

Configurations of seven lines on the real projective plane and the root system of type E_7

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Abstract. Let l_1, l_2, \dots, l_7 be mutually different seven lines on the real projective plane. We consider two conditions; (A) No three of l_1, l_2, \dots, l_7 intersect at a point. (B) There is no conic tangent to any six of l_1, l_2, \dots, l_7 . Cummings [3] and White [16] showed that there are eleven non-equivalent classes of systems of seven lines with condition (A) (cf. [7], Chap. 18). The purposes of this article is to give an interpretation of the classification of Cummings and White in terms of the root system of type E_7 . To accomplish this, it is better to add condition (B) for systems of seven lines. Moreover we need the notion of tetrahedral sets which consist of ten roots modulo signs in the root system of type E_7 and which plays an important role in our study.

1. Introduction.

The configurations of seven lines on the projective plane are closely related with the classical topic on the twenty-eight bitangents of plane quartic curves. As is well-known, the Weyl group of type E_7 is regarded as the group of incidence-preserving automorphisms of the twenty-eight bitangents.

In this article, we shall study the case where the seven lines are defined over the real number field. So let l_1, l_2, \dots, l_7 be mutually different seven lines on $P^2(\mathbf{R})$. We consider the following conditions on these lines:

- (A) No three of l_1, l_2, \dots, l_7 intersect at a point.
- (B) There is no conic tangent to any six of l_1, l_2, \dots, l_7 .

Two systems of seven lines with condition (A) are equivalent provided there exists a one-to-one incidence-preserving correspondence between their polygons. Cummings [3] and White [16] showed that there are eleven non-equivalent classes of systems of seven lines with condition (A) (cf. [7], Chap. 18). One of the purposes of this article is to give an interpretation of the classification of Cummings and White in terms of the root system of type E_7 . To accomplish this, it is better to add condition (B) for systems of seven lines.

We are going to explain the main result of this article. Let $(\xi : \eta : \zeta)$ be a homogeneous coordinate of $P^2(\mathbf{R})$. Then, by a projective linear transformation, we may take

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$$l_1 : \xi = 0,$$

$$l_2 : \eta = 0,$$

$$l_3 : \zeta = 0,$$

$$l_4 : \xi + \eta + \zeta = 0,$$

$$l_5 : \xi + x_1\eta + y_1\zeta = 0,$$

$$l_6 : \xi + x_2\eta + y_2\zeta = 0,$$

$$l_7 : \xi + x_3\eta + y_3\zeta = 0.$$

as defining equations of l_1, l_2, \dots, l_7 . In this manner, we may regard the totality of systems of seven lines with condition (A) as a Zariski open subset of \mathbf{R}^6 . Let $\mathbf{P}_0(2, 7)$ be the subset of \mathbf{R}^6 consisting of systems of seven lines with conditions (A), (B). Then there is an action of $W(E_7)$, the Weyl group of type E_7 , on $\mathbf{P}_0(2, 7)$. This action of $W(E_7)$ naturally induces that on the set \mathcal{P}_7 of connected components of $\mathbf{P}_0(2, 7)$.

Let $\Delta(E_7)$ be the root system of type E_7 . In [11], we introduced the notion of tetrahedral sets which consist of ten roots modulo signs in $\Delta(E_7)$. Let \mathcal{T} be the totality of tetrahedral sets. Then $W(E_7)$ acts on \mathcal{T} in a natural manner.

We now state the main result of this article.

THEOREM. *There is a $W(E_7)$ -equivariant injective map f of \mathcal{T} to \mathcal{P}_7 .*

The symmetric group S_7 on seven letters is regarded as a subgroup of $W(E_7)$ generated by reflections. Noting this, we consider the S_7 -orbital structure of \mathcal{T} .

PROPOSITION. *There are fourteen S_7 -orbits of \mathcal{T} .*

By the $W(E_7)$ -equivariant map f above, we obtain fourteen S_7 -orbits of $f(\mathcal{T})$. Two connected components Ω_1, Ω_2 of $\mathbf{P}_0(2, 7)$ are equivalent if there are equivalent systems L_1, L_2 of seven lines such that $L_i \in \Omega_i$ ($i = 1, 2$). It easily follows from the definition that Ω_1 and Ω_2 are equivalent if they are contained in the same S_7 -orbit. It sometimes happens that Ω_1 and Ω_2 are equivalent unless they are contained in the same S_7 -orbit. As a consequence, we obtain eleven non-equivalent classes in $f(\mathcal{T})$. These eleven equivalent classes correspond to eleven equivalent classes of systems of seven lines classified by Cummings and White. The idea of the proof of the main result is the same as that for the case of six lines developed in [12]. In spite that we do not discuss on the $W(E_7)$ -orbital structure of \mathcal{P}_7 , it is provable (cf. [10]) that the $W(E_7)$ -action on \mathcal{P}_7 is transitive. This implies in particular that the map f is surjective. Our proof for the transitivity is based on a detailed study on a cross ratio variety for the root system of type E_7 (cf. [9]) and we shall treat this subject elsewhere.

We are going to explain the contents of this article briefly. We begin section 2 with recalling the definition of the root system of type E_7 . We next introduce the notion of tetrahedral sets and determine the S_7 -orbital structure of the totality \mathcal{T} of tetrahedral sets. Section 4 is devoted to configurations of n lines on the real projective plane. In particular, we introduce a group W_n acting on the set $\mathbf{P}_0(2, n)$ which consists of systems

of marked n lines on $\mathbf{P}^2(\mathbf{R})$. In section 5, we review the results of [12] on the six lines case. In section 6, we study the space $\mathbf{P}_0(2, 7)$ in detail. We first construct a special system of marked seven lines which contains ten triangles. To these ten triangles, there associates a tetrahedral set. By using this correspondence, we construct a $W(E_7)$ -equivariant injective map f of \mathcal{T} to \mathcal{P}_7 . As a consequence, we can show the main result of this article. In section 7, we discuss the comparison between our results and those of Cummings and White. In the last section, we mention a relationship between tetradiagrams and Shrikhande graphs.

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2. Review on the root system of type E_7 .

We first recall the definition of the root system of type E_7 . Bourbaki [1] is a standard reference on the root system.

Let \tilde{E} be an 8-dimensional Euclidean space with a standard basis $\{\varepsilon_j; 1 \leq j \leq 8\}$. Let $\langle \cdot, \cdot \rangle$ be the inner product on \tilde{E} defined by $\langle \varepsilon_j, \varepsilon_k \rangle = \delta_{jk}$ ($1 \leq j, k \leq 8$) and let E be its linear subspace orthogonal to $\varepsilon_7 + \varepsilon_8$. We define the following sixty-three vectors of E :

$$\begin{aligned}
 & \gamma_1 = \varepsilon_7 - \varepsilon_8, \\
 & \gamma_j = \varepsilon_{j-1} - \gamma_0 + \gamma_1, \quad 1 < j < 8 \\
 & \gamma_{1j} = -\varepsilon_{j-1} + \gamma_0, \quad 1 < j < 8 \\
 & \gamma_{jk} = \varepsilon_{j-1} - \varepsilon_{k-1}, \quad 1 < j < k < 8 \\
 & \gamma_{1jk} = -\varepsilon_{j-1} - \varepsilon_{k-1}, \quad 1 < j < k < 8 \\
 & \gamma_{ijk} = -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + \gamma_0, \quad 1 < i < j < k < 8
 \end{aligned}
 \tag{1}$$

where

$$\gamma_0 = \frac{1}{2} \sum_{j=1}^8 \varepsilon_j - \varepsilon_8.$$

Note that

$$\gamma_i \perp \gamma_{jk}, \quad \gamma_i \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{kl}, \quad \gamma_{ij} \perp \gamma_{ijk}, \quad \gamma_{ij} \perp \gamma_{klm}, \quad \gamma_{ijk} \perp \gamma_{ilm}.$$

The totality $\Delta(E_7)$ of $\pm \gamma_j, \pm \gamma_{jk}, \pm \gamma_{ijk}$ forms a root system of type E_7 .

REMARK 1. The description of E_7 -roots above plays a basic role in the present study (cf. Remark 6). For its geometric meaning, see Shioda [13].

It is clear that

$$\gamma_{12}, \quad \gamma_{123}, \quad \gamma_{23}, \quad \gamma_{34}, \quad \gamma_{45}, \quad \gamma_{56}, \quad \gamma_{67}
 \tag{2}$$

DEFINITION 1. Let A be a set of ten letters a_{ij} ($= a_{ji}$) indexed by numbers i, j ($1 \leq i < j \leq 5$).

- (i) An injection f of A to $\Delta(E_6)$ is called a pentagonal map if the following conditions hold:
 1. $\langle f(a_{ij}), f(a_{i'j'}) \rangle \geq 0 \quad \forall a_{ij}, a_{i'j'} \in A$.
 2. For $a_{ij}, a_{i'j'} \in A$ ($a_{ij} \neq a_{i'j'}$), $\langle f(a_{ij}), f(a_{i'j'}) \rangle = 0$ if and only if $\{i, j\} \cap \{i', j'\} \neq \emptyset$.
- (ii) A subset Γ of $\Delta(E_6)$ is called a pentagonal set if there is a pentagonal map f such that $\Gamma = f(A)^\pm$.

It follows from the definition that A admits a natural action of S_5 , the symmetric group on five letters. If f is a pentagonal map, so is $f \circ \sigma$ for each $\sigma \in S_5$. This implies that the set $f(A)^\pm$ admits an S_5 -action.

To a pentagonal map f , there associates a diagram similar to Dynkin diagrams which we are going to explain.

DEFINITION 2. (i) A pent-diagram consists of ten circles and fifteen segments joining circles as in Figure I with the following properties:

1. There is a bijection between A and the totality of ten circles of the pent-diagram.
2. Two circles corresponding to $a_{ij}, a_{i'j'} \in A$ are joined by a segment if and only if $\{i, j\} \cap \{i', j'\} = \emptyset$.

