeBrun's metrics ([**LB1**]). More precisely, we show the following:

Theorem 1.1. *There exists a self-dual metric g on $4\boldsymbol{CP}^2$ with the following properties*: (i) *the scalar curvature of g is positive type,* (ii) *the identity component of the group of orientation preserving conformal transformations of g is $U(1)$,* (iii) *g is not conformally isometric to the self-dual metrics of LeBrun* ([**LB1**]).

The proof of the theorem is based on an equivariant version of a result of LeBrun–Singer ([**LS**]). We firstly construct a normal crossing 3-fold $Z' = Z_0' \cup 4Z_{EH}'$, where Z_0' and Z_{EH}' are the resolution of the twistor space of M_0 and M_{EH} respectively. Then we will see that there is a $U(1)$-action on Z' which is induced by those on M_0 and M_{EH}. Further we will observe that there exists a $U(1)$-invariant Cartier divisor S' on Z' and show that the pair (Z', S') can be smoothed preserving the $U(1)$-action to give a twistor space Z of $4\boldsymbol{CP}^2$. Then the self-dual metric on $4\boldsymbol{CP}^2$ corresponding to Z is the required one.

We also study an algebraic structure of the above twistor space Z. We recall that there exists a natural square root, which we will denote by $-(1/2)K_Z$, of the anticanonical bundle of Z. Let $|-(1/2)K_Z|$ denote the associated complete linear system on Z. Then we have

Proposition 1.2. *Let Z be the twistor space of $4\boldsymbol{CP}^2$ as above. Then $|-(1/2)K_Z|$ is two-dimensional and has no base locus, and the morphism $\varphi : Z \to \boldsymbol{CP}^2$ induced by $|-(1/2)K_Z|$ is an elliptic fibration.* (*That is, a general fiber of φ is an elliptic curve.*) *Further, the algebraic dimension of Z is two.*

Let us explain another motivation of this paper. Recently, Campana and Kreussler ([**CK**]) has obtained the following result: Let Z be a twistor space over $4\boldsymbol{CP}^2$ and S be a real smooth element of $|-(1/2)K_Z|$. Assume that $|-K_S|$ contains a smooth curve C and let $N_{C/S}$ be the normal bundle of C in S. It is easy to see that the degree of $N_{C/S}$ is zero. Let $a(Z)$ be the algebraic dimension of Z. Then they showed that $(1 \leq)\ a(Z) \leq 2$ and the equality holds if and only if $N_{C/S}$ is of finite order in $\mathrm{Pic}^0 C$. Then they asked which number can be realized by the order of $N_{C/S}$. Our investigation of the twistor spaces via Donaldson–Friedman construction shows that the smallest value $(=1)$ can be realized by the twistor space in Proposition 1.2. See also [**Hon3**].

This paper is organized as follows: In Section 2, we explain an equivariant version of a result of LeBrun–Singer ([**LS**]), which is fundamental for our investigations. Key examples of compact self-dual reflection orbifolds and their equivariant connected sum are also given. In Section 3, the twistor spaces of the two orbifolds M_0 and M_{EH} are described.

Section 4 is a main part of this paper. In §4.1, we construct a pair (Z', S') of a normal crossing 3-fold with a holomorphic $U(1)$-action and an invariant Cartier divisor on it. Then in §4.2–4.5 we show that a $U(1)$-equivariant smoothing of (Z', S') exists and the resulting pair (Z, S) satisfies (i) Z is a twistor space of $4\boldsymbol{CP}^2$, (ii) S is an element of $|-(1/2)K_Z|$ and has a structure of rational elliptic surface (with a $U(1)$-action). Finally in Section 5, we study the algebraic structure of the above pair (Z, S) and complete the proof of Theorem 1.1 and Proposition 1.2.

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2. Equivariant connected sum of self-dual reflection orbifolds.

In their paper ([**DF**]) Donaldson–Friedman developed a general theory for constructing self-dual metrics on the connected sum of compact self-dual manifolds, using twistor theory and deformation theory of compact complex spaces. Later, LeBrun–Singer ([**LS**]) generalized their construction to the case of connected sum of orbifolds whose isotropy groups at the orbifold points are $\{\pm 1\}$. In the first half of this section, we explain an equivariant version of their result. We only give statements, since the proofs are almost parallel to those of Pedersen–Poon ([**PP2**]), who developed an equivariant version of the original construction of Donaldson–Friedman. In the latter half, a key example of an equivariant connected sum is given.

We recall that an orbifold is said to be a *reflection orbifold* ([**LS**]) if the isotropy group at each orbifold point is $\{\pm 1\}$. If M_1 and M_2 are oriented reflection orbifolds of the same dimension and if an orbifold point of each M_i is specified, the connected sum at the orbifold points can be made in an obvious way (cf. [**LS**]) and we denote the resulting orbifold (or manifold) by $M_1 \#_{\boldsymbol{Z}_2} M_2$.

Let (M_1, g_1) and (M_2, g_2) be compact self-dual reflection orbifolds. We assume for simplicity that M_1 and M_2 have unique orbifold points $p_1 \in M_1$ and $p_2 \in M_2$ respectively. Further we suppose that a compact Lie group G is acting on both M_1 and M_2 isometrically and leaving p_i, $i = 1, 2$ fixed. Let Z_i $(i = 1, 2)$ be the twistor space of (M_i, g_i) and L_i be the twistor line over p_i. Z_i has A_1-singularities along L_i (cf. [**LS**]). We have a holomorphic G-action on Z_i which is induced by that on M_i.

Let $\mu_i : Z_i' \to Z_i$ $(i = 1, 2)$ be the blowing-up along L_i and Q_i be the exceptional divisor. Z_i' is non-singular and also has a G-action. It is easy to see that $Q_i \simeq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and $N_{Q_i/Z_i'} \simeq \mathcal{O}(-2, 2)$, where $\mathcal{O}(0, 1)$ denotes the pull-back of $\mathcal{O}_{L_i}(1)$ ([**LS**]).

Let $\phi : Q_1 \to Q_2$ be a biholomorphic map which preserves the real structures and satisfies $\phi^* \mathcal{O}_{Q_2}(1, 0) \simeq \mathcal{O}_{Q_1}(0, 1)$. We further assume that ϕ is G-equivariant. (The existence of such an isomorphism is equivalent to the condition that G-equivariant connected sum can be made at the orbifold points.) Using this isomorphism, we set ([**DF**, **LS**])

$$Z' := Z_1' \bigcup_{\phi} Z_2'.$$

Z' is a normal crossing variety which has a real structure σ' induced by those on Z_1 and Z_2. Then we have

$$\Theta^1_{Z'} \simeq N_{Q_1/Z_1'} \otimes N_{Q_2/Z_2'} \simeq \mathcal{O}_Q,$$

(here we set $Q := Q_1 \simeq Q_2 \subseteq Z'$) and hence Z' is d-semi-stable in the sense of R. Friedman, and we may consider smoothings of Z'.

The following lemma is a key to prove Proposition 2.3 below:

Lemma 2.1. (*cf.* [**PP2**]) *Let $G, (M_i, g_i), p_i, Z_i, Z_i'$ $(i = 1, 2)$ and Z' be as above. Then the action of G on $H^0(\Theta^1_{Z'}) \simeq \boldsymbol{C}$ induced by that on Z' is the trivial action.*

Proposition 2.2. ([**LS**]) *With the notations in Lemma* 2.1, *suppose that*

$$H^2(\Theta_{Z_1', Q_1}) = H^2(\Theta_{Z_2', Q_2}) = 0.$$

Then we have $T^2_{Z'} = 0$.

Proposition 2.3. *With the notations and assumptions in Proposition* 2.2, *let* $p : \mathscr{Z} \to B \subseteq T^1_{Z'}, p^{-1}(0) \simeq Z'$ *be the Kuranishi-family of deformations of* Z', *where we may regard* B *as an open ball in* $T^1_{Z'}$ *containing* 0 *by Proposition* 2.2. *Let* $B^{G,\sigma}$ *denote the subspace of* B *whose points are* G*-invariant and real. Then* $B^{G,\sigma}$ *is at least one-dimensional and for general element* t *of* $B^{G,\sigma}$, $Z_t := p^{-1}(t)$ *has a natural structure of a twistor space of* $M_1 \#_{Z_2} M_2$ *with* G*-symmetry.*

Next we give examples of compact self-dual reflection orbifolds with $U(1)$-actions and their equivariant connected sum. We recall that an action of a Lie group G on a manifold (or an orbifold) is said to be *semi-free* if the isotropy group is either $\{e\}$ or G itself at every point.

Example 2.4. Let $\varpi : M^\circ_{EH} \to \boldsymbol{C}^2/\{\pm 1\}$ be the minimal resolution of the (A_1-) singularity. M°_{EH} has a hyperKähler metric g°_{EH} called *Eguchi–Hanson metric*. It was shown in [**Kr**] that $(M^\circ_{EH}, g^\circ_{EH})$ has a one point compactification (M_{EH}, g_{EH}) as an anti-self-dual reflection orbifold. We reverse the complex orientation and will work on self-duality.

For later references, we introduce a $U(1)$-action on M_{EH}. Let (z, w) be complex coordinates on $\boldsymbol{C}^2$ and consider a $U(1)$-action defined by

$$(z, w) \mapsto (z, tw)$$

for $t \in U(1)$. This induces an isometric $U(1)$-action on M_{EH}. We note that this action is not semi-free. In fact, any point of M_{EH} which lies on the image of $\{z = 0\} \backslash \{(0, 0)\} (\subseteq \boldsymbol{C}^2)$ has the isotropy group $\{\pm 1\}$.

