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On the theory of KM_2O-Langevin equations for stationary flows (1): characterization theorem

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Abstract. In this paper, we introduce a notion of stationarity for the pair of flows in a metric vector space and characterize it in such a way that there exist two relations, called a dissipation-dissipation theorem and a fluctuation-dissipation theorem, among the KM_2O-Langevin matrix associated with the pair of flows.

§1. Introduction.

In a series of papers ([**1**]–[**12**]), we have developped the theory of KM_2O-Langevin equations for the multi-dimensional local and weakly stationary process $\mathbf{X} = (X(n); |n| \le N)$ with discrete time and proposed a method to analyze the inner structure hidden behind a given time series $\mathscr{X} = (\mathscr{X}(n); 0 \le n \le N)$. In particular, we have proposed three tests: one is a stationary test (Test (S)) that checks the weak stationarity of a given time series; the other is a causal test (Test(CS)) that judges the existence and direction of a causal relation between two kinds of time series passing Test(S); the third is a deterministic test (Test(D)) that judges the determinism of the time evolution of time series passing Test(S).

The theoretical background for Test(S) lies in the fluctuation-dissipation theorem (Theorems 3.1 and 4.1) and the construction theorem (Theorem 6.1) obtained in [**1**]. We note that Burg's relation pays an important role in the proof of these two theorems.

In order to get a method that stands the test of analysis for complex time series, however, we have to refine the results of [**1**] for the multi-dimensional local and weakly stationary process $\mathbf{X} = (X(n); 0 \le n \le N)$ whose time domain is restricted to the same set $\{0, 1, \ldots, N\}$ as the time domain of the given time series $\mathscr{X} = (\mathscr{X}(n); 0 \le n \le N)$. Further, there are four points to be reconsidered in two theorems stated above: one is that we took a complicated procedure in the proof of Burg's relation that is needed for the proof of the fluctuation-dissipation theorem; the second is that we put an unsatisfactory assumption in the construction theorem such that the KM_2O-Langevin matrix can be constructed through the fluctuation-dissipation theorem; the third is that there

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was lacked for the consideration for the case $k = n_0$ in Step 15 of the proof of Burg's relation that is needed for the proof of the construction theorem; the fourth is that the construction theorem gives only the stationary solution to the forward KM_2O-Langevin equation. For characterizing the stationary property of the process in terms of the fluctuation-dissipation theorem, we have to take a careful consideration not only for the forward KM_2O-Langevin equation, but also for the backward KM_2O-Langevin equation.

The purpose of this paper and subsequent papers ([**13**], [**14**]) is to solve four problems stated above. From the viewpoint of the algebraic structure of stochastic processes, at first in this paper, we shall reconstruct the theory of KM_2O-Langevin equations for the multi-dimensional flow in a metric vector space and introduce a notion of stationarity for the pair $[\mathbf{X}, \mathbf{Y}]$ of flows $\mathbf{X}$ and $\mathbf{Y}$ in the metric vector space. Next, by rearranging the structure in the proof of two theorems in [**1**] stated above and giving a simplified proof of Burg's relation, we shall characterize a notion of stationarity for the pair of flows in such a way that there exist two relations, called a dissipation-dissipation theorem and a fluctuation-dissipation theorem, among the KM_2O-Langevin matrix associated with the pair of flows. As a byproduct, we will find that Burg's relation holds for any stationary pair of flows. We note that Burg's relation has been proved for the usual weakly stationary process.

Now let us state the contents of the present paper. Let $(W, (\star, *))$ be any metric vector space with an inner product $(\star, *)$ over real field $\boldsymbol{R}$. By a pair of d-dimensional flows in W, we mean a pair $[\mathbf{X}, \mathbf{Y}]$ such that $\mathbf{X} = (X(n); 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell); -N \le \ell \le 0)$ are families of d-dimensional column vectors moving in W with discrete time.

In §2, we shall derive a forward (resp. backward) KM_2O-Langevin fluctuation flow $\nu_+ = (\nu_+(n); 0 \le n \le N)$ (resp. $\nu_- = (\nu_-(\ell); -N \le \ell \le 0)$) associated with the flow $\mathbf{X}$ (resp. $\mathbf{Y}$) by using the innovation method.

The aim of §3 is to derive the KM_2O-Langevin matrix $\mathscr{LM}([\mathbf{X}, \mathbf{Y}])$ associated with the pair $[\mathbf{X}, \mathbf{Y}]$ of flows and the KM_2O-Langevin equations describing the time evolution of $\mathbf{X}$ and $\mathbf{Y}$ that consist of the dissipation part and the fluctuation part, under the independence condition for $\mathbf{X}$ and $\mathbf{Y}$.

In §4, we shall introduce a concept of stationary property for the pair $[\mathbf{X}, \mathbf{Y}]$ of flows. We say that the pair $[\mathbf{X}, \mathbf{Y}]$ of flows satisfies stationary property if there exists a matrix function $R : \{-N, -N+1, \dots, N-1, N\} \to M(d; \boldsymbol{R})$ such that for any m, n $(0 \le m, n \le N)$,

$$(X(m), {}^tX(n)) = R(m-n) \quad \text{and} \quad (Y(-m), {}^tY(-n)) = R(-m+n),$$

where for any two L-dimensional column vectors $x = {}^t(x_1, \dots, x_L)$ and $y = {}^t(y_1, \dots, y_L)$ whose elements x_j, y_j $(1 \le j \le L)$ belong to W, we denote by $(x, {}^ty)$ an $L \times L$ matrix, called to be an inner matrix of x and $y : (x, {}^ty) \equiv ((x_j, y_k))_{1 \le j,k \le L}$. Then we call the pair $[\mathbf{X}, \mathbf{Y}]$ the stationary pair of flows and the matrix function R the covariance matrix function of the stationary pair $[\mathbf{X}, \mathbf{Y}]$ of flows, respectively. We shall state in §4 the main theorem (Theorem 4.2) in this paper that characterizes the notion of stationary property for the pair $[\mathbf{X}, \mathbf{Y}]$ of flows in terms of the dissipation-dissipation theorem (for abbreviation (DDT)) and the fluctuation-dissipation theorem (for abbreviation (FDT)).

