J. Math. Soc. Japan
Vol. 51, No. 4, 1999

Abel's theorem for divisors on an arbitrary compact complex manifold

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(Received Apr. 27, 1998)

Abstract. We prove Abel's theorem for divisors on an arbitrary compact complex manifold by combining the Čech cohomology of sheaves, a **logarithmic residue formula** for 1-forms and de Rham's theory applied to open submanifolds.

0. Introduction.

Let M be a compact complex manifold of dimension n, $\mathrm{Div}(M)$ be the abelian group of divisors on M and $\mathscr{M}(M)$ be the field of meromorphic functions on M. The main purpose of the present paper is to prove the following

ABEL'S THEOREM FOR DIVISORS. *Let $D \in \mathrm{Div}(M)$. Then, D is linearly equivalent to 0 (i.e. $D = (F)$ for some $F \in \mathscr{M}(M)^{\times}$) if and only if the class of D in $H_{2n-2}(M, \boldsymbol{Z})$ is 0 and for any integral $(2n-1)$-chain Q on M with $D = \partial Q$ there exists an integral $(2n-1)$-cycle Γ on M such that for all $[\omega] \in H_{\bar{\partial}}^{n,n-1}(M)$ we have*

$$\int_Q \omega = \int_\Gamma \omega.$$

Notice that $\int_Q \omega$ and $\int_\Gamma \omega$ depend only on the Dolbeault cohomology class $[\omega] \in H_{\bar{\partial}}^{n,n-1}(M)$. If M is Kähler, one can deduce this theorem from results of Kodaira [**K1**] and others. But the validity of it on an arbitrary compact complex manifold M was not known even conjecturally. One proved Abel's Theorem on a compact Kähler manifold M, following Weyl's book "Die Idee der Riemannschen Fläche," 1913 (3rd ed., 1955), by giving a necessary and sufficient condition for a multiplicative function (which was a kind of multi-valued meromorphic function on M) to be single-valued by means of the theory of harmonic integrals.

In 1983 I found out another proof which was based, after Siegel's book [**Si2**], on the consideration of a multiplicative function by means of a logarithmic residue formula, de Rham's theory applied to the open submanifold $M - \mathrm{Supp}\, D$ and the Hodge decomposition and I noticed that the Kählerness of M was not essential for the validity.

In 1990, on the occasion of a joint work with T. Segawa concerning generalizations of Abel's theorem, I recognize that this proof holds true for the necessity on an arbitrary M (where only the residue formula and de Rham's theory are used). After an investigation, I find that this proof holds true for the sufficiency on such an M whose Picard variety (Kodaira [**K4**]) is a complex torus, by an effective use of the cohomology theory of sheaves. Trying to carry out this proof on an arbitrary M is rather

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1991 *Mathematics Subject Classification*. Primary 32A20; Secondary 32C30, 58A12.
Key Words and Phrases. Meromorphic function, logarithmic residue formula, de Rham's theory.

hopeless. Immediately after the finding, I notice a natural method of proof in which the cohomology theory of sheaves is used more effectively. It is to compute directly the composite injection

$$Cl^0(M) \hookrightarrow \mathrm{Pic}^0(M) \cong H^1(M, O_M)/H^1(M, \boldsymbol{Z})$$
$$\cong H_{\bar{\partial}}^{n,n-1}(M)^*/H_{2n-1}(M, \boldsymbol{Z})$$

using the Čech cohomology and to show that the map is induced by

$$\mathrm{Div}^0(M) \ni D \mapsto (H_{\bar{\partial}}^{n,n-1}(M) \ni [\omega] \mapsto \int_Q \omega \in \boldsymbol{C}) \bmod H_{2n-1}(M, \boldsymbol{Z}),$$

where Q is an integral $(2n-1)$-chain on M with $\partial Q = D$ and

$$\mathrm{Div}^0(M) := \{D \in \mathrm{Div}(M) \mid \text{the class of } D \text{ in } H_{2n-2}(M, \boldsymbol{Z}) \text{ is } 0\},$$

$$Cl^0(M) := \mathrm{Div}^0(M)/\{(F) \in \mathrm{Div}(M) \mid F \in \mathscr{M}(M)^\times\}.$$

(If M is Kähler, this fact is essentially shown in Weil [**W2**], p. 893 by means of Kodaira's formula [**K1**], Theorem 3. The key point of my new method is to reverse the order of arguments.) It turns out this new method is available on an arbitrary M. The new method is based, so to speak, on the consideration of a C^∞ solution to a multiplicative Cousin problem (which is single-valued and has 'poles' on M) by means of the residue formula, de Rham's theory and the Serre duality. In §1, I explain the residue formula and prove, as an application of it, the well-known fact that under the isomorphism by Poincaré duality

$$H^2(M, \boldsymbol{Z}) \cong \hat{H}_{2n-2}(M, \boldsymbol{Z}),$$

$c([D])$ corresponds to the homology class of D, where M can be paracompact, $\hat{H}_{2n-2}(M, \boldsymbol{Z})$ is the $(2n-2)$-homology group of infinite chains on M and $c([D])$ is the Chern class of the line bundle $[D] \in \mathrm{Pic}(M)$. This fact plays a fundamental role in the new method and implies in particular that if D is linearly equivalent to 0 then the class of D in $\hat{H}_{2n-2}(M, \boldsymbol{Z})$ is 0. In §2, I accomplish the proof of Abel's Theorem and show how this theorem implies Kodaira's formula mentioned earlier, Abel's theorem for families of effective divisors on a projective algebraic surface due to Severi [**Sev1**] and Igusa's formula in his theory [**I**] of the Picard variety of a projective complex manifold.

I would like to thank T. Segawa for useful conversations during the last period of this work.

