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Generalized Bebutov systems: a dynamical interpretation of shape

Dedicated to Professor Joaquin Arregui on his 70th Birthday

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Abstract. We define a semidynamical system—inspired by some classical dynamical systems studied by Bebutov in function spaces—in the space of approximative maps $A(X, Y)$ between two metric compacta, with a suitable metric. Shape and strong shape morphisms are characterized as invariant subsets of this system. We study their structure and asymptotic properties and use the obtained results to give dynamical characterizations of basic notions in shape theory, like trivial shape, shape domination by polyhedra and internal FANRs.

Introduction.

The theory of shape, introduced by K. Borsuk in 1968 [**7**], has proved to be a successful instrument for the study of the global topological properties of dynamical systems. Shape theory is especially useful when applied to spaces of locally complicated structure, like the attractors of dynamical systems, for which the tools of homotopy theory are not appropriate. The papers [**6**], [**15**], [**17**], [**18**], [**19**], [**28**], [**29**], [**31**], [**32**], [**35**] are good illustrations of the application of shape theory in dynamics. In this paper we establish a new connection between shape and dynamics by adopting a different point of view. The theory of dynamical systems is used here to give a new interpretation of shape. We define a structure of semidynamical system in the space $A(X, Y)$ of approximative maps between two metric compacta X and Y. The metric structure of $A(X, Y)$ is inspired by topologies introduced in previous papers [**16**], [**23**], [**24**], [**30**]. The dynamical structure is inspired by some classical dynamical systems studied by Bebutov in function spaces [**1**], [**2**] (see [**34**, IV.20]). According to this interpretation, shape morphisms and strong shape morphisms are invariant subspaces of the Bebutov space $A(X, Y)$. This means that shape and strong shape morphisms can be viewed as semidynamical systems themselves. The paper is devoted to the study of the structure of the Bebutov system $A(X, Y)$ and in particular to the recognition of Lagrange and Poisson stable orbits and non-wandering points, the determination of the limit sets and the properties of attractors and Lyapunov stable motions. These results are used to give dynamical characterizations of some basic notions in shape theory. For instance:

1) A shape morphism is generated by a map if and only if it is non-dispersive (when viewed as a semidynamical system).

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2) For a metric compactum X the following are equivalent:
i) X has trivial shape
ii) The set of Lagrange stable motions of the Bebutov system $A(X, X)$ is contained in a connected component of $A(X, X)$.
iii) There exists a connected attractor in $A(X, X)$ containing a periodic orbit, and every attractor containing a periodic orbit contains all periodic orbits.
3) A metric compactum X is shape dominated by a polyhedron if and only if its Bebutov system is prolongable.

The reader is referred to the books by Borsuk [**8**], Cordier and Porter [**10**], Dydak and Segal [**12**], and Mardešić and Segal [**25**] for information on shape theory. See also Dydak and Segal [**13**], Edward and Hastings [**14**], Kodama and Ono [**21**], Porter [**26**] and Quigley [**27**] for information on strong shape. For the general theory of dynamical systems we recommend the books [**4**], [**5**], [**33**], [**34**] of Bhatia and Hajek, Bhatia and Szego, Saperstone and Sibirsky respectively. For information on attractors in arbitrary (not necessarily locally compact) Hausdorff spaces, we refer to [**3**]. Some of the results in this paper were obtained while the authors were visiting the Universities of Washington (USA) and Manchester (UK) respectively. The authors are grateful to the Departments of Mathematics of these Universities and in particular to J. Segal and N. Ray for their hospitality. They also thank the referee for useful remarks.

1. A short account of shape theory.

In the first section of this paper we review the characterization of shape theory using approximative maps.

Let X and Y be compact metric spaces such that Y is contained in the Hilbert cube Q. An approximative map from X to Y is a continuous function $F : X \times \boldsymbol{R}_+ \to Q$ such that for every neighborhood V of Y in Q there exists $t_0 \in \boldsymbol{R}_+$ such that $F(X \times [t_0, \infty)) \subset V$. Two approximative maps $F, G : X \times \boldsymbol{R}_+ \to Q$ from X to Y are homotopic if there exists an approximative map $H : X \times [0,1] \times \boldsymbol{R}_+ \to Q$ from $X \times [0,1]$ to Y such that $H(x, 0, t) = F(x, t)$ and $H(x, 1, t) = G(x, t)$ for every $(x, t) \in X \times \boldsymbol{R}_+$. On the other hand, F and G are weakly homotopic if for every neighborhood V of Y in Q there exists $t_0 \in \boldsymbol{R}_+$ such that $F|_{X\times[t_0,\infty)}$ and $G|_{X\times[t_0,\infty)}$ are homotopic in V. We denote by $[F]$ the homotopy class of F and by $[F]_w$ the weak homotopy class of F. Observe that $[F]_w \supset [F]$. We say that F and G are asymptotic if for every $\varepsilon > 0$ there exists $t_0 \in \boldsymbol{R}_+$ such that $d(F(x, t), G(x, t)) < \varepsilon$ for every $(x, t) \in X \times [t_0, \infty)$. If F and G are asymptotic then they are homotopic.

Given $F : X \times \boldsymbol{R}_+ \to Q$ and $G : Y \times \boldsymbol{R}_+ \to Q$ approximative maps from X to Y and from Y to Z respectively, there exists a fundamental map (see [**20**]) $G' : Q \times \boldsymbol{R}_+ \to Q$ from Y to Z being an extension of G. Consider $H : X \times \boldsymbol{R}_+ \to Q$ given by $H(x, t) = G'(F(x, t), t)$. Then H is an approximative map from X to Z whose homotopy class only depends on the homotopy classes of F and G, and whose weak homotopy class only depends on the weak homotopy classes of F and G, being independent of the extension G'. Then the composition of classes is defined as $[G][F] = [H]$ and $[G]_w[F]_w = [H]_w$.

THEOREM 1 (Borsuk). *If we consider the class of the compact metric spaces and the weak homotopy classes of approximative maps between them with the composition of* *weak homotopy classes previously defined we obtain a category isomorphic to the shape category.*

Theorem 2 (Kodama and Ono). *If we consider the class of the compact metric spaces and the homotopy classes of approximative maps between them with the composition of homotopy classes previously defined we obtain a category isomorphic to the strong shape category.*

2. A topology in the space of approximative maps.

Definition 1. Let X and Y be compact metric spaces such that Y is contained as a Z-set (see [**9**]) in the Hilbert cube Q (this is not a restriction since every compactum can be embedded as a Z-set in Q), and denote by $A(X,Y)$ the set of approximative maps from X to Y. Given $F, G \in A(X,Y)$, we define the distance from F to G as

$$\tilde{d}(F,G) = \max\left\{\sum_{k\in \boldsymbol{N}} \frac{d(F|_{X\times[0,k]}, G|_{X\times[0,k]})}{2^k}, \sup_{X\times \boldsymbol{R}_+} |d(F(x,s),Y) - d(G(x,s),Y)|\right\}$$

Then, if $\tilde{d}(F,G) < \varepsilon$, we have that $d(F|_{X\times[0,k]}, G|_{X\times[0,k]}) < 2^k\varepsilon$, for every $k \in \boldsymbol{N}$.

Remark 1. It is easy to see that $\tilde{d}$ is a metric in $A(X,Y)$.

Consider the space $A(X,Y)$ with the topology generated by the distance $\tilde{d}$. It can be seen—in a similar way as in [**24**]—that, with the restriction of Y being a Z-set, it is a topological invariant of the pair (X,Y). Moreover, if α is a homeomorphism from X to X' and β is a homeomorphism from Y to Y', then the function $\gamma : A(X,Y) \to A(X',Y')$ given by $\gamma(F)(x',t) = \tilde{\beta}F(\alpha^{-1}(x'),t)$, is a homeomorphism from $A(X,Y)$ to $A(X',Y')$ (see [**24**]), where $\tilde{\beta}$ is an homeomorphism of Q which is an extension of β (such a extension always exists since Y and Y' are Z-sets in Q (see [**9**])).