These two theorems, (DDT) and (FDT), will be proved in §5 and §6 for the stationary pair of flows, respectively. It will be found that Burg's relation pays an important role in the proof of (FDT), which will be proved in Theorem 6.3. We note that (DDT), (FDT) and Burg's relation have been already proved in [**1**] for the usual weakly stationary process.

In §7, we shall prove that (DDT) and (FDT) imply stationary property for the pair $[\mathbf{X}, \mathbf{Y}]$ of flows by taking the same consideration as in the proof of Burg's relation in Theorem 6.3 of this paper and rearranging the structure in the proof of the construction theorem in [**1**].

From the viewpoint of the fluctuation-dissipation theorem, as a continuation of the present paper, we shall prove a construction theorem for KM_2O-Langevin matrix and an extension theorem for stationary pair of flows in two subsequent papers [**13**] and [**14**], respectively.

§2. KM_2O-Langevin fluctuation flows associated with flows.

Let $(W, (\star, *))$ be any metric vector space with an inner product $(\star, *)$ over real field $\boldsymbol{R}$. By a d-dimensional flow in the space W, we mean a function $\mathbf{Z} = (Z(n); \ell \le n \le r) : \{\ell, \ell+1, \dots, r-1, r\} \to W^d$ such that for each n $(\ell \le n \le r)$,

$$Z(n) = {}^t(Z_1(n), Z_2(n), \dots, Z_d(n)), \tag{2.1}$$

where ℓ and r $(\ell < r)$ are integers and $Z_j(n)$ are vectors in W $(1 \le j \le d, \ell \le n \le r)$. Furthermore, for a d-dimensional flow $\mathbf{Z} = (Z(n); \ell \le n \le r)$ and two integers n_1, n_2 $(\ell \le n_1 \le n_2 \le r)$, we define a subspace $\mathbf{M}_{n_1}^{n_2}(\mathbf{Z})$ of the vector space W by

$$\mathbf{M}_{n_1}^{n_2}(\mathbf{Z}) \equiv \text{the subspace generated by } \{Z_j(m);\ 1 \le j \le d,\ n_1 \le m \le n_2\}. \tag{2.2}$$

Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of d-dimensional flows $\mathbf{X} = (X(n); 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell); -N \le \ell \le 0)$ in the space W. We project for each n $(0 \le n \le N)$ d components of the vector $X(n)$ in the closed subspace $\mathbf{M}_0^{n-1}(\mathbf{X})$ to get a d-dimensional flow $v_+ = (v_+(n); 0 \le n \le N)$ such that

$$v_+(0) \equiv X(0) \tag{2.3$_+$}$$

$$v_+(n) \equiv X(n) - P_{\mathbf{M}_0^{n-1}(\mathbf{X})} X(n) \quad (1 \le n \le N). \tag{2.4$_+$}$$

By rewriting (2.3_+) and (2.4_+), we can get an orthogonal decomposition of the flow $\mathbf{X}$:

$$X(0) = v_+(0) \tag{2.5$_+$}$$

$$X(n) = P_{\mathbf{M}_0^{n-1}(\mathbf{X})} X(n) + v_+(n) \quad (1 \le n \le N). \tag{2.6$_+$}$$

We call this d-dimensional flow v_+ a forward KM_2O-Langevin fluctuation flow associated with the flow $\mathbf{X}$. We define a forward KM_2O-Langevin fluctuation matrix function $V_+ = (V_+(n); 0 \le n \le N)$ associated with the flow $\mathbf{X}$ as follows:

$$V_+(n) \equiv (v_+(n), {}^t v_+(n)) \quad (0 \le n \le N). \tag{2.7$_+$}$$

It is easy to see that the following relations hold between the flow $\mathbf{X}$ and the forward KM_2O-Langevin fluctuation flow ν_+.

Theorem 2.1.

(i) $$\mathbf{M}_0^n(\mathbf{X}) = \mathbf{M}_0^n(\nu_+) \qquad (0 \le n \le N)$$

(ii) $$(X(m), {}^t\nu_+(n)) = 0 \qquad (0 \le m < n \le N)$$

(iii) $$(\nu_+(m), {}^t\nu_+(n)) = \delta_{mn} V_+(n) \quad (0 \le m, n \le N).$$

Moreover, by projecting for each ℓ $(-N \le \ell \le 0)$ d components of the vector $Y(\ell)$ in the closed subspace $\mathbf{M}_{\ell+1}^0(\mathbf{Y})$, we can get a d-dimensional flow $\nu_- = (\nu_-(\ell);\ -N \le \ell \le 0)$ such that

(2.5$_-$) $$Y(0) = \nu_-(0)$$

(2.6$_-$) $$Y(\ell) = P_{\mathbf{M}_{\ell+1}^0(\mathbf{Y})} Y(\ell) + \nu_-(\ell) \quad (-N \le \ell \le -1).$$

We call this d-dimensional flow ν_- a backward KM_2O-Langevin fluctuation flow associated with the flow $\mathbf{Y}$. For each n $(0 \le n \le N)$, we denote by $V_- = (V_-(n);\ 0 \le n \le N)$ the backward KM_2O-Langevin fluctuation matrix function:

(2.7$_-$) $$V_-(n) \equiv (\nu_-(-n), {}^t\nu_-(-n)) \quad (0 \le n \le N).$$

In the same way as in Theorem 2.1, we have

Theorem 2.2.

(i) $$\mathbf{M}_{-n}^0(\mathbf{Y}) = \mathbf{M}_{-n}^0(\nu_-) \qquad (0 \le n \le N)$$

(ii) $$(Y(-m), {}^t\nu_-(-n)) = 0 \qquad (0 \le m < n \le N)$$

(iii) $$(\nu_-(-m), {}^t\nu_-(-n)) = \delta_{mn} V_-(n) \quad (0 \le m, n \le N).$$

Example 2.1. Let $\mathbf{X} = (X(n);\ 0 \le n \le N)$ be any d-dimensional flow in the metric vector space W. We define a d-dimensional flow $\tilde{\mathbf{X}} = (\tilde{X}(\ell);\ -N \le \ell \le 0)$ by

(2.8) $$\tilde{X}(\ell) \equiv X(N + \ell).$$

We call this pair $[\mathbf{X}, \tilde{\mathbf{X}}]$ of flows a natural pair of flows associated with the flow $\mathbf{X}$.