(ii) If f is a pentagonal map, the pent-diagram for $f(A)^\pm$ is the pent-diagram of which the circle corresponding to a_{ij} is attached with the roots $\pm f(a_{ij})$.

REMARK 2. Pent-diagrams are usually called Petersen graphs in literatures.

EXAMPLE 1. There is a pentagonal map f such that $\Gamma = f(A)^\pm$, where

$$f(A) = \{\gamma_{124}, \gamma_{126}, \gamma_{245}, \gamma_{134}, \gamma_{156}, \gamma_{135}, \gamma_{235}, \gamma_{456}, \gamma_{236}, \gamma_{346}\}.$$

The following lemma is easy to prove.

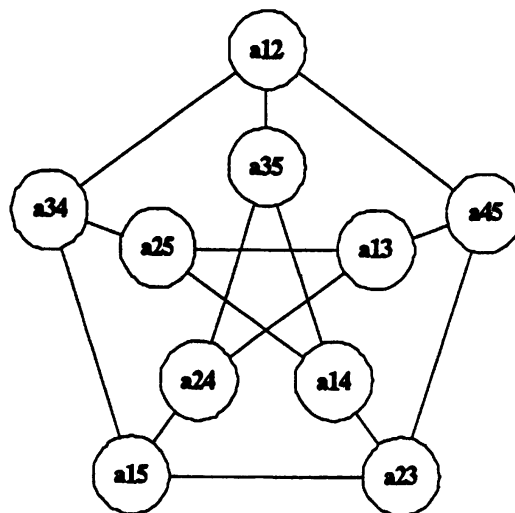


Figure I: A pent-diagram

LEMMA 1. Let α_j ($j = 1, 2, \dots, 6$) be the simple roots of $\Delta(E_6)$ defined by $\alpha_1 = \gamma_{12}, \alpha_2 = \gamma_{123}, \alpha_3 = \gamma_{23}, \alpha_4 = \gamma_{34}, \alpha_5 = \gamma_{45}, \alpha_6 = \gamma_{56}$. If f is a pentagonal map such that $f(\mathbf{A})$ contains

$$\alpha_1, -\alpha_2, -\alpha_3, \alpha_4, -\alpha_5, \alpha_6,$$

then

$$f(\mathbf{A}) = \{\alpha_1, -\alpha_2, -\alpha_3, \alpha_4, -\alpha_5, \alpha_6, -\gamma_7, \gamma_{16}, \gamma_{156}, \gamma_{345}\}.$$

In particular, $f(\mathbf{A})$ is uniquely determined.

This implies the following.

THEOREM 1. The group $W(E_6)$ acts on the set of pentagonal sets transitively.

Let G_0 be the subgroup of $W(E_6)$ generated by $s_{j,j+1}$ ($j = 1, 2, 3, 4, 5$) as in section 2. In our application, the G_0 -orbital structure of pentagonal sets is important.

THEOREM 2. The totality of pentagonal sets is decomposed into four S_6 -orbits whose representatives are given below:

(4)	P_O	$\{\gamma_{12}, -\gamma_{23}, \gamma_{34}, -\gamma_{45}, \gamma_{56}, \gamma_{16}, -\gamma_{123}, -\gamma_{156}, -\gamma_{345}, \gamma_7\}^\pm$
	P_I	$\{\gamma_{126}, -\gamma_{135}, -\gamma_{145}, -\gamma_{235}, \gamma_{246}, \gamma_{346}, -\gamma_{15}, \gamma_{26}, -\gamma_{35}, \gamma_{46}\}^\pm$
	P_{II}	$\{-\gamma_{145}, \gamma_{356}, \gamma_{235}, \gamma_{234}, \gamma_{246}, \gamma_{126}, \gamma_{136}, -\gamma_{24}, \gamma_{16}, -\gamma_{35}\}^\pm$
	P_{III}	$\{\gamma_{234}, \gamma_{456}, \gamma_{126}, \gamma_{125}, \gamma_{145}, \gamma_{134}, \gamma_{136}, \gamma_{356}, \gamma_{235}, \gamma_{246}\}^\pm$

3.2. Tetrahedral sets and tetradiagrams.

For the results of this subsection, see Sekiguchi and Tanabata [11].

DEFINITION 3. Let \mathbf{B} be a set of ten letters a_i ($i = 1, 2, 3, 4$), b_{ij} ($1 \leq i < j \leq 4$) (Assume that $b_{ij} = b_{ji}$ for all i, j).

(i) An injection f of \mathbf{B} to $\Delta(E_7)$ is called a tetrahedral map if the following conditions hold:

1. $\langle f(a_i), f(a_j) \rangle = 0$ ($i \neq j$).
2. $\langle f(b_{ij}), f(b_{i'j'}) \rangle = 0$ ($\{i, j\} \neq \{i', j'\}$).
3. $\langle f(a_i), f(b_{jk}) \rangle = 0$ if and only if $i \notin \{j, k\}$.

(ii) A subset Γ of $\Delta(E_7)$ is called a tetrahedral set if there is a tetrahedral map such that $\Gamma = f(\mathbf{B})^\pm$.

It follows from the definition that \mathbf{B} admits a natural action of S_4 , the symmetric group on four letters. If f is a tetrahedral map, so is $f \circ \sigma$ for each $\sigma \in S_4$. This implies that the set $f(\mathbf{B})$ admits an S_4 -action.

To a tetrahedral map f , there associates a diagram similar to Dynkin diagrams which we are going to explain.

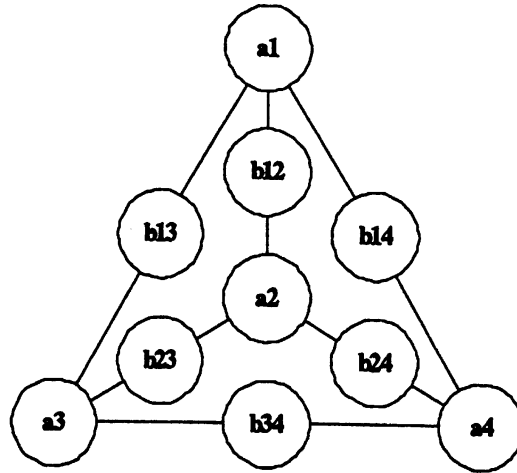


Figure II: A tetradiagram

DEFINITION 4. (i) A tetradiagram consists of ten circles and twelve segments joining circles as in Figure II with the following properties:

1. There is a bijection between \mathbf{B} and the totality of the circles of the tetradiagram.
2. Two circles corresponding to $a_i, a_{i'} \in \mathbf{B}$ (resp. $b_{ij}, b_{i'j'} \in \mathbf{B}$) are not joined by any segment.
3. Two circles corresponding to $a_i, b_{jk} \in \mathbf{B}$ are joined by a segment if and only if $i \in \{j, k\}$.

(ii) If f is a tetrahedral map, the tetradiagram for $\Gamma = f(\mathbf{B})^\pm$ is the tetradiagram of which the circle corresponding to a_i (resp. b_{jk}) is attached with the roots $\pm f(a_i)$ (resp. $\pm f(b_{jk})$).

EXAMPLE 2. There is a tetrahedral map f such that

$$f(\mathbf{B})^\pm = \{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^\pm.$$

In particular, the correspondence

$$\begin{array}{llll} a_1 & \longrightarrow & \gamma_{345} & b_{12} & \longrightarrow & \gamma_{135} & b_{23} & \longrightarrow & \gamma_{124} \\ a_2 & \longrightarrow & \gamma_{123} & b_{13} & \longrightarrow & \gamma_{167} & b_{24} & \longrightarrow & \gamma_{236} \\ a_3 & \longrightarrow & \gamma_{146} & b_{14} & \longrightarrow & \gamma_{347} & b_{34} & \longrightarrow & \gamma_{257} \\ a_4 & \longrightarrow & \gamma_{256} & & & & & & \end{array}$$

induces a tetradiagram for $f(\mathbf{B})^\pm$.

REMARK 3. You can find twenty-four extended Dynkin diagrams of type E_7 embedded in a tetradiagram.

The following lemma is shown by a direct computation.

LEMMA 2. If a tetrahedral set Γ contains

$$\gamma_{12}, \gamma_{23}, \gamma_{34}, \gamma_{45}, \gamma_{56}, \gamma_{67}, \gamma_{123}$$

Table I

Name	Tetrahedral set	Isotropy
A	$\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^{\pm}$	\mathbf{Z}_3
B1	$\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{25}, \gamma_{167}, \gamma_{347}, \gamma_{34}, \gamma_{16}, \gamma_{257}\}^{\pm}$	\mathbf{Z}_3
B2	$\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{14}, \gamma_2, \gamma_{57}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^{\pm}$	1
B3	$\{\gamma_{14}, \gamma_{25}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^{\pm}$	1
B4	$\{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_6, \gamma_{167}, \gamma_{23}, \gamma_{17}, \gamma_{236}, \gamma_{45}\}^{\pm}$	1
B5	$\{\gamma_2, \gamma_{123}, \gamma_{47}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^{\pm}$	1
C1	$\{\gamma_{15}, \gamma_{24}, \gamma_{36}, \gamma_7, \gamma_{25}, \gamma_{167}, \gamma_{17}, \gamma_{34}, \gamma_{346}, \gamma_6\}^{\pm}$	1
C2	$\{\gamma_{25}, \gamma_{14}, \gamma_{146}, \gamma_{256}, \gamma_{345}, \gamma_5, \gamma_{27}, \gamma_{34}, \gamma_{16}, \gamma_{257}\}^{\pm}$	1
C3	$\{\gamma_{36}, \gamma_7, \gamma_{15}, \gamma_{24}, \gamma_{246}, \gamma_{167}, \gamma_{347}, \gamma_{356}, \gamma_{145}, \gamma_{257}\}^{\pm}$	\mathbf{Z}_3
C4	$\{\gamma_6, \gamma_{37}, \gamma_{146}, \gamma_{256}, \gamma_{157}, \gamma_{26}, \gamma_{347}, \gamma_{34}, \gamma_{267}, \gamma_{15}\}^{\pm}$	1
D1	$\{\gamma_{56}, \gamma_{123}, \gamma_{13}, \gamma_{256}, \gamma_{136}, \gamma_{157}, \gamma_{57}, \gamma_{124}, \gamma_{24}, \gamma_1\}^{\pm}$	1
D2	$\{\gamma_{47}, \gamma_{123}, \gamma_2, \gamma_{256}, \gamma_{14}, \gamma_{146}, \gamma_{57}, \gamma_6, \gamma_{236}, \gamma_{23}\}^{\pm}$	1
D3	$\{\gamma_{46}, \gamma_{123}, \gamma_{146}, \gamma_{23}, \gamma_{14}, \gamma_{147}, \gamma_{367}, \gamma_7, \gamma_{25}, \gamma_{257}\}^{\pm}$	1
D4	$\{\gamma_{37}, \gamma_6, \gamma_{146}, \gamma_{256}, \gamma_{157}, \gamma_{136}, \gamma_{57}, \gamma_{124}, \gamma_1, \gamma_{24}\}^{\pm}$	1