The following theorem, which can be proved using techniques and ideas of [**16**] and [**24**], states the main properties of the space $A(X,Y)$.

Theorem 3. *$A(X,Y)$ satisfies the following*:

i) *Two approximative maps from X to Y are homotopic if and only if they are in the same path-connected component of $A(X,Y)$. As a consequence, the morphisms from X to Y in the strong shape category can be identified with the path-connected components of $A(X,Y)$.*

ii) *Two approximative maps from X to Y are weakly homotopic if and only if they are in the same connected component of $A(X,Y)$. Therefore, the morphisms from X to Y in the shape category can be identified with the connected components of $A(X,Y)$.*

iii) *Given $F \in A(X,Y)$, if we consider the sets*

$$a(F) = \{G \in A(X,Y) \mid G \text{ asymptotic to } F\}$$

$$c(F) = \{G \in A(X,Y) \mid F|_{X\times[t_0,\infty)} = G|_{X\times[t_0,\infty)} \text{ for some } t_0 \in \boldsymbol{R}_+\},$$

then $c(F) \subset a(F) \subset [F] \subset [F]_w$ and $\overline{c(F)} = \overline{a(F)} = \overline{[F]} = [F]_w$. In particular, every morphism in the shape category is the closure of a morphism in the strong shape category.

iv) *Given* $F \in A(X,Y)$, *given* $\varepsilon > 0$, *if we define*

$$[F]_\varepsilon = \{G \in A(X,Y) \mid F|_{X\times[k_G,\infty)} \simeq G|_{X\times[k_G,\infty)} \text{ in } B_\varepsilon(Y), k_G \in \boldsymbol{R}_+\}$$

we have that $[F]_\varepsilon$ *is open and closed in* $A(X,Y)$ *and that* $[F]_w = \bigcap_{\varepsilon>0}[F]_\varepsilon$. *Therefore, the morphisms from* X *to* Y *in the shape category can also be identified with the connected quasicomponents of* $A(X,Y)$.

3. The semidynamical system $(A(X,Y), \boldsymbol{R}_+, \pi)$

Let M be a metric space. A semidynamical system on M is a triad $(M, \boldsymbol{R}_+, \pi)$ where $\pi : M \times \boldsymbol{R}_+ \to M$ is a continuous function such that

i) $\pi(x,0) = x$ for every $x \in M$,

ii) $\pi(\pi(x,t),s) = \pi(x,t+s)$ for every $x \in M$ and every $t,s \in \boldsymbol{R}_+$.

Given $x \in M$ we denote by $\pi_x : \boldsymbol{R}_+ \to M$ the positive motion through x defined as $\pi_x(t) = \pi(x,t)$.

Given $x \in M$, we denote by $\gamma^+(x) = \{\pi(x,t) \mid t \in \boldsymbol{R}_+\}$ the positive semi-trajectory of x and by $\gamma^-(x) = \{y \in M \mid x \in \gamma^+(y)\}$ the funnel in x.

The following theorem introduces a semidynamical system in the space $A(X,Y)$.

Theorem 4. *Consider*

$$\begin{aligned} \pi : A(X,Y) \times \boldsymbol{R}_+ &\to A(X,Y) \\ (F,t) &\mapsto \pi(F,t) : X \times \boldsymbol{R}_+ \to Q \\ &\qquad (x,s) \mapsto \pi(F,t)(x,s) = F(x,t+s). \end{aligned}$$

Then $(A(X,Y), \boldsymbol{R}_+, \pi)$ *is a semidynamical system.*

Moreover, if X *is homeomorphic to* X' *and* Y *is homeomorphic to* Y', *then the semidynamical systems* $(A(X,Y), \boldsymbol{R}_+, \pi)$ *and* $(A(X',Y'), \boldsymbol{R}_+, \pi')$ *are isomorphic.*

Proof. We see first that π is continuous. Take $(F_0,t_0) \in A(X,Y) \times \boldsymbol{R}_+$ and $\varepsilon > 0$. Consider $k_0 > t_0$ such that

$$\sum_{k=k_0+1}^{\infty} \frac{\Delta + \varepsilon}{2^k} < \frac{\varepsilon}{2}, \quad \text{with } \Delta = \operatorname{diam}(Y),$$

and such that $d(F_0(x,t),Y) < \varepsilon/4$ for every $x \in X$ and every $t > k_0$. Since F_0 is continuous in $X \times [0,2k_0]$, there exists $\delta > 0$ with $t_0 + \delta < k_0$ such that

$$d(F_0(x,t_0+s), F_0(x,t+s)) < \frac{\varepsilon}{4}$$

for every $x \in X$, every $t \in \boldsymbol{R}_+$ with $|t - t_0| < \delta$ and every $s \in [0,k_0]$, and hence

$$|d(F_0(x,t_0+s),Y) - d(F_0(x,t+s),Y)| < \frac{\varepsilon}{4}.$$

And if $s > k_0$, then $t_0 + s, t + s > k_0$ and

$$|d(F_0(x,t_0+s),Y) - d(F_0(x,t+s),Y)| < 2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

for every $x \in X$.

On the other hand if $\tilde{d}(F, F_0) < (\varepsilon/2^{2k_0+2})$, then for every $x \in X$, every $t \leq k_0$ and every $s \leq k_0$ we have

$$d(F_0(x, t+s), F(x, t+s)) < \frac{\varepsilon}{4}$$

and for every $t, s \in \boldsymbol{R}_+$

$$|d(F_0(x, t+s), Y) - d(F(x, t+s), Y)| < \frac{\varepsilon}{4}.$$

Therefore if $\tilde{d}(F, F_0) < (\varepsilon/2^{2k_0+2})$ and $|t - t_0| < \delta$ then

$$\sum_{k=1}^{\infty} \frac{\sup_{(x,s) \in X \times [0,k]} d(F_0(x, t_0+s), F(x, t+s))}{2^k} < \sum_{k=1}^{k_0} \frac{\varepsilon/2}{2^k} + \sum_{k=k_0+1}^{\infty} \frac{\Delta + \varepsilon}{2^k} < \varepsilon$$

and

$$\sup_{X \times \boldsymbol{R}_+} |d(F_0(x, t_0+s), Y) - d(F(x, t+s), Y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.$$

Hence $\tilde{d}(\pi(F_0, t_0), \pi(F, t)) < \varepsilon$. Therefore π is continuous.

On the other hand, if $F \in A(X, Y)$ then $\pi(F, 0) = F$ and $\pi(\pi(F, t), s) = \pi(F, t+s)$ for every $t, s \in \boldsymbol{R}_+$. Therefore $(A(X, Y), \boldsymbol{R}_+, \pi)$ is a semidynamical system.