Example 2.2. Let $\mathbf{X} = (X(n);\ |n| \le N)$ be any d-dimensional flow in the metric vector space W. We define a d-dimensional flow $\mathbf{X}_+ = (X_+(n);\ 0 \le n \le N)$ by

(2.9) $$X_+(n) \equiv X(n).$$

Further, we define for each s $(0 \le s \le N)$ a d-dimensional flow $\tilde{\mathbf{X}}_-^{(s)} = (\tilde{X}_-^{(s)}(\ell);\ -N \le \ell \le 0)$ by

(2.10) $$\tilde{X}_-^{(s)}(\ell) \equiv X(s + \ell).$$

Then we can obtain $N + 1$ pairs $[\mathbf{X}_+, \tilde{\mathbf{X}}_-^{(s)}]$ of flows.

§3. $\mathbf{KM_2O}$-Langevin equations and $\mathbf{KM_2O}$-Langevin matrix associated with the pair of flows.

Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of d-dimensional flows $\mathbf{X} = (X(n);\, 0 \le n \le N)$ and $\mathbf{Y} = (\mathbf{Y}(\ell);\, -N \le \ell \le 0)$ in the metric vector space W. We shall derive a forward (resp. backward) KM_2O-Langevin equation that governs the forward (resp. backward) time evolution of the flow $\mathbf{X}$ (resp. $\mathbf{Y}$) under the following independence conditions:

(3.1_{N-1}) $\{X_j(n);\, 1 \le j \le d, 0 \le n \le N-1\}$ is linearly independent in W

(3.2_{N-1}) $\{Y_j(-n);\, 1 \le j \le d, 0 \le n \le N-1\}$ is linearly independent in W.

The above conditions imply that there uniquely exist two systems $\{\gamma_+(n,k);\, 1 \le n \le N, 0 \le k < n\}$ and $\{\gamma_-(n,k);\, 1 \le n \le N, 0 \le k < n\}$ of $d \times d$ matrices such that

$$P_{\mathbf{M}_0^{n-1}(\mathbf{X})} X(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k) X(k) \quad (1 \le n \le N) \tag{3.3$_+$}$$

$$P_{\mathbf{M}_{-n+1}^{0}(\mathbf{Y})} Y(-n) = -\sum_{k=0}^{n-1} \gamma_-(n,k) Y(-k) \quad (1 \le n \le N). \tag{3.3$_-$}$$

Immediately from $(2.5_\pm)$ and $(2.6_\pm)$, we have

$$X(0) = \nu_+(0) \tag{3.4$_+$}$$

$$X(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k) X(k) + \nu_+(n) \quad (1 \le n \le N) \tag{3.5$_+$}$$

and

$$Y(0) = \nu_-(0) \tag{3.4$_-$}$$

$$Y(-n) = -\sum_{k=0}^{n-1} \gamma_-(n,k) Y(-k) + \nu_-(-n) \quad (1 \le n \le N). \tag{3.5$_-$}$$

In particular, for each n $(1 \le n \le N)$, we put

$$\delta_\pm(n) \equiv \gamma_\pm(n,0). \tag{3.6$_\pm$}$$

We call equation (3.5_+) (resp. (3.5_-)) with initial condition (3.4_+) (resp. (3.4_-)) a forward (resp. backward) KM_2O-Langevin equation for the d-dimensional flow $\mathbf{X}$ (resp. $\mathbf{Y}$). Further, we call the matrix function $\gamma_+ = (\gamma_+(n,k);\, 0 \le k < n \le N)$ (resp. $\gamma_- = (\gamma_-(n,k);\, 0 \le k < n \le N)$) a forward (resp. backward) KM_2O-Langevin dissipation matrix function associated with the flow $\mathbf{X}$ (resp. $\mathbf{Y}$). In particular, we call the matrix function $\delta_+ = (\delta_+(n);\, 1 \le n \le N)$ (resp. $\delta_- = (\delta_-(n);\, 1 \le n \le N)$) a forward (resp. backward) KM_2O-Langevin partial correlation matrix function associated with the flow $\mathbf{X}$ (resp. $\mathbf{Y}$). We define a system $\mathscr{LM}([\mathbf{X}, \mathbf{Y}])$ by

$$\mathscr{LM}([\mathbf{X}, \mathbf{Y}]) \equiv \{\gamma_\pm(n,k), \delta_\pm(n), V_\pm(m);\, 0 \le k < n \le N, 0 \le m \le N\}$$

and call this system a KM_2O-Langevin matrix associated with the pair $[\mathbf{X}, \mathbf{Y}]$ of d-dimensional flows $\mathbf{X}$ and $\mathbf{Y}$.

When we need to represent explicitly the time domain of the flows $\mathbf{X}$ and $\mathbf{Y}$, we adopt the following notation: for each natural number M less than or equal to N, we restrict the time domain of the flow $\mathbf{X}$ (resp. $\mathbf{Y}$) to the subset $\{0, 1, \dots, M-1, M\}$ (resp. $\{-M, -M+1, \dots, -1, 0\}$) and get the pair $[\mathbf{X}^{(M)}, \mathbf{Y}^{(M)}]$ of flows $\mathbf{X}^{(M)} = (X(n);\ 0 \le n \le M)$ and $\mathbf{Y}^{(M)} = (Y(\ell);\ -M \le \ell \le 0)$. Since the pair $[\mathbf{X}^{(M)}, \mathbf{Y}^{(M)}]$ of flows satisfies also the independence conditions (3.1_{M-1}) and (3.2_{M-1}), we can get the KM_2O-Langevin matrix associated with the pair $[\mathbf{X}^{(M)}, \mathbf{Y}^{(M)}]$ of flows and denote it by

$$\mathscr{LM}([\mathbf{X}^{(M)}, \mathbf{Y}^{(M)}]) \equiv \{\gamma_{\pm}^{(M)}(n,k), \delta_{\pm}^{(M)}(n), V_{\pm}^{(M)}(m);\ 0 \le k < n \le M, 0 \le m \le M\}.$$