Example 2.5. (cf. [**I4**] or Introduction) Let $(S^3 \times S^1, g_3 \oplus g_1)$ be the Riemannian product of the spheres, where g_k denotes the standard metric on S^k. $g_3 \oplus g_1$ is conformally flat. We embed S^3 in $\boldsymbol{C}^2$ as the unit sphere and let $p := (i, 0)$ be a point on S^3. Let R_p denote the reflection with respect to p, that is $R_p(z, w) = (\bar{z}, w)$. Then $\phi_p(z, w) := (-\bar{z}, -w)$ defines an orientation reversing isometric involution whose fixed points are $\{\pm p\}$. Similarly, S^1 has such an isometric involution ϕ_q, $q \in S^1$. We get then an isometric involution τ on $S^3 \times S^1$;

$$\tau(x, y) = (\phi_p(x), \phi_q(y)), \quad (x, y) \in S^3 \times S^1.$$

τ is orientation preserving with the fixed-points $(p, q), (-p, q), (p, -q)$ and $(-p, -q)$.

It follows that $M_0 := S^3 \times S^1/\langle\tau\rangle$ is a reflection orbifold with a conformally flat metric g_0, having four orbifold points. It is clearly seen that M_0 is simply connected and $b_2 = 0$. We also note that if one regards $S^3 \times S^1$ as a quotient of $\boldsymbol{C}^2 \backslash \{0\} = \boldsymbol{H} \backslash \{0\}$ by the $\boldsymbol{Z}$-action defined by $q \mapsto \lambda^n q$ for $q \in \boldsymbol{H} \backslash \{0\}$ and $n \in \boldsymbol{Z}$ ($\lambda > 0$, $\lambda \neq 1$), τ is induced by $q \mapsto q^{-1}$.

Next we introduce a $U(1)$-action. For $t \in U(1)$ and $x = (z, w) \in S^3 \subseteq \boldsymbol{C}^2$ and $y \in S^1$, we set

$$t(x, y) = ((z, tw), y).$$

Then this $U(1)$-action is isometric and commutes with the involution τ, and hence we get an isometric $U(1)$-action on M_0. Since the maps τ and $t = -1 \in U(1)$ coincide on the disjoint two spheres $\{z = -\bar{z}, y = 1\}$ and $\{z = -\bar{z}, y = -1\}$, the $U(1)$-action on M_0 has non-trivial isotropy $\{\pm 1\}$ on their images.

Example 2.6. (cf. [**I4**]) We consider an equivariant connected sum of the above two examples. It is readily seen that we can make a $U(1)$-equivariant connected sum $M_0 \#_{\boldsymbol{Z}_2} 4M_{EH}$ and the resulting $U(1)$-action on $M_0 \#_{\boldsymbol{Z}_2} 4M_{EH}$ is not semi-free. In fact, there exist disjoint two spheres on which $t = -1 \in U(1)$ acts trivially. The 4-manifold $M_0 \#_{\boldsymbol{Z}_2} 4M_{EH}$ is diffeomorphic to $4\boldsymbol{CP}^2$.

Applying Proposition 2.3 to Example 2.6, we can prove the assertions of (ii) and (iii) of Theorem 1.1. But to prove (i), we need to consider additional data: we will observe that there exists a $U(1)$-invariant Cartier divisor S' on Z' (see §4.1) and consider $U(1)$-equivariant deformations of the pair (Z', S'). Due to this consideration, we can obtain a twistor space of $4\boldsymbol{CP}^2$ with a $U(1)$-action which is a special one.

3. Descriptions of the twistor spaces.

3.1. The twistor space of the Eguchi–Hanson space

In this subsection, we use the notations in Example 2.4. The twistor space of the hyperKähler manifold $(M_{EH}^\circ, g_{EH}^\circ)$ is explicitly described in [**Hi**]. But it is more convenient to use the following different description due to Fujiki ([**F**]), which relates the twistor space of (M_{EH}, g_{EH}) with the non-projective abstract algebraic variety constructed by Nagata ([**Na**]).

Let C be the complex projective line and $X := \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times C$ be the product of three complex projective lines. We fix any identification of the first factor with the second one and regard X as the trivial bundle over C. Let σ_X denote the anti-holomorphic involution on X which is defined by

$$\sigma_X(x, y, t) := (\sigma_1(y), \sigma_1(x), \sigma_1(t)),$$

where σ_1 denotes the anti-podal map. We choose any point 0 of C and put $\infty := \sigma_1(0)$.

We fix any non-singular curve $\varDelta$ of bidegree $(1,1)$ on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ which is real with respect to the above real structure and set

$$\varDelta_0 := \varDelta \times \{0\} \subseteq X_0 := \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times \{0\},$$

$$\varDelta_\infty := \varDelta \times \{\infty\} \subseteq X_\infty := \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times \{\infty\},$$

and

$$Q_X := \varDelta \times C \subseteq X.$$

Let

$$\mu_1 : Y \to X$$

denote the blowing-up along $\varDelta_0 \coprod \varDelta_\infty$, and $\tilde{Z}_0'$ and $\tilde{Z}_\infty'(\simeq \Sigma_2$, the Hirzebruch surface of degree 2) the exceptional divisors of $\varDelta_0$ and $\varDelta_\infty$ respectively, and $Q_Y(\simeq Q_X), X_0'(\simeq X_0)$, and $X_\infty'(\simeq X_\infty)$ the proper transforms of Q_X, X_0 and X_∞ respectively. σ_X naturally lifts on Y and defines an anti-holomorphic involution on Y which we denote by σ_Y. The normal bundle of X_0' and X_∞' in Y is isomorphic to $\mathcal{O}(-1,-1)$ and hence X_0' and X_∞' can be blown-down along each projection to $\boldsymbol{CP}^1$. Let

$$\mu_2 : Y \to Z_{EH}'$$

be the blowing-down of X_0' and X_∞' along the different projections. We set $B_0 := \mu_2(X_0')$ $(\simeq \boldsymbol{CP}^1), B_\infty := \mu_2(X_\infty')$ $(\simeq \boldsymbol{CP}^1)$ and $Q := \mu_2(Q_Y)$ $(\simeq Q_Y)$. σ_Y descends to Z_{EH}' and defines an anti-holomorphic involution σ_{EH}'. The projection $X \to C$ induces a surjective morphism $f : Z_{EH}' \to C$. We set $D' := f^{-1}(0)$ and $\bar{D}' := f^{-1}(\infty)$. D' and $\bar{D}'$ are biholomorphic to Σ_2 and all the other fibers are biholomorphic to $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$.

$$\begin{tikzcd}
& \tilde{X}_0, \tilde{X}_\infty \subseteq Y \supseteq \tilde{Z}_0', \tilde{Z}_\infty' \arrow[dl, "\mu_2"'] \arrow[dd] \arrow[dr, "\mu_1"] & \\
B_0, B_\infty \subseteq Z' \arrow[dr, "f'"'] & & X = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times C \supseteq \Delta_0, \Delta_\infty \arrow[dl] \\
& C = \boldsymbol{CP}^1 &
\end{tikzcd}$$

Diagram 1

We can readily see that $N_{Q/Z_{EH}'} \simeq \mathcal{O}_Q(-2,2)$, where $\mathcal{O}_Q(1,0)$ denotes the pull-back of $\mathcal{O}_C(1)$. Let

$$\mu : Z_{EH}' \to Z_{EH}$$

be the contraction of Q along the appropriate projection and set $L_{p_\infty} := \mu(Q)$.

Z_{EH}' is the abstract algebraic variety constructed by Nagata [**Na**] and it was remarked by Fujiki [**F**] that Z_{EH} equipped with the real structure σ_{EH} (which is induced by σ_{EH}') is the twistor space of (M_{EH}, g_{EH}), L_{p_∞} is the twistor line corresponding to the orbifold point p_∞, and the restriction of f on $Z_{EH}' \backslash Q = Z_{EH} \backslash L_{p_\infty}$ is the holomorphic map associated to the hyperKähler structure.

We can show that

$$-\frac{1}{2} K_{Z_{EH}'} \simeq \mathcal{O}(Q) \otimes f^* \mathcal{O}_C(2).$$

The $U(1)$-action which corresponds to that on (M_{EH}, g_{EH}) given in Example 2.9 is induced by the following $U(1)$-action on X via the above birational transform:

$$(x, y, u) \mapsto (tx, ty, tu) \quad \text{for } t \in U(1).$$

3.2. The twistor space of (M_0, g_0)

In this subsection, we use the notations in Example 2.5. We fix a positive real number $\lambda \neq 1$ and let $G := \{\lambda^n | n \in \boldsymbol{Z}\}$ be the infinite cyclic group generated by λ. Let (z, w) be coordinates on $\boldsymbol{C}^2$ and we regard $S^3 \times S^1$ as a quotient space $\boldsymbol{C}^2 \backslash \{0\}/G$, where G acts on $\boldsymbol{C}^2$ by the scalar multiplication. The Hermitian metric

$$h_0 := \frac{dz d\bar{z} + dw d\bar{w}}{|z|^2 + |w|^2}$$

which is defined on $\boldsymbol{C}^2 - \{0\}$ induces a Hermitian metric on $S^3 \times S^1$ and it is conformally isometric to the product metric $g_3 \oplus g_1$. Using this description, the twistor space of $(S^3 \times S^1, g_3 \oplus g_1)$ is given by ([**Pont**])

$$W_0 = (\mathcal{O}(1,-1)\backslash\{0\})/G,$$

where $\mathcal{O}(1,-1)$ denotes the holomorphic line bundle over $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ whose bidegree is $(1,-1)$ and the action of G on $\mathcal{O}(1,-1)$ is given by the scalar multiplication as a vector bundle. By construction, W_0 has a holomorphic fiber bundle map

$$\tilde{\pi} : W_0 \to \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$$

whose fibers are biholomorphic to an elliptic curve

$$E := \boldsymbol{C}^*/G.$$

By a theorem of Campana [**C**], the algebraic dimension of W_0 is two and hence all of the divisors on W_0 are pull-backs of curves on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. We also have $-(1/2)K_{W_0} \simeq \tilde{\pi}^*\mathcal{O}(1,1)$ ([**Pont**]).