1. Logarithmic residue formula for 1-forms.

Our proof of Abel's theorem for divisors can be regarded as a generalization of the proof in Siegel [**Si2**] and is based on the following logarithmic residue formula for 1-forms:

THEOREM 1.1 (Kodaira [**K1**], (24), (30)). *Let U be a paracompact complex manifold of dimension n, $F \in \mathscr{M}(U)^\times$. Then the closed meromorphic 1-form $d \log F$ on U determines a 1-current $[d \log F]$ on U by integrals of absolute convergence and one has*

$$d[d \log F] = 2\pi i(F)$$

as 2*-currents on* U, *where* $(F) \in \mathrm{Div}(U)$ *determines a closed* 2*-current on* U *as an infinite* $(2n-2)$*-cycle on* U.

THEOREM 1.1′ (Kodaira [**K2**], (2.14)). *Let* U, F *as above. Then for any finite* 2*-chain* C *on* U *with* $\mathrm{Supp}\,\partial C \subset U - \mathrm{Supp}(F)$ *one has*

$$\int_{\partial C} d\log F = 2\pi i I(C,(F))_U,$$

where $I(\,\cdot\,,\,\cdot\,)_U$ *denotes the intersection number on* U.

We shall give a proof of the implication Theorem 1.1 $\Rightarrow$ Theorem 1.1′, whose method is used in our proof of Abel's theorem for divisors (Theorem 2.2 below). First of all, from Theorem 1.1 follows that

$$(F) = \partial Q$$

for some infinite real $(2n-1)$-chain Q on U, by de Rham theory ([**R**], Chapitre IV, §23, 6 line after Théorème 19, p. 117).

Put $G := U - \mathrm{Supp}(F)$. Applying de Rham theory to open submanifold G, we can take a C^∞ closed $(2n-1)$-form Ψ on G with compact support which corresponds to the 1-cycle ∂C on G under the isomorphism by Poincaré duality

$$H^{2n-1}(\Gamma_c(G,\mathscr{A}^{\cdot}_G)) \cong H^{2n-1}_c(G,\boldsymbol{C}) \cong H_1(G,\boldsymbol{C}).$$

(For de Rham theory, see also Weil [**W3**], especially §2, p. 127 and §4, p. 139.) This Ψ, considered as a C^∞ closed $(2n-1)$-form on U with compact support, corresponds to the 1-cycle ∂C on U under the isomorphism by Poincaré duality

$$H^{2n-1}(\Gamma_c(U,\mathscr{A}^{\cdot}_U)) \cong H^{2n-1}_c(U,\boldsymbol{C}) \cong H_1(U,\boldsymbol{C}).$$

Since C is a 2-chain on U, ∂C is homologous to 0 on U. Hence,

$$\Psi = d\Phi$$

for some $C^\infty(2n-2)$-form Φ on U with compact support. Then

$$\begin{aligned}
\int_{\partial C} d\log F &= \int_G \Psi \wedge d\log F \quad (\because d\log F, \text{ considered on } G, \text{ is a } C^\infty \text{ closed 1-form})\\
&= \int_U \Psi \wedge d\log F = \int_U d\Phi \wedge d\log F = -\int_U d\log F \wedge d\Phi\\
&= -[d\log F](d\Phi) = -d[d\log F](\Phi) = -2\pi i(F)(\Phi) \quad (\because \text{ Theorem 1.1})\\
&= -2\pi i \partial Q(\Phi) = -2\pi i Q(d\Phi) = -2\pi i Q(\Psi) = -2\pi i \int_Q \Psi\\
&= -2\pi i I(Q,\partial C)_G\\
&\qquad (\because Q, \text{ considered on } G, \text{ is an infinite real } (2n-1)\text{-cycle})\\
&= -2\pi i I(Q,\partial C)_U = 2\pi i I(\partial C,Q)_U = 2\pi i I(C,\partial Q)_U = 2\pi i I(C,(F))_U
\end{aligned}$$

($\because$ In general, for any finite k-chain C and infinite $(d-k+1)$-chain Q on an oriented topological manifold U of dimension d with $\operatorname{Supp} \partial C \cap \operatorname{Supp} \partial Q = \phi$, one has

$$(-1)^k I(\partial C, Q)_U = I(C, \partial Q)_U$$

(see Seifert-Threlfall [**ST**], Zehntes Kapitel, §74, (10)).). $\square$

REMARK 1.2. (i) One can read an equality in Weil [**W1**], p. 114 which is essentially the same as Theorem 1.1$'$. See also Weil [**W2**], p. 874.

(ii) In 1924, Lefschetz already used Theorem 1.1$'$ for a topological proof of the period relations concerning meromorphic functions of n complex variables with $2n$ independent periods (see Siegel [**Si1**], p. 119).

The next fact is well-known (see, e.g., Serre [**Ser**], §II, n^0 6, p. 61 and Grauert-Remmert [**GR**], Einleitung, Nr.1 and Kapitel V, §2, Abschnitt 4), although its proof is rarely found in the literature even if M is compact. (One often attempts to prove it by means of the $\boldsymbol{Z}$-valued pairings with a 2-cycle on M or the $\boldsymbol{C}$-valued pairings with a C^∞ closed $(2n-2)$-form on M with compact support, but this method establishes only the correspondence modulo torsion in case M is compact.)

THEOREM 1.3 (Dolbeault [**D**], Chap. II, §B, n^0 3, Théorème 2.7). *Let M be a paracompact complex manifold of dimension n, $D \in \operatorname{Div}(M)$. Then under the isomorphism by Poincaré duality*

$$H^2(M, \boldsymbol{Z}) \cong \hat{H}_{2n-2}(M, \boldsymbol{Z}),$$

$c([D])$ corresponds to the homology class of D, where $\hat{H}_{2n-2}(M, \boldsymbol{Z})$ is the $(2n-2)$-homology group of infinite chains on M and $c([D])$ is the Chern class of the line bundle $[D] \in \operatorname{Pic}(M) = H^1(M, O_M^\times)$.