By noting that $\mathbf{M}_0^{n-1}(\mathbf{X}^{(N)}) = \mathbf{M}_0^{n-1}(\mathbf{X}^{(M)})$ and $\mathbf{M}_{-n+1}^{0}(\mathbf{Y}^{(N)}) = \mathbf{M}_{-n+1}^{0}(\mathbf{Y}^{(M)})$ for any n $(1 \le n \le M)$, we can show

Theorem 3.1 (Consistency Condition). *For each natural numbers $M, N (M < N)$,*

(i) $$\gamma_{\pm}^{(N)}(n,k) = \gamma_{\pm}^{(M)}(n,k) \quad (0 \le k < n \le M)$$

(ii) $$\delta_{\pm}^{(N)}(n) = \delta_{\pm}^{(M)}(n) \qquad (1 \le n \le M)$$

(iii) $$V_{\pm}^{(N)}(m) = V_{\pm}^{(M)}(m) \qquad (0 \le m \le M).$$

§4. Stationary property and its characterization theorem.

Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of d-dimensional flows $\mathbf{X} = (X(n); 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell);\ -N \le \ell \le 0)$ in the space W. We shall define a concept of stationary property for the pair $[\mathbf{X}, \mathbf{Y}]$ of flows.

Definition 4.1. We say that the pair $[\mathbf{X}, \mathbf{Y}]$ of flows has stationary property if there exists a matrix function $R : \{-N, -N+1, \dots, N-1, N\} \to M(d; \boldsymbol{R})$ such that

$$(X(m), {}^tX(n)) = R(m-n) \qquad (0 \le m, n \le N) \tag{4.1}$$

$$(Y(-m), {}^tY(-n)) = R(-m+n) \quad (0 \le m, n \le N). \tag{4.2}$$

Then we call the pair $[\mathbf{X}, \mathbf{Y}]$ of flows and the matrix function R a d-dimensional stationary pair of flows in the space W and the covariance matrix function of the pair $[\mathbf{X}, \mathbf{Y}]$ of flows, respectively.

Example 4.1. Let $\mathbf{X} = (X(n); 0 \le n \le N)$ be any d-dimensional flow considered in Example 2.1. Moreover, we assume that $\mathbf{X}$ satisfies condition (4.1). Then, the natural pair $[\mathbf{X}, \tilde{\mathbf{X}}]$ of the flow $\mathbf{X}$ has stationary property and its covariance matrix function as the pair of flows is equal to the matrix function R in (4.1).

Example 4.2. Let $\mathbf{X} = (X(n);\ |n| \le N)$ be the same d-dimensional flow as in Example 2.2. Moreover, we asssume that $\mathbf{X}$ satisfies the usual weakly stationary property. Then, for each s $(0 \le s \le N)$, the pair $[\mathbf{X}_+, \tilde{\mathbf{X}}_-^{(s)}]$ defined in (2.9) and (2.10) has stationary property.

Let $[\mathbf{X}, \mathbf{Y}]$ be any stationary pair of flows in the space W such that $\mathbf{X} = (X(n); 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell); -N \le \ell \le 0)$. It is to be noted that

$$ {}^tR(n) = R(-n) \quad (0 \le n \le N). \tag{4.3} $$

By using (4.1) and (4.2), we can introduce some unitary operators which represent certain equivalence between the flows $\mathbf{X}$ and $\mathbf{Y}$.

Theorem 4.1. (i) *There exists a unitary operator* $U : \mathbf{M}_0^N(\mathbf{X}) \to \mathbf{M}_{-N}^0(\mathbf{Y})$ *such that*

$$ U(X_j(m)) = Y_j(-N+m) \quad (1 \le j \le d,\ 0 \le m \le N). \tag{4.4} $$

(ii) *For each integers* ℓ, r, n $(0 \le \ell \le r \le N, 0 \le n \le N-r)$, *there exists a unitary operator* $U_\ell^r(n) : \mathbf{M}_\ell^r(\mathbf{X}) \to \mathbf{M}_{\ell+n}^{r+n}(\mathbf{X})$ *such that*

$$ U_\ell^r(n)(X_j(m)) = X_j(m+n) \quad (1 \le j \le d,\ \ell \le m \le r). \tag{4.5} $$

(iii) *For each integers* ℓ, r, n $(-N \le \ell \le r \le 0,\ 0 \le n \le N+\ell)$, *there exists a unitary operator* $U_\ell^r(-n) : \mathbf{M}_\ell^r(\mathbf{Y}) \to \mathbf{M}_{\ell-n}^{r-n}(\mathbf{Y})$ *such that*

$$ U_\ell^r(-n)(Y_j(m)) = Y_j(m-n) \quad (1 \le j \le d,\ \ell \le m \le r). \tag{4.6} $$

Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of flows in the space W satisfying the independence conditions (3.1_{N-1}) and (3.2_{N-1}). We shall state the following main theorem in this paper that characterizes stationary property, which will be proved in §5, §6 and §7.

Theorem 4.2 (Characterization Theorem for Stationary Property). *The necessary and sufficient condition for the pair* $[\mathbf{X}, \mathbf{Y}]$ *of flows to have stationary property is that*

(DDT) *Dissipation-Dissipation Theorem*: *For each integer* n, k $(1 \le k < n \le N)$,

$$ \text{(DDT-i)} \quad \gamma_+(n,k) = \gamma_+(n-1,k-1) + \delta_+(n)\gamma_-(n-1,n-k-1) $$

$$ \text{(DDT-ii)} \quad \gamma_-(n,k) = \gamma_-(n-1,k-1) + \delta_-(n)\gamma_+(n-1,n-k-1) $$

(FDT) *Fluctuation-Dissipation Theorem*: *For each integer* n $(1 \le n \le N)$,

$$ \text{(FDT-i)} \quad V_+(n) = (I - \delta_+(n)\delta_-(n))V_+(n-1) $$

$$ \text{(FDT-ii)} \quad V_-(n) = (I - \delta_-(n)\delta_+(n))V_-(n-1) $$

$$ \text{(FDT-iii)} \quad V_+(n-1)\,{}^t\delta_-(n) = \delta_+(n)V_-(n-1) $$

$$ \text{(FDT-iv)} \quad V_+(n)\,{}^t\delta_-(n) = \delta_+(n)V_-(n), $$

where

$$ V_+(0) = V_-(0) = (X(0), {}^tX(0)) = (Y(0), {}^tY(0)). \tag{4.7} $$

§5. Dissipation-Dissipation Theorem—(DDT)

Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of the flows $\mathbf{X} = (X(n); 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell); -N \le \ell \le 0)$ satisfying stationary property with the independence conditions (3.1_{N-1}) and (3.2_{N-1}) in the metric vector space W. The aim of this section is to show (DDT) in Theorem 4.2.