Suppose now that α is a homeomorphism from X to X' and β is a homeomorphism from Y to Y'. Since Y and Y' are Z-sets in Q, there exists an homeomorphism $\tilde{\beta} : Q \to Q$ which is an extension of β. Consider $(A(X, Y), \boldsymbol{R}_+, \pi)$ and $(A(X', Y'), \boldsymbol{R}_+, \pi')$ semidynamical systems defined as above. The function $\gamma : A(X, Y) \to A(X', Y')$ given by $\gamma(F)(x', s) = \tilde{\beta}F(\alpha^{-1}(x'), s)$ is a homeomorphism from $A(X, Y)$ to $A(X', Y')$ and, given $(x', s) \in X' \times \boldsymbol{R}_+$

$$\begin{aligned}\gamma(\pi(F, t))(x', s) &= \tilde{\beta}\pi(F, t)(\alpha^{-1}(x'), s) = \tilde{\beta}F(\alpha^{-1}(x'), t+s) = \gamma(F)(x', t+s) \\ &= \pi'(\gamma(F), t)(x', s).\end{aligned}$$

Hence, $\gamma(\pi(F, t)) = \pi'(\gamma(F), t)$ for every $(F, t) \in A(X, Y) \times \boldsymbol{R}_+$. Therefore the semidynamical systems $(A(X, Y), \boldsymbol{R}_+, \pi)$ and $(A(X', Y'), \boldsymbol{R}_+, \pi')$ are isomorphic. □

Remark 2. Denote by

$$\mathscr{C}(X, Y) = \{F \in A(X, Y) | F(x, t) = F(x, 0), \text{ for every } (x, t) \in X \times \boldsymbol{R}_+\}$$

the set of approximative maps generated by a continuous function from X to Y (isometric to the set of continuous functions from X to Y with the supremum distance). This is the set of stable points of $A(X, Y)$. Consider also

$$\mathscr{C}(X \times \boldsymbol{R}_+, Y) = \{F \in A(X, Y) \,|\, F(X \times \boldsymbol{R}_+) \subset Y\}$$

with the induced metric. Then $\mathscr{C}(X, Y)$ and $\mathscr{C}(X \times \boldsymbol{R}_+, Y)$ are positively invariant closed subsets of $A(X, Y)$. The subset whose elements are the periodic approximative maps is also positively invariant. Given $\tau > 0$, the set of periodic approximative maps from X to Y with period less or equal than τ is closed in $A(X, Y)$. The subset of periodic approximative maps is dense in $\mathscr{C}(X \times \boldsymbol{R}_+, Y)$. On the other hand, the subset of approximative maps taking to stable points and the subset of approximative maps taking to cycles are invariant.

PROPOSITION 1. *For every $F \in A(X, Y)$, $\gamma(F) \subset [F]$ and $\overline{\gamma(F)} \subset [F]_w$. Therefore $[F]$ and $[F]_w$ are invariant sets.*

The above considerations imply that the semidynamical system decomposes into two subsystems, the one having as elements those shape morphisms generated by maps from X to Y, and its complement. We will see in the sequel that the dynamical properties of this two subsystems are fairly different. For example, the latter turns out to be dispersive while the former has very natural prolongational limits.

4. Limit sets in $A(X, Y)$.

Let $(M, \boldsymbol{R}_+, \pi)$ be a semidynamical system on a metric space M. The omega limit set of a point $x \in M$ is the set $\Lambda^+(x)$ of the points $y \in M$ such that there is a sequence $\{t_n\}$ in $\boldsymbol{R}_+$ with $t_n \to \infty$ and $\pi(x, t_n) \to y$. π_x is said positively departing if $\Lambda^+(x) = \emptyset$, positively asymptotic if $\Lambda^+(x) \neq \emptyset$ and $\Lambda^+(x) \cap \gamma^+(x) = \emptyset$, and π_x is said positively Poisson stable if $\Lambda^+(x) \cap \gamma^+(x) \neq \emptyset$. π_x is said positively Lagrange stable if $\overline{\gamma^+(x)}$ is compact.

The first positive prolongation of a point $x \in M$ is the set $D^+(x)$ of the points $y \in M$ such that there is a sequence $\{x_n\}$ in M and a sequence $\{t_n\}$ in $\boldsymbol{R}_+$ such that $x_n \to x$ and $\pi(x_n, t_n) \to y$. The first positive prolongational limit of x is the set $J^+(x)$ of the points $y \in M$ such that there is a sequence $\{x_n\}$ in M and a sequence $\{t_n\}$ in $\boldsymbol{R}_+$ such that $x_n \to x$, $t_n \to \infty$ and $\pi(x_n, t_n) \to y$. We say that x is non-wandering if $x \in J^+(x)$.

In this section we show that the main limit sets of this semidynamical system agree with important shape notions. Many of the results deal with the role played by the approximative maps from X to Y whose image is completely contained in Y. Given $F \in A(X, Y)$ we will denote the set $[F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y)$ by $[F]_w^*$.

The following result relates this set with one of the main notions of semidynamical systems.

THEOREM 5. *Consider $F \in A(X, Y)$. Then $J^+(F) = [F]_w^* = [F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y)$.*

As a consequence, $J^+(F) \neq \emptyset$ if and only if $[F]_w$ has a representative generated by a continuous function from X to Y.

PROOF. Suppose that $G \in J^+(F)$. We see first that $G \in \mathscr{C}(X \times \boldsymbol{R}_+, Y)$. Take $(x_0, t_0) \in X \times \boldsymbol{R}_+$ and $\varepsilon > 0$. Consider $t_1 \in \boldsymbol{R}_+$ such that $d(F(x, t_0 + t), Y) < \varepsilon/3$ for every $x \in X$ and every $t \geq t_1$. Since $G \in J^+(F)$, there exists $H \in A(X, Y)$ with $\tilde{d}(F, H) < \varepsilon/3$ and there exists $t_2 \geq t_1$ such that $\tilde{d}(\pi(H, t_2), G) < \varepsilon/3$. Then

$$|d(F(x_0, t_0 + t_2), Y) - d(H(x_0, t_0 + t_2), Y)| < \frac{\varepsilon}{3}, \quad |d(H(x_0, t_0 + t_2), Y) - d(G(x_0, t_0), Y)| < \frac{\varepsilon}{3}.$$

Therefore $|d(F(x_0, t_0 + t_2), Y) - d(G(x_0, t_0), Y)| < 2\varepsilon/3$ and since $d(F(x, t_0 + t_2), Y) < \varepsilon/3$, then $d(G(x_0, t_0), Y) < \varepsilon$. Since this happens for every $\varepsilon > 0$, then $d(G(x_0, t_0), Y) = 0$ and since Y is compact then $G(x_0, t_0) \in Y$.

In order to show that F and G are weakly homotopic consider a neighborhood V of Y in Q (we can suppose without loss of generality that V is an ANR) and take $\varepsilon > 0$ such that $B_\varepsilon(Y) \subset V$ and such that ε-near maps from X to V are homotopic in V. Consider $k_0 \in \boldsymbol{R}_+$ such that $d(F(x,t), Y) < \varepsilon/2$ for every $x \in X$ and $t \geq k_0$. Consider $\delta \leq \varepsilon/2^{k_0}$. Then it is easy to see that $F|_{X\times[k_0,\infty)}$ and $H|_{X\times[k_0,\infty)}$ are homotopic in V for every $H \in A(X,Y)$ with $\tilde{d}(F,H) < \delta$. Moreover we can take δ in such a way that $G|_{X\times[k_0,\infty)}$ and $H'|_{X\times[k_0,\infty)}$ are also homotopic in V for every $H' \in A(X,Y)$ with $\tilde{d}(G,H') < \delta$. Now, since $G \in J^+(F)$ there exists $H \in A(X,Y)$ with $\tilde{d}(F,H) < \delta$ and there exists $t_0 \in \boldsymbol{R}_+$ such that $d(G,\pi(H,t_0)) < \delta$. But then $F|_{X\times[k_0,\infty)}$ and $H|_{X\times[k_0,\infty)}$, and $G|_{X\times[k_0,\infty)}$ and $\pi(H,t_0)|_{X\times[k_0,\infty)}$ are homotopic in V, and since $H(X\times[k_0,\infty)) \subset V$ then $H|_{X\times[k_0,\infty)}$ and $\pi(H,t_0)|_{X\times[k_0,\infty)}$ are also homotopic in V. Hence $F|_{X\times[k_0,\infty)}$ and $G|_{X\times[k_0,\infty)}$ are homotopic in V.

