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On some Hamiltonian structures of Painlevé systems, II

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Abstract. We give a symplectic description of the fiber space for each J-th Painlevé system $(J = \mathrm{V}, \mathrm{IV}, \mathrm{III}, \mathrm{II})$ which was constructed by K. Okamoto. Hamiltonian function in every chart is a polynomial of the canonical coordinates.

§0. Introduction.

This is the second part of the series of our papers. In the preceding paper ([**11**]), we studied a Hamiltonian structure of the sixth Painlevé system (H_{VI}) equivalent to the sixth Painlevé equation P_{VI}. More precisely, we gave a description of the fiber space E_{VI} for the sixth Painlevé system constructed by K. Okamoto ([**7**]) so that each fiber $E_{\mathrm{VI}}(t)$ has a symplectic structure and proved that there exists no other Hamiltonian system holomorphic on the whole space E_{VI} than the sixth Painlevé system. In this paper, we continue the study for other Painlevé systems, namely, we obtain a symplectic description of the fiber space E_J for each $J = \mathrm{V}$, IV, III, II. The uniqueness of holomorphic Hamiltonian systems on each E_J will be proved in the next part ([**12**]). We have not yet completed the study for the first Painlevé system because it has some difficulty to obtain a symplectic description of E_{I}.

Painlevé equations P_J are the equations given by

$$P_{\mathrm{VI}}: \quad \frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\frac{dx}{dt}$$
$$+ \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(x-t)^2}\right],$$

$$P_{\mathrm{V}}: \quad \frac{d^2x}{dt^2} = \left(\frac{1}{2x} + \frac{1}{x-1}\right)\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right)$$
$$+ \gamma\frac{x}{t} + \delta\frac{x(x+1)}{x-1},$$

$$P_{\mathrm{IV}}: \quad \frac{d^2x}{dt^2} = \frac{1}{2x}\left(\frac{dx}{dt}\right)^2 + \frac{3}{2}x^3 + 4tx^2 + 2(t^2-\alpha)x + \frac{\beta}{x},$$

$$P_{\mathrm{III}}: \quad \frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(\alpha x^2 + \beta) + \gamma x^3 + \frac{\delta}{x},$$

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$$P_{\mathrm{II}}: \quad \frac{d^2x}{dt^2} = 2x^3 + tx + \alpha,$$

$$P_{\mathrm{I}}: \quad \frac{d^2x}{dt^2} = 6x^2 + t,$$

where x and t are complex variables, α, β, γ, and δ are complex constants ([**4**]). It is known that each P_J is equivalent to a Hamiltonian system $(H_J): dx/dt = \partial H_J/\partial y$, $dy/dt = -\partial H_J/dx$, where

$$H_{\mathrm{VI}}(x,y,t) = \frac{1}{t(t-1)}[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t) + \kappa_1 x(x-t) + (\kappa_t - 1)x(x-1)\}y + \kappa(x-t)]$$

$$\left(\kappa := \frac{1}{4}[(\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - {\kappa_\infty}^2]\right),$$

$$H_{\mathrm{V}}(x,y,t) = \frac{1}{t}[x(x-1)^2y^2 - \{\kappa_0(x-1)^2 + \kappa_t x(x-1) - \eta tx\}y + \kappa(x-1)]$$

$$\left(\kappa := \frac{1}{4}\{(\kappa_0 + \kappa_t)^2 - \kappa_\infty^2\}\right),$$

$$H_{\mathrm{IV}}(x,y,t) = 2xy^2 - \{x^2 + 2tx + 2\kappa_0\}y + \kappa_\infty x,$$

$$H_{\mathrm{III}}(x,y,t) = \frac{1}{t}[2x^2y^2 - \{2\eta_\infty tx^2 + (2\kappa_0 + 1)x - 2\eta_0 t\}y + \eta_\infty(\kappa_0 + \kappa_\infty)tx],$$

$$H_{\mathrm{II}}(x,y,t) = \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right)y - \left(\alpha + \frac{1}{2}\right)x,$$

$$H_{\mathrm{I}}(x,y,t) = \frac{1}{2}y^2 - 2x^3 - tx.$$

Here the relations between the constants in the equations P_J and the Hamiltonians H_J are given by

$$\alpha = \frac{1}{2}\kappa_\infty^2, \quad \beta = \frac{1}{2}\kappa_0^2, \quad \gamma = \frac{1}{2}\kappa_1^2, \quad \delta = \frac{1}{2}\kappa_t^2$$

for $J = \mathrm{VI}$,

$$\alpha = \kappa_\infty^2/2, \quad \beta = -\kappa_0^2/2, \quad \gamma = -\eta(1 + \kappa_t), \quad \delta = -\eta^2/2$$

for $J = \mathrm{V}$,

$$\alpha = -\kappa_0 + 2\kappa_\infty + 1, \quad \beta = -2\kappa_0^2$$

for $J = \mathrm{IV}$, and

$$\alpha = -4\eta_\infty\kappa_\infty, \quad \beta = 4\eta_0(\kappa_0 + 1), \quad \gamma = 4\eta_\infty^2, \quad \delta = -4\eta_0^2$$

for $J = \mathrm{III}$ ([**4**], [**8**]). The equivalence of P_J and (H_J) means that if we eliminate the variable y in (H_J) then we obtain P_J. We notice that each Hamiltonian H_J is a polynomial of x and y of which the coefficients are rational functions of t holomorphic in B_J where

$$B_{\mathrm{VI}} = \boldsymbol{C} - \{0, 1\}, \quad B_{\mathrm{V}} = B_{\mathrm{III}} = \boldsymbol{C} - \{0\}, \quad B_{\mathrm{IV}} = B_{\mathrm{II}} = B_{\mathrm{I}} = \boldsymbol{C}.$$

The most important property of (H_J) (or P_J) is the so called *Painlevé property* which is stated as: *if* $(x(t), y(t))$ *is a local solution of* (H_J) *determined by an arbitrary initial condition* $x(t_0) = x_0 \in \boldsymbol{C}$, $y(t_0) = y_0 \in \boldsymbol{C}$ *with* $t_0 \in B_J$ *then both* $x(t)$ *and* $y(t)$ *can be meromorphically continued along any curve in* B_J *with a starting point* t_0.

Let $\mathcal{Q}_J = (\boldsymbol{C}^2 \times B_J, \pi_J, B_J)$ be a trivial fiber space over B_J. Then the system (H_J) determines a complex 1-dimensional nonsingular foliation such that every leaf passing through a point in $\boldsymbol{C}^2 \times t (t \in B_J)$ is transversal to the fiber $\boldsymbol{C}^2 \times t$. But this foliation is not *uniform*, namely, for a point $(x_0, y_0, t_0) \in \boldsymbol{C}^2 \times B_J$ and a curve l in B_J with a starting point t_0, l may not be lifted to a leaf in $\boldsymbol{C}^2 \times B_J$ through the point (x_0, y_0, t_0) because $x(t)$ or $y(t)$ may have poles on l where $(x(t), y(t))$ is the solution of (H_J) with $(x(t_0), y(t_0)) = (x_0, y_0)$.

In the paper [**7**], K. Okamoto constructed a fiber space $\mathcal{P}_J = (E_J, \pi_J, B_J)$ such that

(i) $\mathcal{P}_J$ contains $\mathcal{Q}_J$ as a fiber subspace,

(ii) the system (H_J) of differential equations in $\boldsymbol{C}^2 \times B_J$ is holomorphically extended to a system in E_J and it determines a uniform foliation on $\mathcal{P}_J$,

(iii) every leaf in E_J intersects with the total space of $\mathcal{Q}_J$.

He named each fiber $E_J(t) = {\pi_J}^{-1}(t)$ *a space of initial conditions of* (H_J), since there exists a bijection from it to the set of all solutions of (H_J). We can imagine the space E_J by virtue of the following fact: for any simply connected domain U in B_J, ${\pi_J}^{-1}(U)$ is, as a set, a disjoint union of all the extended trajectories determined by (H_J). Each fiber $E_J(t)$ is constructed as follows. We first take a compactification $\bar{\Sigma}_\varepsilon$ (the so called Hirzeburch surface) of $\boldsymbol{C}^2$ where ε is a certain constant depending on the constants in H_J. Next we make finite number of quadric transformations to $\bar{\Sigma}_\varepsilon \times t$ and get $\overline{E_J(t)}$. Lastly we obtain $E_J(t)$ by removing some divisors which consist of vertical leaves and inaccessible singular points. Here, a *vertical* leaf is a leaf contained in a fiber, and an *inaccessible* singular point is a singular point of the foliation through which no solution of (H_J) passes.

The object of this paper is to introduce certain local coordinate systems of each space E_J $(J = \mathrm{V}, \mathrm{IV}, \mathrm{III}, \mathrm{II})$ so that (1°) every fiber $E_J(t)$ has a symplectic structure and (2°) in each chart of E_J, the original Hamiltonian system (H_J) is written as a Hamiltonian system with a Hamiltonian function which is a polynomial of the canonical coordinates.

In Section 1, we state our results in five theorems. We also cite a result in the preceding paper [**11**] which corresponds to those of this paper. In the following sections, we prove these theorems except the last theorem because it can be verified by simple calculations. In case $J = \mathrm{VI}$, we could easily obtain canonical coordinate systems by composing standard coordinate systems of quadric transformations ([**11**]). However, in the other cases, we have to make a certain device, namely, we have to insert a change of variables as (2.9), (3.6), (4.3) or (5.5) in order to make transition functions symplectic.

§1. Main results.

In order to state our results, we explain a definition and a property of a symplectic mapping. Let $\phi : x = x(X, Y, t),\ \ y = y(X, Y, t),\ \ t = t$ be a biholomorphic mapping from a domain D in $\boldsymbol{C}^3 \ni (X, Y, t)$ into $\boldsymbol{C}^3 \ni (x, y, t)$. We say that ϕ is *symplectic*, if for every $t = t_0$, $\phi_{t_0} = \phi|_{t=t_0}$ is a symplectic mapping from $D_{t_0} = D|_{t=t_0}$ to $\phi(D_{t_0})$, namely, if

$$dy \wedge dx = dY \wedge dX$$

for every fixed $t = t_0$. Let ϕ be a symplectic mapping as above. Then any Hamiltonian system $dx/dt = \partial H/\partial y,\ dy/dt = -\partial H/\partial x$ defined in $\phi(D)$ is transformed to $dX/dt = \partial K/\partial Y,\ dY/dt = -\partial K/\partial X$ in D where $K = K(X, Y, t)$ is a function in D satisfying

$$dy \wedge dx - dH \wedge dt = dY \wedge dX - dK \wedge dt. \tag{1.1}$$

We note that the Hamiltonian function $K = K(X, Y, t)$ is uniquely determined modulo functions independent of X and Y.