At first, we shall represent the independence conditons (3.1_{N-1}) and (3.2_{N-1}) in terms of the covariance matrix function R of the pair $[\mathbf{X}, \mathbf{Y}]$ of flows. We define for each integer n $(1 \le n \le N+1)$ a block matrix $T_{\pm}(n) (\in M(nd; \boldsymbol{R}))$, called to be Toeplitz matix, by

$$(5.1_{\pm}) \qquad T_{\pm}(n) \equiv \begin{pmatrix} R(0) & R(\pm 1) & \cdots & R(\pm(n-1)) \\ R(\mp 1) & R(0) & \cdots & R(\pm(n-2)) \\ \vdots & \vdots & \ddots & \vdots \\ R(\mp(n-1)) & R(\mp(n-2)) & \cdots & R(0) \end{pmatrix}.$$

It follows from (4.3) that

$$(5.2) \qquad T_{+}(1) = T_{-}(1) = R(0)$$

$$(5.3_{\pm}) \qquad {}^{t}T_{\pm}(n) = T_{\pm}(n) \quad (1 \le n \le N+1).$$

In the sequel, we shall treat the case where the following condition holds:

$$(5.4) \qquad R(0) \in GL(d; \boldsymbol{R}).$$

For each integer n $(1 \le n \le N+1)$, we define two nd-dimensional column vectors $Z_{\pm}(n)$ by

$$(5.5_{+}) \qquad Z_{+}(n) \equiv ({}^{t}X(n-1), {}^{t}X(n-2), \ldots, {}^{t}X(0))$$

$$(5.5_{-}) \qquad Z_{-}(n) \equiv ({}^{t}X(0), {}^{t}X(1), \ldots, {}^{t}X(n-1)).$$

It can be seen that

Lemma 5.1.

$$T_{\pm}(n) = (Z_{\pm}(n), {}^{t}Z_{\pm}(n)) \quad (1 \le n \le N+1).$$

By using Lemma 5.1, we shall prove

Lemma 5.2. *Either of the following* (i) *and* (ii) *holds*:

(i) $$T_{+}(n), T_{-}(n) \in GL(nd; \boldsymbol{R}) \quad (1 \le n \le N+1).$$

(ii) *There exists some* n_0 $(1 \le n_0 \le N)$ *such that*

$$\begin{cases} T_{+}(n), T_{-}(n) \in GL(nd; \boldsymbol{R}) & (1 \le n \le n_0) \\ T_{+}(n), T_{-}(n) \notin GL(nd; \boldsymbol{R}) & (n_0 + 1 \le n \le N+1). \end{cases}$$

Proof. Since it follows from Lemma 5.1 that for each n $(1 \le n \le N+1), T_{+}(n)$ is a regular matrix if and only if $T_{-}(n)$ is a regular matrix, it suffices to show the statements in (i) and (ii) for the matrices $T_{+}(n)$ $(1 \le n \le N+1)$. Suppose that (i) does not hold. We define n_0 by $n_0 \equiv \min\{2 \le n \le N+1; T_{+}(n) \notin GL(d; \boldsymbol{R})\} - 1$. When $n_0 = N$, (ii) holds. Let n_0 be any integer such that $1 \le n_0 \le N-1$. It follows from the definition of n_0 that $T_{+}(n_0+1) \notin GL(nd; \boldsymbol{R})$ and $T_{+}(n) \in GL(nd; \boldsymbol{R})$ for any

n $(1 \le n \le n_0)$. Further, we find that for any n $(n_0 + 2 \le n \le N + 1)$, $T_+(n) \notin GL(nd; \boldsymbol{R})$, because if $T_+(n) \in GL(nd; \boldsymbol{R})$, it then follows from Lemma 5.1 that $T_+(n_0 + 1) \in GL(n_{n_0+1} d; \boldsymbol{R})$. □

Immediately from Lemmas 5.1 and 5.2, we have

Proposition 5.1. (i) *The independence condition* (3.1_{N-1}) *holds if and only if the independence condition* (3.2_{N-1}) *holds.*

(ii) *The independence condition* (3.1_{N-1}) *is equivalent to the following Toeplitz conditon* (5.6_N):

$$(5.6_N) \qquad T_+(n), T_-(n) \in GL(nd; \boldsymbol{R}) \quad (1 \le n \le N).$$

(iii) *The following independence condition* (3.1_N):

$$(3.1_N) \qquad \{X_j(n);\ 1 \le j \le d, 0 \le n \le N\} \text{ is linearly independent in } W$$

is equivalent to the following Toeplitz condition (5.6_{N+1}):

$$(5.6_{N+1}) \qquad T_+(n), T_-(n) \in GL(nd; \boldsymbol{R}) \quad (1 \le n \le N + 1).$$

To prove (DDT), we shall prepare one more lemma.

Lemma 5.3. *For each* n, ℓ $(1 \le n \le N, 0 \le \ell \le n - 1)$,

$$\text{(i)} \qquad R(n - \ell) = -\sum_{k=0}^{n-1} \gamma_+(n,k) R(k - \ell)$$

$$\text{(ii)} \qquad {}^tR(n - \ell) = -\sum_{k=0}^{n-1} \gamma_-(n,k)\, {}^tR(k - \ell).$$

Proof. By taking the inner product of the both-hand sides in the forward KM_2O-Langevin equation (3.5_+) and the vector $X(\ell)$, we have (i). Noting (4.2), similarly, we can show (ii). □

By virtue of Proposition 5.1 and Lemma 5.3, we can show the following (DDT) by using the same procedure as in Theorem 3.1 in [**1**] and so omit its proof.