We have proved that $J^+(F) \subset [F]_w \cap \mathscr{C}(X\times\boldsymbol{R}_+, Y)$. Consider now $G \in [F]_w \cap \mathscr{C}(X\times\boldsymbol{R}_+, Y)$. We see first that for every $\varepsilon > 0$ and every $t_0 \in \boldsymbol{R}_+$, there exist $G' \in B_\varepsilon(F)$ and $t \geq t_0$ such that $\pi(G',t) = G$. Given $\varepsilon > 0$ consider $k_0 \geq t_0$ such that

$$\sum_{k=k_0+1}^{\infty} \frac{\varDelta + \varepsilon}{2^k} < \varepsilon, \quad \text{where } \varDelta = \operatorname{diam}(Y),$$

and such that $F|_{X\times[k_0,\infty)}$ and $G|_{X\times[k_0,\infty)}$ are homotopic in $B_{(\varepsilon/2)}(Y)$. Then there exists a continuous function $h : X\times[0,1] \to B_{(\varepsilon/2)}(Y)$ such that $h_0 = F|_{X\times\{k_0\}}$ and $h_1 = G|_{X\times\{k_0\}}$. Define $G' : X\times\boldsymbol{R}_+ \to Q$ as

$$G'(x,t) = \begin{cases} F(x,t) & \text{if } t \leq k_0 \\ h(x, t-k_0) & \text{if } k_0 \leq t \leq k_0+1 \\ G(x, 2k_0+1-t) & \text{if } k_0+1 \leq t \leq 2k_0+1 \\ G(x, t-2k_0-1) & \text{if } 2k_0+1 \leq t. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} \frac{\sup_{(x,s)\in X\times[0,k]} d(F(x,s), G'(x,s))}{2^k} \leq \sum_{k=1}^{k_0} \frac{0}{2^k} + \sum_{k=k_0+1}^{\infty} \frac{\varDelta+\varepsilon}{2^k} < \varepsilon.$$

On the other hand, it is easy to see that $|d(G'(x,t),Y) - d(F(x,t),Y)| = |d(F(x,t),Y)| < \varepsilon$ for every $(x,t) \in X\times\boldsymbol{R}_+$. Hence $G' \in B_\varepsilon(F)$ and $\pi(G', 2k_0+1) = G$ with $2k_0+1 \geq t_0$. Therefore $G \in J^+(F)$. □

Corollary 1. *Consider $F \in A(X,Y)$. Then*

$$D^+(F) = \gamma^+(F) \cup J^+(F) \subset [F] \cup [F]_w^* \subset [F]_w.$$

Consequently, $\mathscr{C}(X\times\boldsymbol{R}_+, Y)$ is a stable set in the sense of Bhatia and Hájek [**4**]. *Moreover, for every $F \in A(X,Y)$, the sets $[F]_w$ and $[F]_w^*$ are stable.*

On the other hand, if we consider

$$S = \{F \in A(X,Y) \mid [F]_w^* = \varnothing\},$$

then every positively invariant closed subset of S is stable. In particular, $[F]_w$ and $\overline{\gamma^+(F)} = \gamma^+(F)$ are stable for every $F \in S$.

THEOREM 6. *Consider $F \in A(X, Y)$. Then $\Lambda^+(F) \subset [F]_w^*$.*

Moreover $\lim_{t\to\infty}\pi(F, t) = G$ if and only if $a(F) \cap \mathscr{C}(X, Y) = \{G\}$, where $a(F)$ is the set of approximative maps asymptotic to F.

PROOF. The first statement is a consequence of the above theorem. In order to prove the second statement consider $s_0 \in \boldsymbol{R}_+$ and $\varepsilon > 0$. Take $k_0 > s_0$. Since $\lim_{t\to\infty}\pi(F, t) = G$ there exists $t_0 > s_0$ such that

$$\tilde{d}(\pi(F, t), G) < \frac{\varepsilon}{4} \quad \text{and} \quad \tilde{d}(\pi(F, t - s_0), G) < \frac{\varepsilon}{2^{k_0+1}}$$

for every $t \geq t_0$, and hence for every $x \in X$

$$d(F(x, t), G(x, 0)) < \frac{\varepsilon}{2}, \quad d(F(x, t), G(x, s_0)) < \frac{\varepsilon}{2}.$$

Then $d(G(x, s_0), G(x, 0)) < \varepsilon$ for every $\varepsilon > 0$. Hence $G(x, s_0) = G(x, 0)$ for every $s_0 \in \boldsymbol{R}_+$. This also implies that F and G are asymptotic and hence homotopic. The proof of the converse statement is left to the reader. $\square$

The following example shows that $\Lambda^+(F)$ does not necessarily have to be contained in $[F]$ nor in $\mathscr{C}(X, Y)$.

EXAMPLE 1. Consider $F, G : \{0\} \times \boldsymbol{R}_+ \to S^1$ given by $F(0, t) = (1, 0)$ and

$$G(0, t) = \begin{cases} e^{i2\pi t} & \text{if } 2^k - 1 \leq t \leq 2^k \\ (1, 0) & \text{rest.} \end{cases}$$

Then $F \in \Lambda^+(G)$ but is not homotopic to G. Moreover, varying slightly F we can also get $F \notin \mathscr{C}(X, Y)$ with the same properties.

COROLLARY 2. *Let $F \in A(X, Y)$ satisfy any of the following conditions:*
i) $\Lambda^+(F) \neq \emptyset$,
ii) π_F *positively asymptotic,*
iii) π_F *positively Poisson stable,*
iv) π_F *positively Lagrange stable.*
Then $[F]_w$ has a representative generated by a continuous function from X to Y.

On the other hand, if $[F]_w$ has not a representative generated by a continuous function from X to Y, then π_F is positively departing.

THEOREM 7. *A positive motion π_F is positively Lagrange stable if and only if F is uniformly continuous.*

PROOF. Suppose that π_F is positively Lagrange stable but F is not uniformly continuous. Then there exists $\varepsilon > 0$ such that for every $n \in \boldsymbol{N}$ there exist $(x_n, t_n), (y_n, s_n) \in X \times \boldsymbol{R}_+$ such that $d(x_n, y_n) < 1/n$, $t_n \leq s_n < t_n + (1/n)$ and $d(F(x_n, t_n), F(y_n, s_n)) > \varepsilon$.

By the compactness of $\overline{\gamma^+(F)}$ there exists a sequence $\{t_{k_n}\}$ such that $\{\pi(F, t_{k_n})\}$ converges. Hence given $\varepsilon > 0$, there exists $n_0 \in \boldsymbol{N}$ such that $\tilde{d}(\pi(F, t_{k_n}), \pi(F, t_{k_{n_0}})) < \varepsilon/6$

for every $n \geq n_0$, and hence

$$d(F(x, t_{k_n} + t), F(x, t_{k_{n_0}} + t)) < \frac{\varepsilon}{3}$$

for every $(x, t) \in X \times [0, 1]$. Since $\pi(F, t_{k_{n_0}})$ is continuous in $X \times [0, 1]$, there exists $0 < \delta < 1$ such that for every $x, y \in X$ with $d(x, y) < \delta$ and every $s \in \boldsymbol{R}_+$ with $0 < s < \delta$ we have

$$d(F(x, t_{k_{n_0}}), F(y, t_{k_{n_0}} + s)) < \frac{\varepsilon}{3}.$$

Therefore, for every $n \geq n_0$ and for every $x, y \in X$ with $d(x, y) < \delta$ and every $s \in \boldsymbol{R}_+$ with $0 < s < \delta$ we have

$$d(F(x, t_{k_n}), F(y, t_{k_n} + s)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore for every $n \geq n_0$ such that $1/k_n < \delta$ we have that $d(x_{k_n}, y_{k_n}) < 1/k_n < \delta$ and $0 < s_{k_n} - t_{k_n} < 1/k_n < \delta$ and hence

$$d(F(x_{k_n}, t_{k_n}), F(y_{k_n}, s_{k_n})) < \varepsilon,$$

and this is a contradiction. Hence F is uniformly continuous.