REMARK 1.4. Kodaira [**K2**], pp. 851–852, p. 854 already contained a special case of Theorem 1.3 where M was compact, the correspondence was modulo torsion and D was a canonical divisor apparently. Kodaira [**K6**], §3.6, (3.166) gives another proof of this case for the general D. (Kodaira-Spencer [**KS**], p. 876 contained the case where M was compact and torsion was included, but their proof was incorrect. The next proof gives just a correction of it in case M is compact; in [**KS**] they claimed $I(s_{ijk}, D)_M = c_{ijk}$ instead of $I(s_{ijk}, D)_M = c_{ijk} + (\delta a)_{ijk}$.)

We shall give a proof of Theorem 1.3 as an application of Theorem 1.1$'$, following Dolbeault [**D**], pp. 112–115 and Kodaira [**K6**], pp. 167–170 in substance.

Take a sufficiently fine simplicial decomposition of M, whose vertices, 1-simplices, 2-simplices, ... are denoted by p_k, s_{jk} $(j<k)$, s_{ijk} $(i<j<k), \ldots$ and satisfy $\partial s_{jk} = p_k - p_j$, $\partial s_{ijk} = s_{jk} - s_{ik} + s_{ij}, \ldots$. Let U_k be the open star of p_k. One gets an open cover $\{U_k\}$ of M. One may assume that $D|_{U_k} = (F_k)$ for some $F_k \in \mathscr{M}(U_k)^\times$. Then one has

$$[D] = [\{f_{jk}\}] \in H^1(\{U_k\}, O_M^\times) \subset H^1(M, O_M^\times) \quad \text{with } f_{jk} = F_j/F_k$$

and has

$$c([D]) = [\{c_{ijk}\}] \in H^2(\{U_k\}, \boldsymbol{Z}) = H^2(M, \boldsymbol{Z})$$

with

$$c_{ijk} = \frac{1}{2\pi\sqrt{-1}} \ (\log f_{jk} - \log f_{ik} + \log f_{ij}).$$

On the other hand, one may also assume that $\operatorname{Supp} D$ does not meet any 1-simplex s_{jk}. Denoting by $\bar{\beta} \in H^2(M, \boldsymbol{Z})$ the cohomology class which corresponds to the homology class of D in $\hat{H}_{2n-2}(M, \boldsymbol{Z})$ under the isomorphism by Poincaré duality, one has

$$\bar{\beta} = [\{b_{ijk}\}] \in H^2(\{U_k\}, \boldsymbol{Z}) = H^2(M, \boldsymbol{Z})$$

with

$$b_{ijk} = I(s_{ijk}, D)_M.$$

What is required is to show that $\bar{\beta} = c([D])$, i.e. there is a 1-cochain

$$a = \{a_{jk}\} \in C^1(\{U_k\}, \boldsymbol{Z})$$

such that

$$\begin{aligned} b_{ijk} - c_{ijk} &= (\delta a)_{ijk} \\ &= a_{jk} - a_{ik} + a_{ij} \end{aligned}$$

for all $U_i \cap U_j \cap U_k \neq \varnothing$.

In order to calculate $b_{ijk} = I(s_{ijk}, D)_M$ for any 2-simplex s_{ijk}, denoting by $p_{ijk}, p_{jk}, p_{ik}, p_{ij}$ the barycenter of $s_{ijk}, s_{jk}, s_{ik}, s_{ij}$ respectively and joining p_{jk}, p_{ik}, p_{ij} to p_{ijk} by three line segments, one decomposes the 2-simplex s_{ijk} into three 2-cells $e_{ijk} \subset U_i$, $e_{jki} \subset U_j$, $e_{kij} \subset U_k$ (see Figure 5 in Kodaira [**K6**], p. 169). One may assume that $\operatorname{Supp} D$ does not meet the three line segments. Then

$$\begin{aligned} I(s_{ijk}, D)_M &= I(e_{ijk}, D)_{U_i} + I(e_{jki}, D)_{U_j} + I(e_{kij}, D)_{U_k} \\ &= I(e_{ijk}, (F_i))_{U_i} + I(e_{jki}, (F_j))_{U_j} + I(e_{kij}, (F_k))_{U_k}. \end{aligned}$$

Applying Theorem 1.1′ to U_i, U_j, U_k respectively, we get

$$\begin{aligned} 2\pi\sqrt{-1} I(s_{ijk}, D)_M &= \int_{\partial e_{ijk}} d \log F_i + \int_{\partial e_{jki}} d \log F_j + \int_{\partial e_{kij}} d \log F_k \\ &= \int_{p_{ij}}^{p_{ijk}} d \log F_i - \int_{p_{ik}}^{p_{ijk}} d \log F_i + \int_{p_{jk}}^{p_{ijk}} d \log F_j \\ &\quad - \int_{p_{ij}}^{p_{ijk}} d \log F_j + \int_{p_{ik}}^{p_{ijk}} d \log F_k - \int_{p_{jk}}^{p_{ijk}} d \log F_k + (\delta x)_{ijk} \\ &= \int_{p_{jk}}^{p_{ijk}} d \log f_{jk} - \int_{p_{ik}}^{p_{ijk}} d \log f_{ik} + \int_{p_{ij}}^{p_{ijk}} d \log f_{ij} + (\delta x)_{ijk}, \end{aligned}$$