Let $(z_0 : z_1 : z_2 : z_3)$ be homogeneous coordinates on $\boldsymbol{CP}^3$ (which we regard as the twistor space of the standard 4-sphere) and

$$\tilde{\tau} : (z_0 : z_1 : z_2 : z_3) \mapsto (z_2 : z_3 : z_0 : z_1)$$

a holomorphic involution on $\boldsymbol{CP}^3$. If we regard $\mathcal{O}(1,-1)\backslash\{0\}$ as an open subset of $\boldsymbol{CP}^3$ via the rational map

$$(z_0 : z_1 : z_2 : z_3) \mapsto ((z_0 : z_1), (z_2 : z_3)),$$

$\tilde{\tau}$ induces a holomorphic involution on W_0 which we also denote by $\tilde{\tau}$. $\tilde{\tau}$ on W_0 is nothing but the lifted involution of τ on $S^3 \times S^1$. The set of fixed points of $\tilde{\tau}$ on W_0 consists of four twistor lines $\tilde{L}_{01}, \tilde{L}_{02}, \tilde{L}_{03}$ and $\tilde{L}_{04}$ which correspond to the four fixed points of τ. Let

$$f_0 : W_0 \to Z_0 := W_0/\langle\tilde{\tau}\rangle$$

be the quotient map and set $L_{0i} := f_0(\tilde{L}_{0i})$ for $1 \leq i \leq 4$. Z_0 is the twistor space of M_0 and has A_1-singularities along L_{0i}, and L_{0i} $(1 \leq i \leq 4)$ is the twistor line corresponding to the orbifold points of M_0. $\tilde{\tau}$ (on W_0) induces a holomorphic involution on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and we denote it by ι. ι is explicitly given by

$$\iota : ((u_0 : u_1), (v_0 : v_1)) \mapsto ((v_0 : v_1), (u_0 : u_1)),$$

where $((u_0 : u_1), (v_0 : v_1))$ denote appropriate bihomogeneous coordinates on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and hence the resulting quotient space of $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ is biholomorphic to $\boldsymbol{CP}^2$. Let $\tilde{\Delta} \subseteq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ denote the set of fixed points of $\iota, \alpha : \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \to \boldsymbol{CP}^2$ the quotient map and set $\hat{\Delta} := \alpha(\tilde{\Delta})$, a conic. Let

$$\pi : Z_0 \to \boldsymbol{CP}^2$$

be the morphism induced by $\tilde{\pi}$. We have

$$\tilde{\pi}^{-1}(\tilde{\Delta}) \simeq \tilde{\Delta} \times E, \quad \pi^{-1}(\hat{\Delta}) \simeq \hat{\Delta} \times \boldsymbol{CP}^1$$

and $\tilde{\pi}|_{\tilde{L}_{0i}}$ (resp. $\pi|_{L_{0i}}$) is a biholomorphic map onto $\tilde{\Delta}$ (resp. $\hat{\Delta}$) for each $1 \le i \le 4$.

In summary, we have the following commutative diagrams:

$$\begin{array}{ccccc} \tilde{L}_{0i} \subseteq W_0 & \longrightarrow & N_0 \\ \downarrow \wr \quad \downarrow f_0 & & \downarrow \\ L_{0i} \subseteq Z_0 & \longrightarrow & M_0, \end{array} \qquad \begin{array}{ccc} W_0 & \xrightarrow{\tilde{\pi}} & \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \\ \downarrow f_0 & & \downarrow \alpha \\ Z_0 & \xrightarrow{\pi} & \boldsymbol{CP}^2 \end{array}$$

Diagram 2

The $U(1)$-action on $\boldsymbol{CP}^3$ given by $(z_0 : z_1 : z_2 : z_3) \mapsto (z_0 : tz_1 : z_2 : tz_3)$ descends to W_0 and Z_0, which correspond to the $U(1)$-actions given in Example 2.5. They also define $U(1)$-action on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and on $\boldsymbol{CP}^2$, respectively. Let $\tilde{\mathscr{C}}_0$ be the curve on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ which is uniquely determined by the following conditions: (i) $\tilde{\mathscr{C}}_0$ is an ι-invariant element of $|\mathscr{O}(1,1)|$, (ii) $\tilde{\mathscr{C}}_0$ is real with respect to the real structure induced by that on W_0, (iii) $\tilde{\mathscr{C}}_0$ is not equal to $\tilde{\Delta}$.

Next we see that there is a $U(1)$-invariant divisor on W_0 and Z_0 which will play important roles in our investigations. We set $R_0 := \tilde{\pi}^{-1}(\tilde{\mathscr{C}}_0)$, which is a real $U(1)$-invariant element of $|-(1/2)K_{W_0}|$. Then clearly $R_0 \simeq \tilde{\mathscr{C}}_0 \times E$. Further, we set $\mathscr{C}_0 := \pi(\tilde{\mathscr{C}}_0)$, a real line on $\boldsymbol{CP}^2$, and $S_0 := \pi^{-1}(\mathscr{C}_0)$. That is, S_0 is obtained as a quotient of $R_0 \simeq \boldsymbol{CP}^1 \times E$ with $E \simeq \boldsymbol{C}^*/\boldsymbol{Z}$ by the involution

$$((z_0 : z_1), w) \mapsto ((z_0 : -z_1), 1/w), \tag{$*$}$$

where $(z_0 : z_1)$ denote homogeneous coordinates on $\boldsymbol{CP}^1$ and w is a holomorphic coordinate on $\boldsymbol{C}^* \subseteq \boldsymbol{C}$.

S_0 is clearly a $U(1)$-invariant real element of $|-(1/2)K_{Z_0}| = |\pi^*\mathscr{O}_{\boldsymbol{CP}^2}(1)|$. Moreover, we observe that S_0 intersects each L_{0i} $(1 \le i \le 4)$ with two points $\{p_{0i}, \bar{p}_{0i}\}$ and these 8 points are A_1-singularities of S_0. S_0 is a rational elliptic surface with a $U(1)$-action. Let $\mu_0 : S_0' \to S_0$ be the minimal resolution of singularities and $l_{0i} := \mu_0^{-1}(p_{0i})$ and $\bar{l}_{0i} := \mu_0^{-1}(\bar{p}_{0i})$ $(1 \le i \le 4)$ the exceptional curves. As an elliptic surface, S_0' has two singular fibers, both of which are type I_0^* in Kodaira's notation, and every non-singular fiber is the same elliptic curve E. Moreover, the elliptic fibration $S_0' \to \boldsymbol{CP}^1$ is induced by the anticanonical system.

Alternatively, S_0' can be also obtained by blown-down to $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ as in the following figure and the resulting $U(1)$-action on $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ is given by

$$((\xi_0 : \xi_1), (\eta_0 : \eta_1)) \mapsto ((\xi_0 : \xi_1), (\eta_0 : t\eta_1)),$$

where $(\xi_0 : \xi_1)$ denote homogeneous coordinates in the direction of fibers of $S_0' \to \tilde{\Delta}$. To see these, it suffices to check that the self-intersection numbers of the $U(1)$-invariant (rational) curves are indicated in Figure 1. But this fact can be proved in elementary ways. Then following up the figure in a reverse order, we see that S_0' can be blown-

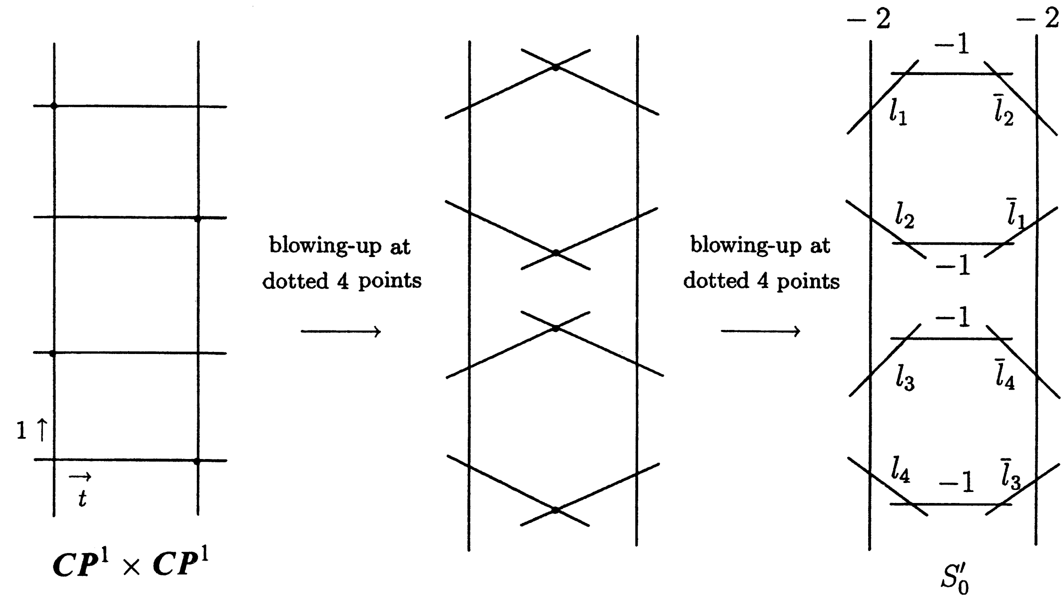

Figure 1

down to get $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. The claim for the $U(1)$-action can also be checked in a similar way.