We see now that if F is uniformly continuous then π_F is positively Lagrange stable. To prove this we are going to show that every sequence in $\overline{\gamma^+(F)}$ has a convergent subsequence and it suffices to show it for sequences in $\gamma^+(F)$, i.e., for sequences of the kind $\{\pi(F, t_n)\}$. Finally by the continuity of π it is enough to consider the case $t_n \to \infty$.

Since X is compact there exists a countable dense subset E of $X \times \boldsymbol{R}_+$. By the compactness of Q there exists $\{t_{k_n}\}$ such that $\{\pi(F, t_{k_n})\}$ converges pointwise in E (if $E = \{(x_i, t_i)\}$ we take a subsequence $\{\pi(F, t_n^1)\}$ pointwise convergent in (x_1, t_1), this subsequence has a subsequence $\{\pi(F, t_n^2)\}$ pointwise convergent in (x_1, t_1) and (x_2, t_2), and so on. Then the diagonal subsequence $\{\pi(F, t_n^n)\}$ converges pointwise in E).

Moreover, if $(x, t) \in (X \times \boldsymbol{R}_+) \backslash E$ then given $\varepsilon > 0$, by the uniform continuity of F, there exists $(y, s) \in E$ such that $d(F(x, t_{k_n} + t), F(y, t_{k_n} + s)) < \varepsilon/3$ for every $n \in \boldsymbol{N}$. On the other hand, there exists $n_0 \in \boldsymbol{N}$ such that $d(F(y, t_{k_n} + s), F(y, t_{k_{n_0}} + s)) < \varepsilon/3$ for every $n \geq n_0$. Therefore, for every $n \geq n_0$,

$$d(F(x, t_{k_n} + t), F(x, t_{k_{n_0}} + t)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence for every $(x, t) \in (X \times \boldsymbol{R}_+) \backslash E$, $\{F(x, t_{k_n} + t)\}$ is a Cauchy sequence, and since Q is compact, it is a convergent sequence. Therefore $\{\pi(F, t_{k_n})\}$ converges pointwise in $X \times \boldsymbol{R}_+$ to a function $G : X \times \boldsymbol{R}_+ \to Q$.

Moreover, for every $(x, t) \in X \times \boldsymbol{R}_+$, given $\varepsilon > 0$, there exists $n_0 \in \boldsymbol{N}$ such that, for every $n \geq n_0$, $d(F(x, t_{k_n} + t), G(x, t)) < \varepsilon/2$, and there exists $n \geq n_0$ such that $d(F(x, t_{k_n} + t), Y) < \varepsilon/2$. Hence $d(G(x, t), Y) < \varepsilon$. Therefore $G(X \times \boldsymbol{R}_+) \subset Y$.

We see now that G is uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(F(x, t), F(y, s)) < \varepsilon/3$ for every $(x, t), (y, s) \in X \times \boldsymbol{R}_+$ such that $d((x, t), (y, s)) < \delta$. On the other hand given $(x, t), (y, s)$ there exists $n_0 \in \boldsymbol{N}$ such that $d(F(x, t_{k_{n_0}} + t), G(x, t))$

$< \varepsilon/3$ and $d(F(y, t_{k_{n_0}} + s), G(y, s)) < \varepsilon/3$. Therefore, if $(x, t), (y, s) \in X \times \boldsymbol{R}_+$ are such that $d((x, t), (y, s)) < \delta$, we have

$$d(G(x, t), G(y, s)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore G is uniformly continuous and, in particular, $G \in A(X, Y)$.

Finally we see that $\{\pi(F, t_{k_n})\}$ converges to G in $A(X, Y)$. Given $\varepsilon > 0$ take k_0 such that

$$\sum_{k=k_0+1}^{\infty} \frac{\Delta + \varepsilon}{2^k} < \frac{\varepsilon}{2} \quad \text{where } \Delta = \operatorname{diam}(Y).$$

Since F and G are uniformly continuous, there exists $\delta > 0$ such that

$$d(F(x, t), F(y, s)) < \frac{\varepsilon}{6}, \quad d(G(x, t), G(y, s)) < \frac{\varepsilon}{6}$$

for every $(x, t), (y, s) \in X \times \boldsymbol{R}_+$ with $d((x, t), (y, s)) < \delta$. On the other hand, there exist $\{(x_1, t_1), \dots, (x_r, t_r)\} \subset X \times [0, k_0]$ such that $X \times [0, k_0] \subset B_\delta(x_1, t_1) \cup \cdots \cup B_\delta(x_r, t_r)$. Consider $n_0 \in \boldsymbol{N}$ such that $d(F(x, t_{k_n} + t), Y) < \varepsilon$ for every $(x, t) \in X \times \boldsymbol{R}_+$ and every $n \geq n_0$, and such that

$$d(F(x_i, t_{k_n} + t_i), G(x_i, t_i)) < \frac{\varepsilon}{6},$$

for every $i \in \{1, \dots, r\}$ and every $n \geq n_0$. Then, for every $(x, t) \in X \times [0, k_0]$ there exists $i \in \{1, \dots, r\}$ such that $(x, t) \in B_\delta(x_i, t_i)$ and hence, for every $n \geq n_0$,

$$d(F(x, t_{k_n} + t), G(x, t)) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

Therefore, taking into account that $G(X \times \boldsymbol{R}_+) \subset Y$, we have that, for every $n \geq n_0$,

$$\sum_{k=1}^{\infty} \frac{\sup_{(x,t) \in X \times [0,k]} d(F(x, t_{k_n} + t), G(x, t))}{2^k} < \sum_{k=1}^{k_0} \frac{\varepsilon/2}{2^k} + \sum_{k=k_0+1}^{\infty} \frac{\Delta + \varepsilon}{2^k} < \varepsilon,$$

and

$$\sup_{X \times \boldsymbol{R}_+} |d(F(x, t_{k_n} + t), Y) - d(G(x, t), Y)| < \varepsilon.$$

Then $\tilde{d}(\pi(F, t_{k_n}), G) < \varepsilon$ for every $n \geq n_0$, and $\{\pi(F, t_{k_n})\}$ converges to G in $A(X, Y)$. □

COROLLARY 3. *If $F \in A(X, Y)$ is uniformly continuous, then $[F]_w$ has a representative generated by a continuous function from X to Y.*

THEOREM 8. *Denote by $\mathscr{M}$ the set of non-wandering points. Then $\mathscr{M} = \mathscr{C}(X \times \boldsymbol{R}_+, Y)$.*

PROOF. If $F \in \mathscr{M}$, then $F \in J^+(F) \subset \mathscr{C}(X \times \boldsymbol{R}_+, Y)$. Conversely, if $F \in \mathscr{C}(X \times \boldsymbol{R}_+, Y)$, then $F \in [F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y) = J^+(F)$. □

5. Stability and attraction properties.

In this section we are going to study properties concerning stability and attraction for closed sets. We will use the approach given in [**5**], where these notions are defined for closed subsets L in arbitrary metric spaces, in terms of neighborhoods of the kind $B_\varepsilon(L)$. In other books on dynamical systems these concepts are defined in terms of arbitrary neighborhoods of L.