where we put

$$x_{jk} := \int_{p_j}^{p_{jk}} d\log F_j - \int_{p_k}^{p_{jk}} d\log F_k.$$

Hence

$$\begin{aligned}
2\pi\sqrt{-1}I(s_{ijk}, D)_M &= \log f_{jk}(p_{ijk}) - \log f_{jk}(p_{jk}) - \log f_{ik}(p_{ijk}) \\
&\quad + \log f_{ik}(p_{ik}) + \log f_{ij}(p_{ijk}) - \log f_{ij}(p_{ij}) + (\delta x)_{ijk} \\
&= 2\pi\sqrt{-1}c_{ijk} - (\delta y)_{ijk} + (\delta x)_{ijk} \\
&= 2\pi\sqrt{-1}c_{ijk} + (\delta z)_{ijk},
\end{aligned}$$

where we put

$$\begin{aligned}
y_{jk} &:= \log f_{jk}(p_{jk}), \\
z_{jk} &:= \int_{p_j}^{p_{jk}} d\log F_j - \int_{p_k}^{p_{jk}} d\log F_k - \log f_{jk}(p_{jk}).
\end{aligned}$$

Since

$$\begin{aligned}
\exp z_{jk} &= F_j(p_{jk})F_j(p_j)^{-1}F_k(p_{jk})^{-1}F_k(p_k)f_{jk}(p_{jk})^{-1} \\
&= F_j(p_j)^{-1}F_k(p_k),
\end{aligned}$$

putting

$$a_{jk} := \frac{1}{2\pi\sqrt{-1}}(z_{jk} + \log F_j(p_j) - \log F_k(p_k)) \in \boldsymbol{Z},$$

we have

$$I(s_{ijk}, D)_M = c_{ijk} + (\delta a)_{ijk}$$

as required. □

For Theorem 1.3 and the second Cousin problem, see Remark 2.3.

2. Abel's theorem for divisors.

Let M be a compact complex manifold of dimension n throughout this section. The diagram in the next proposition is important in our proof of Abel's theorem for divisors.

Proposition 2.1. *The following diagram is commutative*:

$$\begin{array}{ccccc}
H^1(M, O_M) & \cong & H^{0,1}_{\bar\partial}(M) & \cong & H^{n,n-1}_{\bar\partial}(M)^* \\
\uparrow & & & & \uparrow \\
H^1(M, \boldsymbol{Z}) & & \cong & & H_{2n-1}(M, \boldsymbol{Z}),
\end{array}$$

where the top row is Dolbeault isomorphism and Serre duality, the bottom row is the isomorphism by Poincaré duality, the left-hand injection is induced by the exact sequence $0 \to \boldsymbol{Z}_M \to O_M \to O_M^\times \to 0$ *and the right-hand arrow is defined as follows*:

$$H_{2n-1}(M, \boldsymbol{Z}) \ni [\Gamma] \mapsto \left(H_{\bar{\partial}}^{n,n-1}(M) \ni [\omega] \mapsto \int_\Gamma \omega \in \boldsymbol{C} \right).$$

Proof. One has only to compute the composite map α:

$$H_{2n-1}(M, \boldsymbol{Z}) \cong H^1(M, \boldsymbol{Z}) \hookrightarrow H^1(M, O_M) \cong H_{\bar{\partial}}^{0,1}(M) \cong H_{\bar{\partial}}^{n,n-1}(M)^*.$$

Let $[\Gamma] \in H_{2n-1}(M, \boldsymbol{Z})$. The image of it by the composite map

$$H_{2n-1}(M, \boldsymbol{Z}) \cong H^1(M, \boldsymbol{Z}) \hookrightarrow H^1(M, \boldsymbol{C}) \cong H^1(\Gamma(M, \mathscr{A}_M^{\cdot}))$$

is represented by C^∞ closed 1-form θ such that $\int_\Gamma \Psi = \int_M \theta \wedge \Psi$ for all C^∞ closed $(2n-1)$-form Ψ. The image of $[\theta]$ by the composite map

$$H^1(\Gamma(M, \mathscr{A}_M^{\cdot})) \cong H^1(M, \boldsymbol{C}) \to H^1(M, O_M) \cong H_{\bar{\partial}}^{0,1}(M)$$

is represented by $\bar{\partial}$-closed $(0,1)$-form $\theta^{0,1} = (0,1)$-part of θ. ($\because$ $[\theta]$ corresponds to $[\{c_{ij}\}] \in H^1(M, \boldsymbol{C})$, where 1-cocycle $\{c_{ij}\} \in Z^1(\{U_i\}, \boldsymbol{C})$ with some open cover $\{U_i\}$ of M satisfies $\theta = d\phi_i$ for some $\phi_i \in \Gamma(U_i, \mathscr{A}_M^0)$ and $c_{ij} = \phi_j - \phi_i$ on $U_i \cap U_j \neq \emptyset$. The image of $[\{c_{ij}\}]$ by the composite map $H^1(M, \boldsymbol{C}) \to H^1(M, O_M) \cong H_{\bar{\partial}}^{0,1}(M)$ is represented by $\bar{\partial}$-closed $(0,1)$-form φ with $\varphi = \bar{\partial}\phi_i$ on U_i. Then $\varphi = \theta^{0,1}$.)