(these are the simple roots in (3)), then Γ coincides with

$$\{\gamma_{12}, \gamma_{23}, \gamma_{34}, \gamma_{45}, \gamma_{56}, \gamma_{67}, \gamma_{123}, \gamma_1, \gamma_7, \gamma_{167}\}^{\pm}.$$

In virtue of this lemma, the classification of tetrahedral sets is essentially reduced to that of fundamental systems of roots of $\Delta(E_7)$, which is well-known, and we get

PROPOSITION 1. *If Γ and Γ' are tetrahedral sets, there exists $w \in W(E_7)$ such that $w\Gamma = \Gamma'$.*

The G_1 -orbit structure will be also important. For this purpose, we will introduce fourteen tetrahedral sets.

If Γ is a tetrahedral set which has the name M in TABLE I, we denote by O_M the S_7 -orbit of Γ . If Γ' is a tetrahedral set such that $\Gamma' \in O_M$, we call Γ' a tetrahedral set of type M. Similarly, we call a tetradiagram for Γ' that of type M.

We now concentrate our attention to the tetrahedral set (cf. Example 2)

$$\Gamma = \{\gamma_{345}, \gamma_{123}, \gamma_{146}, \gamma_{256}, \gamma_{135}, \gamma_{167}, \gamma_{347}, \gamma_{124}, \gamma_{236}, \gamma_{257}\}^{\pm}.$$

Put

$$\sigma_1 = s_3 s_{15} s_{24} s_{67},$$

$$\sigma_2 = s_1 s_{24} s_{36} s_{57},$$

$$\sigma_3 = s_7 s_{14} s_{25} s_{36},$$

$$c = s_1 s_{23} s_{123} s_{45} s_{145} s_{67} s_{167}.$$

Then c is the non-trivial central element of $W(E_7)$ and it is easy to show that

$$\sigma_1 \Gamma = \sigma_2 \Gamma = \sigma_3 \Gamma = \Gamma.$$

More precisely, we find that there is a tetrahedral map f of \mathbf{B} to $\Delta(E_7)$ such that

$$f(a_1) = \gamma_{345}, \quad f(a_2) = \gamma_{123}, \quad f(a_3) = \gamma_{146}, \quad f(a_4) = \gamma_{256}.$$

Then σ_1 (resp. σ_2, σ_3) corresponds to a permutation between a_1 and a_2 (resp. that between a_2 and a_3 and that between a_3 and a_4). Therefore the group generated by $\sigma_i (i = 1, 2, 3)$ is isomorphic to S_4 by the correspondence

$$\sigma_1 \rightarrow (12), \quad \sigma_2 \rightarrow (23), \quad \sigma_3 \rightarrow (34).$$

PROPOSITION 2. *The isotropy subgroup of Γ in $W(E_7)$ is isomorphic to the group $S_4 \times \mathbf{Z}_2$.*

As a consequence of Propositions 1, 2, we obtain a classification of S_7 -orbital structure of tetrahedral sets.

THEOREM 3. *The totality \mathcal{T} of tetrahedral sets is decomposed into fourteen S_7 -orbits $O_M (M \in \{A, B1, \dots, B5, C1, \dots, C4, D1, \dots, D4\})$.*

4. Configurations of n lines on the real projective plane.

The main subject of this article is the study on configurations of seven lines on the real projective plane. But we start this section with discussing on a general case, namely, the case of configurations of n lines, since the formulation is almost same. We here note that there are many articles treated configurations of six, seven, eight lines (cf. [3], [4], [6], [7], [12], [15], [16]).

A set of mutually different n lines on $\mathbf{P}^2(\mathbf{R})$ indexed by numbers $1, 2, \dots, n$ is called a system of marked n lines (on $\mathbf{P}^2(\mathbf{R})$) in this article.

Let $\{l_1, l_2, \dots, l_n\}$ be a system of marked n lines. We give a condition on these n lines:

- (A) No three of l_1, l_2, \dots, l_n intersect at a point.

We define p -gons for a system of marked n lines $\{l_1, l_2, \dots, l_n\}$. Each connected component of $\mathbf{P}^2(\mathbf{R}) - \bigcup_{j=1}^n l_j$ is called a polygon. If it is surrounded by p lines $l_{k_1}, l_{k_2}, \dots, l_{k_p}$, it is called a p -gon and is written as $l_{k_1}l_{k_2} \cdots l_{k_p}$.

The totality of systems of marked n lines on $\mathbf{P}^2(\mathbf{R})$ with condition (A) forms the configuration space $\mathbf{P}(2, n)$; the space $\mathbf{P}(2, n)$ is defined by

$$\mathbf{P}(2, n) = GL(3, \mathbf{R}) \backslash M'(3, n) / (\mathbf{R}^\times)^n,$$

where $M'(3, n)$ is the set of $3 \times n$ real matrices of which no 3-minor vanishes. Permutations on the n lines l_1, l_2, \dots, l_n induce a biregular S_n -action on $\mathbf{P}(2, n)$.

We are going to define ‘‘an action of a group W_n (defined later) on $\mathbf{P}(2, n)$ ’’ in a concrete manner. Let $L = \{l_1, l_2, \dots, l_n\}$ be a system of marked n lines. We assume that the j -th line l_j is defined by an equation

$$l_j : a_{1j}\xi + a_{2j}\eta + a_{3j}\zeta = 0,$$

where $(\xi : \eta : \zeta)$ is a homogeneous coordinate of $\mathbf{P}^2(\mathbf{R})$. To L , we associate a $3 \times n$

matrix

$$X = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \end{pmatrix}.$$

By a projective linear transformation, we may take as l_1, l_2, l_3, l_4 the lines defined by $\xi = 0, \eta = 0, \zeta = 0, \xi + \eta + \zeta = 0$, respectively. Moreover, we may divide the equation of l_j by a_{1j} ($5 \leq j \leq n$). In this manner, it is possible to choose as a representative of any system of marked n lines a matrix of the form

$$(5) \quad X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & \cdots & x_{n-4} \\ 0 & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{n-4} \end{pmatrix}.$$

Therefore $P(2, n)$ is identified with an open subset of $\mathbf{R}^{2(n-4)}$ by the correspondence

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & \cdots & x_{n-4} \\ 0 & 0 & 1 & 1 & y_1 & y_2 & \cdots & y_{n-4} \end{pmatrix} \rightarrow (x_1, x_2, \dots, x_{n-4}, y_1, y_2, \dots, y_{n-4}).$$

We introduce the following n birational transformations $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$ on $\mathbf{R}^{2(n-4)}$ with the coordinate (x, y) :

$$\begin{aligned} \sigma_0 : x_i &\rightarrow \frac{1}{x_i}, \quad y_i \rightarrow \frac{1}{y_i} \quad (i = 1, \dots, n-4) \\ \sigma_1 : (x_1, x_2, \dots, x_{n-4}, y_1, y_2, \dots, y_{n-4}) &\rightarrow \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_{n-4}}, \frac{y_1}{x_1}, \frac{y_2}{x_2}, \dots, \frac{y_{n-4}}{x_{n-4}} \right) \\ \sigma_2 : (x_1, x_2, \dots, x_{n-4}, y_1, y_2, \dots, y_{n-4}) &\rightarrow (y_1, y_2, \dots, y_{n-4}, x_1, x_2, \dots, x_{n-4}) \\ \sigma_3 : (x_1, x_2, \dots, x_{n-4}, y_1, y_2, \dots, y_{n-4}) &\rightarrow (x'_1, x'_2, \dots, x'_{n-4}, y'_1, y'_2, \dots, y'_{n-4}) \\ \sigma_4 : (x_1, x_2, \dots, x_{n-4}, y_1, y_2, \dots, y_{n-4}) &\rightarrow \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_{n-4}}{x_1}, \frac{1}{y_1}, \frac{y_2}{y_1}, \dots, \frac{y_{n-4}}{y_1} \right) \\ \sigma_{k+4} : x_k &\leftrightarrow x_{k+1}, \quad y_k \leftrightarrow y_{k+1} \quad (k = 1, 2, \dots, n-5), \end{aligned}$$

where

$$x'_j = \frac{x_j - y_j}{1 - y_j}, \quad y'_j = \frac{y_j}{y_j - 1}, \quad j = 1, 2, \dots, n-4.$$

Let W_n be the group generated by n elements $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$. In the cases $n = 6, 7$, the correspondence

$$s_{12} \rightarrow \sigma_1, \quad s_{123} \rightarrow \sigma_0, \quad s_{j,j+1} \rightarrow \sigma_{j-1} \quad (j = 3, \dots, n-1)$$

induces a surjective homomorphism $p_{W(E_n)}$ of $W(E_n)$ to the group W_n . In the sequel, we frequently confuse $g \in W(E_n)$ with $p_{W(E_n)}(g)$ and subgroups of $W(E_n)$ with their images by $p_{W(E_n)}$ for simplicity.