Conversely, if a rational surface X is obtained by blowing-up $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ 8 times as in Figure 1, then X is also given by the quotient of $\boldsymbol{CP}^1 \times E$ by the involution $(*)$ and then resolving all of the (A_1)-singularities.

Further, we note that $U(1)$-action on S_0' is not semi-free, since there exist four (disjoint) (-1)-curves on S_0' on which $t = -1 \in U(1)$ acts trivially.

4. Equivariant smoothings.

4.1. A construction of a pair with a $U(1)$-action

Let Z_0 and Z_{EH} be the twistor space described in the previous section. In this subsection, using Z_0 and Z_{EH}, we construct a normal crossing variety Z' with $U(1)$-action and invariant Cartier divisor S' on Z'. We use the notations of the previous section, unless otherwise stated.

First we consider the twistor space Z_0 which was described in §3.2. Z_0 is the twistor space of the conformally flat reflection orbifold (M_0, g_0) and has four distinguished twistor lines L_{0i} $(1 \leq i \leq 4)$ along which Z_0 has A_1-singularities. Let $\mu_0 : Z_0' \to Z_0$ be the blowing up along $L_{01} \coprod L_{02} \coprod L_{03} \coprod L_{04}$ and Q_{0i} $(1 \leq i \leq 4)$ be the exceptional divisor of L_{0i} and put $Q_0 := \sum_{i=1}^{4} Q_{0i}$. Then Z_0' is non-singular and $Q_{0i} \simeq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and $N_{Q_{0i}/Z_0'} \simeq \mathcal{O}(-2, 2)$ where $\mathcal{O}(0, 1)$ denotes the pull-back of $\mathcal{O}_{L_{0i}}(1)$ ([LS]). Further, let S_0' denote the proper transform of S_0. Then μ_0 is the minimal resolution of S_0.

On the other hand, let $\tilde{\mu}_0 : W_0' \to W_0$ be the blowing-up of W_0 along $\tilde{L}_{01} \coprod \tilde{L}_{02} \coprod \tilde{L}_{03} \coprod \tilde{L}_{04}$ and $\tilde{Q}_{0i}$ $(1 \leq i \leq 4)$ be the exceptional divisor of $\tilde{L}_{0i}$. ($\tilde{L}_{0i}$ is the twistor line corresponding to L_{0i}.) We have $\tilde{Q}_{0i} \simeq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and $N_{\tilde{Q}_{0i}/W_0'} \simeq \mathcal{O}(-1, 1)$ where $\mathcal{O}(0, 1)$ denotes the pull-back of $\mathcal{O}_{\tilde{L}_{0i}}(1)$. Moreover, let R_0' be the proper transform of R_0. $\tilde{\mu}_0|_{R_0'}$ is 8-points blowing-up of the non-singular surface R_0. We have the following commutative diagram:

$$\begin{array}{ccccc} W_0' & \xrightarrow{\tilde{\mu}_0} & W_0 & \xrightarrow{\tilde{\pi}} & \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \\ {\scriptstyle f_0'}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Z_0' & \xrightarrow{\mu_0} & Z_0 & \xrightarrow{\pi} & \boldsymbol{CP}^2. \end{array}$$

We note that $U(1)$ acts on the whole of the diagram.

Let Z_i', $1 \le i \le 4$, be copies of the blown-up twistor space Z_{EH}' described in §3.1. Z_i' is obtained by blowing-up the twistor space Z_{EH} of the anti-self-dual reflection orbifold (M_{EH}, g_{EH}) along the singular twistor line.

We recall that there exists a surjective holomorphic map $f_i : Z_i' \to C = \boldsymbol{CP}^1$ (cf. §3.1). Further we have a divisor $Q_i \subseteq Z_i'$ which is biholomorphic to $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. We set

$$D_i' := f_i^{-1}(0), \quad \bar{D}_i' := f_i^{-1}(\infty),$$

and

$$l_i := D_i \cap Q_i, \quad \bar{l}_i := \bar{D}_i \cap Q_i.$$

D_i and $\bar{D}_i$ are biholomorphic to Σ_2, and l_i and $\bar{l}_i$ are non-singular rational curves.

Next we construct a pair of normal crossing varieties which is the main object we study in this section. First we choose four biholomorphic maps

$$\phi_i : Q_{0i} \to Q_i, \quad 1 \le i \le 4$$

which preserve both the real structures and $U(1)$-actions and satisfy $\phi_i(l_{0i}) = l_i$ and $\phi_i(\bar{l}_{0i}) = \bar{l}_i$. Using these isomorphisms, we set

$$Z' := Z_0' \cup \left(\coprod_{i=1}^{4} Z_i' \right),$$

$$S' := S_0' \cup \left(\coprod_{i=1}^{4} \left(D_i' \coprod \bar{D}_i' \right) \right).$$

It is obvious that S' is a real Cartier divisor on Z'. Further, Z' has a $U(1)$-action under which S' is invariant.

These varieties are illustrated as follows:

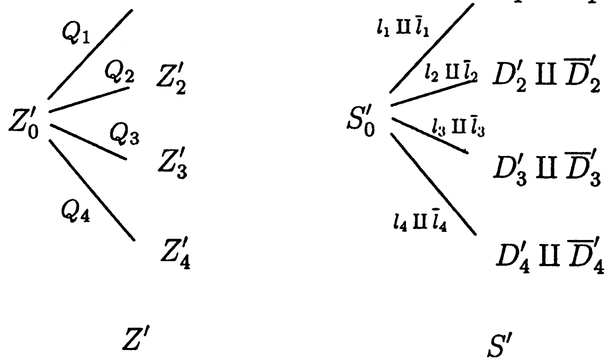

Figure 2

4.2. Calculations of obstructions

Let Z_0, L_{0i} $(\subseteq Z_0)$, $\mu_0 : Z_0' \to Z_0$, $Q_0 = \sum_{i=1}^4 Q_{0i}$, Z_i' and Q_i $(1 \leq i \leq 4)$ have the meanings of the previous subsection. Let R_0 $(\subseteq W_0)$, R_0' $(\subseteq W_0')$, S_0 $(\subseteq Z_0)$, S_0' $(\subseteq Z_0')$, $D_i, \bar{D}_i$ $(\subseteq Z_i)$ and $D_i', \bar{D}_i'$ $(\subseteq Z_i')$ also denote the $U(1)$-invariant divisors defined in §§3 and 4.1. In this subsection, we show that the cohomology groups $H^2(\Theta_{Z_0', Q_0}(-S_1'))$ and $H^2(\Theta_{Z_i', Q_i}(-D_i' - \bar{D}_i'))$ vanish. These results will be needed to prove the unobstructedness of deformations of the pair (Z', S').

4.2.1 The case of (M_0, g_0).

In this subsection, we show that the cohomology group $H^2(\Theta_{Z_0', Q_0}(-S_0'))$ vanishes. The proof is the same line as that of [**LS**, Lemma 1]; that is, we reduce it to the vanishing of a cohomology group of the double cover. In the proofs of the following lemmas and propositions, we omit the subscript 0 for simplicity.

We begin with the following:

Lemma 4.1. *We have $H^j(\Theta_{W_0}(-R_0)) = 0$ for any $j \geq 0$.*

Proof. We recall that $R = \tilde{\pi}^{-1}(\tilde{\mathscr{C}})$, where $\tilde{\mathscr{C}} \subseteq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ is a (real) non-singular curve of bidegree $(1,1)$. We have the following exact sequence of sheaves on W ([**Pont**]):

$$0 \to \mathcal{O}_{W_0} \to \Theta_W \to \tilde{\pi}^* \Theta_{\boldsymbol{CP}^1 \times \boldsymbol{CP}^1} \to 0. \tag{1}$$

Tensoring (1) with $\mathcal{O}_W(-R) \simeq \tilde{\pi}^* \mathcal{O}_{\boldsymbol{CP}^1 \times \boldsymbol{CP}^1}(-1,-1)$, we have

$$0 \to \tilde{\pi}^* \mathcal{O}(-1,-1) \to \Theta_W(-R) \to \tilde{\pi}^*(\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1)) \to 0. \tag{2}$$

It is easy to see that for any $j \geq 0$, we have

$$H^j(\tilde{\pi}^* \mathcal{O}(-1,-1)) = H^j(\tilde{\pi}^*(\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1))) = 0.$$

Hence by the cohomology exact sequence of (2), we have

$$H^j(\Theta_W(-R)) = 0 \quad \text{for any } j \geq 0. \qquad \square$$

Next we prove

Lemma 4.2. *We have $H^j(\Theta_{W_0', \tilde{Q}_0}(-R_0')) = 0$ for any $j \geq 0$.*

Proof. We have the following exact sequence of sheaves on W':

$$0 \longrightarrow \Theta_{W', \tilde{Q}} \xrightarrow{\tilde{\mu}_*} \tilde{\mu}^* \Theta_W \longrightarrow \tilde{\mu}^* \left(\bigoplus_{i=1}^4 N_{\tilde{L}_i/W} \right) \longrightarrow 0. \tag{3}$$

Here, $\tilde{L}_i = \tilde{L}_{0i}$ $(1 \leq i \leq 4)$ is the center of the blowing-up $\mu = \mu_0$. Since $N_{\tilde{L}_i/W} \simeq \mathcal{O}(1)^{\oplus 2}$, we have $\tilde{\mu}^* N_{\tilde{L}_i/W} \simeq \mathcal{O}_{\tilde{Q}_i}(0,1)^{\oplus 2}$, where $\mathcal{O}_{\tilde{Q}_i}(0,1) := \tilde{\mu}^* \mathcal{O}_{L_i}(1)$. Then tensoring $\mathcal{O}_{W'}(-R') \simeq \tilde{\mu}^* \mathcal{O}_W(-R)$ with (3), we have the exact sequence

$$0 \longrightarrow \Theta_{W', \tilde{Q}}(-R') \xrightarrow{\tilde{\mu}_*} \tilde{\mu}^*(\Theta_W(-R)) \longrightarrow \bigoplus_{i=1}^4 \mathcal{O}_{\tilde{Q}_i}(0,-1)^{\oplus 2} \longrightarrow 0. \tag{4}$$

Hence by the cohomology exact sequence of (4), we have

$$H^j(\Theta_{W',\tilde{Q}}(-R')) \simeq H^j(\tilde{\mu}^*(\Theta_W(-R))) \quad \text{for any } j \geq 0. \tag{5}$$

Since μ is a blowing-up, we have

$$H^j(\tilde{\mu}^*(\Theta_W(-R))) \simeq H^j(\Theta_W(-R)) \quad \text{for any } j \geq 0. \tag{6}$$

Then the claim follows from (5), (6) and Lemma 4.1. □

The following lemma can be directly verified using local coordinates.