Therefore, the image of $[\Gamma]$ by the composite map α is given by the linear form

$$\begin{aligned} H_{\bar{\partial}}^{n,n-1}(M) \ni [\omega] \mapsto \int_M \theta^{0,1} \wedge \omega &= \int_M \theta \wedge \omega \\ &= \int_\Gamma \omega \in \boldsymbol{C} \ \ (\because d\omega = \bar{\partial}\omega = 0). \end{aligned}$$ □

Now we shall finish preparation. Put

$$\mathrm{Div}^0(M) := \{D \in \mathrm{Div}(M) \,|\, \text{the class of } D \text{ in } H_{2n-2}(M, \boldsymbol{Z}) \text{ is } 0\},$$

$$\mathrm{Pic}^0(M) := \{L \in \mathrm{Pic}(M) = H^1(M, O_M^\times) \,|\, H^2(M, \boldsymbol{Z}) \ni c(L) = 0\}.$$

It follows from Theorem 1.3 that the inverse image of $\mathrm{Pic}^0(M)$ by the map

$$\mathrm{Div}(M) \ni D \mapsto [D] \in \mathrm{Pic}(M)$$

is equal to $\mathrm{Div}^0(M)$. Hence, by

$$\{D \in \mathrm{Div}(M) \,|\, \mathrm{Pic}(M) \ni [D] = 0\} = \{(F) \in \mathrm{Div}(M) \,|\, F \in \mathscr{M}(M)^\times\},$$

this map induces an injection $Cl^0(M) \hookrightarrow \mathrm{Pic}^0(M)$, where we put

$$Cl^0(M) := \mathrm{Div}^0(M)/\{(F) \in \mathrm{Div}(M) \,|\, F \in \mathscr{M}(M)^\times\}.$$

The exponential exact sequence $0 \to \boldsymbol{Z}_M \to O_M \to O_M^\times \to 0$ gives rise to

$$H^1(M, O_M)/H^1(M, \boldsymbol{Z}) \cong \mathrm{Pic}^0(M).$$

By Proposition 2.1, the map $H_{2n-1}(M,\boldsymbol{Z}) \to H^{n,n-1}_{\bar\partial}(M)^*$ is an injection and

$$H^1(M,O_M)/H^1(M,\boldsymbol{Z}) \cong H^{n,n-1}_{\bar\partial}(M)^*/H_{2n-1}(M,\boldsymbol{Z}).$$

Theorem 2.2 (Abel's Theorem for Divisors). *In the above circumstances, the composite injection*

$$\begin{aligned} Cl^0(M) \hookrightarrow \mathrm{Pic}^0(M) &\cong H^1(M,O_M)/H^1(M,\boldsymbol{Z}) \\ &\cong H^{n,n-1}_{\bar\partial}(M)^*/H_{2n-1}(M,\boldsymbol{Z}) \end{aligned}$$

is induced by

$$\mathrm{Div}^0(M) \ni D \mapsto \left(H^{n,n-1}_{\bar\partial}(M) \ni [\omega] \mapsto \int_Q \omega \in \boldsymbol{C}\right) \bmod H_{2n-1}(M,\boldsymbol{Z}),$$

where Q is an integral $(2n-1)$-chain on M with $\partial Q = D$.

Proof. Let $D \in \mathrm{Div}^0(M)$. Then for some sufficiently fine open cover $\{U_\lambda\}$ of M and some

$$[\{h_{\lambda\mu}\}] \in H^1(\{U_\lambda\},O_M) \subset H^1(M,O_M),$$

one has $D|_{U_\lambda} = (F_\lambda)$ for some $F_\lambda \in \mathscr{M}(U_\lambda)^\times$ and $F_\lambda/F_\mu = \exp 2\pi i h_{\lambda\mu}$ on $U_\lambda \cap U_\mu \neq \varnothing$. By $H^1(M,O_M) \cong H^{0,1}_{\bar\partial}(M)$, $[\{h_{\lambda\mu}\}]$ corresponds to $[\varphi]$, where $\bar\partial$-closed $(0,1)$-form φ satisfies $h_{\lambda\mu} = \phi_\mu - \phi_\lambda$ on $U_\lambda \cap U_\mu \neq \varnothing$ for some $\phi_\lambda \in \Gamma(U_\lambda, \mathscr{A}^{0,0}_M)$ and $\varphi = \bar\partial\phi_\lambda$ on U_λ. Then one gets a non-vanishing C^∞ function $\tilde F$ on $M - \mathrm{Supp}\, D$ with the property

$$\tilde F|_{U_\lambda} = F_\lambda \exp 2\pi i \phi_\lambda.$$

$d\log\tilde F$ is a C^∞ closed 1-form on $M - \mathrm{Supp}\,D$ and satisfies

$$d\log\tilde F|_{U_\lambda} = d\log F_\lambda + 2\pi i d\phi_\lambda.$$

Now, for any $[\omega] \in H^{n,n-1}_{\bar\partial}(M)$ one has

$$\int_M \varphi \wedge \omega = \frac{1}{2\pi i}\int_M d\log\tilde F \wedge \omega$$

$$(\because (0,1)\text{-part of } \frac{1}{2\pi i} d\log\tilde F \text{ is equal to } \varphi).$$

Claim 2.2.

$$\frac{1}{2\pi i}\int_M d\log\tilde F \wedge \omega = \int_Q \omega + \sum_j \int_{\Gamma_j} \omega \cdot \left[\frac{1}{2\pi i}\int_{\gamma_j} d\log\tilde F - I(\gamma_j, Q)_M\right],$$

where $[\gamma_j] \in H_1(M,\boldsymbol{Z})$ $(1 \leqq j \leqq b_1(M))$ constitute a basis of $H_1(M,\boldsymbol{Z})$/torsion, each representative 1-cycle γ_j is taken such that $\mathrm{Supp}\,\gamma_j \subset M - \mathrm{Supp}\,D$ *and $[\Gamma_j]$ $(1 \leqq j \leqq b_{2n-1}(M))$ is the basis of $H_{2n-1}(M,\boldsymbol{Z})$ dual to the basis $[\gamma_j]$* mod *torsion $(1 \leqq j \leqq b_1(M))$ by Poincaré duality*

$$H_{2n-1}(M, \boldsymbol{Z}) \cong H^1(M, \boldsymbol{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\boldsymbol{Z}}(H_1(M, \boldsymbol{Z})/torsion, \boldsymbol{Z}),$$

i.e. $I(\gamma_j, \Gamma_k)_M = \delta_{jk}$ (*Kronecker delta*).