REMARK 4. In spite that in this article, we do not treat the Weyl group $W(E_8)$ of type E_8 , for the case $n = 8$, the argument above goes well for this case, namely, there is a surjective homomorphism of $W(E_8)$ to W_8 .

We now define a set R_n consisting of irreducible polynomials f_1, \dots, f_N of variables (x, y) with the following properties:

1. Neither f_j nor f_i/f_j are constants if $i \neq j$.
2. $f_1 = x_1$.
3. For each $i = 1, \dots, N, j = 0, 1, \dots, n - 1$,

$$f_i \circ \sigma_j = c_{ij} f_1^{m_1} \cdots f_N^{m_N},$$

where c_{ij} is a non-zero constant and $m_k \in \mathbf{Z}$ ($k = 1, \dots, N$).

4. The number N is minimal under the conditions 1, 2, 3.

Let $\mathbf{P}_0(2, n)$ be the intersection of $f_j \neq 0$ ($j = 1, 2, \dots, N$) in $\mathbf{R}^{2(n-4)}$. Then W_n acts on $\mathbf{P}_0(2, n)$ biregularly.

LEMMA 3. (i) If $n = 6, 7$, then $N = 15, 35$, respectively.

(ii) Assume that $n = 6, 7$. Let $L = \{l_1, l_2, \dots, l_n\}$ be a system of marked n lines on $\mathbf{P}^2(\mathbf{R})$. Then L is contained in $\mathbf{P}_0(2, n)$ if and only if the following conditions hold.

- (A) No three of l_1, l_2, \dots, l_n intersect at a point.
- (B) There is no conic tangent to any six of l_1, l_2, \dots, l_n .

REMARK 5. In the case $n = 8$, it is possible to describe the condition for $\{l_1, l_2, \dots, l_8\} \in \mathbf{P}(2, 8)$ to be contained in $\mathbf{P}_0(2, 8)$ if we treat the dual points. Namely, let

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1), \quad P_4 = (1 : 1 : 1),$$

$$P_5 = (1 : x_1 : y_1), \quad P_6 = (1 : x_2 : y_2), \quad P_7 = (1 : x_3 : y_3), \quad P_8 = (1 : x_4 : y_4)$$

be eight points of $\mathbf{P}^2(\mathbf{R})$. Then the 3×8 matrix defined by P_1, \dots, P_8 is contained in $\mathbf{P}_0(2, 8)$ if and only if the following conditions hold:

1. The points P_1, \dots, P_8 are mutually different.
2. There is no line passing through any three of P_1, \dots, P_8 .
3. There is no conic passing through any six of P_1, \dots, P_8 .
4. There is no cubic passing through all of P_1, \dots, P_8 and having one of P_1, \dots, P_8 as a double point.

For the case $n = 7$, we are going to construct the polynomials of R_n concretely. Let X be the matrix of the form (4), namely,

$$(6) \quad X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 \end{pmatrix}.$$

We write $X = (x_1x_2x_3x_4x_5x_6x_7)$ for simplicity. Let D_{ijk} be the determinant of the 3×3 minor $(x_i x_j x_k)$ of X ($1 \leq i < j < k \leq 7$). Clearly $D_{123}, D_{124}, D_{134}, D_{23k}$ ($k > 3$) are constants but the remaining ones are irreducible polynomials. As a result, we obtain $7!/(3! \cdot 4!) - 7 = 28$ polynomials. Moreover, we have to construct seven polynomials. Let $L = \{l_1, l_2, \dots, l_7\}$ be a system of marked seven lines defined by the matrix X . We pick out a line l_j from L . Then it is easy to write down an equation $p_j = 0$ for the condition that there is a conic tangent to the six lines l_k ($k \neq j$). Polynomials p_1, \dots, p_7 are given as follows (cf. [9]):

$$\begin{aligned}
 p_1(x_1, x_2, x_3, y_1, y_2, y_3) &= (1 - y_2)(x_1 - x_2)(x_3 - x_1 + y_1 - y_3 + x_1y_3 - x_3y_1) \\
 &\quad - (1 - y_3)(x_1 - x_3)(x_2 - x_1 + y_1 - y_2 + x_1y_2 - x_2y_1), \\
 p_2(x_1, x_2, x_3, y_1, y_2, y_3) &= x_3(x_2 - y_2)(x_1 - x_2)(x_3 - x_1 + y_1 - y_3 + x_1y_3 - x_3y_1) \\
 &\quad - x_2(x_3 - y_3)(x_1 - x_3)(x_2 - x_1 + y_1 - y_2 + x_1y_2 - x_2y_1), \\
 p_3(x_1, x_2, x_3, y_1, y_2, y_3) &= y_3(x_2 - y_2)(y_1 - y_2)(x_3 - x_1 + y_1 - y_3 + x_1y_3 - x_3y_1) \\
 &\quad - y_2(x_3 - y_3)(y_1 - y_3)(x_2 - x_1 + y_1 - y_2 + x_1y_2 - x_2y_1), \\
 p_4(x_1, x_2, x_3, y_1, y_2, y_3) &= x_2y_3(x_1 - x_3)(y_1 - y_2) - x_3y_2(x_1 - x_2)(y_1 - y_3), \\
 p_5(x_1, x_2, x_3, y_1, y_2, y_3) &= x_2y_3(1 - x_3)(1 - y_2) - x_3y_2(1 - x_2)(1 - y_3), \\
 p_6(x_1, x_2, x_3, y_1, y_2, y_3) &= x_1y_3(1 - x_3)(1 - y_1) - x_3y_1(1 - x_1)(1 - y_3), \\
 p_7(x_1, x_2, x_3, y_1, y_2, y_3) &= x_1y_2(1 - x_2)(1 - y_1) - x_2y_1(1 - x_1)(1 - y_2).
 \end{aligned}$$

We may take R_7 as the set consisting of D_{ijk} ($((i, j, k) \neq (1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, l) (l > 3))$ and p_j ($j = 1, \dots, 7$).

We return to the general case. Let \mathcal{P}_n be the set of connected components of $\mathbf{P}_0(2, n)$. The action of W_n on $\mathbf{P}_0(2, n)$ naturally induces that on \mathcal{P}_n . Then it is interesting to attack the problem below:

PROBLEM 1. Determine the W_n -orbital structure of \mathcal{P}_n .

5. Geometry of six lines on the real projective plane.

In this section, we review the results of Sekiguchi and Yoshida [12]. For the details of the statements of this section, see [12].

We first state an answer to Problem 1 in the previous section for the case $n = 6$.

- THEOREM 4. (i) *There are 432 connected components of $\mathbf{P}_0(2, 6)$.*
 (ii) *The Weyl group $W(E_6)$ acts on the set \mathcal{P}_6 transitively.*
 (iii) *For each connected component of $\mathbf{P}_0(2, 6)$, its isotropy subgroup in $W(E_6)$ is isomorphic to the symmetric group S_5 .*
 (iv) *The set \mathcal{P}_6 is in a one to one correspondence with that of pentagonal sets.*

Table II

Types	triangles	rectangles	pentagon	hexagon
O	6	9	0	1
I	6	8	2	0
II	7	6	3	0
III	10	0	6	0

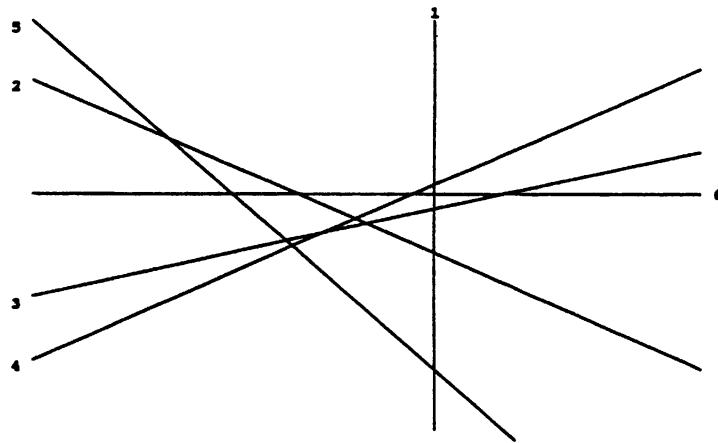


Figure III

Let $L = \{l_1, \dots, l_6\} \in \mathbf{P}_0(2, 6)$ be a system of marked six lines. The totality of the systems in $\mathbf{P}_0(2, 6)$ which can be obtained by continuous deformations of the l_j 's form a connected component of $\mathbf{P}_0(2, 6)$, which is a 4-dimensional cell. There are four S_6 -orbits of the sets of connected components of $\mathbf{P}_0(2, 6)$, referred to as O, I, II, III. Each orbit is characterized by the numbers of polygons so that any of the systems in it cut out (cf. [7]):

A system of marked six lines is said to be of type $T \in \{O, I, II, III\}$ if it belongs to the orbit T.

Let L be the system of marked six lines given in FIGURE III. Then there are ten triangles given as follows:

$$l_1l_2l_3, l_1l_2l_5, l_1l_3l_6, l_1l_4l_6, l_2l_3l_4, l_2l_4l_6, l_2l_5l_6, l_3l_4l_5, l_3l_5l_6, l_1l_4l_5.$$

You can find that there is a bijective map between the set of ten triangles given above and the pentagonal set in Example 1. This gives a correspondence between \mathcal{P}_6 and the totality of pentagonal sets in Theorem 4 (iv).