Lemma 4.3. *Let $g : X \to Y$ be a ramified double covering of complex manifold branched along a smooth divisor D. Let $\tilde{D} := g^{-1}(D)$ denote the ramification divisor. Then we have the following natural isomorphism of $\mathcal{O}_X$-modules*:

$$g^*\Theta_{Y,D} \simeq \Theta_{X,\tilde{D}}.$$

Applying this lemma to $f_0' : W_0' \to Z_0'$, we have

Lemma 4.4. *We have a natural isomorphism $f_0'^*\Theta_{Z_0',Q_0} \simeq \Theta_{W_0',\tilde{Q}_0}$.*

Using these lemmas, we show the following:

Proposition 4.5. *We have $H^2(\Theta_{Z_0',Q_0}(-S_0')) = 0$.*

Proof. First we note that since $f' : W' \to Z'$ is the quotient map of the action of $\boldsymbol{Z}_2$ with fixed locus $\tilde{Q}_1 \coprod \tilde{Q}_2 \coprod \tilde{Q}_3 \coprod \tilde{Q}_4$, we have a holomorphic line bundle F on Z' and a holomorphic section ξ of $\mathcal{O}_{Z'}(Q)$ which satisfies

$$F \otimes F \simeq \mathcal{O}_{Z'}(Q)$$

and

$$W' \simeq \{\eta \in F \mid \eta^2 = \xi \in \mathcal{O}_{Z'}(Q)\}.$$

Then we have (cf. [**BPV**] for example)

$$R^q f_*' \mathcal{O}_{W'} \simeq \begin{cases} \mathcal{O}_{Z'} \oplus F^{-1} & q = 0, \\ 0 & q \geq 1. \end{cases} \tag{7}$$

On the other hand, by Lemma 4.4 we have

$$\begin{aligned} f'^*(\Theta_{Z',Q}(-S')) &\simeq \Theta_{W',\tilde{Q}} \otimes f'^*\mathcal{O}_{Z'}(-S') \\ &\simeq \Theta_{W',\tilde{Q}}(-R'). \end{aligned}$$

Therefore using (7), we get

$$H^j(W', \Theta_{W',\tilde{Q}}(-R')) \simeq H^j(Z', \Theta_{Z',Q}(-S')) \oplus H^j(Z', \Theta_{Z',Q}(-S') \otimes F^{-1})$$

for any $j \geq 0$. But now we have $H^2(W', \Theta_{W',\tilde{Q}}(-R')) = 0$ by Lemma 4.2. Hence we have $H^2(Z', \Theta_{Z',Q}(-S')) = 0$. □

4.2.2. The case of Eguchi–Hanson space

In this subsection, we show that the cohomology group $H^2(\Theta_{Z_i', Q_i}(-D_i' - \bar{D}_i'))$ vanishes, where Z_i' $(1 \le i \le 4)$ is the resolution of the twistor space of Eguchi–Hanson space (cf. §§2 and 3.1). We omit the subscript i for simplicity in this subsection. Recall that there exists the following diagram:

$$
\begin{tikzcd}
 & Y \arrow[dl, "\mu_2"'] \arrow[dr, "\mu_1"] \arrow[dd] & \\
Z' \arrow[dr, "f'"'] & & X = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times C \arrow[dl] \\
 & C = \boldsymbol{CP}^1 &
\end{tikzcd}
$$

Diagram 3

D' and $\bar{D}'$ are fibers of f over 0 and $\infty \in C$ respectively and Q is the image of $Q_X = \Delta \times C$ $(\subseteq X)$ by the rational map $\mu_2 \cdot \mu_1^{-1}$, where Δ is a real non-singular curve of bidegree $(1,1)$ in $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. We have $D' \simeq \Sigma_2 \simeq \bar{D}', Q \simeq \boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ and $N_{Q/X} \simeq \mathcal{O}(-2,2)$ where $\mathcal{O}(1,0)$ denotes the pull-back of $\mathcal{O}_C(1)$.

The following two lemmas are easy to prove:

Lemma 4.6. *We have a natural isomorphism*

$$\mu_2^* \mathcal{O}_{Z'}(-Q - D' - \bar{D}') \simeq \mathcal{O}_Y(-Q_Y - \tilde{Z}_0' - \tilde{Z}_\infty' - \tilde{X}_0 - \tilde{X}_\infty),$$

where $\tilde{Z}_0', \tilde{Z}_\infty', \tilde{X}_0, \tilde{X}_\infty$ *and* Q_Y *are the proper transforms of* $D' =: Z_0', \bar{D}' =: Z_\infty', X_0 = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times \{0\}$ $(\subseteq X)$, $X_\infty = \boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \times \{\infty\}$ $(\subseteq X)$, *and* Q_X *respectively.*

Lemma 4.7. *With the notations of the previous lemma, we have a natural isomorphism*

$$\mathcal{O}_Y(-Q_Y - \tilde{Z}_0' - \tilde{Z}_\infty' - \tilde{X}_0 - \tilde{X}_\infty) \simeq \mathcal{O}_Y(-\tilde{Z}_0' - \tilde{Z}_\infty') \otimes \mu_1^* \mathcal{O}_X(-Q_X),$$

where $Q_X = \mu_1(Q_Y)$ *as before.*

Next we prove the following

Lemma 4.8. *For any* $j \ge 0$, *we have a natural isomorphism*

$$H^j(\Theta_{Z'}(-Q - D' - \bar{D}')) \simeq H^j(\Theta_Y(-Q_Y - \tilde{Z}_0' - \tilde{Z}_\infty' - \tilde{X}_0 - \tilde{X}_\infty)).$$

Proof. Since μ_2 is the blowing-up of Z' along $B_0 \coprod B_\infty$ (cf. §3.1), we have the following exact sequence of sheaves on Y:

$$0 \to \Theta_Y \to \mu_2^* \Theta_{Z'} \to (\Theta_{\tilde{X}_0/B_0} \otimes \mathcal{O}_{\tilde{X}_0}(-1)) \oplus (\Theta_{\tilde{X}_\infty/B_\infty} \otimes \mathcal{O}_{\tilde{X}_\infty}(-1)) \to 0, \tag{8}$$

where $\Theta_{\tilde{X}_0/B_0}$ denotes the sheaf of relative tangent vector field with respect to the projection $\tilde{X}_0 \to B_0$ and $\mathcal{O}_{\tilde{X}_0}(-1)$ denotes the tautological line bundle over $\tilde{X}_0$, where we regard $\tilde{X}_0$ as the projectified normal bundle $P(N_{B_0/Z'})$. Since $N_{B_0/Z'} \simeq \mathcal{O}(-1)^{\oplus 2}$, we have

$$\Theta_{\tilde{X}_0/B_0} \simeq \mathcal{O}(2,0)$$

and

$$\mathcal{O}_{\tilde{X}_0}(-1) \simeq \mathcal{O}(-1,-1),$$

where $\mathcal{O}(0,1)$ denotes the pull-back of $\mathcal{O}_{B_0}(1)$. Hence (8) becomes

$$0 \to \Theta_Y \to \mu_2^* \Theta_{Z'} \to \mathcal{O}_{\tilde{X}_0}(1,-1) \oplus \mathcal{O}_{\tilde{X}_\infty}(1,-1) \to 0. \tag{9}$$

Tensoring $\mathcal{O}_Y(-Q_Y - \tilde{Z}'_0 - \tilde{Z}'_\infty - \tilde{X}_0 - \tilde{X}_\infty)$ with (9) and using Lemma 4.6 and the isomorphisms

$$\mathcal{O}_Y(Q_Y)|_{\tilde{X}_0} \simeq \mathcal{O}_{\tilde{X}_0},$$

$$\mathcal{O}_Y(\tilde{Z}'_0)|_{\tilde{X}_0} \simeq \mathcal{O}_{\tilde{X}_0}(1,1),$$

$$\mathcal{O}_Y(\tilde{X}_0)|_{\tilde{X}_0} \simeq N_{\tilde{X}_0/Y} \simeq \mathcal{O}_{\tilde{X}_0}(-1,-1),$$

together with the same isomorphisms for the sheaves on $\tilde{X}_\infty$, we get an exact sequence

$$\begin{aligned} 0 &\to \Theta_Y(-Q_Y - \tilde{Z}'_0 - \tilde{Z}'_\infty - \tilde{X}_0 - \tilde{X}_\infty) \to \mu_2^*(\Theta_{Z'}(-Q - D' - \bar{D}')) \\ &\to \mathcal{O}_{\tilde{X}_0}(1,-1) \oplus \mathcal{O}_{\tilde{X}_\infty}(1,-1) \to 0. \end{aligned} \tag{10}$$