Then the first assertion of this paper is stated as

THEOREM 1. *The space E_{V} for the fifth Painlevé system (H_{V}) is obtained by glueing five copies of $\boldsymbol{C}^2 \times B_{\mathrm{V}}$:*

$$V(00) \times B_{\mathrm{V}} = \boldsymbol{C}^2 \times B_{\mathrm{V}} \ni (x, y, t) = (x(00), y(00), t),$$

$$V(0\infty) \times B_{\mathrm{V}} = \boldsymbol{C}^2 \times B_{\mathrm{V}} \ni (x(0\infty), y(0\infty), t),$$

$$V(1\infty) \times B_{\mathrm{V}} = \boldsymbol{C}^2 \times B_{\mathrm{V}} \ni (x(1\infty), y(1\infty), t),$$

$$V(\infty 0+) \times B_{\mathrm{V}} = \boldsymbol{C}^2 \times B_{\mathrm{V}} \ni (x(\infty 0+), y(\infty 0+), t),$$

$$V(\infty 0-) \times B_{\mathrm{V}} = \boldsymbol{C}^2 \times B_{\mathrm{V}} \ni (x(\infty 0-), y(\infty 0-), t)$$

via the following symplectic transformations

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{1.2}$$

$$x(00) = 1 + x(1\infty), \quad y(00) = -\frac{\eta t}{x(1\infty)^2} + \frac{\kappa_t + 1}{x(1\infty)} + y(1\infty), \tag{1.3}$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\varepsilon(+) - x(\infty 0+)y(\infty 0+)), \tag{1.4}$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-) \tag{1.5}$$

where

$$B_{\mathrm{V}} = \boldsymbol{C} - \{0\}, \tag{1.6}$$

$$\varepsilon(+) = (\kappa_0 + \kappa_t + \kappa_\infty)/2, \tag{1.7}$$

and $V(00) \times B_{\mathrm{V}}$ is the original space in which the Hamiltonian function $H_{\mathrm{V}}(x, y, t)$ is defined.

THEOREM 2. *The space E_{IV} for the fourth Painlevé system (H_{IV}) is obtained by glueing four copies of $\boldsymbol{C}^2 \times B_{\mathrm{IV}}$:*

$$V(00) \times B_{\mathrm{IV}} = \boldsymbol{C}^2 \times B_{\mathrm{IV}} \ni (x, y, t) = (x(00), y(00), t),$$

$$V(0\infty) \times B_{\mathrm{IV}} = \boldsymbol{C}^2 \times B_{\mathrm{IV}} \ni (x(0\infty), y(0\infty), t),$$

$$V(\infty 0) \times B_{\mathrm{IV}} = \boldsymbol{C}^2 \times B_{\mathrm{IV}} \ni (x(\infty 0), y(\infty 0), t),$$

$$V(\infty\infty) \times B_{\mathrm{IV}} = \boldsymbol{C}^2 \times B_{\mathrm{IV}} \ni (x(\infty\infty), y(\infty\infty), t)$$

via the following symplectic transformations

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{1.8}$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\kappa_\infty - x(\infty 0)y(\infty 0)), \tag{1.9}$$

$$\begin{aligned} &x(\infty 0) = x(\infty\infty), \\ &y(\infty 0) = -\frac{1/2}{x(\infty\infty)^3} - \frac{t}{x(\infty\infty)^2} + \frac{2\kappa_\infty - \kappa_0 + 1}{x(\infty\infty)} + y(\infty\infty) \end{aligned} \tag{1.10}$$

where

$$B_{\mathrm{IV}} = \boldsymbol{C}, \tag{1.11}$$

and $V(00) \times B_{\mathrm{IV}}$ is the original space in which the Hamiltonian function $H_{\mathrm{IV}}(x, y, t)$ is defined.

THEOREM 3. *The space E_{III} for the third Painlevé system (H_{III}) is obtained by glueing four copies of $\boldsymbol{C}^2 \times B_{\mathrm{III}}$:*

$$V(00) \times B_{\mathrm{III}} = \boldsymbol{C}^2 \times B_{\mathrm{III}} \ni (x, y, t) = (x(00), y(00), t),$$

$$V(0\infty) \times B_{\mathrm{III}} = \boldsymbol{C}^2 \times B_{\mathrm{III}} \ni (x(0\infty), y(0\infty), t),$$

$$V(\infty 0) \times B_{\mathrm{III}} = \boldsymbol{C}^2 \times B_{\mathrm{III}} \ni (x(\infty 0), y(\infty 0), t),$$

$$V(\infty\eta_\infty t) \times B_{\mathrm{III}} = \boldsymbol{C}^2 \times B_{\mathrm{III}} \ni (x(\infty\eta_\infty t), y(\infty\eta_\infty t), t),$$

via the following symplectic transformations

$$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0 + 1}{x(0\infty)} + y(0\infty), \tag{1.12}$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\varepsilon - x(\infty 0)y(\infty 0)), \tag{1.13}$$

$$x(\infty 0) = x(\infty\eta_\infty t), \quad y(\infty 0) = -\frac{\eta_\infty t}{x(\infty\eta_\infty t)^2} + \frac{\kappa_\infty}{x(\infty\eta_\infty t)} + y(\infty\eta_\infty t) \tag{1.14}$$

where

$$B_{\mathrm{III}} = \boldsymbol{C} - \{0\}, \tag{1.15}$$

$$\varepsilon = (\kappa_0 + \kappa_\infty)/2, \tag{1.16}$$

and $V(00) \times B_{\mathrm{III}}$ *is the original space in which the Hamiltonian function* $H_{\mathrm{III}}(x, y, t)$ *is defined.*

THEOREM 4. *The space* E_{II} *for the second Painlevé system* (H_{II}) *is obtained by glueing three copies of* $\boldsymbol{C}^2 \times B_{\mathrm{II}}$:

$$V(00) \times B_{\mathrm{II}} = \boldsymbol{C}^2 \times B_{\mathrm{II}} \ni (x, y, t) = (x(00), y(00), t),$$

$$V(\infty 0) \times B_{\mathrm{II}} = \boldsymbol{C}^2 \times B_{\mathrm{II}} \ni (x(\infty 0), y(\infty 0), t),$$

$$V(\infty\infty) \times B_{\mathrm{II}} = \boldsymbol{C}^2 \times B_{\mathrm{II}} \ni (x(\infty\infty), y(\infty\infty), t)$$

via the following symplectic transformations

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\varepsilon - x(\infty 0) y(\infty 0)), \tag{1.17}$$

$$\begin{aligned} &x(\infty 0) = x(\infty\infty), \\ &y(\infty 0) = -\frac{2}{x(\infty\infty)^4} - \frac{t}{x(\infty\infty)^2} - \frac{2\alpha}{x(\infty\infty)} + y(\infty\infty) \end{aligned} \tag{1.18}$$

where

$$B_{\mathrm{II}} = \boldsymbol{C}, \tag{1.19}$$

$$\varepsilon = -\alpha - \frac{1}{2}, \tag{1.20}$$

and $V(00) \times B_{\mathrm{II}}$ *is the original space in which the Hamiltonian function* $H_{\mathrm{II}}(x, y, t)$ *is defined.*

The second assertion of this paper is

THEOREM 5. *For every* $J = \mathrm{V}, \mathrm{IV}, \mathrm{III}, \mathrm{II}$, *the Hamiltonian function* $H_J(*) = H_J(*; x(*), y(*), t)$ *in every chart* $V(*) \times B_J$ *is a polynomial of* $x(*)$ *and* $y(*)$ *of which the coefficients are rational functions of* t *holomorphic in* B_J.

For the reader's convenience, we cite here a result in [**11**] corresponding the above theorems: *The space* E_{VI} *is obtained by glueing six copies of* $\boldsymbol{C}^2 \times B_{\mathrm{VI}}$ *via the following symplectic transformations*

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty) y(0\infty)), \quad y(00) = 1/y(0\infty),$$

$$x(00) = 1 + y(1\infty)(\kappa_1 - x(1\infty) y(1\infty)), \quad y(00) = 1/y(1\infty),$$

$$x(00) = t + y(t\infty)(\kappa_t - x(t\infty) y(t\infty)), \quad y(00) = 1/y(t\infty),$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\varepsilon(+) - x(\infty 0+) y(\infty 0+)),$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-) y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-),$$

where $\varepsilon(+) = (\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty)/2$. *Moreover, the Hamiltonian function in each chart is a function of a polynomial of the coordinates of which the coefficients are rational functions of* t *holomorphic in* B_{VI}.

§2. Proof of Theorem 1.

In the following sections, we prove Theorems from 1 to 4 by reviewing the construction of each fiber $E_J(t)$ $(t \in B_J, J = \mathrm{V}, \dots, \mathrm{II})$ ([**7**]) and by suitably choosing local canonical coordinate systems.

For every J, we begin our study with a minimal compactification $\bar{\Sigma}_\varepsilon$ of $\boldsymbol{C}^2$ obtained by glueing four $U_i = \boldsymbol{C}^2 \ni (x_i, y_i), i = 0, 1, 2, 3$, via the following identifications:

$$x_0 = x_1, \quad y_0 = 1/y_1, \tag{2.1}$$

$$x_0 = 1/x_2, \quad y_0 = x_2(\varepsilon - x_2 y_2), \tag{2.2}$$

$$x_2 = x_3, \quad y_2 = 1/y_3, \tag{2.3}$$

where ε is a complex constant. This manifold is known as Hirzeburch surface, which is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ if $\varepsilon \neq 0$, and to a compactification of the cotangent bundle over $\boldsymbol{P}^1$ if $\varepsilon = 0$. We consider each U_i or $U_i \times B_J$ as a chart of $\bar{\Sigma}_\varepsilon$ or $\bar{\Sigma}_\varepsilon \times B_J$ respectively. Note that $y_1 = 0$ in U_1 corresponds to $y_3 = 0$ in U_3 because

$$x_1 = 1/x_3, \quad y_1 = y_3/[x_3(\varepsilon y_3 - x_3)].$$

In the present case where $J = \mathrm{V}$, we take the constant ε as $\varepsilon = \varepsilon(+)$ given by (1.7):