6. Systems of marked seven lines.

In this section, we construct a $W(E_7)$ -equivariant map of the totality of tetrahedral sets into the set \mathcal{P}_7 . For this purpose, we first take a 3×7 matrix

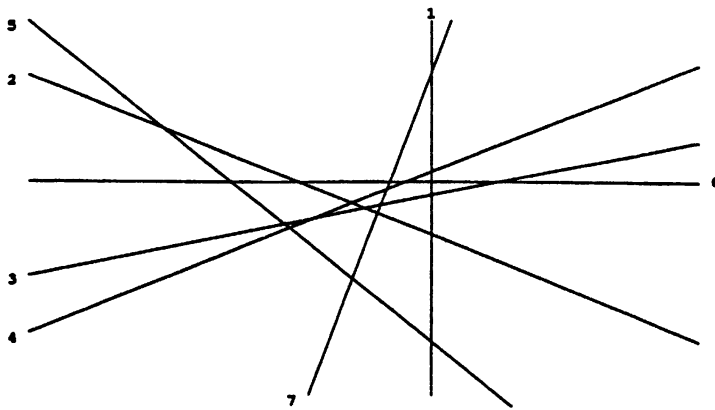


Table III

T_1	$l_1 l_2 l_5$	γ_{125}
T_2	$l_1 l_3 l_6$	γ_{136}
T_3	$l_1 l_4 l_6$	γ_{146}
T_4	$l_1 l_5 l_7$	γ_{157}
T_5	$l_2 l_3 l_4$	γ_{234}
T_6	$l_2 l_3 l_7$	γ_{237}
T_7	$l_2 l_5 l_6$	γ_{256}
T_8	$l_3 l_4 l_5$	γ_{345}
T_9	$l_3 l_5 l_6$	γ_{356}
T_{10}	$l_4 l_6 l_7$	γ_{467}

Figure IV

$$(7) \quad X_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{2}{3} & \frac{1}{2} & -3 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & \frac{5}{8} & -\frac{7}{4} \end{pmatrix}.$$

In other words, $(x, y) = (a_1, a_2, a_3, b_1, b_2, b_3)$, where

$$(a_1, a_2, a_3, b_1, b_2, b_3) = \left(\frac{2}{3}, \frac{1}{2}, -3, \frac{1}{3}, \frac{5}{8}, -\frac{7}{4} \right).$$

In the sequel, we write $(a, b) = (a_1, a_2, a_3, b_1, b_2, b_3)$ for simplicity. Let $L_0 = \{l_1, l_2, \dots, l_7\}$ be the system of marked seven lines corresponding to X_0 (cf. FIGURE IV). There are ten triangles T_j ($j = 1, 2, \dots, 10$) for L_0 which are given in TABLE III: From TABLE III, we define a set of roots of $\Delta(E_7)$

$$(8) \quad \Gamma_0 = \{\gamma_{125}, \gamma_{136}, \gamma_{146}, \gamma_{157}, \gamma_{234}, \gamma_{237}, \gamma_{256}, \gamma_{345}, \gamma_{356}, \gamma_{467}\}^{\pm}.$$

It is clear from the definition that Γ_0 is a tetrahedral set.

REMARK 6. The correspondence between ten triangles for L_0 and roots of $\Delta(E_7)$ in TABLE III is easily obtained once we write down roots of $\Delta(E_7)$ as in section 2.

We now consider the polynomials

$$(9) \quad \begin{aligned} D_{125} &= y_1, & D_{237} &= 1, \\ D_{136} &= -x_2, & D_{256} &= y_2 - y_1, \\ D_{146} &= y_2 - x_2, & D_{345} &= x_1 - 1, \\ D_{157} &= x_1 y_3 - x_3 y_1, & D_{356} &= x_2 - x_1, \\ D_{234} &= 1, & D_{467} &= x_2 y_3 - x_3 y_2 - x_2 + x_3 + y_2 - y_3, \end{aligned}$$

which are 3×3 minors of the matrix X (cf. (6)) and whose indices are same as those of the triangles for L_0 . Since D_{234}, D_{237} are constants, we neglect these two in the subsequent arguments.

It is easy to see that

$$D_{125}(a, b) > 0, D_{136}(a, b) < 0, D_{146}(a, b) > 0, D_{157}(a, b) < 0,$$

$$D_{256}(a, b) > 0, D_{345}(a, b) < 0, D_{356}(a, b) < 0, D_{467}(a, b) < 0.$$

Noting these, we consider an open subset F of (x, y) -space defined by the inequalities of the eight polynomials in (9) with signatures same as the values at $(x, y) = (a, b)$, namely,

$$F = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbf{R}^6; D_{125} > 0, \dots, D_{467} < 0\}.$$

Our goal of this section is to find a connected component Ω of F so that the correspondence $\Gamma_0 \rightarrow \Omega$ extends to a $W(E_7)$ -equivariant map of \mathcal{T} to \mathcal{P}_7 .

We begin the study with investigating the properties of F . It follows from the definition that

$$F = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbf{R}^6; (x_1, x_2, y_1, y_2) \in F_0, \right. \\ \left. \frac{1 - y_2}{1 - x_2} x_3 + \frac{y_2 - x_2}{1 - x_2} < y_3 < \frac{y_1}{x_1} x_3 \right\},$$

where

$$F_0 = \{(x_1, x_2, y_1, y_2) \in \mathbf{R}^4; 0 < x_2 < x_1 < 1, 0 < y_1 < y_2, x_2 < y_2\}.$$

It is easy to see that there are 14 polynomials of the forms D_{ijk} which are not constant and are independent of x_3, y_3 . Let R'_6 be the set of these 14 polynomials. We put

$$E = \{(x_1, x_2, y_1, y_2) \in \mathbf{R}^4; p(x_1, x_2, y_1, y_2) \neq 0 \ (\forall p \in R'_6)\}.$$

Then the connected components of E contained in F_0 are the following six open subsets:

$$E_1 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 < x_1, y_2 < 1\},$$

$$E_2 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 < x_1, y_2 > 1\},$$

$$E_3 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 > x_1, y_2 < 1\},$$

$$E_4 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 < 1, y_2 > 1\},$$

$$E_5 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 > 1, (1 - x_1)y_2 < (1 - y_1)x_2 + y_1 - x_1\},$$

$$E_6 = \{(x_1, x_2, y_1, y_2) \in F_0; y_1 < y_2, (1 - x_1)y_2 > (1 - y_1)x_2 + y_1 - x_1\}.$$

We now consider a 3×6 matrix

$$X' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix}$$

and the corresponding system of marked six lines which we denote by $L' = \{l_1, \dots, l_6\}$.

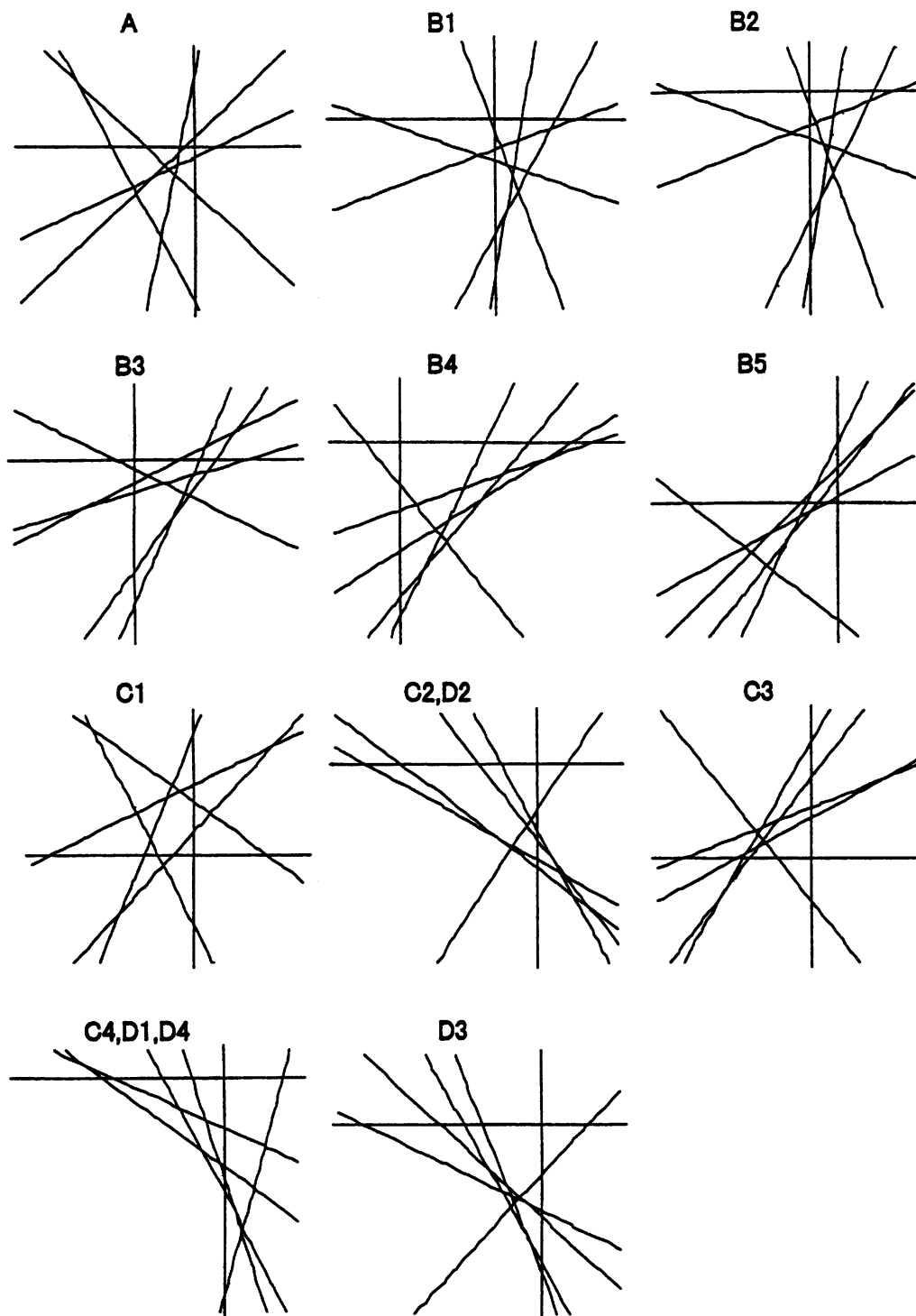


Figure V

LEMMA 4. (i) The type of the system L' is III, II, II, I, I, O if (x_1, x_2, y_1, y_2) is contained in $E_1, E_2, E_3, E_4, E_5, E_6$, respectively.