Theorem 5.1 (Dissipation-Dissipation Theorem). *For each integer* $n, k (1 \le k < n \le N)$,

$$\text{(i)} \qquad \gamma_+(n,k) = \gamma_+(n-1, k-1) + \delta_+(n)\gamma_-(n-1, n-k-1)$$

$$\text{(ii)} \qquad \gamma_-(n,k) = \gamma_-(n-1, k-1) + \delta_-(n)\gamma_+(n-1, n-k-1).$$

§6. Fluctuation-Dissipation Theorem—(FDT).

Let $[\mathbf{X}, \mathbf{Y}]$ be any stationary pair of flows with the independence conditions (3.1_{N-1}) and (3.2_{N-1}) in the metric vector space W. In this section, we shall show (FDT) in Theorem 4.2. For that purpose, we shall prepare some lemmas. From Theorems 2.1 and 2.2, we have

LEMMA 6.1.

(i) $$(X(n), {}^t v_+(n)) = V_+(n) \qquad (0 \le n \le N)$$

(ii) $$(Y(-n), {}^t v_-(-n)) = V_-(n) \quad (0 \le n \le N).$$

LEMMA 6.2.

(i) $$\det T_\pm(n) = \prod_{k=0}^{n-1} \det V_\pm(k) \quad (1 \le n \le N+1)$$

(ii) $$V_+(n), V_-(n) \in GL(d; \boldsymbol{R}) \quad (0 \le n \le N-1).$$

PROOF. By taking the inner product of the both-hand sides in the forward KM_2O-Langevin equation (3.5_+) and the vector $X(n)$, we find from (4.1), (4.3) and Lemma 6.1 (i) that

$$R(0) = -\sum_{k=0}^{n-1} \gamma_+(n,k)\, {}^t R(n-k) + V_+(n). \tag{6.1}$$

Representing (6.1) and n relations in Lemma 5.3 (i) in the form of block matrices, we have

$$\begin{pmatrix} I & \gamma_+(n,n-1) & \cdots & \cdots & \gamma_+(n,0) \\ 0 & I & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I \end{pmatrix} T_+(n+1) = \begin{pmatrix} V_+(n) & 0 & \cdots & 0 \\ {}^t R(1) & & & \\ \vdots & & T_+(n) & \\ \vdots & & & \\ {}^t R(n) & & & \end{pmatrix}$$

and so that $\det T_+(n+1) = \det V_+(n) \det T_+(n)$. As $T_+(1) = R(0) = V_+(0)$, we obtain the plus part in (i). The minus part in (i) is also similarly proved. Since it follows from Proposition 5.1 (i) that Toeplitz condition (5.6_N) holds, (ii) comes from (i). □

LEMMA 6.3.

(i) $$V_\pm(n) = \sum_{k=1}^{n} \gamma_\pm(n,n-k) R(\pm k) + R(0) \quad (0 \le n \le N)$$

(ii) $$R(\pm(n+1)) = -\sum_{k=0}^{n-1} \gamma_\pm(n,k) R(\pm(k+1)) - \delta_\pm(n+1) V_\mp(n) \quad (0 \le n \le N-1).$$

PROOF. (i) follows immediately from (6.1) in the proof of Lemma 6.2. By replacing ℓ and n in Lemma 5.3 (i) by 0 and $n+1$, respectively, and using Theorem 5.1 (i), we have

$$R(n+1) = -\sum_{k=1}^{n} \gamma_+(n,k-1) R(k) - \delta_+(n+1)\left\{\sum_{k=1}^{n} \gamma_-(n,n-k) R(k) + R(0)\right\}.$$

Further, by substituting Lemma 6.3 (i) into the second term of the right-hand side, we find that the plus part in (ii) holds. In a similar way, the minus part in (ii) is proved. □

By Lemmas 6.2 (ii) and 6.3 (ii), we can obtain a formula for two KM_2O-Langevin partial correlation matrix functions $\delta_+ = \delta_+(*)$ and $\delta_- = \delta_-(*)$.

Theorem 6.1 (Formula for KM_2O-Langevin Partial Correlation Matrix Function). *For each integer n $(0 \le n \le N-1)$,*

(i)
$$\delta_+(n+1) = -\left\{ R(n+1) + \sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1) \right\} V_-(n)^{-1}$$

(ii)
$$\delta_-(n+1) = -\left\{ {}^tR(n+1) + \sum_{k=0}^{n-1} \gamma_-(n,k)\, {}^tR(k+1) \right\} V_+(n)^{-1}.$$

Next, we shall prove the following fluctuation-dissipation theorem.

Theorem 6.2 (Fluctuation-Dissipation Theorem-1).

(i)
$$V_+(n) = (I - \delta_+(n)\delta_-(n))V_+(n-1) \quad (1 \le n \le N)$$

(ii)
$$V_-(n) = (I - \delta_-(n)\delta_+(n))V_-(n-1) \quad (1 \le n \le N).$$

Proof. By Lemma 6.3 (i) and Theorem 5.1 (i), we have

$$V_+(n) = \delta_+(n)\left\{ {}^tR(n) + \sum_{k=1}^{n-1} \gamma_+(n-1,k-1)\, {}^tR(k) \right\}$$
$$+ \sum_{k=1}^{n-1} \gamma_+(n-1,n-k-1)\, {}^tR(k) + R(0).$$

Therefore, we can see from (i) and (ii) in Lemma 6.3 that (i) holds. In a similar way, we have (ii). □

Next, we shall show that Burg's relation holds for the stationary pair $[\mathbf{X},\mathbf{Y}]$ of flows.

Theorem 6.3 (Burg's relation).

$$\sum_{k=0}^{n-1} R(k+1)\, {}^t\gamma_-(n,k) = \sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1) \quad (1 \le n \le N).$$

In order to prove Burg's relation, we shall define $d \times d$ matrices A_n $(1 \le n \le N)$ by

(6.2)
$$A_n \equiv \sum_{k=0}^{n-1} R(k+1)\, {}^t\gamma_-(n,k) - \sum_{k=0}^{n-1} \gamma_+(n,k)R(k+1).$$

Immediately from Lemma 6.3 (ii), we have

Lemma 6.4. $A_n = \delta_+(n+1)V_-(n) - V_+(n)\,{}^t\delta_-(n+1)$ $(1 \le n \le N-1)$.