$$\varepsilon = (\kappa_0 + \kappa_t + \kappa_\infty)/2.$$

2.1. We extend the system (H_{V}) defined in $U_0 \times B_{\mathrm{V}} \ni (x_0, y_0, t) = (x, y, t)$ to a Pfaffian system defined in the whole space $\bar{\Sigma}_\varepsilon \times B_{\mathrm{V}}$ and we observe the foliation of $\bar{\Sigma}_\varepsilon \times B_{\mathrm{V}}$ defined by the Pfaffian system. We see that, in $U_i \times B_{\mathrm{V}}, i = 0, 2$, the foliation has no singular points and every leaf is transversal with fibers. However, in $U_i \times B_{\mathrm{V}}$, $i = 1, 3$, the foliation has both singular points and vertical leaves. Recall that a vertical leaf is, by definition, a leaf contained in a fiber. Set

$$D^{(0)}(t) = (U_1(y_1 = 0) \times t) \cup (U_3(y_3 = 0) \times t) \cong \boldsymbol{P}^1,$$

$$a_\nu^{(0)}(t) = \{(x_1, y_1, t) = (\nu, 0, t)\}, \quad \nu = 0, 1,$$

$$a_\nu^{(0)}(t) = \{(x_3, y_3, t) = (0, 0, t)\}, \quad \nu = \infty,$$

where $U_i(y_i = 0)$ denotes the set $\{(x_i, y_i) \in U_i \mid y_i = 0\}$. Then $D^{(0)}(t) - \bigcup_\nu \{a_\nu^{(0)}(t)\}$ is a vertical leaf and the three points $a_\nu^{(0)}(t), \nu = 0, 1, \infty$ are the singular points of the foliation, which is verified, for example, by

$$\frac{dy_1}{dx_1} = \frac{[3x_1^2 - 2x_1 + 1 + O(y_1)]y_1}{2x_1(x_1 - 1)^2 + O(y_1)}, \quad \frac{dt}{dx_1} = \frac{ty_1}{2x_1(x_1 - 1)^2 + O(y_1)}$$

where $O(y_1)$ denotes a polynomial of x_1, y_1, t with a factor y_1. In the following, $O(r)$ always denotes a polynomial of some three variables which has a factor r and the superscript (k) of a letter indicates that it is concerned with a k-th quadric transformation.

2.2. Quadric transformations with centers $a_v^{(0)}(t)$ and $a_v^{(1)}(t)$ for arbitrarily fixed $t \in B_{\mathrm{V}}$ and $v = 0, \infty$. In order to completely separate the leaves passing through the point $a_v^{(0)}(t)$, we make quadric transformations two times successively. We denote the quadric transformation with center a by Q_a.

2.2.1. The first quadric transformation with center $a_v^{(0)}(t)$. Let $(z_v^{(1)}, w_v^{(1)}) \in \boldsymbol{C}^2$ and $(Z_v^{(1)}, W_v^{(1)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_0^{(1)}(t) = Q_{a_0^{(0)}(t)}(U_1 \times t)$ for $v = 0$ or of $V_\infty^{(1)}(t) = Q_{a_\infty^{(0)}(t)}(U_3 \times t)$ for $v = \infty$ defined by

$$
(2.4)\qquad \begin{aligned} x_1 &= z_0^{(1)}, \quad y_1 = z_0^{(1)} w_0^{(1)}, \\ x_1 &= Z_0^{(1)} W_0^{(1)}, \quad y_1 = W_0^{(1)} \end{aligned}
$$

for $v = 0$, or

$$
(2.5)\qquad \begin{aligned} x_3 &= z_\infty^{(1)}, \quad y_3 = z_\infty^{(1)} w_\infty^{(1)}, \\ x_3 &= Z_\infty^{(1)} W_\infty^{(1)}, \quad y_3 = W_\infty^{(1)} \end{aligned}
$$

for $v = \infty$, then the exceptional curve is given by

$$
\begin{aligned} D_v^{(1)}(t) &:= Q_{a_v^{(0)}(t)}(a_v^{(0)}(t)) \\ &= \{(z_v^{(1)}, w_v^{(1)}, t) \mid z_v^{(1)} = 0\} \cup \{(Z_v^{(1)}, W_v^{(1)}, t) \mid W_v^{(1)} = 0\} \end{aligned}
$$

and our system is written as

$$
\frac{dW}{dZ} = \frac{(1 + O(W))W}{Z - \kappa_v + O(W)}, \quad \frac{dt}{dZ} = \frac{tW}{Z - \kappa_v + O(W)}
$$

with $(Z, W) = (Z_v^{(1)}, W_v^{(1)})$ in a neighborhood of $D_v^{(1)}(t) = \{W_v^{(1)} = 0\}$, or

$$
w\frac{dz}{dt} = \frac{1}{t}[2 + O(z) + O(w)], \quad z\frac{dw}{dt} = \frac{1}{t}[-1 + O(z) + O(w)]
$$

with $(z, w) = (z_v^{(1)}, w_v^{(1)})$ in a neighborhood of $(z_v^{(1)}, w_v^{(1)}, t) = (0, 0, t)$. Therefore, we see that

$$
\begin{aligned} a_v^{(1)}(t) &= \{(Z_v^{(1)}, W_v^{(1)}, t) = (\kappa_v, 0, t)\} \in D_v^{(1)}(t), \\ b_v^{(1)}(t) &= \{(z_v^{(1)}, w_v^{(1)}, t) = (0, 0, t)\} \in D_v^{(1)}(t) \cap D^{(0)}(t) \end{aligned}
$$

are singular points of the foliation and $D_v^{(1)}(t) - \{a_v^{(1)}(t), b_v^{(1)}(t)\}$ is a vertical leaf. Here we also denote by the $D^{(0)}(t)$ the proper image of $D^{(0)}(t)$ by the above quadric transformations. The similar convention is made throughout the paper. We see moreover that the point $b_v^{(1)}(t)$ is a singular point through which no solution of (H_{V}) passes by virtue of Painlevé property and the above form of the system near $(z_v^{(1)}, w_v^{(1)}, t) = (0, 0, t)$. We call such a singular point an *inaccessible* singular point.

2.2.2. The second quadric transformation with center $a_v^{(1)}(t)$. Let $(z_v^{(2)}, w_v^{(2)}) \in \boldsymbol{C}^2$ and $(Z_v^{(2)}, W_v^{(2)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_0^{(2)}(t) = Q_{a_0^{(1)}(t)}(V_0^{(1)}(t))$ for $v = 0$ or of $V_\infty^{(2)}(t) = Q_{a_\infty^{(1)}(t)}(V_\infty^{(1)}(t))$ for $v = \infty$ defined by

$$
(2.6)\qquad
\begin{aligned}
Z_v^{(1)} &= \kappa_v + z_v^{(2)}, \quad W_v^{(1)} = z_v^{(2)} w_v^{(2)}, \\
Z_v^{(1)} &= \kappa_v + Z_v^{(2)} W_v^{(2)}, \quad W_v^{(1)} = W_v^{(2)},
\end{aligned}
$$

then

$$D_v^{(2)}(t) := Q_{a_v^{(1)}(t)}(a_v^{(1)}(t)) = \{z_v^{(2)} = 0\} \cup \{W_v^{(2)} = 0\}.$$

We can verify that the Pfaffian system is written as

$$
\begin{aligned}
t dZ_v^{(2)} - P_v(Z_v^{(2)}, W_v^{(2)}, t)\, dt = 0, \\
t dW_v^{(2)} - Q_v(Z_v^{(2)}, W_v^{(2)}, t)\, dt = 0
\end{aligned}
$$

in the coordinates $Z_v^{(2)}$, $W_v^{(2)}$ and t where P_v, Q_v are certain polynomials of $Z_v^{(2)}, W_v^{(2)}$ and t. This means that the foliation has no singular points in $(Z_v^{(2)}, W_v^{(2)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{V}}$ and every leaf in the space is transversal with fibers. On the other hand, the point $(z_v^{(2)}, w_v^{(2)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes the point is the vertical leaf $D_v^{(1)}(t) - \{b_v^{(1)}(t)\}$, because our system is written as

$$\frac{dw}{dz} = \frac{w}{1 + O(w)}, \quad \frac{dt}{dz} = \frac{tw}{1 + O(w)}$$

with $(z, w) = (z_v^{(2)}, w_v^{(2)})$ in a neighborhood of $(z_v^{(2)}, w_v^{(2)}, t) = (0, 0, t)$.

2.3. Quadric transformations with centers $a_1^{(0)}(t), \ldots, a_1^{(3)}(t)$ for arbitrarily fixed $t \in B_{\mathrm{V}}$. In order to separate the leaves passing through the point $a_1^{(0)}(t)$, we make quadric transformations four times successively.

2.3.1. The first quadric transformation with center $a_1^{(0)}(t)$. Let $(z_1^{(1)}, w_1^{(1)}) \in \boldsymbol{C}^2$ and $(Z_1^{(1)}, W_1^{(1)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_1^{(1)}(t) = Q_{a_1^{(0)}(t)}(U_1 \times t)$ defined by

$$
(2.7)\qquad
\begin{aligned}
x_1 &= 1 + z_1^{(1)}, \quad y_1 = z_1^{(1)} w_1^{(1)}, \\
x_1 &= 1 + Z_1^{(1)} W_1^{(1)}, \quad y_1 = W_1^{(1)},
\end{aligned}
$$

then

$$D_1^{(1)}(t) := Q_{a_1^{(0)}(t)}(a_1^{(0)}(t)) = \{z_1^{(1)} = 0\} \cup \{W_1^{(1)} = 0\},$$

and our system is expressed as

$$\frac{dz}{dw} = \frac{(O(z) + O(w))z}{(-\eta t w + O(z))w}, \quad \frac{dt}{dw} = \frac{tz}{-\eta t w + O(z)}$$

in a neighborhood of $D_1^{(1)}(t) = \{z_1^{(1)} = 0\}$ where $(z, w) = (z_1^{(1)}, w_1^{(1)})$, or

$$\frac{dW}{dZ} = \frac{O(W)W}{\eta t + O(W)}, \quad \frac{dt}{dZ} = \frac{tW}{\eta t + O(W)},$$

in a neighborhood of $(Z_1^{(1)}, W_1^{(1)}, t) = (0, 0, t)$ where $(Z, W) = (Z_1^{(1)}, W_1^{(1)})$. Hence we see that the point

$$a_1^{(1)}(t) = \{(z_1^{(1)}, w_1^{(1)}, t) = (0, 0, t)\} \in D_1^{(1)}(t) \cap D_1^{(0)}(t)$$

is a singular point and $D_1^{(1)}(t) - \{a_1^{(1)}(t)\}$ is a vertical leaf.