(ii) The triangles for L' are given by the table below if (x_1, x_2, y_1, y_2) is contained in $E_1, E_2, E_3, E_4, E_5, E_6$.

E_1	$l_1l_2l_3, l_1l_2l_5, l_1l_3l_6, l_1l_4l_5, l_1l_4l_6, l_2l_3l_4, l_2l_4l_6, l_2l_5l_6, l_3l_4l_5, l_3l_5l_6$
E_2	$l_1l_2l_3, l_1l_2l_5, l_1l_3l_6, l_1l_4l_5, l_2l_4l_6, l_3l_4l_5, l_3l_5l_6$
E_3	$l_1l_2l_3, l_1l_3l_6, l_1l_4l_5, l_2l_3l_4, l_2l_4l_6, l_2l_5l_6, l_3l_5l_6$
E_4	$l_1l_2l_3, l_1l_3l_6, l_1l_4l_5, l_2l_4l_5, l_2l_4l_6, l_3l_5l_6$
E_5	$l_1l_2l_3, l_1l_2l_4, l_1l_3l_6, l_2l_5l_6, l_3l_4l_5, l_4l_5l_6$
E_6	$l_1l_2l_3, l_1l_2l_4, l_1l_3l_5, l_2l_4l_6, l_3l_5l_6, l_4l_5l_6$

The proof of this lemma is easy but is a little lengthy. Therefore we omit it. As an easy consequence of Lemma 4, we find the following.

PROPOSITION 3. *Let $L \in \mathbf{P}_0(2, 7)$ be a system of marked seven lines obtained from L' by adding a seventh line l_7 . If (x_1, x_2, y_1, y_2) is a point of one of E_2, \dots, E_6 , the set of triangles for L does not coincide with $\{T_1, T_2, \dots, T_{10}\}$.*

PROOF. It follows from Lemma 4 that if (x_1, x_2, y_1, y_2) is contained in one of E_2, \dots, E_6 , there is no triangle $l_1l_4l_6$ for L . This implies the proposition. \square

Noting this proposition, we focus our attention on an open subset F' of F defined by

$$F' = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbf{R}^6; (x_1, x_2, y_1, y_2) \in E_1, \right. \\ \left. \frac{1 - y_2}{1 - x_2}x_3 + \frac{y_2 - x_2}{1 - x_2} < y_3 < \frac{y_1}{x_1}x_3 \right\}.$$

If $(x_1, x_2, x_3, y_1, y_2, y_3)$ is contained in F' , then

$$\frac{1 - y_2}{1 - x_2}x_3 + \frac{y_2 - x_2}{1 - x_2} < y_3 < \frac{y_1}{x_1}x_3, \quad \frac{y_2 - x_2}{1 - x_2} > 0,$$

which imply

$$\left(\frac{1 - y_2}{1 - x_2} - \frac{y_1}{x_1} \right) x_3 < -\frac{y_2 - x_2}{1 - x_2} < 0.$$

Therefore F' is decomposed into two parts F'_+, F'_- of F' defined by

$$F'_+ = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in F'; x_3 > 0, \frac{1 - y_2}{1 - x_2} - \frac{y_1}{x_1} < 0 \right\}, \\ F'_- = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in F'; x_3 < 0, \frac{1 - y_2}{1 - x_2} - \frac{y_1}{x_1} > 0 \right\}.$$

REMARK 7. We explain a geometric meaning of the condition $(1 - y_2)/(1 - x_2) - (y_1/x_1) = 0$. Let S be the surface obtained by six points P_j ($j = 1, 2, \dots, 6$) blowing up

of \mathbf{P}^2 , where

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1), \quad P_4 = (1 : 1 : 1),$$

$$P_5 = (1 : x_1 : y_1), \quad P_6 = (1 : x_2 : y_2).$$

Then S is a cubic surface in \mathbf{P}^3 . The condition above corresponds to the condition that S has an Eckardt point.

At the present stage, we need the following lemma.

LEMMA 5. (i) *The two sets F'_+ and F'_- are connected and mutually disjoint. Moreover $\dim \overline{F'_+} \cap \overline{F'_-} < 5$.*

(ii) *The triangles for the system of marked seven lines corresponding to $(x, y) \in F'_+$ are*

$$l_1l_2l_3, l_1l_3l_6, l_1l_4l_5, l_1l_5l_7, l_2l_3l_7, l_2l_4l_6, l_2l_5l_6, l_3l_4l_5, l_3l_5l_6, l_4l_6l_7.$$

(iii) *The triangles for the system of marked seven lines corresponding to any $(x, y) \in F'_-$ are T_1, T_2, \dots, T_{10} .*

PROOF. (i) It is easy to reduce the claim on the connectivity of F'_+, F'_- to the fact that E_1 is connected. We omit the details.

As to the dimension of $\overline{F'_+} \cap \overline{F'_-}$, the claim follows from computations of inequalities defining both F'_+, F'_- .

(ii) It is easy to see that

$$\left(\frac{2}{3}, \frac{1}{2}, 10, \frac{1}{3}, \frac{4}{5}, \frac{24}{5}\right) \in F'_+.$$

Since F'_+ is connected, it suffices to determine the triangles for a system of marked seven lines corresponding to $(2/3, 1/2, 10, 1/3, 4/5, 24/5)$ and the result follows from a direct computation.

(iii) Noting that $(a, b) \in F'_-$, we can easily show the claim. □

By direct computation, we find that the codimension one boundaries of F'_- are the hypersurfaces

$$D_{125} = 0, D_{136} = 0, D_{146} = 0, D_{157} = 0, D_{256} = 0, D_{345} = 0, D_{356} = 0, D_{467} = 0.$$

LEMMA 6. *For each polynomial p of R_7 , $p(x, y)/p(a, b) > 0$ for all $(x, y) \in F'_-$.*

PROOF. It follows from the definition of F'_- that

$$D_{125} = y_1 > 0, \quad D_{135} = -x_2 < 0, \quad D_{136} = -x_3 > 0,$$

$$D_{145} = -x_1 + y_1 < 0, \quad D_{146} = -x_2 + y_2 > 0, \quad D_{157} = x_1y_3 - x_3y_1 < 0,$$

$$D_{246} = 1 - y_2 > 0, \quad D_{156} = y_1 - y_2 < 0, \quad D_{345} = 1 - x_1 > 0,$$

$$D_{356} = -x_1 + x_2 < 0, \quad D_{467} = x_2y_3 - x_3y_2 - x_2 + x_3 + y_2 - y_3 < 0.$$

It is easy to check that the signature of each polynomial of the form D_{ijk} contained in R_7 is invariant on the set F'_- and same as that of $D_{ijk}(a, b)$. More concretely, we have the following inequalities on F'_- :

$$\begin{aligned} D_{125} > 0, \quad D_{126} > 0, \quad D_{127} < 0, \quad D_{135} < 0, \quad D_{136} < 0, \quad D_{137} > 0, \quad D_{145} < 0, \\ D_{146} > 0, \quad D_{147} > 0, \quad D_{156} > 0, \quad D_{157} < 0, \quad D_{167} > 0, \quad D_{245} > 0, \quad D_{246} > 0, \\ D_{247} > 0, \quad D_{256} < 0, \quad D_{257} > 0, \quad D_{267} > 0, \quad D_{345} < 0, \quad D_{346} < 0, \quad D_{347} < 0, \\ D_{356} < 0, \quad D_{357} < 0, \quad D_{367} < 0, \quad D_{456} < 0, \quad D_{457} < 0, \quad D_{467} < 0, \quad D_{567} > 0. \end{aligned}$$

Next we show that the signature of each p_j is invariant on F'_- . For this purpose, we first note:

$$\begin{aligned} p_7 &= (1 - x_1)(x_1y_2 - x_2y_1) + y_2(x_1 - x_2)(x_1 - y_1), \\ p_6 &= (1 - x_1)(x_1y_3 - x_3y_1) + y_3(x_1 - x_3)(x_1 - y_1), \\ p_5 &= (1 - x_2)(x_2y_3 - x_3y_2) + y_3(x_2 - x_3)(x_2 - y_2), \\ p_1 &= (x_3 - x_1)(y_3 - y_1)D_{456} + (1 - x_2)(1 - y_1)(y_2 - y_1)(x_3 - x_1) \\ &\quad - (1 - x_1)(1 - y_2)(x_2 - x_1)(y_3 - y_1), \\ p_2 &= \frac{-x_3D_{467}p_7 + x_1D_{456}p_5}{1 - y_2}, \\ p_3 &= \frac{y_3D_{467}p_7 - y_1D_{456}p_5}{1 - x_2}, \\ p_4 &= x_3y_1y_2(x_2 - x_1) - y_3x_1x_2(y_2 - y_1) + x_3y_3(x_1y_2 - x_2y_1). \end{aligned}$$

From these identities, we easily conclude that

$$p_1 < 0, p_2 < 0, p_3 > 0, p_4 > 0, p_5 > 0, p_6 < 0, p_7 > 0$$

on F'_- . □

Summarizing the arguments above, we obtain the theorem below:

THEOREM 5. *An open subset Ω of \mathbf{R}^6 defined by the inequalities*

$$0 < x_2 < x_1 < 1, 0 < y_1 < y_2, y_2 < x_2, \frac{1 - y_2}{1 - x_2}x_3 + \frac{y_2 - x_2}{1 - x_2} < y_3 < \frac{y_1}{x_1}x_3, \underline{x_3 < 0}$$

has the following properties.