Let $(M, \boldsymbol{R}_+, \pi)$ be a semidynamical system on a metric space M. A closed subset L of M is stable if for every $x \in L$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $\pi(B_\delta(x) \times \boldsymbol{R}_+) \subset B_\varepsilon(L)$. L is uniformly stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\pi(B_\delta(L) \times \boldsymbol{R}_+) \subset B_\varepsilon(L)$.

If L is a nonempty closed subset of M, the region of weak attraction of L is the set $A_\omega(L)$ of the points $x \in M$ such that for every $\varepsilon > 0$ and every $\tau \in \boldsymbol{R}_+$ there exists $t > \tau$ such that $\pi(x, t) \in B_\varepsilon(L)$. The region of attraction of L is the set $A(L)$ of the points $x \in M$ such that for every $\varepsilon > 0$ there exists $t \in \boldsymbol{R}_+$ such that $\pi(\{x\} \times [t, \infty)) \subset B_\varepsilon(L)$. The region of strong attraction of L is the set $A_s(L)$ of the points $x \in M$ such that for every $\varepsilon > 0$ there exist $\delta > 0$ and $t \in \boldsymbol{R}_+$ such that $\pi(B_\delta(x) \times [t, \infty)) \subset B_\varepsilon(L)$.

L is a weak attractor, attractor or strong attractor if there exists $\delta > 0$ such that $B_\delta(L)$ is contained in $A_\omega(L)$, $A(L)$ or $A_s(L)$, respectively. L is asymptotically stable if and only if L is a uniformly stable attractor.

Proposition 2. *For every* $F \in A(X, Y)$,

$$A_\omega([F]_w) = A([F]_w) = [F]_w.$$

Proof. Take $G \in A_\omega([F]_w)$ and $\varepsilon > 0$. Consider $0 < \varepsilon' < \varepsilon$ such that ε'-near maps from X to $B_{\varepsilon'}(Y)$ are homotopic in $B_\varepsilon(Y)$. There exist $t_0 \in \boldsymbol{R}_+$ and $F' \in [F]_w$ such that $\tilde{d}(\pi(G, t_0), F') < \varepsilon'/2$ and $G(X \times [t_0, \infty)) \subset B_{\varepsilon'/2}(Y)$, and hence $F'(X \times [0, \infty)) \subset B_{\varepsilon'}(Y)$. In particular $F'(X \times \{0\}) \subset B_{\varepsilon'}(Y)$, $G(X \times \{t_0\}) \subset B_{\varepsilon'/2}(Y)$ and $d(F'|_{X \times \{0\}}, G|_{X \times \{t_0\}}) < \varepsilon'/2$. Therefore $F'|_{X \times \{0\}}$ and $G|_{X \times \{t_0\}}$ are homotopic in $B_\varepsilon(Y)$ and hence F' and $\pi(G, t_0)$ are also homotopic in $B_\varepsilon(Y)$. On the other hand, there exists $t_1 \in \boldsymbol{R}_+$ such that $G|_{X \times [t_1, \infty)}$ and $\pi(G, t_0)|_{X \times [t_1, \infty)}$, and $F|_{X \times [t_1, \infty)}$ and $F'|_{X \times [t_1, \infty)}$ are homotopic in $B_\varepsilon(Y)$. Hence $F|_{X \times [t_1, \infty)}$ and $G|_{X \times [t_1, \infty)}$ are homotopic in $B_\varepsilon(Y)$. Therefore $G \in [F]_w$ and $A_\omega([F]_w) \subset [F]_w$.

In order to prove that $[F]_w \subset A([F]_w)$, it suffices to observe that if $G \in [F]_w$ then $\pi(\{G\} \times [0, \infty)) \subset [F]_w$. □

Corollary 4. *Given* $F \in A(X, Y)$ *with* $[F]_w^* \neq \emptyset$, *then*

$$[F]_w^* \subset a([F]_w^*) \subset A([F]_w^*) \subset A_\omega([F]_w^*) \subset [F]_w,$$

where the first inclusion is strict.

Moreover, given a closed subset L *of* $A(X, Y)$, *then* $[F]_w^* \subset A(L)$ *if and only if* $[F]_w^* \subset L$.

Proof. The first part is an immediate consequence of the above proposition. The fact of the first inclusion being strict is a consequence of Y being a Z-set in Q.

Moreover if $[F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y) \subset L$ then

$$[F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y) \subset A([F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y)) \subset A(L).$$

Conversely, suppose that there exists $G \in [F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y)$ such that $G \notin L$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(L) \cap B_\varepsilon(G) = \emptyset$. Consider $H : X \times \boldsymbol{R}_+ \to Y$ given by

$$H(x,t) = \begin{cases} G(x,t) & \text{if } 0 \le t \le 1 \\ G(x, n^2 + n - t) & \text{if } n \le t \le n^2 + n,\ n \in \boldsymbol{N} \\ G(x, t - n^2 - n) & \text{if } n^2 + n \le t \le (n+1)^2,\ n \in \boldsymbol{N}. \end{cases}$$

Then $H \in [F]_w \cap \mathscr{C}(X \times \boldsymbol{R}_+, Y)$ but $H \notin A(L)$ since $\pi(\{H\} \times [t, \infty)) \cap B_\varepsilon(G) \ne \emptyset$ for every $t \in \boldsymbol{R}_+$ and hence $\pi(\{H\} \times [t, \infty)) \not\subset B_\varepsilon(L)$. □

The following example shows, however, that none of the inclusions can be replaced, in general, by an equality, not even in the case of compacta with trivial shape.

EXAMPLE 2. Consider $X = [0,1]$ and

$$Y = \left\{ (x,y) \in \boldsymbol{R}^2 \mid x \in [-1,1] - \{0\},\ y = \sin\left(\frac{1}{x}\right) \right\} \cup (\{0\} \times [-1,1]).$$

Then $[F]_w = A(X, Y)$ for every $F \in A(X, Y)$ and

$$a(\mathscr{C}(X \times \boldsymbol{R}_+, Y)) \ne A(\mathscr{C}(X \times \boldsymbol{R}_+, Y)) \ne A_\omega(\mathscr{C}(X \times \boldsymbol{R}_+, Y)) \ne A(X, Y).$$

Moreover, none of this regions of attraction is a neighborhood of $\mathscr{C}(X \times \boldsymbol{R}_+, Y)$.
Moreover, $A_s(\mathscr{C}(X \times \boldsymbol{R}_+, Y)) = \emptyset$.