2.3.2. The second quadric transformation with center $a_1^{(1)}(t)$. Let $(z_1^{(2)}, w_1^{(2)}) \in \boldsymbol{C}^2$ and $(Z_1^{(2)}, W_1^{(2)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_1^{(2)}(t) = Q_{a_1^{(1)}(t)}(V_1^{(1)}(t))$ defined by

$$\begin{aligned} z_1^{(1)} &= z_1^{(2)}, \quad w_1^{(1)} = z_1^{(2)} w_1^{(2)}, \\ z_1^{(1)} &= Z_1^{(2)} W_1^{(2)}, \quad w_1^{(1)} = W_1^{(2)}, \end{aligned} \tag{2.8}$$

then

$$D_1^{(2)}(t) := Q_{a_1^{(1)}(t)}(a_1^{(1)}(t)) = \{z_1^{(2)} = 0\} \cup \{W_1^{(2)} = 0\},$$

and our system is written as

$$\frac{dz}{dw} = \frac{(O(1) + O(z) + O(w))z}{(-2 - 2\eta tw + O(z))w}, \quad \frac{dt}{dw} = \frac{tz}{-2 - 2\eta tw + O(z)}$$

with $(z, w) = (z_1^{(2)}, w_1^{(2)})$ in a neighborhood of $D_1^{(2)}(t) = \{z_1^{(2)} = 0\}$, or

$$w\frac{dz}{dt} = \frac{1}{t}[2 + O(z) + O(w)], \quad z\frac{dw}{dt} = \frac{1}{t}[-2 + O(z) + O(w)]$$

in a neighborhood of $(z_1^{(2)}, w_1^{(2)}, t) = (0, 0, t)$, or

$$W\frac{dZ}{dt} = \frac{1}{t}[2\eta t + O(Z) + O(W)], \quad Z\frac{dW}{dt} = \frac{1}{t}[-2\eta t + O(Z) + O(W)]$$

with $(Z, W) = (Z_1^{(2)}, W_1^{(2)})$ in a neighborhood of $(Z_1^{(2)}, W_1^{(2)}, t) = (0, 0, t)$. Therefore we see that the points

$$a_1^{(2)}(t) = \{(z_1^{(2)}, w_1^{(2)}, t) = (0, -1/(\eta t), t)\} \in D_1^{(2)}(t),$$

$$b_{10}^{(2)}(t) = \{(z_1^{(2)}, w_1^{(2)}, t) = (0, 0, t)\} \in D_1^{(2)}(t) \cap D^{(0)}(t),$$

$$b_{1\infty}^{(2)}(t) = \{(Z_1^{(2)}, W_1^{(2)}, t) = (0, 0, t)\} \in D_1^{(2)}(t) \cap D_1^{(1)}(t)$$

are singular points, $b_{10}^{(2)}(t)$ and $b_{1\infty}^{(2)}(t)$ are inaccessible singular points, and $D_1^{(2)}(t) - \{a_1^{(2)}(t), b_{10}^{(2)}(t), b_{1\infty}^{(2)}(t)\}$ is a vertical leaf.

2.3.3. The third quadric transformation with center $a_1^{(2)}(t)$. Here we insert a change of variables

$$z_1^{(2)} = z_1^{(2)}, \quad w_1^{(2)} = 1/v_1^{(2)}, \tag{2.9}$$

namely, a change of local coordinates near the point $a_1^{(2)}(t)$. The change of variables is necessary for making transition functions in a description of E_{V} symplectic.

Let $(z_1^{(3)}, w_1^{(3)}) \in \boldsymbol{C}^2$ and $(Z_1^{(3)}, W_1^{(3)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_1^{(3)}(t) = Q_{a_1^{(2)}(t)}(V_1^{(2)}(t))$ defined by

$$\begin{aligned} z_1^{(2)} &= z_1^{(3)}, \quad v_1^{(2)} = -\eta t + z_1^{(3)} w_1^{(3)}, \\ z_1^{(2)} &= Z_1^{(3)} W_1^{(3)}, \quad v_1^{(2)} = -\eta t + W_1^{(3)}, \end{aligned} \tag{2.10}$$

then

$$D_1^{(3)}(t) := Q_{a_1^{(2)}(t)}(a_1^{(2)}(t)) = \{z_1^{(3)} = 0\} \cup \{W_1^{(3)} = 0\}.$$

We see that our system is expressed as

$$\frac{dz}{dw} = \frac{(-\eta t + O(z))z}{2\eta t((\kappa_t + 1) - w) + O(z)}, \quad \frac{dt}{dw} = \frac{tz}{2\eta t((\kappa_t + 1) - w) + O(z)}$$

in a neighborhood of $D_1^{(3)}(t) = \{z_1^{(3)} = 0\}$ where $(z, w) = (z_1^{(3)}, w_1^{(3)})$, or

$$W\frac{dZ}{dt} = \frac{1}{t}[\eta t + O(Z) + O(W)], \quad Z\frac{dW}{dt} = \frac{1}{t}[-2\eta t + O(Z) + O(W)]$$

in a neighborhood of $(Z_1^{(3)}, W_1^{(3)}, t) = (0, 0, t)$ where$(Z, W) = (Z_1^{(3)}, W_1^{(3)})$. Therefore,

$$a_1^{(3)}(t) = \{(z_1^{(3)}, w_1^{(3)}, t) = (0, \kappa_t + 1, t)\} \in D_1^{(3)}(t),$$

$$b_1^{(3)}(t) = \{(Z_1^{(3)}, W_1^{(3)}, t) = (0, 0, t)\} \in D_1^{(3)}(t) \cap D_1^{(2)}(t)$$

are singular points, $b_1^{(3)}(t)$ is an inaccessible singular point, and $D_1^{(3)}(t) - \{a_1^{(3)}(t), b_1^{(3)}(t)\}$ is a vertical leaf.

2.3.4. The fourth quadric transformation with center $a_1^{(3)}(t)$. Let $(z_1^{(4)}, w_1^{(4)}) \in \boldsymbol{C}^2$ and $(Z_1^{(4)}, W_1^{(4)}) \in \boldsymbol{C}^2$ be coordinate systems of $V_1^{(4)}(t) = Q_{a_1^{(3)}(t)}(V_1^{(3)}(t))$ defined by

$$\begin{aligned} z_1^{(3)} &= z_1^{(4)}, \quad w_1^{(3)} = (\kappa_t + 1) + z_1^{(4)} w_1^{(4)}, \\ z_1^{(3)} &= Z_1^{(4)} W_1^{(4)}, \quad w_1^{(3)} = (\kappa_t + 1) + W_1^{(4)}. \end{aligned} \tag{2.11}$$

We can verify that the Pfaffian system is written as

$$t dz_1^{(4)} - P_1(z_1^{(4)}, w_1^{(4)}, t)\, dt = 0,$$

$$t dw_1^{(4)} - Q_1(z_1^{(4)}, w_1^{(4)}, t)\, dt = 0$$

in the coordinates $z_1^{(4)}$, $w_1^{(4)}$, and t where P_1, Q_1 are certain polynomials of $z_1^{(4)}, w_1^{(4)}$ and t. This means that the foliation has no singular points in $(z_1^{(4)}, w_1^{(4)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{V}}$ and every leaf in the space is transversal with fibers. On the other hand,we can verify that

$$\frac{dZ}{dW} = \frac{ZO(Z)}{-\eta t + O(Z)}, \quad \frac{dt}{dW} = \frac{tZ}{-\eta t + O(Z)}$$

in a neighborhood of $(Z_1^{(4)}, W_1^{(4)}, t) = (0, 0, t)$ where $(Z, W) = (Z_1^{(4)}, W_1^{(4)})$, which shows that the point $(Z_1^{(4)}, W_1^{(4)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes the point is the vertical leaf $D_1^{(3)}(t) - \{b_1^{(3)}(t)\}$.

2.4. The space E_{V}. Denote by Φ_t the composition of all the above eight quadric transformations. Then the space constructed by K. Okamoto ([**7**]) is the space defined by

$$E_{\mathrm{V}} = \bigcup_{t \in B_{\mathrm{V}}} E_{\mathrm{V}}(t) \times t,$$

where

$$E_{\mathrm{V}}(t) = \overline{E_{\mathrm{V}}(t)} - D^{(0)}(t) \cup \bigcup_{\nu=0,\infty} D_\nu^{(1)}(t) \cup \bigcup_{k=1,2,3} D_1^{(k)}(t), \quad \overline{E_{\mathrm{V}}(t)} = \Phi_t(\bar{\Sigma}_\varepsilon \times t).$$

We can verify that the extended system of (H_{V}) defines a uniform foliation on E_{V}.

By following the above procedure, we see that E_{V} is a 3-dimensional complex manifold obtained by glueing

$$\{(x_0, y_0, t) \in \boldsymbol{C}^2 \times B_{\mathrm{V}}\}, \quad \{(x_2, y_2, t) \in \boldsymbol{C}^2 \times B_{\mathrm{V}}\},$$

$$\{(Z_\nu^{(2)}, W_\nu^{(2)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{V}}\}, \quad \nu = 0, \infty,$$

$$\{(z_1^{(4)}, w_1^{(4)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{V}}\}$$

via the coordinate transformations (2.1)–(2.11). It is easy to see that

$$dy_0 \wedge dx_0 = dy_2 \wedge dx_2,$$

$$dy_0 \wedge dx_0 = -dW_0^{(2)} \wedge dZ_0^{(2)}, \quad dy_2 \wedge dx_2 = -dW_\infty^{(2)} \wedge dZ_\infty^{(2)},$$

$$dy_0 \wedge dx_0 = dw_1^{(4)} \wedge dz_1^{(4)}.$$

Therefore, by choosing new coordinate systems as

$$(x(00), y(00)) = (x_0, y_0),$$

$$(x(0\infty), y(0\infty)) = (-Z_0^{(2)}, W_0^{(2)}), \quad (x(1\infty), y(1\infty)) = (z_1^{(4)}, w_1^{(4)}),$$

$$(x(\infty 0+), y(\infty 0+)) = (x_2, y_2), \quad (x(\infty 0-), y(\infty 0-)) = (-Z_\infty^{(2)}, W_\infty^{(2)}),$$

we obtain a description of E_{V} given in THEOREM 1. Thus we have proved THEOREM 1.