- (0) Ω is a connected component of $\mathbf{P}_0(2, 7)$.
- (i) Ω is a connected component of the open subset F defined by

$$D_{125} > 0, D_{136} < 0, D_{146} > 0, D_{157} < 0, D_{256} > 0, D_{345} < 0, D_{356} < 0, D_{467} < 0.$$

The codimension one boundary of Ω is the union of non-empty open subsets of hypersurfaces

$$D_{125} = 0, D_{136} = 0, D_{146} = 0, D_{157} = 0, D_{256} = 0, D_{345} = 0, D_{356} = 0, D_{467} = 0.$$

- (ii) For each $(x, y) \in \Omega$, the triangles for the system of marked seven lines corresponding to (x, y) are T_1, T_2, \dots, T_{10} .
- (iii) There is no connected component of $\mathbf{P}_0(2, 7)$ other than Ω satisfying the conditions same as (i), (ii).

PROOF. Since $\Omega = F'_-$, the theorem follows from the arguments before in this section.

We now put

$$h_1 = s_5s_{13}s_{26}s_{47}, h_2 = s_6s_{13}s_{27}s_{45}, h_3 = s_7s_{16}s_{23}s_{45}, c = s_1s_{23}s_{123}s_{45}s_{145}s_{67}s_{167}.$$

Then it is easy to show the following.

- LEMMA 7. (i) $h_j^2 = 1$ ($j = 1, 2, 3$), $(h_1h_2)^3 = (h_2h_3)^3 = 1$, $(h_1h_3)^2 = 1$.
 (ii) The group H' generated by h_1, h_2, h_3 is isomorphic to S_4 .
 (iii) $H = H' \times \langle c \rangle$ is the stabilizer of the tetrahedral set Γ_0 in $W(E_7)$.

PROOF. Easy.

We now study the action of H on the set Ω .

LEMMA 8. $h\Omega = \Omega$ for all $h \in H$.

PROOF. Since Ω is connected, so is $h\Omega$ ($\forall h \in H$). In virtue of that h_1, h_2, h_3, c generate H , it suffices to show that $h_j \cdot (a_1, a_2, a_3, b_1, b_2, b_3) \in \Omega$ ($j = 1, 2, 3$), $c \cdot (a_1, a_2, a_3, b_1, b_2, b_3) \in \Omega$. By direct computation, we find that

$$h_1 \cdot (a, b) = \left(\frac{524}{711}, \frac{1}{12}, -\frac{19}{3}, \frac{262}{3441}, \frac{5}{16}, -\frac{7}{4} \right),$$

$$h_2 \cdot (a, b) = \left(\frac{25}{33}, \frac{131}{374}, -\frac{21}{11}, \frac{1}{3}, \frac{917}{2294}, -1 \right),$$

$$h_3 \cdot (a, b) = \left(\frac{1}{3}, \frac{1}{5}, -\frac{1501}{524}, \frac{1}{6}, \frac{2}{5}, -\frac{553}{338} \right),$$

$$c \cdot (a, b) = (a, b).$$

Then it is easy to see that $h_j \cdot (a, b) \in \Omega$ ($j = 1, 2, 3$), $c \cdot (a, b) \in \Omega$. So the lemma follows. □

REMARK 8. The central element c acts on the (x, y) -space as an identity.

THEOREM 6. Let Γ_0 be the tetrahedral set defined in (8). If $\Gamma = g\Gamma_0$ for some $g \in W(E_7)$, we define $f(\Gamma)$ as the connected component of $\mathbf{P}_0(2, 7)$ containing $g \cdot (a, b)$. Then f defines a $W(E_7)$ -equivariant injection of \mathcal{T} to \mathcal{P}_7 .

PROOF. Let \tilde{H} be the isotropy subgroup of Γ_0 in $W(E_7)$. Then Lemma 8 implies that H is contained in \tilde{H} . Therefore, as a $W(E_7)$ -equivariant map, f is well-defined.

What we have to show is that f is injective. Our proof of this fact is based on the classification of the S_7 -orbital structure of \mathcal{T} . In virtue of Theorem 3, we find that

there are fourteen S_7 -orbits of \mathcal{F} . If Γ is a tetrahedral set of type M, we also call the system of marked seven lines corresponding to $f(\Gamma)$ of type M. It follows from TABLE IV given in the next section that if there are ten triangles for a system Γ of marked seven lines, the type of Γ is A. This easily implies that f is injective and the theorem follows.

REMARK 9. In spite that we do not discuss on the $W(E_7)$ -orbital structure of \mathcal{P}_7 , it is provable (cf. [10]) that the $W(E_7)$ -action on \mathcal{P}_7 is transitive. Our proof for this statement is based on a detailed study on a cross ratio variety for the root system of type E_7 (cf. [9]) and we shall treat this subject elsewhere.

7. Fourteen systems of marked seven lines.

Cummings [3], White [16] (cf. Grünbaum [7], Chap. 18) showed that there are eleven equivalent classes of systems of seven lines with condition (A). In this section, we discuss the relationship between our results and their studies.

We first compute representative matrices for all types of systems of marked seven lines:

$$(10) \quad X_A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{37}{100} & \frac{1961}{260} & \frac{2479}{1840} \\ 0 & 0 & 1 & 1 & \frac{37}{10} & \frac{148}{13} & \frac{481}{184} \end{pmatrix}$$

$$X_{B1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{100}{37} & \frac{260}{1961} & \frac{1840}{2479} \\ 0 & 0 & 1 & 1 & \frac{10}{37} & \frac{13}{148} & \frac{184}{481} \end{pmatrix}$$

$$X_{B2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{100}{37} & \frac{265}{13} & \frac{335}{92} \\ 0 & 0 & 1 & 1 & 10 & \frac{373}{13} & \frac{3041}{368} \end{pmatrix}$$

$$(11) \quad X_{B3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{37}{63} & \frac{148}{13} & \frac{999}{644} \\ 0 & 0 & 1 & 1 & \frac{37}{7} & \frac{1184}{65} & \frac{4329}{1288} \end{pmatrix}$$

$$X_{B4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{3041}{2600} & \frac{312577}{62530} & \frac{67}{130} \\ 0 & 0 & 1 & 1 & \frac{92}{65} & \frac{736}{169} & \frac{184}{481} \end{pmatrix}$$

$$X_{B5} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{3438347}{12436070} & \frac{312577}{62530} & \frac{312577}{312577} \\ 0 & 0 & 1 & 1 & \frac{3438347}{1243607} & \frac{312577}{32227} & \frac{312577}{180412} \end{pmatrix}$$

$$X_{C1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{27}{13} & \frac{45}{53} & \frac{87543}{312577} \\ 0 & 0 & 1 & 1 & \frac{1}{5} & -\frac{5}{3} & -\frac{2386381}{8439579} \end{pmatrix}$$

$$X_{C2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{80}{27} & -\frac{31738}{33611} & -\frac{252}{13} \\ 0 & 0 & 1 & 1 & -\frac{170}{201} & -\frac{44387}{100833} & -\frac{1071}{890} \end{pmatrix}$$

(12)

$$X_{C3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\frac{293193}{1269692} & -\frac{727553}{2085640} & \frac{141167}{453744} \\ 0 & 0 & 1 & 1 & -\frac{488655}{754952} & \frac{3637765}{1251384} & -\frac{705835}{403328} \end{pmatrix}$$

$$X_{C4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{29471}{331970} & -\frac{999747}{243860} & \frac{4379844}{6766225} \\ 0 & 0 & 1 & 1 & -\frac{88413}{1161895} & -\frac{999747}{2438600} & -\frac{1459948}{1353245} \end{pmatrix}$$

$$X_{D1} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{170}{371} & \frac{44387}{145220} & \frac{1071}{1961} \\ 0 & 0 & 1 & 1 & \frac{80}{53} & \frac{373}{265} & \frac{312577}{180412} \end{pmatrix}$$

$$X_{D2} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{504160}{312577} & \frac{49320}{15869} & \frac{27400}{18277} \\ 0 & 0 & 1 & 1 & \frac{2672048}{1562885} & \frac{548}{185} & \frac{274}{175} \end{pmatrix}$$

(13)

$$X_{D3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{540}{71} & \frac{100}{37} & \frac{80}{27} \\ 0 & 0 & 1 & 1 & \frac{211491}{43436} & \frac{373}{148} & \frac{373}{108} \end{pmatrix}$$

Table IV

Types	triangles	rectangles	pentagons	hexagon	heptagon
A	10	6	6	0	0
B1	7	12	3	0	0
B2	9	8	5	0	0
B3	8	10	4	0	0
B4	8	10	4	0	0
B5	11	5	5	1	0
C1	7	14	0	0	1
C2,D2	7	13	1	1	0
C3	9	9	3	1	0
C4,D1,D4	8	11	2	1	0
D3	7	12	3	0	0

$$X_{D4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{575527}{1825835} & -\frac{88413}{1161895} & \frac{907497}{2157805} \\ 0 & 0 & 1 & 1 & -\frac{1431}{445} & -\frac{6801}{3115} & -\frac{33611}{5785} \end{pmatrix}$$

The index M of the matrix X_M means the type of the system of marked seven lines corresponding to X_M for $M \in \{A, B1, \dots, D4\}$.

Observing the configurations of the fourteen systems of marked seven lines (cf. FIGURE V), we conclude that the numbers of polygons for each system are given in TABLE IV.