Since

$$\frac{1}{2\pi i}\int_{\gamma_j} d\log \tilde{F} \in \boldsymbol{Z} \quad \text{for all } j,$$

it follows from Claim 2.2 that

$$\left(H_{\bar{\partial}}^{n,n-1}(M) \ni [\omega] \mapsto \int_M \varphi \wedge \omega \in \boldsymbol{C}\right)$$

$$\equiv \left(H_{\bar{\partial}}^{n,n-1}(M) \ni [\omega] \mapsto \int_Q \omega \in \boldsymbol{C}\right) \bmod H_{2n-1}(M, \boldsymbol{Z}),$$

as required.

Proof of Claim 2.2 (by a method used in the proof of the Riemann-Roch theorem in Kodaira [**K5**]):

Put $G := M - \operatorname{Supp} D$. In the same way as in the proof of the implication Theorem 1.1 $\Rightarrow$ Theorem 1.1$'$, we can take a C^∞ closed $(2n-1)$-form $k_G(\gamma_j)$ on G with compact support which corresponds to the 1-cycle γ_j on G. Then, $k_G(\gamma_j)$ $(1 \leqq j \leqq b_1(M))$ considered on M constitute a basis of $H^{2n-1}(\Gamma(M, \mathscr{A}_M^{\cdot}))$. Hence

$$\omega = \sum_j a_j k_G(\gamma_j) + d\Phi$$

for some $a_j \in \boldsymbol{C}$ and some $C^\infty(2n-2)$-form Φ on M ($\because d\omega = \bar{\partial}\omega = 0$), where

$$\int_{\Gamma_k} \omega = \sum_j a_j \int_{\Gamma_k} k_G(\gamma_j)$$

$$= \sum_j a_j I(\Gamma_k, \gamma_j)_M = -\sum_j a_j I(\gamma_j, \Gamma_k)_M = -a_k$$

for all k. Then

$$\sum_j a_j \frac{1}{2\pi i}\int_{\gamma_j} d\log \tilde{F}$$

$$= \sum_j a_j \frac{1}{2\pi i}\int_G k_G(\gamma_j) \wedge d\log \tilde{F} \quad (\because d\log \tilde{F} \text{ is } C^\infty \text{ closed on } G)$$

$$= -\frac{1}{2\pi i}\int_M d\log \tilde{F} \wedge \omega + \frac{1}{2\pi i}\int_M d\log \tilde{F} \wedge d\Phi.$$

Hence

$$\frac{1}{2\pi i}\int_M d\log\tilde{F}\wedge\omega+\sum_j a_j\frac{1}{2\pi i}\int_{\gamma_j} d\log\tilde{F}$$

$$=\frac{1}{2\pi i}\int_M d\log\tilde{F}\wedge d\Phi$$

$$=\frac{1}{2\pi i}[d\log\tilde{F}](d\Phi)=\frac{1}{2\pi i}d[d\log\tilde{F}](\Phi)$$

$$=D(\Phi)\ \ (\because\text{Theorem 1.1 on } U_\lambda)$$

$$=\partial Q(\Phi)=Q(d\Phi)=Q(\omega)-\sum_j a_jQ(k_G(\gamma_j))$$

$$=\int_Q\omega-\sum_j a_jI(Q,\gamma_j)_G\ \ (\because Q \text{ is an infinite cycle on } G)$$

$$=\int_Q\omega+\sum_j a_jI(\gamma_j,Q)_M. \qquad \square$$

Remark 2.3. The function $\tilde{F}$ appearing in the proof of Theorem 2.2 is just a C^∞ solution to the multiplicative Cousin problem with the data $D\in\mathrm{Div}^0(M)$. Recall that given a data $D\in\mathrm{Div}(X)$, where X is a paracompact complex manifold of dimension n, the problem has a continuous (in fact, a C^∞) solution if and only if $c([D])=0$ (or, by Theorem 1.3, the class of D in $\hat{H}_{2n-2}(X,\boldsymbol{Z})$ is 0). When $H^1(X,O_X)=0$, this condition implies that $[D]=0$, i.e. the problem has an analytic solution. For the second Cousin problem and Oka's principle, see Serre [**Ser**], §II and Grauert-Remmert [**GR**], Kapitel V, §§2–3. We refer the reader to Nagashima [**N**], §2 for relations between Theorem 1.1′, Theorem 1.3 and the solubility of the second Cousin problem.

Remark 2.4. (i) If the canonical injection $H^1(M,\boldsymbol{R})\hookrightarrow H^1(M,O_M)$ is surjective or, equivalently, the Picard variety $H^1(M,O_M)/H^1(M,\boldsymbol{Z})$ of M is a complex torus (Kodaira [**K4**], pp. 13–15; these are valid if M is Kähler, see Kodaira-Spencer [**KS**], p. 872), then we have: $D=(F)$ for some multiplicative function F on M whose multiplier

$$\chi_F\in\mathrm{Hom}_{\boldsymbol{Z}}(H_1(M,\boldsymbol{Z}),U(1))\xleftarrow{\sim}H^1(M,U(1)),\quad \chi_F([\gamma])=\exp\int_\gamma d\log F,$$

belongs to the subgroup

$$\mathrm{Hom}_{\boldsymbol{Z}}(H_1(M,\boldsymbol{Z})/\text{torsion},U(1))\xleftarrow{\sim}H^1(M,\boldsymbol{R})/H^1(M,\boldsymbol{Z})$$