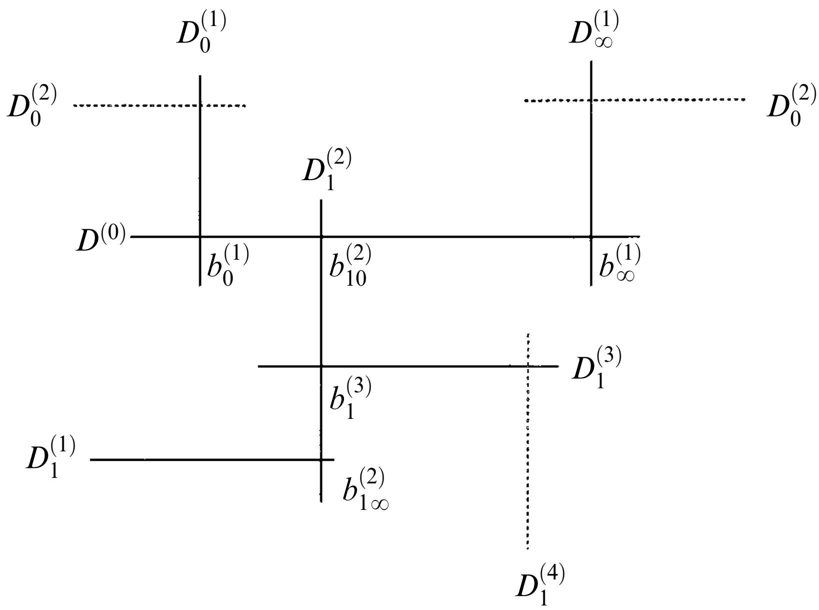

Figure 1. $J = \mathrm{V}$

§3. Proof of Theorem 2.

In the following sections, we only give the exact forms of our transformations, because the verification of the transformations is the same as that in the preceding section, §2.

In the case of $J = \mathrm{IV}$, we take ε for $\bar{\Sigma}_\varepsilon$ as $\varepsilon = \kappa_\infty$.

3.1. Extend the system (H_{IV}) defined in $U_0 \times B_{\mathrm{IV}} \ni (x_0, y_0, t) = (x, y, t)$ to a Pfaffian system defined in the whole space $\bar{\Sigma}_\varepsilon \times B_{\mathrm{IV}}$. Then, in $U_i \times B_{\mathrm{IV}}, i = 0, 2$, the foliation defined by the Pfaffian system has no singular points and every leaf is transversal with fibers, however, in $U_i \times B_{\mathrm{IV}}, i = 1, 3$, the foliation has both singular points and vertical leaves. We see that, for any fixed $t \in B_{\mathrm{VI}}$, $D^{(0)}(t) - \bigcup_v \{a_v^{(0)}(t)\}$ is a vertical leaf and the two points $a_v^{(0)}(t), v = 0, \infty$ are singular points of the foliation, where

$$D^{(0)}(t) = (U_1(y_1 = 0) \times t) \cup (U_3(y_3 = 0) \times t) \cong \boldsymbol{P}^1,$$

$$a_0^{(0)}(t) = \{(x_1, y_1, t) = (0, 0, t)\}, \quad a_\infty^{(0)}(t) = \{(x_3, y_3, t) = (0, 0, t)\}.$$

3.2. Quadric transformations with centers $a_0^{(0)}(t)$ and $a_0^{(1)}(t)$ for any fixed $t \in B_{\mathrm{IV}}$. In order to separate the leaves passing through the point $a_0^{(0)}(t)$, we make quadric transformations two times successively.

3.2.1. The first quadric transformation with center $a_0^{(0)}(t)$. Let

$$x_1 = z_0^{(1)}, \quad y_1 = z_0^{(1)} w_0^{(1)},$$

$$(3.1) \qquad x_1 = Z_0^{(1)} W_0^{(1)}, \quad y_1 = W_0^{(1)},$$

then

$$D_0^{(1)}(t) := Q_{a_0^{(0)}(t)}(a_0^{(0)}(t)) = \{z_0^{(1)} = 0\} \cup \{W_0^{(1)} = 0\},$$

the points

$$a_0^{(1)}(t) = \{(Z_0^{(1)}, W_0^{(1)}, t) = (\kappa_0, 0, t)\} \in D_0^{(1)}(t),$$

$$b_0^{(1)}(t) = \{(z_0^{(1)}, w_0^{(1)}, t) = (0, 0, t)\} \in D^{(0)}(t) \cap D_0^{(1)}(t)$$

are singular points of the foliation, $D_0^{(1)}(t) - \{a_0^{(1)}(t), b_0^{(1)}(t)\}$ is a vertical leaf, and $b_0^{(1)}(t)$ is an inaccessible singular point.

3.2.2. The second quadric transformation with center $a_0^{(1)}(t)$. Let

$$(3.2) \qquad \begin{aligned} Z_0^{(1)} &= \kappa_0 + z_0^{(2)}, \quad W_0^{(1)} = z_0^{(2)} w_0^{(2)}, \\ Z_0^{(1)} &= \kappa_0 + Z_0^{(2)} W_0^{(2)}, \quad W_0^{(1)} = W_0^{(2)}, \end{aligned}$$

then

$$D_0^{(2)}(t) := Q_{a_0^{(1)}(t)}(a_0^{(1)}(t)) = \{z_0^{(2)} = 0\} \cup \{W_0^{(2)} = 0\}.$$

We see that, in the $(Z_0^{(2)}, W_0^{(2)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{IV}}$, the Pfaffian system has no singular points, every leaf is transversal with the fibers, and the point $(z_0^{(2)}, w_0^{(2)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes it is the vertical leaf $D_0^{(1)}(t) - \{b_0^{(1)}(t)\}$.

3.3. Quadric transformations with centers $a_\infty^{(0)}(t), \dots, a_\infty^{(5)}(t)$ for any fixed $t \in B_{\mathrm{IV}}$. To separate the leaves passing through the point $a_\infty^{(0)}(t)$, we make quadric transformations six times successively.

3.3.1. The first quadric transformation with center $a_\infty^{(0)}(t)$. Let

$$(3.3) \qquad \begin{aligned} x_3 &= z_\infty^{(1)}, \quad y_3 = z_\infty^{(1)} w_\infty^{(1)}, \\ x_3 &= Z_\infty^{(1)} W_\infty^{(1)}, \quad y_3 = W_\infty^{(1)}, \end{aligned}$$

then

$$D_\infty^{(1)}(t) := Q_{a_\infty^{(0)}(t)}(a_\infty^{(0)}(t)) = \{z_\infty^{(1)} = 0\} \cup \{W_\infty^{(1)} = 0\},$$

the point

$$a_\infty^{(1)}(t) = \{(z_\infty^{(1)}, w_\infty^{(1)}, t) = (0, 0, t)\} \in D_\infty^{(1)}(t)$$

is a singular point of the foliation, and $D_\infty^{(1)}(t) - \{a_\infty^{(1)}(t)\}$ is a vertical leaf.

3.3.2. The second quadric transformation with center $a_\infty^{(1)}(t)$. Let

$$(3.4) \qquad \begin{aligned} z_\infty^{(1)} &= z_\infty^{(2)}, \quad w_\infty^{(1)} = z_\infty^{(2)} w_\infty^{(2)}, \\ z_\infty^{(1)} &= Z_\infty^{(2)} W_\infty^{(2)}, \quad w_\infty^{(1)} = W_\infty^{(2)}, \end{aligned}$$

then

$$D_\infty^{(2)}(t) := Q_{a_\infty^{(1)}(t)}(a_\infty^{(1)}(t)) = \{z_\infty^{(2)} = 0\} \cup \{W_\infty^{(2)} = 0\},$$

the points

$$a_\infty^{(2)}(t) = \{(z_\infty^{(2)}, w_\infty^{(2)}, t) = (0, 0, t)\} \in D_\infty^{(2)}(t),$$

$$b_\infty^{(2)}(t) = \{(Z_\infty^{(2)}, W_\infty^{(2)}, t) = (0, 0, t)\} \in D_\infty^{(1)}(t) \cap D_\infty^{(2)}(t)$$

are singular points of the foliation, the point $b_\infty^{(2)}(t)$ is an inaccessible singular point, and $D_\infty^{(2)}(t) - \{a_\infty^{(2)}(t), b_\infty^{(2)}(t)\}$ is a vertical leaf.

3.3.3. The third quadric transformation with center $a_\infty^{(2)}(t)$. Let

$$z_\infty^{(2)} = z_\infty^{(3)}, \quad w_\infty^{(2)} = z_\infty^{(3)} w_\infty^{(3)},$$
$$z_\infty^{(2)} = Z_\infty^{(3)} W_\infty^{(3)}, \quad w_\infty^{(2)} = W_\infty^{(3)}, \tag{3.5}$$

then

$$D_\infty^{(3)}(t) := Q_{a_\infty^{(2)}(t)}(a_\infty^{(2)}(t)) = \{z_\infty^{(3)} = 0\} \cup \{W_\infty^{(3)} = 0\}.$$

We see that

$$a_\infty^{(3)}(t) = \{(z_\infty^{(3)}, w_\infty^{(3)}, t) = (0, -2, t)\} \in D_\infty^{(3)}(t),$$

$$b_{\infty 0}^{(3)}(t) = \{(z_\infty^{(3)}, w_\infty^{(3)}, t) = (0, 0, t)\} \in D^{(0)}(t) \cap D_\infty^{(3)}(t),$$

$$b_{\infty\infty}^{(3)}(t) = \{(Z_\infty^{(3)}, W_\infty^{(3)}, t) = (0, 0, t)\} \in D_\infty^{(2)}(t) \cap D_\infty^{(3)}(t)$$

are singular points of the foliation, the points $b_{\infty 0}^{(3)}(t)$ and $b_{\infty\infty}^{(3)}(t)$ are inaccesssible singular points, and $D_\infty^{(3)}(t) - \{a_\infty^{(3)}(t), b_{\infty 0}^{(3)}(t), b_{\infty\infty}^{(3)}(t)\}$ is a vertical leaf.

3.3.4. The fourth quadric transformation with center $a_\infty^{(3)}(t)$. Here we make a change of coordinate systems near the point $a_\infty^{(3)}(t)$ given by

$$z_\infty^{(3)} = z_\infty^{(3)}, \quad w_\infty^{(3)} = 1/v_\infty^{(3)}. \tag{3.6}$$

Let

$$z_\infty^{(3)} = z_\infty^{(4)}, \quad v_\infty^{(3)} = -1/2 + z_\infty^{(4)} w_\infty^{(4)},$$
$$z_\infty^{(3)} = Z_\infty^{(4)} W_\infty^{(4)}, \quad v_\infty^{(3)} = -1/2 + W_\infty^{(4)}, \tag{3.7}$$

then

$$D_\infty^{(4)}(t) := Q_{a_\infty^{(3)}(t)}(a_\infty^{(3)}(t)) = \{z_\infty^{(4)} = 0\} \cup \{W_\infty^{(4)} = 0\},$$

the points

$$a_\infty^{(4)}(t) = \{(z_\infty^{(4)}, w_\infty^{(4)}, t) = (0, -t, t)\} \in D_\infty^{(4)}(t),$$

$$b_\infty^{(4)}(t) = \{(Z_\infty^{(4)}, W_\infty^{(4)}, t) = (0, 0, t)\} \in D_\infty^{(3)}(t) \cap D_\infty^{(4)}(t)$$

are singular points of the foliation, $b_\infty^{(4)}(t)$ is an inaccessible singular point, and $D_\infty^{(4)}(t) - \{a_\infty^{(4)}(t), b_\infty^{(4)}(t)\}$ is a vertical leaf.