We now discuss the relationship between these fourteen systems of marked seven lines and simple 2-arrangements of seven lines (for the definition of simple 2-arrangements of seven lines, see Grünbaum [7]). Let L be a system of marked seven lines with conditions (A), (B). Forgetting the indices of lines of L , we obtain a simple 2-arrangement of seven lines on the real projective plane. Two 2-arrangements are *equivalent* provided there exists a one-to-one incidence-preserving correspondence between their polygons. Comparing our results with those in [7], Chap. 18, we observe the following.

LEMMA 9. *Let Γ be a tetrahedral set and let L be a system of marked seven lines contained in $f(\Gamma)$. Then, for $j = 1, 2, \dots, 7$, Γ contains a root γ_j of $\Delta(E_7)$ if and only if the system L_j of marked six lines obtained from L by taking off the j -th line is of type O , in other words, there is no hexagon for L_j .*

As easy consequences of this lemma and other observations, we find the following.

1. If there is no hexagon for the system of marked six lines obtained from L by taking off any line of L , then type of L is one of A, B1, B3.
2. Systems of marked seven lines of types B1 and D3 are not distinguished each other by the numbers of polygons. If L is of type D3, there is a hexagon for one of systems of marked six lines obtained from L by taking off a line and if L is of type B1, this doesn't happen.
3. Systems of marked seven lines of types B3 and B4 are not distinguished each other by the numbers of polygons. If L is of type B4, there is a hexagon for one of systems of marked six lines obtained from L by taking off a line and if L is of type B3, this doesn't happen.
4. Systems of marked seven lines of types C2 and D2 are equivalent.
5. Systems of marked seven lines of types C4, D1, D4 are equivalent.
6. Eleven systems of marked seven lines of types A, B1, B2, B3, B4, B5, C1, C2, C3, C4, D3 are neither equivalent.

We stress here that of the statements above, the last one actually agrees with the classification of simple 2-arrangements of seven lines by Cummings and White.

REMARK 10. Needless to say, we have to study properties of conics tangent to five lines of seven lines of 2-arrangements to distinguish systems of types C2 and D2 each other and also those of types C4, D1, D4.

8. Tetradiagrams and Shrikhande graphs.

In this section, being irrelevant to the arguments of the previous sections, we mention a relationship between tetradiagrams and Shrikhande graphs. For the definition and properties of Shrikhande graphs, see [2], [14]. Most of the statements of this section are obtained by the discussions with A. Munemasa.

A Shrikhande graph consists of sixteen vertices which correspond to roots of $\Delta(E_7)$ (see FIGURE VI). To treat it convenient for our purpose, we put the vertices on the lattice points

$$\Sigma = \{(\sqrt{3}(m-n), m+n); m, n \in \mathbf{Z}\}$$

and identify two vertices modulo

$$\Sigma_0 = \{(\sqrt{3}(m-n), m+n); m, n \in 4\mathbf{Z}\}.$$

Moreover we consider the lines

$$(14) \quad x = \frac{\sqrt{3}m}{2}, y = \pm\sqrt{3}x + m, (\forall m \in \mathbf{Z}).$$

These connect the vertices.

LEMMA 10. *There is a subset S of $\Delta(E_7)$ with the following conditions:*

1. *The order of S is 16.*
2. *There is a bijection h of S to Σ/Σ_0 such that for $\alpha, \beta \in S$, $\langle \alpha, \beta \rangle \neq 0$ if and only if $h(\alpha)$ and $h(\beta)$ are contained in one of the lines defined by equations (14) and are adjacent to each other.*

DEFINITION 5. A set S (resp. a map h) of Lemma 10 is called a Shrikhande set (resp. a Shrikhande map).

We are going to construct a Shrikhande set S from a tetrahedral set Γ with the following condition: If G_S (resp. G_Γ) is the isotropy subgroup of S^\pm (resp. Γ) in $W(E_7)$, then G_Γ is a subgroup of G_S . We note that the order of G_S is 192. We first recall the set \mathbf{B} and a tetrahedral map f of Definition 3. We put

$$\mathbf{B}_1 = \{a_j; j = 1, 2, 3, 4\}, \quad \mathbf{B}_2 = \{b_{ij}; 1 \leq i < j \leq 4\}.$$

Then $f(\mathbf{B}_1)^\perp = V_1^\pm \cup W_1^\pm$, where

$$V_1 = \{\beta_j; j = 1, 2, 3, 4\}, \quad W_1 = \{\delta_j; j = 1, 2, 3, 4\}$$

and $\langle \beta_j, \beta_k \rangle = 0$, $\langle \delta_j, \delta_k \rangle = 0$ ($j \neq k$). We may assume that $U_1 = f(\mathbf{B}_1)$, V_1, W_1 are subsets of $\Delta(E_7)^+$. It is easy to see that the union $U_1^\pm \cup V_1^\pm \cup W_1^\pm$ is a root system of type D_4 and that there is a unique root, say δ_1 , of $V_1 \cup W_1$ which is orthogonal to the set $f(\mathbf{B}_2)$. Then we find that $Z = \{\alpha \in \Delta(E_7)^+; \langle \alpha, \delta_1 \rangle \neq 0\}$ consists of thirty-two elements. For each $j = 2, 3, 4$, we consider the set $Z_j = \{\alpha \in Z; \langle \alpha, \delta_j \rangle \neq 0\}$. Then $Z_j = U_j^\pm \cup V_j^\pm$, where

$$U_j = \{\alpha_{j,k}; k = 1, 2, 3, 4\}, \quad V_j = \{\beta_{j,k}; k = 1, 2, 3, 4\},$$

and $\langle \alpha_{j,k}, \alpha_{j,k'} \rangle = 0$, $\langle \beta_{j,k}, \beta_{j,k'} \rangle = 0$, ($k \neq k'$). Let G_{U_1} be the group of automorphisms of U_1^\pm in $W(E_7)$. It follows that the set $\{U_j^\pm, V_j^\pm; j = 2, 3, 4\}$ is decomposed into two G_{U_1} -orbits consisting of three sets. We may take $\{U_j^\pm; j = 2, 3, 4\}$ and $\{V_j^\pm; j = 2, 3, 4\}$ as the two G_{U_1} -orbits. Then one of $\bigcup_{j=1}^4 U_j$, $U_1 \cup (\bigcup_{j=2}^4 V_j)$ is a Shrikhande set and the other is not. So we may assume that $U = \bigcup_{j=1}^4 U_j$ is a Shrikhande set. Then $V = \bigcup_{j=1}^4 V_j$ is also a Shrikhande set and $s_{\delta_1}(U) = V$, where s_{δ_1} is a reflection with respect to δ_1 .

PROPOSITION 4. Let G_U (resp. G_V) be the group of automorphisms of U^\pm (resp. V^\pm) in $W(E_7)$. Then $G_U = G_V$ and $G_{U_1} \subset G_U$.

PROOF. It is easy to show the proposition by using the following example.

EXAMPLE 3. We consider the case where for a tetrahedral map f ,

$$f(\mathbf{B}_1) = \{\gamma_{135}, \gamma_{146}, \gamma_{236}, \gamma_{245}\},$$

$$f(\mathbf{B}_2) = \{\gamma_{14}, \gamma_{147}, \gamma_{25}, \gamma_{257}, \gamma_{36}, \gamma_{367}\}.$$

Then $\delta_1 = \gamma_7$ and

$$U_1 = \{\gamma_{135}, \gamma_{146}, \gamma_{236}, \gamma_{245}\}, \quad V_1 = \{\gamma_{136}, \gamma_{145}, \gamma_{235}, \gamma_{246}\},$$

$$U_2 = \{\gamma_1, \gamma_{27}, \gamma_{134}, \gamma_{156}\}, \quad V_2 = \{\gamma_2, \gamma_{17}, \gamma_{234}, \gamma_{256}\},$$

$$U_3 = \{\gamma_3, \gamma_{47}, \gamma_{123}, \gamma_{356}\}, \quad V_3 = \{\gamma_4, \gamma_{37}, \gamma_{124}, \gamma_{456}\},$$

$$U_4 = \{\gamma_5, \gamma_{67}, \gamma_{125}, \gamma_{345}\}, \quad V_4 = \{\gamma_6, \gamma_{57}, \gamma_{126}, \gamma_{346}\}.$$

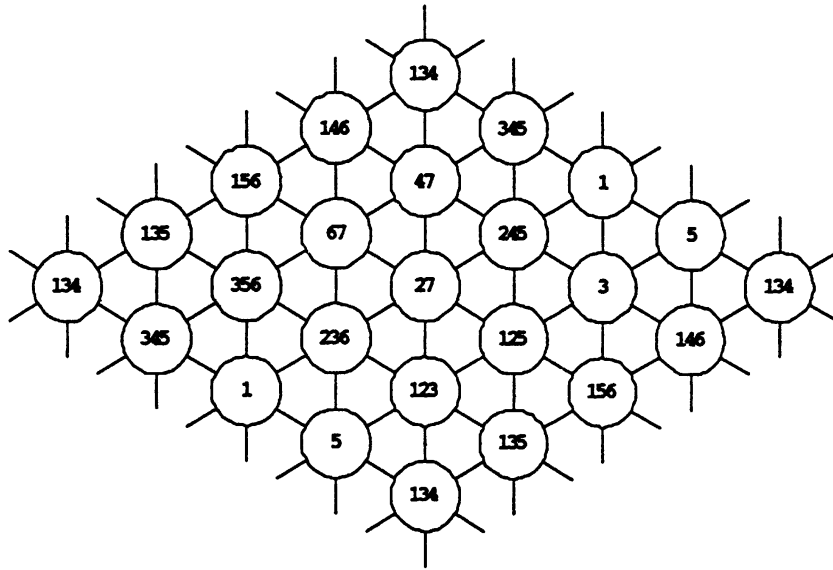


Figure VI: Shrikhande graph

A correspondence of the Shrikhande set $U = \bigcup_{j=1}^4 U_j$ and the totality of the vertices of Σ/Σ_0 is given in FIGURE VI.

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