PROPOSITION 3. *Let L be a non empty closed subset of $A(X, Y)$ and $F \in A(X, Y)$ such that $F \in A_s(L)$. Then $[F]_w \subset A(L)$.*

PROOF. Consider $G \in [F]_w$ and $\varepsilon > 0$. Since $F \in A_s(L)$, there exist $\delta > 0$ and $t \in \boldsymbol{R}_+$ such that $\pi(\{F'\} \times [t, \infty)) \subset B_\varepsilon(L)$ for every $F' \in B_\delta(F)$. On the other hand, since $G \in [F]_w = \overline{c(F)}$ (see Theorem 3), there exists $G' \in B_\delta(F)$ such that $\pi(G, t_G) = \pi(G', t_G)$ for some $t_G \in \boldsymbol{R}_+$. Therefore $G \in A(L)$. □

PROPOSITION 4. *Let $L \subset A(X, Y)$ be a weak attractor. Then $A_\omega(L)$ is a union of weak homotopy classes. If moreover L is an attractor then*

$$L \supset \{[F]_w^* \mid F \in A(L)\}.$$

PROOF. Let L be a weak attractor and take $F \in A_\omega(L)$. Since $A_\omega(L)$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(F) \subset A_\omega(L)$. Then given $G \in [F]_w$ there exists $G' \in B_\varepsilon(F)$ such that $\pi(G, t_G) = \pi(G', t_G)$ for some $t_G \in \boldsymbol{R}_+$. Hence, since $G' \in B_\varepsilon(F) \subset A_\omega(L)$, then $G \in A_\omega(L)$.
On the other hand, if L is an attractor and $F \in A(L)$, then $[F]_w \subset A(L)$, and by the above proposition, $[F]_w^* \subset L$. □

Notions of attraction and stability can also be established for positive motions; we adopt here the definitions given in [**33**]. Given a semidynamical system $(M, \boldsymbol{R}_+, \pi)$ on a metric space M and given $x \in M$, the positive motion π_x is positively Lyapunov stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\pi(x,t),\pi(y,t)) < \varepsilon$ for every $y \in B_\delta(x)$ and every $t \in \boldsymbol{R}_+$. π_x is uniformly positively Lyapunov stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\pi(x,t+\tau),\pi(y,t)) < \varepsilon$ for every $y \in B_\delta(\pi(x,\tau))$ and every $t,\tau \in \boldsymbol{R}_+$. Given $x \in M$, the region of orbital attraction of π_x is the set $\mathscr{A}^+(x)$ of the points $y \in M$ such that there exists $t_0 \in \boldsymbol{R}_+$ such that $\lim_{t\to\infty} d(\pi(x,t+t_0),\pi(y,t)) = 0$. π_x is an orbital attractor if $\mathscr{A}^+(x)$ is a neighborhood of $\overline{\gamma^+(x)}$. π_x is orbitally asymptotically stable if π_x is an uniformly positively Lyapunov stable orbital attractor.

Proposition 5. *A positive motion π_G is orbitally attracted by a positive motion π_F if and only if there exists $t_0 \in \boldsymbol{R}_+$ such that $\pi(F,t_0)$ and G are asymptotic. Hence*

$$\mathscr{A}^+(F) = \bigcup_{t\in\boldsymbol{R}_+} a(\pi(F,t)) \subset [F],$$

where $a(\pi(F,t)) = \{G \in A(X,Y) | G$ asymptotic to $\pi(F,t)\}$. Moreover, if π_F is an orbital attractor, then

$$\mathscr{A}^+(F) = \bigcup_{t\in\boldsymbol{R}_+} a(\pi(F,t)) = [F] = [F]_w.$$

Proof. The first statement is straightforward. In order to prove the second statement, suppose that $\mathscr{A}^+(F)$ is a neighborhood of $\overline{\gamma^+(F)}$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(F) \subset \mathscr{A}^+(F)$. Consider $G \in [F]_w$. Then, given $\varepsilon > 0$, consider k_0 such that

$$\sum_{k=k_0+1}^{\infty} \frac{\Delta+\varepsilon}{2^k} < \varepsilon, \quad \text{where } \Delta = \operatorname{diam}(Y),$$

and $F|_{X\times[k_0,\infty)}$ and $G|_{X\times[k_0,\infty)}$ are homotopic in $B_{\varepsilon/2}(Y)$. It is easy to see that then there exists $h: X\times[0,1] \to B_{\varepsilon/2}(Y)$ such that $h_0 = F|_{X\times\{k_0\}}, h_1 = G|_{X\times\{k_0+1\}}$. Consider $H: X\times\boldsymbol{R}_+ \to Q$ given by

$$H(x,t) = \begin{cases} F(x,t) & \text{if } t \le k_0 \\ h(x,t-k_0) & \text{if } k_0 \le t \le k_0+1 \\ G(x,t) & \text{if } k_0+1 \le t. \end{cases}$$

Then $H \in B_\varepsilon(F) \subset \mathscr{A}^+(F)$ and hence there exists $t \in \boldsymbol{R}_+$ such that H is asymptotic to $\pi(F,t)$. On the other hand, since $\pi(H,k_0+1) = \pi(G,k_0+1)$, then also G is asymptotic to $\pi(F,t)$. Hence $G \in \mathscr{A}^+(F)$. Therefore $[F]_w \subset \mathscr{A}^+(F) \subset [F] \subset [F]_w$. $\square$

Proposition 6. *Consider $F \in A(X,Y)$ such that π_F is positively Lyapunov stable. Then*

$$[F]_w = [F] = a(F) = \{G \in A(X,Y) \mid G \text{ asymptotic to } F\}.$$

Proof. Take $G \in [F]_w$ and $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\tilde{d}(\pi(F,t),\pi(H,t)) < \varepsilon$$

for every $H \in B_\delta(F)$ and every $t \in \boldsymbol{R}_+$. On the other hand, since $G \in [F]_w$, we saw in the proof of the above proposition that given $\delta > 0$ there exist $H \in B_\delta(F) \cap [F]_w$ and $t_0 \in \boldsymbol{R}_+$ such that $\pi(G, t_0) = \pi(H, t_0)$. Then $\tilde{d}(\pi(F,t), \pi(G,t)) < \varepsilon$, for every $t \geq t_0$ and hence $d(F(x,t), G(x,t)) < \varepsilon$ for every $x \in X$ and every $t \geq t_0$. Therefore F and G are asymptotic. □

6. A Lyapunov function in $(A(X, Y), \boldsymbol{R}_+, \pi)$.

Given a semidynamical system $(M, \boldsymbol{R}_+, \pi)$ on a metric space M, a Lyapunov function on M is a continuous function $\varphi : M \to \boldsymbol{R}_+$ such that $\varphi(\pi(x,t)) \leq \varphi(x)$ for every $x \in M$ and every $t \in \boldsymbol{R}_+$.

REMARK 3. The function $\varphi : A(X, Y) \to \boldsymbol{R}_+$ given by

$$\varphi(F) = \max\{d(F(x,t), Y) \mid (x,t) \in X \times \boldsymbol{R}_+\}$$

for every $F \in A(X, Y)$, is a Lyapunov function in $A(X, Y)$.

Observe that $\varphi(F) \leq \tilde{d}(F, \mathscr{C}(X \times \boldsymbol{R}_+, Y))$ for every $F \in A(X, Y)$ and that

$$B_\varepsilon(\mathscr{C}(X \times \boldsymbol{R}_+, Y)) \subset \varphi^{-1}([0, \varepsilon)) \subset \varphi^{-1}([0, \varepsilon]) = \mathscr{C}(X \times \boldsymbol{R}_+, \bar{B}_\varepsilon(Y))$$

for every $\varepsilon > 0$.

7. Equivalence of dynamic properties and shape properties.

Given compact metric spaces X and Y we may consider the Bebutov semidynamical system in $A(X, Y)$. We have already mentioned that, since every shape morphism is invariant, it can be considered as a semidynamical system itself. In this section we present characterizations of shape properties in dynamical terms. The proofs of the theorems are, in general, simple consequences of the results developed in previous sections.

THEOREM 9. *Given a shape morphism $[F]_w$, the following are equivalent*:
i) *$[F]_w$ is generated by a map*,
ii) *the semidynamical subsystem restricted to $[F]_w$ is non-dispersive*,
iii) *there exists an orbit in $[F]_w$ not agreeing with its first positive prolongation.*

PROOF. The equivalence of i) and ii) is a direct consequence of [**5**, Theorem IV.1.8]. To prove that i) implies iii) suppose that $[F]_w$ is generated by a map. Then we just have to take $G \in [F]_w$ such that $G(X \times \{t_n\}) \not\subset Y$ for some divergent sequence $\{t_n\} \subset \boldsymbol{R}_+$. Then G satisfies the required condition. To see that it is always possible to choose such a G consider $f : X \to Y$ generating $[F]_w$. Consider $y_0 \notin Y$ (since Y is a Z-set in Q, $Y \neq Q$). Then there exists $h : X \times [0,1] \to Q$ such that $h_0(x) = y_0$ for every $x \in X$ and such that $h_1 = f$. Then there exists $b \in (0,1]$ such that $h_t(X) \not\subset Y$ for values of $t \in [0, b)$ arbitrary close to b and such that h_b is homotopic to f in Y. We define $G(x,t) = h(x, bt/(1+t))$.