3.3.5. The fifth quadric transformation with center $a_\infty^{(4)}(t)$. Let

$$\begin{aligned} z_\infty^{(4)} &= z_\infty^{(5)}, \quad w_\infty^{(4)} = -t + z_\infty^{(5)} w_\infty^{(5)}, \\ z_\infty^{(4)} &= Z_\infty^{(5)} W_\infty^{(5)}, \quad w_\infty^{(4)} = -t + W_\infty^{(5)}, \end{aligned} \tag{3.8}$$

then

$$D_\infty^{(5)}(t) := Q_{a_\infty^{(4)}(t)}(a_\infty^{(4)}(t)) = \{z_\infty^{(5)} = 0\} \cup \{W_\infty^{(5)} = 0\},$$

the points

$$a_\infty^{(5)}(t) = \{(z_\infty^{(5)}, w_\infty^{(5)}, t) = (0, 1 - \kappa_0 + 2\kappa_\infty, t)\} \in D_\infty^{(5)}(t),$$

$$b_\infty^{(5)}(t) = \{(Z_\infty^{(5)}, W_\infty^{(5)}, t) = (0, 0, t)\} \in D_\infty^{(4)}(t) \cap D_\infty^{(5)}(t)$$

are singular points of the foliation, $b_\infty^{(5)}(t)$ is an inaccessible singular point, and $D_\infty^{(5)}(t) - \{a_\infty^{(5)}(t), b_\infty^{(5)}(t)\}$ is a vertical leaf.

3.3.6. The sixth quadric transformation with center $a_\infty^{(5)}(t)$. Let

$$\begin{aligned} z_\infty^{(5)} &= z_\infty^{(6)}, \quad w_\infty^{(5)} = (1 - \kappa_0 + 2\kappa_\infty) + z_\infty^{(6)} w_\infty^{(6)}, \\ z_\infty^{(5)} &= Z_\infty^{(6)} W_\infty^{(6)}, \quad w_\infty^{(5)} = (1 - \kappa_0 + 2\kappa_\infty) + W_\infty^{(6)}, \end{aligned} \tag{3.9}$$

then

$$D_\infty^{(6)}(t) := Q_{a_\infty^{(5)}(t)}(a_\infty^{(5)}(t)) = \{z_\infty^{(6)} = 0\} \cup \{W_\infty^{(6)} = 0\}.$$

We can verify that our system has no singular points and every leaf is transversal with the fibers in $(z_\infty^{(6)}, w_\infty^{(6)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{IV}}$, moreover, the point $(Z_\infty^{(6)}, W_\infty^{(6)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes it is the vertical leaf $D_\infty^{(5)}(t) - \{b_\infty^{(5)}(t)\}$.

3.4. The space E_{IV}. Denote by Φ_t the composition of all the above eight quadric transformations. Then the space constructed by K. Okamoto is the space defined by

$$E_{\mathrm{IV}} = \bigcup_{t \in B_{\mathrm{IV}}} E_{\mathrm{IV}}(t) \times t$$

where

$$E_{\mathrm{IV}}(t) = \overline{E_{\mathrm{IV}}(t)} - D^{(0)}(t) \cup D_0^{(1)}(t) \bigcup_{1 \le k \le 5} D_0^{(k)}(t), \quad \overline{E_{\mathrm{IV}}(t)} = \Phi_t(\bar{\Sigma}_\varepsilon \times t).$$

By the above procedure, we see that E_{IV} is a 3-dimensional complex manifold obtained by glueing

$$\{(x_0, y_0, t) \in \boldsymbol{C}^2 \times B_{\mathrm{IV}}\}, \quad \{(x_2, y_2, t) \in \boldsymbol{C}^2 \times B_{\mathrm{IV}}\},$$

$$\{(Z_0^{(2)}, W_0^{(2)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{IV}}\}, \quad \{(z_\infty^{(6)}, w_\infty^{(6)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{IV}}\}$$

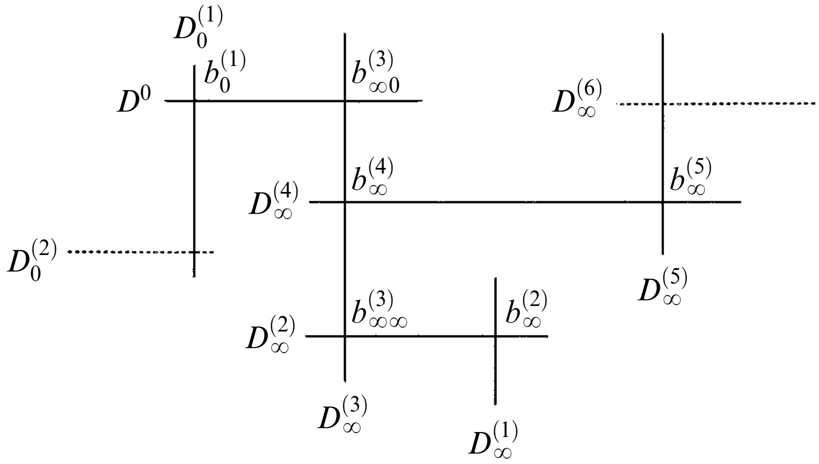

Figure 2. $J = \mathrm{IV}$

via the coordinate transformations (3.1)–(3.9), and

$$dy_0 \wedge dx_0 = dy_2 \wedge dx_2, \quad dy_0 \wedge dx_0 = -dW_0^{(2)} \wedge dZ_0^{(2)}, \quad dy_2 \wedge dx_2 = dw_\infty^{(6)} \wedge dz_\infty^{(6)}.$$

Therefore, by choosing new coordinate systems as

$$(x(00), y(00)) = (x_0, y_0), \quad (x(0\infty), y(0\infty)) = (-Z_0^{(2)}, W_0^{(2)}),$$

$$(x(\infty 0), y(\infty 0)) = (x_2, y_2), \quad (x(\infty\infty), y(\infty\infty)) = (z_\infty^{(6)}, w_\infty^{(6)}),$$

we obtain an expression of E_{IV} given in Theorem 2, which completes the proof of the theorem.

§4. Proof of Theorem 3.

In the case of $J = \mathrm{III}$, we take ε for $\bar{\Sigma}_\varepsilon$ as (1.16).

4.1. For any fixed $t \in B_{\mathrm{III}}$, our extended Pfaffian system has two singular points $a_\nu^{(0)}(t), \nu = 0, \infty$ and a vertical leaf $D^{(0)}(t) - \bigcup_\nu \{a_\nu^{(0)}(t)\}$ on a fiber $\bar{\Sigma}_\varepsilon \times t$ where

$$D^{(0)}(t) = (U_1(y_1 = 0) \times t) \cup (U_3(y_3 = 0) \times t) \cong \boldsymbol{P}^1,$$

$$a_0^{(0)}(t) = \{(x_1, y_1, t) = (0, 0, t)\}, \quad a_\infty^{(0)}(t) = \{(x_3, y_3, t) = (0, 0, t)\}.$$

4.2. Quadric transformations with centers $a_0^{(0)}(t), \ldots, a_0^{(3)}(t)$ for any fixed $t \in B_{\mathrm{III}}$. To separate the leaves passing through the point $a_0^{(0)}(t)$, we make quadric transformations four times successively.

4.2.1. The first quadric transformation with center $a_0^{(0)}(t)$. Let

$$\text{(4.1)} \qquad \begin{aligned} x_1 &= z_0^{(1)}, \quad y_1 = z_0^{(1)} w_0^{(1)}, \\ x_1 &= Z_0^{(1)} W_0^{(1)}, \quad y_1 = W_0^{(1)}, \end{aligned}$$

then

$$D_0^{(1)}(t) := Q_{a_0^{(0)}(t)}(a_0^{(0)}(t)) = \{z_0^{(1)} = 0\} \cup \{W_0^{(1)} = 0\},$$

the point

$$a_0^{(1)}(t) = \{(z_0^{(1)}, w_0^{(1)}, t) = (0, 0, t)\} \in D_0^{(1)}(t)$$

is a singular point of the foliation, and $D_0^{(1)}(t) - \{a_0^{(1)}(t)\}$ is a vertical leaf.

4.2.2. The second quadric transformation with center $a_0^{(1)}(t)$. Let

$$z_0^{(1)} = z_0^{(2)}, \quad w_0^{(1)} = z_0^{(2)} w_0^{(2)}, \qquad (4.2)$$
$$z_0^{(1)} = Z_0^{(2)} W_0^{(2)}, \quad w_0^{(1)} = W_0^{(2)},$$

then

$$D_0^{(2)}(t) := Q_{a_0^{(1)}(t)}(a_0^{(1)}(t)) = \{z_0^{(2)} = 0\} \cup \{W_0^{(2)} = 0\},$$

the points

$$a_0^{(2)}(t) = \{(z_0^{(2)}, w_0^{(2)}, t) = (0, -1/(\eta_0 t), t)\} \in D_0^{(2)}(t),$$

$$b_{00}^{(2)}(t) = \{(z_0^{(2)}, w_0^{(2)}, t) = (0, 0, t)\} \in D^{(0)}(t) \cap D_0^{(2)}(t),$$

$$b_{0\infty}^{(2)}(t) = \{(Z_0^{(2)}, W_0^{(2)}, t) = (0, 0, t)\} \in D_0^{(1)}(t) \cap D_0^{(2)}(t)$$

are singular points of the foliation, $b_{00}^{(2)}(t), b_{0\infty}^{(2)}(t)$ are inaccessible singular points, and $D_0^{(2)}(t) - \{a_0^{(2)}(t), b_{00}^{(2)}(t), b_{0\infty}^{(2)}(t)\}$ is a vertical leaf.