$\Leftrightarrow D\in\mathrm{Div}^0(M)$. (In general, we see as in the proof of Theorem 1.3 that $D=(F)$ for some multiplicative function F on M with $\chi=\chi_F$ if and only if χ maps to $[D]$ under $H^1(M,U(1))\hookrightarrow H^1(M,O_M^\times)$ (see Nagashima [**N**], §2 for details). Hence the implication $\Rightarrow$ holds on an arbitrary M. $\Leftarrow$ is shown by taking $h_{\lambda\mu}$ as real constants in the proof of Theorem 2.2 and then F is given by $d\log F|_{U_\lambda}=d\log F_\lambda$. In case M is Kähler, Kodaira has given a result ([**K1**], Theorem 1) which says that $D=(F)$ for some multiplicative function F on M if and only if the class of D in $H_{2n-2}(M,\boldsymbol{Z})$ is a torsion element and then Igusa [**I**], pp. 13–14 has treated the torsion element using the duality of finite abelian groups

$$\mathrm{Hom}_{\boldsymbol{Z}}(T_1(M), \boldsymbol{Q}/\boldsymbol{Z}) \xleftarrow{\sim} T^2(M) \cong T_{2n-2}(M)$$

(where $T_j(M) \subset H_j(M, \boldsymbol{Z})$, $T^k(M) \subset H^k(M, \boldsymbol{Z})$ are the torsion subgroups; cf. Seifert-Threlfall [**ST**], Zehntes Kapitel, §77, Aufgabe 2 and Hattori [**Ha**], Chapter 8, Problem 7, p. 307) and obtained the equivalence $\Leftrightarrow$; see also Weil [**W1**], [**W2**].)

In that case, we can show by the same method as in the proof of Claim 2.2 the following formula for all $[\omega] \in H_{\bar{\partial}}^{n,n-1}(M)$:

$$(*) \qquad \int_Q \omega = -\sum_j \int_{\Gamma_j} \omega \cdot \left[\frac{1}{2\pi i}\int_{\gamma_j} d\log F - I(\gamma_j, Q)_M\right], \quad \partial Q = (F).$$

Theorem 2.2 follows also from $(*)$ and gives a necessary and sufficient condition for a multiplicative function to be single-valued.

(ii) If M is Kähler, then for any $\omega \in H^{n,n-1}(M) := \{$harmonic $(n, n-1)$-form on $M\}$ we have

$$\omega = \sum_j a_j H\gamma_j$$

(H denotes the harmonic part of a current and a_j is the same as in the proof of Claim 2.2). Hence, in this case $(*)$ in (i) is seen to be equivalent to Kodaira's formula ([**K1**], Theorem 3, [**K3**], §9):

$$\frac{1}{2\pi i}\int_{\gamma_j} d\log F = I(\gamma_j, Q)_M + \int_Q H\gamma_j \quad \text{for all } j$$

by means of the Hodge decomposition

$$H^{2n-1}(M, \boldsymbol{C}) \cong H^{n,n-1}(M) \oplus \overline{H^{n,n-1}(M)}.$$

(iii) When M is Kähler, denoting by Ω the Kähler form on M, we have

$$H^0(M, \Omega_M^1) \xrightarrow{\sim} H^{n,n-1}(M); \quad A \mapsto A \wedge \Omega^{n-1}$$

by Hodge theory. Hence $\int_Q A \wedge \Omega^{n-1}$ appears in Theorem 2.2, which corresponds to $Q_f(A \wedge \Omega^{n-1})$ appearing in a condition for additive functions to be single-valued (Kodaira [**K2**], (3.32), [**K3**], §7). When M is projective, for Ω obtained from a hyperplane section E we get a formula

$$\int_Q A \wedge \Omega^{n-1} = \int_{Q\,.\,E^{n-1}} A$$

(where E^{n-1} is a linear space section of codimension $n-1$) which corresponds to a formula rewriting $Q_f(A \wedge \Omega^{n-1})$ (Kodaira [**K2**], (7.2), [**K3**], §7). Then Theorem 2.2 says: $D \in \mathrm{Div}^0(M)$ is linearly equivalent to 0 if and only if for any integral $(2n-1)$-chain Q on M with $D = \partial Q$ there exists an integral $(2n-1)$-cycle Γ on M such that for all $A \in H^0(M, \Omega_M^1)$

$$\int_{Q\,.\,E^{n-1}} A = \int_{\Gamma\,.\,E^{n-1}} A.$$

Especially, in case of $n = 2$ this formulation implies the following result of Severi [**Sev2**], §4 for algebraically equivalent effective divisors D_1, D_2 on M: if for any integral 1-chain t on E with $D_1\,.\,E - D_2\,.\,E = \partial t$ there exists an integral 1-cycle γ on M (or, by Lefschetz's theorem, an integral 1-cycle γ on E) such that $\int_t A = \int_\gamma A$ for all $A \in H^0(M, \Omega_M^1)$, then dD_1 is linearly equivalent to dD_2 for some $d \in \boldsymbol{N}$ which depends only on M. (To see this for $n \geqq 2$, take an integral $(2n-1)$-chain Q_0 on M with $D_1 - D_2 = \partial Q_0$ and put $t = Q_0\,.\,E^{n-1}$ on E^{n-1} and then notice that, by the strong Lefschetz's theorem

$$H_{2n-1}(M, \boldsymbol{Q}) \xrightarrow{\sim} H_1(M, \boldsymbol{Q}); \quad [\Gamma] \mapsto [\Gamma\,.\,E^{n-1}],$$

the free $\boldsymbol{Z}$-module $H_{2n-1}(M, \boldsymbol{Z})$ is embedded into the free $\boldsymbol{Z}$-module $H_1(M, \boldsymbol{Z})$/torsion of the same rank. Hence, for some $d \in \boldsymbol{N}$ (which depends only on M) the class of $d\gamma$ in $H_1(M, \boldsymbol{Z})$/torsion is equal to the class of $\Gamma\,.\,E^{n-1}$ with some integral $(2n-1)$-cycle Γ on M. Put $Q = dQ_0$.) Abel's theorem for a family $\{D_s\}_{s \in S}$ of effective divisors due to [**Sev1**] (see also Zariski [**Z**], p. 104, p. 164) follows from this result by the fact that if dD_s are linearly equivalent for all $s \in S$ then D_s are also, provided that S is connected. (This fact is readily seen by means of the Picard variety of M.) In the works of Severi, E is assumed only to be an irreducible member of a continuous system of ∞^1 with $(E^2) > 0$.