Moreover, by taking the same procedure as Step 4 in the proof of Lemma 4.3 in [**1**], we can show

Lemma 6.5. *For each natural number* n $(2 \le n \le N)$,

$$A_n = \delta_+(n)I_n{}^t\delta_-(n) + \delta_+(n)II_n + III_n{}^t\delta_-(n) + IV_n,$$

where

$$I_n = -V_-(n-1)\,{}^t\delta_+(n-1) + \delta_-(n-1)V_+(n-1)$$

$$\begin{aligned} II_n = &-R(1) - \sum_{j=1}^{n-2}\gamma_-(n-1,n-1-j)R(j+1) \\ &+\delta_-(n-1)\left\{\sum_{j=0}^{n-2}R(j+1)\,{}^t\gamma_-(n-1,j)\right\} - V_-(n-1)\,{}^t\gamma_-(n-1,n-2) \end{aligned}$$

$$\begin{aligned} III_n = &R(1) + \sum_{j=1}^{n-2}R(j+1)\,{}^t\gamma_+(n-1,n-1-j) \\ &-\left\{\sum_{j=0}^{n-2}\gamma_+(n-1,j)R(j+1)\right\}{}^t\delta_+(n-1) + \gamma_+(n-1,n-2)V_+(n-1) \end{aligned}$$

$$\begin{aligned} IV_n = &-\sum_{j=0}^{n-3}\gamma_+(n-1,j)R(j+2) + \sum_{j=0}^{n-3}R(j+2)\,{}^t\gamma_-(n-1,j) \\ &-\left\{\sum_{j=0}^{n-3}\gamma_+(n-1,j)R(j+1)\right\}{}^t\gamma_-(n-1,n-2) \\ &+\gamma_+(n-1,n-2)\left\{\sum_{j=0}^{n-3}R(j+1)\,{}^t\gamma_-(n-1,j)\right\}. \end{aligned}$$

After the above preparations, we shall show Theorem 6.3.

Proof of Theorem 6.3. We shall show that $A_n = 0$ for each n $(1 \le n \le N)$ by induction on n. We treat the case $n = 1$. We see from (6.2) that $A_1 = R(1)\,{}^t\delta_-(1) - \delta_+(1)R(1)$. Since it follows from Lemma 6.3 (ii) that $\delta_\pm(1) = -R(\pm 1)R(0)^{-1}$, we find that $A_1 = 0$. For any fixed n_0 $(2 \le n_0 \le N)$, we assume that $A_m = 0$ for each m $(1 \le m \le n_0 - 1)$. It then follows from Lemma 6.4 that

$$\delta_+(m)V_-(m-1) = V_+(m-1)\,{}^t\delta_-(m) \quad (1 \le m \le n_0). \tag{6.3}$$

Further, by combining (6.3) with Theorem 6.2, we can show

$$\delta_+(m)V_-(m) = V_+(m)\,{}^t\delta_-(m) \quad (1 \le m \le n_0). \tag{6.4}$$

Next, we shall show that $I_m = II_m = III_m = IV_m = 0$ for each m $(2 \le m \le n_0)$. Immediately from (6.4), it follows that $I_m = 0$ for each m $(2 \le m \le n_0)$. By using Theorem 5.1 (i) (ii), Theorem 6.2 and the assumption of mathematical induction, we see that for each m $(2 \le m \le n_0)$,

$$II_m = -(I - \delta_-(m-1)\delta_+(m-1))\left\{R(1) + \sum_{j=1}^{m-2}\gamma_-(m-2, m-2-j)R(j+1) + V_-(m-2)\,{}^t\gamma_-(m-1, m-2)\right\}.$$

By using Theorem 6.1 (i), Theorem 5.1 (ii) and (6.3), we can get

$$R(1) + \sum_{j=1}^{m-2}\gamma_-(m-2, m-2-j)R(j+1) + V_-(m-2)\,{}^t\gamma_-(m-1, m-2) = -II_{m-1}.$$

Hence, we see that $II_m = (I - \delta_-(m-1)\delta_+(m-1))II_{m-1}$ and so

$$II_m = (I - \delta_-(m-1)\delta_+(m-1))\cdots(I - \delta_-(2)\delta_+(2))II_2. \tag{6.5}$$

Since a simple computation gives us that $II_2 = 0$, we conclude from (6.5) that $II_m = 0$ for each m $(2 \le m \le n_0)$. In the same way, we can show that $III_m = 0$ for each m $(2 \le m \le n_0)$.

Noting that for each j $(0 \le j \le m-3)$, $j+2 = m-1-(m-3-j)$ and $0 \le m-3-j \le (m-1)-1$, we can apply the case where $n = m-1$ and $\ell = m-3-j$ in Lemma 5.3 (i) to see that for each j $(0 \le j \le m-3)$,

$$\begin{aligned} R(j+2) &= R(m-1-(m-3-j)) \\ &= -\gamma_+(m-1, m-2)R(j+1) - \sum_{k=0}^{m-3}\gamma_+(m-1, k)R(j-(m-3-k)). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\sum_{j=0}^{m-3} R(j+2)\,{}^t\gamma_-(m-1, j) \\ &= -\gamma_+(m-1, m-2)\left\{\sum_{j=0}^{m-3} R(j+1)\,{}^t\gamma_-(m-1, j)\right\} \\ &\quad - \sum_{k=0}^{m-3}\gamma_+(m-1, k)\left\{\sum_{j=0}^{m-3} R(j-(m-3-k))\,{}^t\gamma_-(m-1, j)\right\}. \end{aligned} \tag{6.6}$$

By noting $R(j+2) = {}^t({}^tR(j+2))$ and using Lemma 5.3 (ii), similarly, we have

$$\text{(6.7)}\qquad \sum_{j=0}^{m-3} \gamma_+(m-1,j)R(j+2)$$

$$= -\left\{\sum_{j=0}^{m-3} \gamma_+(m-1,j)R(j+1)\right\} {}^t\gamma_-(m-1,m-2)$$

$$- \sum_{j=0}^{m-3} \gamma_+(m-1,j)\left\{\sum_{k=0}^{m-3} R(j-(m-3-k))\, {}^t\gamma_-(m-1,j)\right\}.$$

Therefore, we can conclude from (6.6) and (6.7) that $IV_m = 0$ for each m $(2 \le m \le n_0)$.