4.2.3. The third quadric transformation with center $a_0^{(2)}(t)$. We insert here the transformation

$$z_0^{(2)} = z_0^{(2)}, \quad w_0^{(2)} = 1/v_0^{(2)}. \qquad (4.3)$$

Let

$$z_0^{(2)} = z_0^{(3)}, \quad v_0^{(2)} = -\eta_0 t + z_0^{(3)} w_0^{(3)}, \qquad (4.4)$$
$$z_0^{(2)} = Z_0^{(3)} W_0^{(3)}, \quad v_0^{(2)} = -\eta_0 t + W_0^{(3)},$$

then

$$D_0^{(3)}(t) := Q_{a_0^{(2)}(t)}(a_0^{(2)}(t)) = \{z_0^{(3)} = 0\} \cup \{W_0^{(3)} = 0\},$$

the points

$$a_0^{(3)}(t) = \{(z_0^{(3)}, w_0^{(3)}, t) = (0, \kappa_0 + 1, t)\} \in D_0^{(3)}(t),$$

$$b_0^{(3)}(t) = \{(Z_0^{(3)}, W_0^{(3)}, t) = (0, 0, t)\} \in D_0^{(2)}(t) \cap D_0^{(3)}(t)$$

are singular points of the foliation, the point $b_0^{(3)}(t)$ is an inaccessible singular point, and $D_0^{(3)}(t) - \{a_0^{(3)}(t), b_0^{(3)}(t)\}$ is a vertical leaf.

4.2.4. The fourth quadric transformation with center $a_0^{(3)}(t)$. Let

$$z_0^{(3)} = z_0^{(4)}, \quad w_0^{(3)} = (\kappa_0 + 1) + z_0^{(4)} w_0^{(4)}, \qquad (4.5)$$
$$z_0^{(3)} = Z_0^{(4)} W_0^{(4)}, \quad w_0^{(3)} = (\kappa_0 + 1) + W_0^{(4)},$$

then

$$D_0^{(4)}(t) = Q_{a_0^{(3)}(t)}(a_0^{(3)}(t)) = \{z_0^{(4)} = 0\} \cup \{W_0^{(4)} = 0\}.$$

We see that our system has no singular points and every leaf is transversal with the fibers in $(z_0^{(4)}, w_0^{(4)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{III}}$, moreover, the point $(Z_0^{(4)}, W_0^{(4)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes it is the vertical leaf $D_0^{(3)}(t) - \{b_0^{(3)}(t)\}$.

4.3. Quadric transformations with centers $a_\infty^{(0)}(t), \ldots, a_\infty^{(3)}(t)$ for any fixed $t \in B_{\mathrm{III}}$. This procedure is the same as that given in the preceding section 4.2 provided the constants $\kappa_0, \kappa_\infty, \eta_0, \eta_\infty$ are replaced by $\kappa_\infty - 1, \kappa_0 + 1, \eta_\infty, \eta_0$ respectively.

4.4. The space E_{III}. Let Φ_t denote the composition of all the above eight quadric transformations. Then the space constructed by K. Okamoto is the space defined by

$$E_{\mathrm{III}} = \bigcup_{t \in B_{\mathrm{III}}} E_{\mathrm{III}}(t) \times t$$

where

$$E_{\mathrm{III}}(t) = \overline{E_{\mathrm{III}}(t)} - D^{(0)}(t) \cup \bigcup_{v=0,\,\infty,\,1 \le k \le 3} D_v^{(k)}(t), \quad \overline{E_{\mathrm{III}}(t)} = \Phi_t(\bar{\Sigma}_\varepsilon \times t).$$

We see that E_{III} is a 3-dimensional complex manifold obtained by glueing

$$\{(x_0, y_0, t) \in \boldsymbol{C}^2 \times B_{\mathrm{III}}\}, \quad \{(x_2, y_2, t) \in \boldsymbol{C}^2 \times B_{\mathrm{III}}\},$$

$$\{(z_0^{(4)}, w_0^{(4)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{III}}\}, \quad \{(z_\infty^{(4)}, w_\infty^{(4)}, t) \in \boldsymbol{C}^2 \times B_{\mathrm{III}}\}$$

via the coordinate transformations (4.1)–(4.5) for $v = 0$ and the corresponding ones for $v = \infty$, and

$$dy_0 \wedge dx_0 = dy_2 \wedge dx_2, \quad dy_0 \wedge dx_0 = dw_0^{(4)} \wedge dz_0^{(4)}, \quad dy_2 \wedge dx_2 = dw_\infty^{(4)} \wedge dz_\infty^{(4)}.$$

Therefore, by taking new coordinate systems as

$$(x(00), y(00)) = (x_0, y_0), \quad (x(0\infty), y(0\infty)) = (z_0^{(4)}, w_0^{(4)}),$$

$$(x(\infty 0), y(\infty 0)) = (x_2, y_2), \quad (x(\infty\eta_\infty t), y(\infty\eta_\infty t)) = (z_\infty^{(4)}, w_\infty^{(4)}),$$

we obtain an expression of E_{III} given in Theorem 3, which proves the theorem.

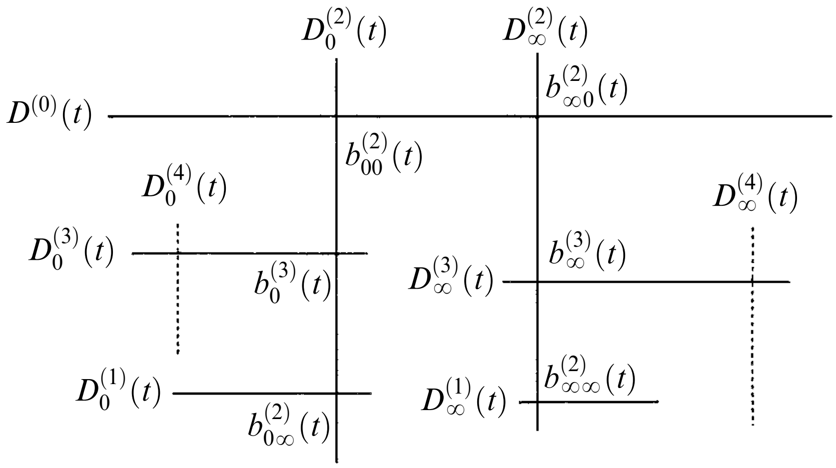

Figure 3. $J = \mathrm{III}$

§5. Proof of THEOREM 4.

In the case of $J = \mathrm{II}$, we take ε for $\bar{\Sigma}_\varepsilon$ as (1.20).

5.1. For any fixed $t \in B_{\mathrm{III}}$, our extended Pfaffian system has a singular point $a_\infty^{(0)}(t)$ and a vertical leaf $D^{(0)}(t) - \{a_\infty^{(0)}(t)\}$ on a fiber $\bar{\Sigma}_\varepsilon \times t$ where

$$D^{(0)}(t) = (U_1(y_1 = 0) \times t) \cup (U_3(y_3 = 0) \times t) \cong \boldsymbol{P}^1,$$

$$a_\infty^{(0)}(t) = \{(x_3, y_3, t) = (0, 0, t)\}.$$

5.2. Quadric transformations with centers $a_\infty^{(0)}(t), \dots, a_\infty^{(7)}(t)$ for any fixed $t \in B_{\mathrm{II}}$. To separate the solutions which pass through the point $a_\infty^{(0)}(t)$, we make quadric transformations eight times successively.

5.2.1. The first quadric transformation with center $a_\infty^{(0)}(t)$. Let

$$\begin{aligned} &x_3 = z_\infty^{(1)}, \quad y_3 = z_\infty^{(1)} w_\infty^{(1)}, \\ &x_3 = Z_\infty^{(1)} W_\infty^{(1)}, \quad y_3 = W_\infty^{(1)}, \end{aligned} \tag{5.1}$$

then

$$D_\infty^{(1)}(t) := Q_{a_\infty^{(0)}(t)}(a_\infty^{(0)}(t)) = \{z_\infty^{(1)} = 0\} \cup \{W_\infty^{(1)} = 0\},$$

the point

$$a_\infty^{(1)}(t) = \{(z_\infty^{(1)}, w_\infty^{(1)}, t) = (0, 0, t)\} \in D_\infty^{(1)}(t)$$

is a singular point of the foliation and $D_\infty^{(1)}(t) - \{a_\infty^{(1)}(t)\}$ is a vertical leaf.

5.2.2. The second quadric transformation with center $a_\infty^{(1)}(t)$. Let

$$\begin{aligned} &z_\infty^{(1)} = z_\infty^{(2)}, \quad w_\infty^{(1)} = z_\infty^{(2)} w_\infty^{(2)}, \\ &z_\infty^{(1)} = Z_\infty^{(2)} W_\infty^{(2)}, \quad w_\infty^{(1)} = W_\infty^{(2)}, \end{aligned} \tag{5.2}$$

then

$$D_\infty^{(2)}(t) := Q_{a_\infty^{(1)}(t)}(a_\infty^{(1)}(t)) = \{z_\infty^{(2)} = 0\} \cup \{W_\infty^{(2)} = 0\},$$

the points

$$a_\infty^{(2)}(t) = \{(z_\infty^{(2)}, w_\infty^{(2)}, t) = (0, 0, t)\} \in D_\infty^{(2)}(t),$$

$$b_\infty^{(2)}(t) = \{(Z_\infty^{(2)}, W_\infty^{(2)}, t) = (0, 0, t)\} \in D_\infty^{(1)}(t) \cap D_\infty^{(2)}(t)$$

are singular points of the foliation, the point $b_\infty^{(2)}(t)$ is an inaccessible singular point, and $D_\infty^{(2)}(t) - \{a_\infty^{(2)}(t), b_\infty^{(2)}(t)\}$ is a vertical leaf.

5.2.3. The third quadric transformation with center $a_\infty^{(2)}(t)$. Let

$$\begin{aligned} &z_\infty^{(2)} = z_\infty^{(3)}, \quad w_\infty^{(2)} = z_\infty^{(3)} w_\infty^{(3)}, \\ &z_\infty^{(2)} = Z_\infty^{(3)} W_\infty^{(3)}, \quad w_\infty^{(2)} = W_\infty^{(3)}, \end{aligned} \tag{5.3}$$

then

$$D_{\infty}^{(3)}(t) := Q_{a_{\infty}^{(2)}(t)}(a_{\infty}^{(2)}(t)) = \{z_{\infty}^{(3)} = 0\} \cup \{W_{\infty}^{(3)} = 0\},$$

the points

$$a_{\infty}^{(3)}(t) = \{(z_{\infty}^{(3)}, w_{\infty}^{(3)}, t) = (0, 0, t)\} \in D_{\infty}^{(3)}(t),$$

$$b_{\infty}^{(3)}(t) = \{(Z_{\infty}^{(3)}, W_{\infty}^{(3)}, t) = (0, 0, t)\} \in D_{\infty}^{(2)}(t) \cap D_{\infty}^{(3)}(t)$$

are singular points of the foliation, the point $b_{\infty}^{(3)}(t)$ is an inaccessible singular point, and $D_{\infty}^{(3)}(t) - \{a_{\infty}^{(3)}(t), b_{\infty}^{(3)}(t)\}$ is a vertical leaf.