(iv) When M is projective, we find that Igusa's formula ([**I**], p. 15, the last line) is essentially equivalent to the following:

$$(*') \qquad \int_{Q\,.\,E^{n-1}} A = -\sum_j \int_{\Gamma_j\,.\,E^{n-1}} A \cdot \left[\frac{1}{2\pi i}\int_{\gamma_j} d\log F - I(\gamma_j, Q)_M\right],$$

which follows from $(*)$ in (i) in the same way as in (iii), with the substitution of

$$-\int_{\Gamma_j\,.\,E^{n-1}} A = \sum_k \int_{\gamma_k} A \cdot I(\Gamma_k\,.\,\Gamma_j, E^{n-1})_M.$$

If one denotes by A_α $(1 \leqq \alpha \leqq h^{1,0}(M))$ a basis of $H^0(M, \Omega_M^1)$, then a period matrix of the Albanese variety of M is given by $\int_{\gamma_k} A_\alpha$ and we see that a period matrix of the Picard variety of M is given by $\int_{\Gamma_j\,.\,E^{n-1}} A_\alpha$. The intersection numbers $I(\Gamma_k\,.\,\Gamma_j, E^{n-1})_M$ and the periods $\int_{\Gamma_j\,.\,E^{n-1}} A_\alpha$ have already appeared in a generalization of the Riemann period relations by Hodge [**Ho**], p. 114.

In fact, using his notation, Igusa's formula is essentially the same as

$$\left(\frac{1}{2\pi i}\int_\gamma d\log F - I(\gamma, Q; V)\right) = \varepsilon(m)^t E^{-1},$$

where Q is that of our notation. This can be rewritten as

$$\int_{Q\,.\,C(\mathbf{M})}(\Phi)=\sum_{i=1}^{\varepsilon}\sum_{j=1}^{q}\int_{P'_{ij}}^{P_{ij}}(\Phi)=\left(\frac{1}{2\pi i}\int_{\gamma}d\log F-I(\gamma,Q;V)\right)\cdot{}^tE^t\omega,$$

that is to say

$$\int_{Q\,.\,C(\mathbf{M})}\Phi_{\alpha}=\sum_{j,k=1}^{2q}\omega_{\alpha k}E_{kj}\cdot\left[\frac{1}{2\pi i}\int_{\gamma_j}d\log F-I(\gamma_j,Q;V)\right],\quad 1\leqq\alpha\leqq q,$$

where $\omega_{\alpha k}=\int_{\gamma_k}\Phi_{\alpha}$. Now, Igusa's $\boldsymbol{Z}$-matrix E ([**I**], pp. 8–9) given by

$${}^tI_{\beta}^{-1}=\left(\frac{E|O}{O|*}\right)\quad\text{and}\quad(I_{\beta})_{ij}=I(\beta_i,\beta_j;C(\mathbf{M}))\in\boldsymbol{Q},\quad 1\leqq i,j\leqq 2p$$

(where β_j $(1\leqq j\leqq 2q)$ is the homology basis of rational "invariant cycles" on $C(\mathbf{M})$ such that $\beta_j\sim\gamma_j$ (in V mod. $\boldsymbol{Q}$), β_j $(2q+1\leqq j\leqq 2p)$ is that of rational "vanishing cycles" on $C(\mathbf{M})$ and p is the genus of the curve $C(\mathbf{M})$; see Zariski [**Z**], Chapter VI and Appendix to it) is expressed simply as

$$E_{kj}=I(\Gamma_k\,.\,\Gamma_j,C(\mathbf{M});V)\in\boldsymbol{Z},$$

where Γ_j $(1\leqq j\leqq 2q)$ denotes that of our Claim 2.2, since we have

$$\begin{aligned}&I(\beta_l,\Gamma_k\,.\,C(\mathbf{M});C(\mathbf{M}))\\&\quad=I(\beta_l,i^*\Gamma_k;C(\mathbf{M}))=I(i_*\beta_l,\Gamma_k;V)\\&\quad=I(\gamma_l,\Gamma_k;V)=\delta_{lk},\quad 1\leqq l\leqq 2q\end{aligned}$$

(where $i:C(\mathbf{M})\hookrightarrow V$ is the inclusion) and

$$\Gamma_k\,.\,C(\mathbf{M})\sim\sum_{j=1}^{2q}\beta_j\cdot I(\Gamma_k\,.\,\Gamma_j,C(\mathbf{M});V)\ \ (\text{on}\ \ C(\mathbf{M})\,\text{mod.}\,\boldsymbol{Q})$$

($\because\Gamma_k.C(\mathbf{M})$ is obviously an "invariant cycle" on $C(\mathbf{M})$ and hence is homologous on $C(\mathbf{M})$ to a linear combination of β_j $(1\leqq j\leqq 2q)$. The coefficients of it are determined by

$$\Gamma_k.C(\mathbf{M})\sim\sum_{j=1}^{2q}\gamma_j\cdot I(\Gamma_k\,.\,\Gamma_j,C(\mathbf{M});V)\ \ (\text{in}\ \ V\ \ \text{mod.}\ \ \boldsymbol{Q})$$

which follows at once from $I(\Gamma_k\,.\,C(\mathbf{M}),\Gamma_j;V)=I(\Gamma_k\,.\,\Gamma_j,C(\mathbf{M});V))$.

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