Conversely if an orbit $\gamma^+(G)$ in $[F]_w$ doesn't agree with its first positive prolongation then $J^+(G) \neq \varnothing$ and the result follows. □

The following result is a consequence of Theorems 8 and 9.

THEOREM 10. *Let X be a compact metric space. The following are equivalent*:

i) *X has trivial shape.*

ii) *The set of non-wandering points of $A(X,X)$ is contained in a connected component of $A(X,X)$.*

iii) *The set of Lagrange stable motions of $A(X,X)$ is contained in a connected component of $A(X,X)$.*

PROOF. If X has trivial shape then every pair of morphisms from X to X are homotopic, hence $A(X,X)$ is connected and this implies ii) and iii). Since the set of non-wandering points of $A(X,X)$ agrees with $\mathscr{C}(X \times \boldsymbol{R}_+, X)$, ii) implies that the identity map is in the same connected component that, and hence is weakly homotopic to, any constant map. Therefore ii) implies i). Finally, since the identity map and any constant map are both uniformly continuous, iii) implies that they have to be weakly homotopic. Therefore iii) implies that X has trivial shape too. $\square$

If $X \subset X'$, we can consider the Bebutov semidynamical systems defined in $A(X, Y)$ and in $A(X', Y)$. Given $F \in A(X, Y)$ and $G \in A(X', Y)$, we say that the orbit through F is the restriction of the orbit through G if $\pi(F, t) = \pi(G, t)$ for every t, when the latter is restricted to X. This is equivalent to $G|_{X \times \boldsymbol{R}_+} = F$. We say that a Bebutov system $A(X, Y)$ is prolongable if for every X' containing X, there exists a neighborhood W of X in X' such that every orbit in $A(X, Y)$ is the restriction of an orbit in $A(W, Y)$.

THEOREM 11. *Let X be a metric space. Then X is shape dominated by a polyhedron if and only if $A(X,X)$ is prolongable.*

PROOF. Is a consequence of the fact that X is shape dominated by a polyhedron if and only if X is a FANR (see [**8**, p. 350]), and this is equivalent to X having a neighborhood U in X' such that X is a shape retract of U. $\square$

THEOREM 12. *Let X be a compact metric space shape dominated by a polyhedron. The following are equivalent*:

i) *X is an internal FANR.*

ii) *Every shape morphism from an arbitrary compactum Z to X is generated by a map* (*from Z to X*).

iii) *Every connected component of $A(Z,X)$ contains a Lagrange stable orbit*, *for any compactum Z.*

iv) *Every connected component of $A(Z,X)$ contains a non-wandering orbit*, *for any compactum Z.*

PROOF. The implication i) $\Rightarrow$ ii) was proved in [**11**]. Conversely, if X is a FANR such that every shape morphism from an arbitrary compactum Z to X is generated by a continuous map, then X is internally movable and hence (see [**22**]) an internal FANR. The rest of the equivalences are consequences of the fact that shape morphisms from Z to X can be identified with the connected components of $A(Z,X)$ and the results in the previous sections. $\square$

The following result relates shape notions with attraction properties in $A(X, Y)$.

THEOREM 13. *Let X be a compact metric space. Then the following are equivalent*:

i) *X has trivial shape.*

ii) *There exists a connected attractor in $A(X,X)$, and every attractor contains the set $\mathscr{C}(X \times \boldsymbol{R}_+, X)$.*

iii) *There exists a connected attractor in $A(X,X)$ containing a periodic orbit, and every attractor containing a periodic orbit contains all periodic orbits.*

PROOF. If X has trivial shape then $A(X,X)$ is a connected attractor containing $\mathscr{C}(X \times \boldsymbol{R}_+, X)$. Moreover, any attractor must contain some approximative map F and then, by proposition 4, it contains $[F]_w^* = \mathscr{C}(X \times \boldsymbol{R}_+, X)$. Therefore i) implies ii).

Obviously ii) implies iii). Suppose now that iii) is satisfied. Then there exists a connected attractor L in $A(X,X)$ containing a periodic orbit. Since L is connected, there exists $F \in A(X,X)$ such that $L \subset [F]_w$. Then iii) implies that also every periodic orbit is contained in $[F]_w$ and, since the periodic orbits are dense in $\mathscr{C}(X \times \boldsymbol{R}_+, X)$, then $\mathscr{C}(X \times \boldsymbol{R}_+, X) \subset [F]_w$. Therefore the set of non-wandering points is contained in a connected component of $A(X,X)$ and the result follows from Theorem 12. $\square$

The necessary conditions for an orbit to be an orbital attractor or Lyapunov Stable stated in Section 5 are really strong conditions. In fact we have the following result.

THEOREM 14. *Let X be a continuum and Y a compact metric space. Then*

i) *Every orbit in $A(X,Y)$ is an orbital attractor if and only if Y is finite.*

ii) *Every orbit in $A(X,Y)$ is Lyapunov stable if and only if Y is 0-dimensional.*

PROOF. We only prove ii), leaving i) to the reader. Suppose that Y is 0-dimensional. Then it is totally disconnected. Let F be an approximative map from X to Y and consider $\varepsilon > 0$. There exist $\{U_1, U_2, \ldots, U_n\}$ pairwise disjoint open and closed subsets of Y with diameter less than $\varepsilon/5$, such that $Y = U_1 \cup \cdots \cup U_n$. There exists $0 < \delta < \varepsilon/5$ such that $d(U_i, U_j) > 5\delta$ for every $i \neq j$. Then $B_\delta(Y) = B_\delta(U_1) \cup \cdots \cup B_\delta(U_n)$, union of open balls in Q with $d(B_\delta(U_i), B_\delta(U_j)) > 3\delta$. Consider now $k_0 \in \boldsymbol{N}$ such that $F(x,t) \in B_\delta(Y)$ for every $x \in X$ and every $t \geq k_0$. Then there exists i_0 such that $F(x,t) \in B_\delta(U_{i_0})$ for every $x \in X$ and every $t \geq k_0$. Let G be any approximative map with $\tilde{d}(F,G) < \delta/2^{k_0}$. Then $d(F(x,t), G(x,t)) < \delta$ for every $x \in X$ and every $t \leq k_0$. On the other hand $|d(F(x,t), Y) - d(G(x,t), Y)| < \delta$ for every $x \in X$ and every $t \in \boldsymbol{R}_+$, and since $F(x,t) \in B_\delta(U_{i_0})$, this implies that $G(x,t) \in B_{2\delta}(U_{i_0})$ for every $x \in X$ and every $t \geq k_0$. Then it is easy to see that $d(F(x,t), G(x,t)) < \varepsilon$ for every $x \in X$ and every $t \in \boldsymbol{R}_+$. Therefore, $\tilde{d}(\pi(F,t), \pi(G,t)) < \varepsilon$ for every $t \in \boldsymbol{R}_+$. Hence π_F is Lyapunov stable.

Conversely, since every two constant approximative maps with image in the same connected component are homotopic, it follows from Proposition 6 that if every orbit in $A(X,Y)$ is Lyapunov stable then all connected components must be points. Hence Y has to be totally disconnected. $\square$

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