5.2.4. The fourth quadric transformation with center $a_{\infty}^{(3)}(t)$. Let

(5.4)
$$z_{\infty}^{(3)} = z_{\infty}^{(4)}, \quad w_{\infty}^{(3)} = z_{\infty}^{(4)} w_{\infty}^{(4)},$$
$$z_{\infty}^{(3)} = Z_{\infty}^{(4)} W_{\infty}^{(4)}, \quad w_{\infty}^{(3)} = W_{\infty}^{(4)},$$

then

$$D_{\infty}^{(4)}(t) := Q_{a_{\infty}^{(3)}(t)}(a_{\infty}^{(3)}(t)) = \{z_{\infty}^{(4)} = 0\} \cup \{W_{\infty}^{(4)} = 0\},$$

the points

$$a_{\infty}^{(4)}(t) = \{(z_{\infty}^{(4)}, w_{\infty}^{(4)}, t) = (0, -1/2, t)\} \in D_{\infty}^{(4)}(t),$$

$$b_{\infty 0}^{(4)}(t) = \{(z_{\infty}^{(4)}, w_{\infty}^{(4)}, t) = (0, 0, t)\} \in D^{(0)}(t) \cap D_{\infty}^{(4)}(t),$$

$$b_{\infty\infty}^{(4)}(t) = \{(Z_{\infty}^{(4)}, W_{\infty}^{(4)}, t) = (0, 0, t)\} \in D_{\infty}^{(3)}(t) \cap D_{\infty}^{(4)}(t)$$

are singular points of the foliation, the points $b_{\infty 0}^{(4)}(t), b_{\infty\infty}^{(4)}(t)$ are inaccessible singular points, and $D_{\infty}^{(4)}(t) - \{a_{\infty}^{(4)}(t), b_{\infty 0}^{(4)}(t), b_{\infty\infty}^{(4)}(t)\}$ is a vertical leaf.

5.2.5. The fifth quadric transformation with center $a_{\infty}^{(4)}(t)$. We insert here a transformation given by

(5.5)
$$z_{\infty}^{(4)} = z_{\infty}^{(4)}, \quad w_{\infty}^{(4)} = 1/v_{\infty}^{(4)}.$$

Let

(5.6)
$$z_{\infty}^{(4)} = z_{\infty}^{(5)}, \quad v_{\infty}^{(4)} = -2 + z_{\infty}^{(5)} w_{\infty}^{(5)},$$
$$z_{\infty}^{(4)} = Z_{\infty}^{(5)} W_{\infty}^{(5)}, \quad v_{\infty}^{(4)} = -2 + W_{\infty}^{(5)},$$

then

$$D_{\infty}^{(5)}(t) := Q_{a_{\infty}^{(4)}(t)}(a_{\infty}^{(4)}(t)) = \{z_{\infty}^{(5)} = 0\} \cup \{W_{\infty}^{(5)} = 0\},$$

the points

$$a_{\infty}^{(5)}(t) = \{(z_{\infty}^{(5)}, w_{\infty}^{(5)}, t) = (0, 0, t)\} \in D_{\infty}^{(5)}(t),$$

$$b_{\infty}^{(5)}(t) = \{(Z_{\infty}^{(5)}, W_{\infty}^{(5)}, t) = (0, 0, t)\} \in D^{(0)}(t) \cap D_{\infty}^{(5)}(t)$$

are singular points of the foliation, the point $b_\infty^{(5)}(t)$ is an inaccessible singular point, and $D_\infty^{(5)}(t) - \{a_\infty^{(5)}(t), b_\infty^{(5)}(t)\}$ is a vertical leaf.

5.2.6. The sixth quadric transformation with center $a_\infty^{(5)}(t)$. Let

$$z_\infty^{(5)} = z_\infty^{(6)}, \quad w_\infty^{(5)} = z_\infty^{(6)} w_\infty^{(6)}, \tag{5.7}$$
$$z_\infty^{(5)} = Z_\infty^{(6)} W_\infty^{(6)}, \quad w_\infty^{(5)} = W_\infty^{(6)},$$

then

$$D_\infty^{(6)}(t) := Q_{a_\infty^{(5)}(t)}(a_\infty^{(5)}(t)) = \{z_\infty^{(6)} = 0\} \cup \{W_\infty^{(6)} = 0\},$$

the points

$$a_\infty^{(6)}(t) = \{(z_\infty^{(6)}, w_\infty^{(6)}, t) = (0, -t, t)\} \in D_\infty^{(6)}(t),$$
$$b_\infty^{(6)}(t) = \{(Z_\infty^{(6)}, W_\infty^{(6)}, t) = (0, 0, t)\} \in D_\infty^{(5)}(t) \cap D_\infty^{(6)}(t)$$

are singular points of the foliation, the point $b_\infty^{(6)}(t)$ is an inaccessible singular point, and $D_\infty^{(6)}(t) - \{a_\infty^{(6)}(t), b_\infty^{(6)}(t)\}$ is a vertical leaf.

5.2.7. The seventh quadric transformation with center $a_\infty^{(6)}(t)$. Let

$$z_\infty^{(6)} = z_\infty^{(7)}, \quad w_\infty^{(6)} = -t + z_\infty^{(7)} w_\infty^{(7)}, \tag{5.8}$$
$$z_\infty^{(6)} = Z_\infty^{(7)} W_\infty^{(7)}, \quad w_\infty^{(6)} = -t + W_\infty^{(7)},$$

then

$$D_\infty^{(7)}(t) := Q_{a_\infty^{(6)}(t)}(a_\infty^{(6)}(t)) = \{z_\infty^{(7)} = 0\} \cup \{W_\infty^{(7)} = 0\},$$

the points

$$a_\infty^{(7)}(t) = \{(z_\infty^{(7)}, w_\infty^{(7)}, t) = (0, -2\alpha, t)\} \in D_\infty^{(7)}(t),$$
$$b_\infty^{(7)}(t) = \{(Z_\infty^{(7)}, W_\infty^{(7)}, t) = (0, 0, t)\} \in D_\infty^{(6)}(t) \cap D_\infty^{(7)}(t)$$

are singular points of the foliation, the point $b_\infty^{(7)}(t)$ is an inaccessible singular point, and $D_\infty^{(7)}(t) - \{a_\infty^{(7)}(t), b_\infty^{(7)}(t)\}$ is a vertical leaf.

5.2.8. The eighth quadric transformation with center $a_\infty^{(7)}(t)$. Let

$$z_\infty^{(7)} = z_\infty^{(8)}, \quad w_\infty^{(7)} = -2\alpha + z_\infty^{(8)} w_\infty^{(8)}, \tag{5.9}$$
$$z_\infty^{(7)} = Z_\infty^{(8)} W_\infty^{(8)}, \quad w_\infty^{(7)} = -2\alpha + W_\infty^{(8)},$$

then

$$D_\infty^{(8)}(t) := Q_{a_\infty^{(7)}(t)}(a_\infty^{(7)}(t)) = \{z_\infty^{(8)} = 0\} \cup \{W_\infty^{(8)} = 0\}.$$

We see that, in $(z_\infty^{(8)}, w_\infty^{(8)}, t)$-space $\boldsymbol{C}^2 \times B_{\mathrm{II}}$, our system has no singular points and every leaf is transversal with the fibers, moreover, the point $(Z_\infty^{(8)}, W_\infty^{(8)}, t) = (0, 0, t)$ is not a singular point of the foliation and the leaf which passes the point is the vertical leaf $D_\infty^{(7)}(t) - \{b_\infty^{(7)}(t)\}$.

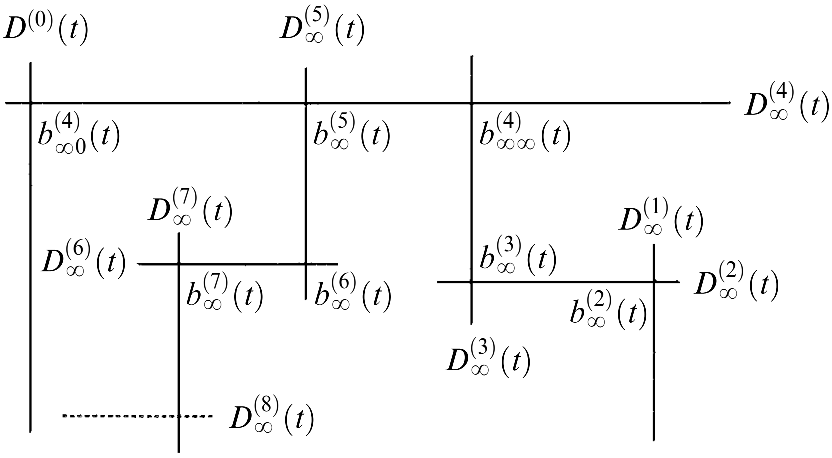

Figure 4. $J = \mathrm{II}$

5.3. The space E_{II}. Let Φ_t denote the composition of all the above eight quadric transformations. Then a space defined by

$$E_{\mathrm{II}} = \bigcup_{t \in B_{\mathrm{II}}} E_{\mathrm{II}}(t) \times t$$

with

$$E_{\mathrm{II}}(t) = \overline{E_{\mathrm{II}}(t)} - D^{(0)}(t) \cup \bigcup_{1 \le k \le 7} D^{(k)}_\infty(t), \quad \overline{E_{\mathrm{II}}(t)} = \Phi_t(\bar{\Sigma}_\varepsilon \times t)$$

is the space constructed by K. Okamoto.

By the above procedure, we can verify that E_{II} is a 3-dimensional complex manifold obtained by glueing

$$\{(x_0, y_0, t) \in \boldsymbol{C}^2 \times B_{\mathrm{II}}\}, \ \{(x_2, y_2, t) \in \boldsymbol{C}^2 \times B_{\mathrm{II}}\}, \ \{(z^{(8)}_\infty, w^{(8)}_\infty, t) \in \boldsymbol{C}^2 \times B_{\mathrm{II}}\}$$

via the coordinate transformations (5.1)–(5.9), and

$$dy_0 \wedge dx_0 = dy_2 \wedge dx_2, \quad dy_2 \wedge dx_2 = dw^{(8)}_\infty \wedge dz^{(8)}_\infty.$$

Therefore, by choosing coordinate systems as

$$(x(00), y(00)) = (x_0, y_0), \quad (x(\infty 0), y(\infty 0)) = (x_2, y_2),$$

$$(x(\infty\infty), y(\infty\infty)) = (z^{(8)}_\infty, w^{(8)}_\infty),$$

we obtain an expression of E_{II} given in Theorem 4, which shows the theorem.

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