Then the desired isomorphism follows from the cohomology exact sequence of (10) and a Leray spectral sequence for μ_2. □

In the same way, using Lemma 4.7, we can show the following

Lemma 4.9. *For any $j \geq 0$, we have a natural isomorphism*

$$H^j(\Theta_Y(-Q_Y - \tilde{Z}'_0 - \tilde{Z}'_\infty - \tilde{X}_0 - \tilde{X}_\infty)) \simeq H^j(\Theta_X(-Q_X - X_0 - X_\infty)).$$

By Lemmas 4.8 and 4.9, we have

Lemma 4.10. *For any $j \geq 0$, we have a natural isomorphism*

$$H^j(\Theta_{Z'}(-Q - D' - \bar{D}')) \simeq H^j(\Theta_X(-Q_X - X_0 - X_\infty)) \quad \text{for any } j \geq 0.$$

Next we have

Lemma 4.11. *For any $j \geq 0$, we have*

$$H^j(\Theta_{Z'}(-Q - D' - \bar{D}')) = 0.$$

Proof. By lemma 4.10, it suffices to show that $H^i(\Theta_X(-Q_X - X_0 - X_\infty)) = 0$ for any $i \geq 0$. Let $\mathcal{O}_X(0,0,1)$ denote the pull-back of $\mathcal{O}_C(1)$. Then we have

$$\Theta_X \simeq \mathcal{O}_X(2,0,0) \oplus \mathcal{O}_X(0,2,0) \oplus \mathcal{O}_X(0,0,2)$$

and

$$\mathcal{O}_X(-Q_X - X_0 - X_\infty) \simeq \mathcal{O}_X(-1,-1,-2).$$

From these isomorphisms, we easily get the desired results. □

Next we show the following:

Proposition 4.12. *We have*

$$H^2(\Theta_{Z',Q}(-D' - \bar{D}')) = 0.$$

Proof. We consider the exact sequence

$$0 \to \Theta_{Z'}(-Q - D' - \bar{D}') \to \Theta_{Z',Q}(-D' - \bar{D}') \to \Theta_Q(-l - \bar{l}) \to 0, \tag{11}$$

where $l = D' \cap Q$ and $\bar{l} = \bar{D}' \cap Q$ as before. From the cohomology sequence of (11) and Lemma 4.11, we have

$$H^2(\Theta_{Z',Q}(-D' - \bar{D}')) \simeq H^2(\Theta_Q(-l - \bar{l}))$$

and since we have $\Theta_Q(-l - \bar{l}) \simeq \mathcal{O}(2,-2) \oplus \mathcal{O}$, we get the desired result. □

4.3. $U(1)$-equivariant smoothings of S'

In this subsection, we investigate deformations of S' (cf. §4.1) and show that we can deform S' preserving the $U(1)$-action into a non-singular rational elliptic surface. This result is needed to know what kind of complex surface we obtain when one can smooth the pair (Z', S') $U(1)$-equivariantly. We continue to use the notations of §4.1. We only give outlines of proofs of the lemmas and propositions below, since they are something standard.

We have the following lemma:

Lemma 4.13. *We have the following isomorphisms*

$$\Theta^q_{S'} \simeq \begin{cases} \bigoplus_{i=1}^{4} (\mathcal{O}_{l_i} \oplus \mathcal{O}_{\bar{l}_i}) & q = 1, \\ 0 & q \geq 2. \end{cases}$$

Moreover, the induced $U(1)$-action on $H^0(\Theta^1_{S'}) \simeq \boldsymbol{C}^8$ is trivial.

For the proof, we use the facts that S' is normal crossing, $l_i^2 = \bar{l}_i^2 = -2$ on S'_0 and $l_i^2 = \bar{l}_i^2 = 2$ on D'_i for any $1 \leq i \leq 4$.

Proposition 4.14. *We have $H^2(\Theta_{S'}) = T^2_{S'} = 0$. In particular, deformations of S' are unobstructed.*

Proof. (outline) Using the description of S'_0 in Figure 1, we can show that

$$H^2\left(\Theta_{S'_0, \Sigma_{i=1}^4 (l_i + \bar{l}_i)}\right) = 0. \tag{12}$$

On the other hand, it is easy to see that $H^2(\Theta_{D'_i, l_i}) = H^2(\Theta_{\bar{D}'_i, \bar{l}_i}) = 0$, recalling that l_i (resp. $\bar{l}_i$) is a $(+2)$-section of $D_i \simeq \Sigma_2$ (resp. $\bar{D}_i \simeq \Sigma_2$). Then (12) and the cohomology exact sequence of

$$0 \to \Theta_{S'} \to \nu_*\left(\Theta_{S'_0,\Sigma_{i=1}^4(l_i+\bar{l}_i)} \oplus \left(\bigoplus_{i=1}^4 (\Theta_{D'_i,l_i} \oplus \Theta_{\bar{D}'_i,\bar{l}_i})\right)\right) \to \bigoplus_{i=1}^4 (\Theta_{l_i} \oplus \Theta_{\bar{l}_i}) \to 0,$$

show that $H^2(\Theta_{S'}) = 0$. (Here, ν denotes the normalization of S'.) Then Lemma 4.13 and a local to global spectral sequence shows that there exists an exact sequence

$$(13) \qquad 0 \longrightarrow H^1(\Theta_{S'}) \longrightarrow T^1_{S'} \xrightarrow{\ r\ } \bigoplus_{i=1}^4 H^0(\mathcal{O}_{l_i} \oplus \mathcal{O}_{\bar{l}_i}) \longrightarrow H^2(\Theta_{S'}) \longrightarrow T^2_{S'} \longrightarrow 0.$$

Hence we get $T^2_{S'} = 0$. □

Let $p : \mathscr{S} \to B$, $p^{-1}(0) \simeq S'$ be the Kuranishi family of deformations of S', where B can be identified with an open ball in $T^1_{S'}$ containing 0 by Proposition 4.14. Let $v \in T^1_{S'}$ be any real element such that all of 8 components of $r(v)$ (see (13)) are non-zero, and let B' be any non-singular holomorphic curve in B whose tangent vector at 0 is v. Then by (13) $S_t = p^{-1}(t), t \in B' \backslash \{0\}$ is a non-singular complex surface, at least if we choose B' sufficiently small. Further, such S_t is rational and satisfies $c_1^2(S_t) = 0$. To see this, first we recall that S'_1 is a 8 points blowing-up of $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. Let $\beta : \mathscr{S}' \to S'_1 \times \boldsymbol{C}$ be the blowing-up along $\amalg_{i=1}^4((l_i \times 0) \amalg (\bar{l}_i \times 0))$ and put $q := \mathrm{pr} \cdot \beta$, where pr denotes the projection from $S'_1 \times \boldsymbol{C}$ to $\boldsymbol{C}$. Then $q^{-1}(0)$ is biholomorphic to S', from which we get $c_1^2(S_t) = 0$. The rationality of S_t follows easily from the rationality of irreducible components of S', Castelnuovo's criterion and upper-semi-continuity of dimensions of cohomology groups.

Next we study $U(1)$-equivariant smoothings of S'. Let $v \in T^1_{S'}$ be any real and $U(1)$-invariant element such that all of 8 components of $r(v)$ (see (13)) is non-zero. (Such an element exists by the last claim of Lemma 4.13.) Let B' be any non-singular curve in B whose tangent vector at 0 is v. Then $p|_{B'} : \mathscr{S}|_{B'} \to B'$, the restriction of the Kuranishi family on B', gives a $U(1)$-equivariant smoothing of S'. Now we recall that S' can be illustrated as follows:

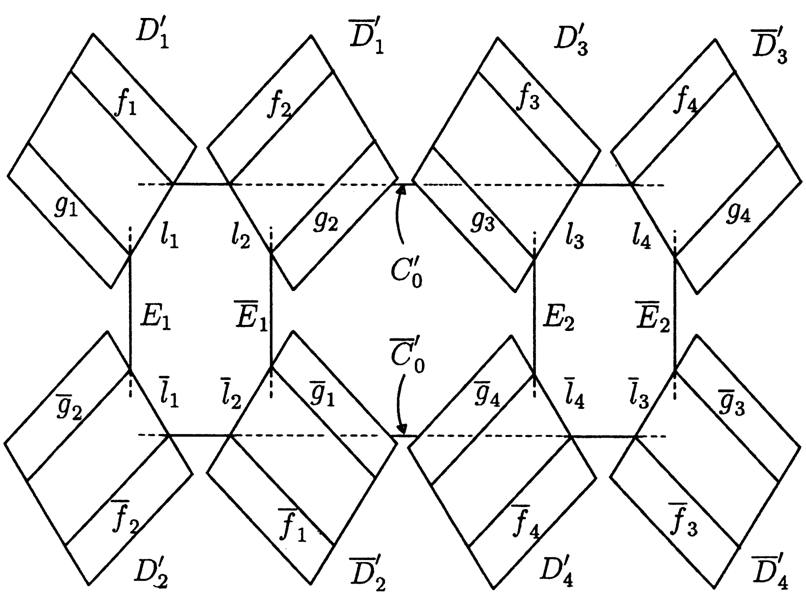

Figure 3

Using the notations in this figure, we put

$$C' := C_0' \cup (f_1 \amalg f_2 \amalg f_3 \amalg f_4), \quad \bar{C}' := \bar{C}_0' \cup (\bar{f}_2 \amalg \bar{f}_1 \amalg \bar{f}_4 \amalg \bar{f}_3),$$

$$E_1' := E_1 \cup (g_1 \amalg \bar{g}_2), \quad \bar{E}_1' := \bar{E}_1 \cup (g_2 \amalg \bar{g}_1),$$

$$E_2' := E_1 \cup (g_3 \amalg \bar{g}_4), \quad \bar{E}_2' := \bar{E}_2 \cup (g_4 \amalg \bar{g}_3).$$

We note that all of these six curves are connected and $U(1)$-invariant Cartier divisors on S'. Then we have

Proposition 4.15. *All of these six curves on S' are stable under the above $U(1)$-equivariant smoothing $p|_{B'} : \mathscr{S}|_{B'} \to B'$ and also deformed into irreducible non-singular rational curves.*

Proof. (outline) We put $A := C' + \bar{C}' + E_1' + \bar{E}_1' + E_2' + \bar{E}_2'$ for simplicity. We can show that $T^2_{S',A} = 0$ by similar calculations in the proof of Proposition 4.14. In particular, deformations of the pair (S', A) are unobstructed.

Next, the same argument as in the proof of Lemma 5.4 in [**Hon1**] shows that there exists a $U(1)$-equivariant exact sequence

$$\cdots \to T^1_{S',A} \to T^1_{S'} \to H^1(\mathscr{O}_A(A)) \to T^2_{S',A} \to \cdots.$$

Thus to prove the stability of A under $U(1)$-equivariant deformations, it is sufficient to show that $H^1(\mathscr{O}_A(A))^{U(1)} = 0$. This in turn can be shown by careful calculations using standard methods.

The second claim of the proposition can be proved by the same argument of the proof of Proposition 2.3 of [**Hon2**]. □

Let $C_t, \bar{C}_t, E_{1t}, \bar{E}_{1t}, E_{2t}$ and $\bar{E}_{2t}$ be the preserved irreducible non-singular $U(1)$-invariant rational curves on $S_t := p^{-1}(t)$. Then a slight modification of the argument in the proof of Proposition 2.3 in [**Hon2**] shows that they satisfy

$$C_t^2 = \bar{C}_t^2 = -2,$$

$$E_{it}^2 = \bar{E}_{it}^2 = -1 \quad \text{for } i = 1, 2,$$

$$C_t \cdot E_{it} = C_t \cdot \bar{E}_{it} = \bar{C}_t \cdot E_{it} = \bar{C}_t \cdot \bar{E}_{it} = 0 \quad \text{for } i = 1, 2.$$

Further, if one notices that S_t has a morphism onto $\boldsymbol{CP}^1$ whose generic fiber is $\boldsymbol{CP}^1$ and that each of $E_{1t}, \bar{E}_{1t}, E_{2t}, \bar{E}_{2t}$ is one of the irreducible components of distinct fibers of the morphism, it is easy to see that there exist non-singular $U(1)$-invariant rational curves l_{it} and $\bar{l}_{it}$ $(1 \le i \le 4)$ such that the configuration of the curves are exactly the same as S_0' (see Figure 1). Then again by following up the procedure in Figure 1 in the reverse order, we can conclude that S_t is also obtained by blowing $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ up at 8 points as in the Figure 1 preserving the $U(1)$-action. Thus we get

Proposition 4.16. *Let S' be the normal crossing surface with a $U(1)$-action constructed in §4.1. Then for any $U(1)$-equivariant smoothing $p : \mathscr{S} \to B', 0 \in B' \subseteq \boldsymbol{C}$*

with $p^{-1}(0) \simeq S'$, *there exists an open neighbourhood* B' *of* 0 *with* $B' \subseteq B$ *such that* $S_t = p^{-1}(t)$, $t \in B' \backslash \{0\}$ *is a rational elliptic surface obtained by the procedure in Figure* 1.

4.4. $U(1)$**-equivariant smoothings of** (Z', S')

Let (Z', S') be the pair of normal crossing varieties constructed in §4.1. Z' is a normal crossing 3-fold with a holomorphic $U(1)$-action and S' is a $U(1)$-invariant Cartier divisor on Z'. Further, S' is real with respect to the real structure on Z'. In this subsection, we investigate deformations of (Z', S') preserving the $U(1)$-action.

First we show the unobstructedness of deformations of (Z', S').

The following lemma can be proved exactly in the same way as Lemma 4.8 in [**Hon1**] and we omit the proof.

Lemma 4.17. *We have the following isomorphisms*

$$\Theta^q_{Z',S'} \simeq \Theta^q_{Z'} \simeq \begin{cases} \bigoplus_{i=1}^{4} \mathcal{O}_{Q_i} & q = 1\ , \\ 0 & q \geq 2. \end{cases}$$

Using the results of §4.2, we show the following:

Proposition 4.18. *Let* (Z', S') *be the pair of normal crossing varieties as above. Then we have* $T^2_{Z',S'} = H^2(\Theta_{Z'}(-S')) = 0$. *In particular, deformations of the pair* (Z', S') *are unobstructed.*

Proof. First we show that $H^2(\Theta_{Z'}(-S')) = 0$. Let $\nu : Z'_1 \coprod Z'_2 \to Z' = Z'_1 \cup Z'_2$ denote the normalization of Z'. Then we have an exact sequence

$$(14) \qquad 0 \to \Theta_{Z'}(-S') \to \nu_*\left(\Theta_{Z'_0,Q_0}(-S'_0) \oplus \left(\bigoplus_{i=1}^{4} \Theta_{Z'_i,Q_i}(-D'_i - \bar{D}'_i)\right)\right) \to \bigoplus_{i=1}^{4} \Theta_{Q_i}(-l_i - \bar{l}_i) \to 0.$$

Now by Propositions 4.5 and 4.12, we easily get

$$H^2\left(\nu_*\left(\Theta_{Z'_0,Q_0}(-S'_0) \oplus \left(\bigoplus_{i=1}^{4} \Theta_{Z'_i,Q_i}(-D'_i - \bar{D}'_i)\right)\right)\right) = 0.$$

On the other hand, by considering the exact sequence

$$0 \to \Theta_{Z'_i}(-Q_i - D'_i - \bar{D}'_i) \to \Theta_{Z'_i,Q_i}(-D'_i - \bar{D}'_i) \to \Theta_{Q_i}(-l_i - \bar{l}_i) \to 0$$

for $1 \leq i \leq 4$ and using Lemma 4.11 for $j = 2$, the map

$$H^1(\Theta_{Z'_i,Q_i}(-D'_i - \bar{D}'_i)) \to H^1(\Theta_{Q_i}(-l_i - \bar{l}_i))$$

is surjective. Therefore by the cohomology exact sequence of (14), we get

$$H^2(\Theta_{Z'}(-S')) = 0.$$

Combining this and Proposition 4.14 with the cohomology exact sequence of

$$0 \to \Theta_{Z'}(-S') \to \Theta_{Z',S'} \to \Theta_{S'} \to 0,$$

we get

$$H^2(\Theta_{Z',S'}) = 0. \tag{15}$$

Finally, the local to global spectral sequence

$$E_2^{p,q} := H^p(\Theta^q_{Z',S'}) \quad \Rightarrow \quad T^{p+q}_{Z',S'}$$

with the aid of Lemma 4.17 for $q \geq 2$ induces an exact sequence

$$0 \longrightarrow H^1(\Theta_{Z',S'}) \longrightarrow T^1_{Z',S'} \xrightarrow{r} H^0(\Theta^1_{Z',S'}) \longrightarrow H^2(\Theta_{Z',S'}) \longrightarrow T^2_{Z',S'} \longrightarrow H^1(\Theta^1_{Z',S'}). \tag{16}$$

Now using Lemma 4.17 for $q = 1$, we get

$$H^1(\Theta^1_{Z',S'}) \simeq \bigoplus_{i=1}^4 H^1(\mathcal{O}_{Q_i}) = 0.$$

Hence by (15) and (16), we have $T^2_{Z',S'} = 0$. $\square$

Let $\{\mathscr{Z} \xrightarrow{p} B, \mathscr{S} \xrightarrow{q} B$ and $\mathscr{S} \hookrightarrow \mathscr{Z}\}$ be the Kuranishi family of deformations of the pair (Z', S'). Proposition 4.18 implies that B can be regarded as a small open neighborhood of 0 in the vector space $T^1_{Z',S'}$. Let $v \in T^1_{Z',S'}$ be any real $U(1)$-invariant element such that $r(v)$ (see (16)) is non-zero and let B' be any $U(1)$-invariant real non-singular holomorphic curve in B whose tangent vector at 0 is v. By choosing B' sufficiently small, we have

Proposition 4.19. *Let $\{\mathscr{Z}' \xrightarrow{p'} B', \mathscr{S}' \xrightarrow{q'} B'\}$ be the restriction of the Kuranishi family on B'. Then if $t \in B' \backslash \{0\}$,* (i) *and* (ii) *below hold. In addition, if such a t is real,* (iii)–(vi) *below also hold.* (i) *both $Z_t = p^{-1}(t)$ and $S_t = q^{-1}(t)$ are non-singular,* (ii) *Z_t has a $U(1)$-action under which S_t is invariant,* (iii) *Z_t has a real structure σ_t and (Z_t, σ_t) has a structure of twistor space of $4\boldsymbol{CP}^2$,* (iv) *σ_t commutes with the $U(1)$-action,* (v) *S_t is a σ_t-invariant* (*i.e. real*) *element of $|-(1/2)K_{Z_t}|$ which is also $U(1)$-invariant,* (vi) *S_t has a structure of rational elliptic surface obtained by blowing-up $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ 8 times as in Figure* 1.

Proof. We note that there exists a $U(1)$-equivariant commutative diagram [cf. **Hon1**, Lemma 4.14]

$$\begin{array}{ccc} T^1_{Z',S'} & \longrightarrow & \bigoplus_{i=1}^4 H^0(\mathcal{O}_{Q_i}) \\ \downarrow & & \downarrow \\ T^1_{S'} & \longrightarrow & \bigoplus_{i=1}^4 (H^0(\mathcal{O}_{l_i}) \oplus H^0(\mathcal{O}_{\bar{l}_i})). \end{array}$$

Then by the choice of $v \in T^1_{Z',S'}$, the claims of (i) are clear. (ii) is also obvious since we choose B' as $U(1)$-invariant. (iii) follows from Donaldson–Friedman. (iv) and (v) are clear. (vi) immediately follows from Proposition 4.16. □

In particular, we get the following

Theorem 4.20. *Let (Z', S') be the pair of normal crossing varieties with the $U(1)$-action constructed in §4.1. Then there exists a $U(1)$-equivariant smoothing (Z_t, S_t) of (Z', S') such that Z_t is a twistor space of $4\boldsymbol{CP}^2$, S_t is a real element of $|-(1/2)K_{Z_t}|$ and that S_t has a structure of a rational elliptic surface obtained by the procedure in Figure* 1.

5. An algebraic structure of the twistor spaces

We continue to use the notations in the previous section. That is,

(Z', S'): the pair of normal crossing varieties constructed in §4.1,

$\{\mathscr{X} \xrightarrow{p} B, \mathscr{S} \xrightarrow{q} B \text{ with } \mathscr{S} \hookrightarrow \mathscr{X}\}$: the Kuranishi family of deformations of (Z', S') (if we set $Z_t := p^{-1}(t)$, $S_t := q^{-1}(t)$ as before, they satisfy $Z_0 \simeq Z'$ and $S_0 \simeq S'$).

Further, let $B' \subseteq B$ be a $U(1)$-invariant real non-singular holomorphic curve such that the restriction of the Kuranishi family $\{\mathscr{X}' \xrightarrow{p'} B', \mathscr{S}' \xrightarrow{q'} B'\}$ gives a smoothing of (Z', S') as in Proposition 4.19. We denote by $(B')^\sigma$ the subset of B' consisting of a real element with respect to the real structure on $T^1_{Z', S'}$. $(B')^\sigma$ is isomorphic to an open interval in $\boldsymbol{R}$. We recall that if $t \in (B')^\sigma$ is non-zero, Z_t is a twistor space of $4\boldsymbol{CP}^2$ with a $U(1)$-action, and S_t is a $U(1)$-invariant real element of $|-(1/2)K_{Z_t}|$. Further we recall that the anticanonical system of S_t induces a morphism onto $\boldsymbol{CP}^1$ whose generic fiber is an elliptic curve.

The cohomology sequence of the exact sequence

$$0 \to \mathscr{O}_{Z_t} \to -\frac{1}{2}K_{Z_t} \to -K_{S_t} \to 0,$$

together with the simply connectivity of $4\boldsymbol{CP}^2$ implies the exactness of the sequence

$$0 \to H^0(\mathscr{O}_{Z_t}) \to H^0\left(-\frac{1}{2}K_{Z_t}\right) \to H^0(-K_{S_t}) \to 0.$$

Hence $|-(1/2)K_{Z_t}|$ induces a surjective morphism $\varphi_t : Z_t \to \boldsymbol{CP}^2$ and $\varphi_t|_{S_t}$ is the elliptic fibration induced by $|-K_{S_t}|$. On the other hand, it is easy to see that $\kappa^{-1}(S_t) = 1$, where $\kappa^{-1}(S_t)$ denotes the anti-Kodaira dimension of S_t (cf. [**S**]). Further, we have $a(Z_t) \le 1 + \kappa^{-1}(S_t)$ by [**C**], where $a(Z_t)$ denotes the algebraic dimension of Z_t. Therefore we can conclude that $a(Z_t) = 2$. So a generic fiber of φ_t must be an elliptic curve. Thus we obtain

Proposition 5.1. *The complete linear system $|-(1/2)K_{Z_t}|$ is two dimensional and has no base locus. The morphism $\varphi_t : Z_t \to \boldsymbol{CP}^2$ induced by $|-(1/2)K_{Z_t}|$ gives an algebraic reduction of Z_t.*

It is easy to see that $|\mathscr{O}_{Z'}(S')|$ is two dimensional and base point free. Let $\varphi_0 : Z' \to \boldsymbol{CP}^2$ denote the morphism induced by $|\mathscr{O}_{Z'}(S')|$. We can readily see that $\varphi_0|_{Z'_0} : Z'_0 \to \boldsymbol{CP}^2$ is f_0 (see §3.2), the morphism induced by $|-(1/2)K_{Z_0}| = |f_0^*\mathscr{O}_{\boldsymbol{CP}^2}(1)|$.

Moreover, $\varphi_0|_{Z_i'}$ $(1 \leq i \leq 4)$ is the morphism induced by $|f_i^* \mathcal{O}_C(2)|$, where $f_i : Z_i' \to C = \boldsymbol{CP}^1$ is the morphism described in §3.1. The image of $\varphi_0|_{Z_i'}$ is the conic $\hat{\Delta} \subseteq \boldsymbol{CP}^2$, which is the branch locus of the double covering map $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1 \to \boldsymbol{CP}^2$.

Let $p'_* \mathcal{O}_{\mathscr{X}}(\mathscr{S})$ denote the direct image sheaf (on B') and consider the relative morphism $\Phi : \mathscr{X}' \to \boldsymbol{P}(p'_* \mathcal{O}_{\mathscr{X}}(\mathscr{S}))$ (cf. [**U**, Chapter 1]). Then for $t \in (B')^\sigma \backslash \{0\}$, we have $\mathcal{O}_{\mathscr{X}}(\mathscr{S})|_{Z_t} \simeq -(1/2)K_{Z_t}$ and $|-(1/2)K_{Z_t}|$ is two dimensional, and hence the rank of $p'_* \mathcal{O}_{\mathscr{X}}(\mathscr{S})$ is three. Therefore the restriction of Φ over $(B')^\sigma$ induces a commutative diagram

$$\begin{array}{ccc} (\mathscr{X}_0')^\sigma & \xrightarrow{\Phi^\sigma} & (B')^\sigma \times \boldsymbol{CP}^2 \\ \downarrow & & \downarrow \\ (B')^\sigma & \xrightarrow{\simeq} & (B')^\sigma \end{array}$$

such that for $t \in (B')^\sigma$, $\Phi^\sigma|_{Z_t} : Z_t \to \boldsymbol{CP}^2$ is φ_t. In brief, the morphism φ_0 extends over generic fiber to give an elliptic fibration $\varphi_t : Z_t \to \boldsymbol{CP}^2$. With this observation in hand, we complete the proof of Theorem 1.1.

Theorem 5.2. *There exists a self-dual metric g_t on $4\boldsymbol{CP}^2$ with the following properties*: (i) *the scalar curvature of g_t is positive type,* (ii) *the identity component of the group of orientation preserving conformal transformations of g_t is $U(1)$,* (iii) *g_t is not conformally isometric to the self-dual metrics of LeBrun* ([**LB1**]).

Proof. Let g_t be one of the self-dual metrics on $4\boldsymbol{CP}^2$ which correspond to Z_t. Then a theorem of Gauduchon ([**G**, Théorèm 2]) and the existence of the net $|S_t|$ imply that the scalar curvature of g_t is positive type. Hence we get (i).

Next let $\mathscr{C}^+$ denote the identity component of orientation preserving conformal transformations of $(4\boldsymbol{CP}^2, g_t)$. Obata's theorem implies that $\mathscr{C}^+$ is compact. On the other hand, a theorem of Poon ([**P2**, Theorem 1.3]) and the positivity of the scalar curvature imply that $\mathscr{C}^+$ is at most two dimensional. We now suppose that $\mathscr{C}^+$ is two dimensional; that is $\mathscr{C}^+ = U(1)^2$, the two dimensional torus. The action of $U(1)^2$ on $(4\boldsymbol{CP}^2, g_t)$ induces a holomorphic action of $U(1)^2$ on Z_t and we complexify it to get an action of $(\boldsymbol{C}^*)^2$ on Z_t. $(\boldsymbol{C}^*)^2$ also acts on the linear system $|-(1/2)K_{Z_t}|$ and we obtain an action of $(\boldsymbol{C}^*)^2$ on $\boldsymbol{CP}^2$. But since there exists the above diagram, the $(\boldsymbol{C}^*)^2$-action on $\boldsymbol{CP}^2$ is just the one on the central fiber Z' and hence preserves $\hat{\Delta}$, a conic on which singular fibers of $\varphi_0 : Z' \to \boldsymbol{CP}^2$ lie. Hence there exists a $\boldsymbol{C}^*$-subgroup of $(\boldsymbol{C}^*)^2$ which acts trivially on $\boldsymbol{CP}^2$. But such $\boldsymbol{C}^*$-action on Z_t must be trivial, since $\varphi_t : Z_t \to \boldsymbol{CP}^2$ is a non-trivial elliptic fibration. Therefore, $U(1)^2$ cannot act on Z_t effectively. On the other hand, we know that there exists a $U(1)$-action on Z_t. Hence we have $\mathscr{C}^+ = U(1)$ and get (ii).

Finally, we recall that the $U(1)$-action on S_t is not semi-free. Hence by [**LB2**, Corollary 1], g_t is not one of self-dual metrics of LeBrun ([**LB1**]). □

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Nobuhiro HONDA
Department of Mathematics,
Graduate School of Science,
Osaka University,
560-0043, Japan
E-mail: honda@math.sci.hiroshima-u.ac.jp

Current address:
Department of Mathematics,
Faculty of Science,
Hiroshima University, 739-8526, Japan
E-mail: honda@math.sci.hiroshima-u.ac.jp

Mitsuhiro ITOH
Institute of Mathematics,
University of Tsukuba,
305-8571, Japan
E-mail: itohm@sakura.cc.tsukuba.ac.jp