Thus we have proved that $A_m = 0$ for each m $(2 \le m \le n_0)$, which completes the proof of Theorem 6.3. □

Remark 6.1. In §4 of [**1**], we have proved Theorem 6.3 through thirteen steps by expanding the term IV_n in A_n of Lemma 6.5 successively with respect to KM_2O-Langevin partial correlation matrix functions.

We shall also call relations (6.3) and (6.4) the fluctuation-dissipation theorem.

Theorem 6.4 (Fluctuation-Dissipation Theorem-2).

$$\text{(i)}\qquad \delta_+(n)V_-(n-1) = V_+(n-1)\, {}^t\delta_-(n) \quad (1 \le n \le N)$$

$$\text{(ii)}\qquad \delta_+(n)V_-(n) = V_+(n)\, {}^t\delta_-(n) \qquad (1 \le n \le N).$$

§7. Characterizaton theorem for stationary flow.

The aim of this section is to complete the proof of Theorem 4.2. Let $[\mathbf{X}, \mathbf{Y}]$ be any pair of the flows $\mathbf{X} = (X(n);\, 0 \le n \le N)$ and $\mathbf{Y} = (Y(\ell);\, -N \le \ell \le 0)$ satisfying the independence conditions (3.1_{N-1}) and (3.2_{N-1}) in the metric vector space W. Further, we shall assume that (DDT) and (FDT) hold. As we have found in (6.4), we note that (FDT-iv) comes from (FDT-i), (FDT-ii) and (FDT-iii).

We shall introduce some notations. For each m, n $(0 \le m, n \le N)$, we denote by $R_+(m,n)$ (resp. $R_-(-m,-n)$) the inner product of $X(m)$ and $X(n)$ (resp. the inner product of $Y(-m)$ and $Y(-n)$):

$$(7.1_+)\qquad R_+(m,n) \equiv (X(m), {}^tX(n))$$

$$(7.1_-)\qquad R_-(-m,-n) \equiv (Y(-m), {}^tY(-n)).$$

It is to be noted that

$$(7.2_\pm)\qquad {}^tR_\pm(\pm m, \pm n) = R_\pm(\pm n, \pm m).$$

Further, we define a $d \times d$ matrix V by

$$(7.3)\qquad V \equiv R_+(0,0) = R_-(0,0) = (X(0), {}^tX(0)) = (Y(0), {}^tY(0)).$$

In order to complete the proof of Theorem 4.2, we have only to show the following Proposition 7.1.

PROPOSITION 7.1. *If* (DDT) *and* (FDT) *hold, then the pair* $[\mathbf{X}, \mathbf{Y}]$ *of flows satisfies the following properties*:

(i) $R_{\pm}(\pm m \pm \ell, \pm n \pm \ell) = R_{\pm}(\pm m, \pm n) \quad (0 \le n \le m \le N, 0 \le \ell \le N - m)$

(ii) $R_{+}(m,n) = R_{-}(-n,-m) \qquad (0 \le n \le m \le N).$

In the sequel, we shall prove Proposition 7.1 by showing seven lemmas consisting of eighteen claims.

[Step 1] By using Theorems 2.1, 2.2 and KM_2O-Langevin equations $(3.5_{\pm})$, we have

LEMMA 7.1.

(i) $$(X(m), {}^t v_{+}(n)) = \delta_{m,n} V_{+}(n) \quad (0 \le m \le n \le N)$$

(ii) $$(Y(-m), {}^t v_{-}(-n)) = \delta_{m,n} V_{-}(n) \quad (0 \le m \le n \le N).$$

By taking the same procedure as in Lemmas 5.3 and 6.3, we obtain

LEMMA 7.2.

(i) $$R_{\pm}(\pm n, \pm \ell) = -\sum_{k=0}^{n-1} \gamma_{\pm}(n,k) R_{\pm}(\pm k, \pm \ell) \qquad (0 \le \ell < n \le N)$$

(ii) $$R_{\pm}(\pm n, \pm n) = -\sum_{k=0}^{n-1} \gamma_{\pm}(n,k) R_{\pm}(\pm k, \pm n) + V_{\pm}(n) \quad (1 \le n \le N).$$

[Step 2] By a direct calculation, we can show

LEMMA 7.3.

(i) $$V_{+}(0) = V_{-}(0) = V$$

(ii) $$R_{\pm}(\pm 1, 0) = -\delta_{\pm}(1) V$$

(iii) $$R_{+}(1,0) = R_{-}(0,-1)$$

(iv) $$R_{\pm}(\pm 1, \pm 1) = R_{\pm}(0,0) = V.$$

LEMMA 7.4. *For each* n $(1 \le n \le N)$,

(i) $$R_{+}(n,0) = -\delta_{+}(n) V_{-}(n-1) - \sum_{k=1}^{n-1} \gamma_{+}(n-1, k-1) R_{+}(k,0)$$

(ii) $$V_{-}(n) = V + \sum_{k=1}^{n} R_{+}(0,k)\, {}^t\gamma_{-}(n, n-k).$$

Proof. By induction on n, we shall show (i) together with (ii). When $n = 1$, (i) (resp. (ii)) follows from Lemma 7.3 (i) (resp. Lemmas 7.2 (i) and 7.3 (iv)). For any fixed n_0 $(2 \le n_0 \le N-1)$, let us assume that (i) and (ii) hold for $n = n_0 - 1$. By Lemma 7.2 (i) and (DDT-i), we have

$$R_+(n_0, 0) = -\sum_{k=1}^{n_0-1} \gamma_+(n_0 - 1, k-1) R_+(k, 0) - \delta_+(n_0)\left\{ V + \sum_{k=1}^{n_0-1} \gamma_-(n_0 - 1, n_0 - 1 - k) R_+(k, 0) \right\}.$$

By applying (ii) with $n = n_0 - 1$ into the second term of the above right-hand side, we find that (i) holds for $n = n_0$.

Next, we shall show (ii) for $n = n_0$. Since $V_-(\star)$ is symmetric, we can transpose the both-hand sides of (FDT-ii) with (FDT-iii) to get.

$$V_-(n_0) = V_-(n_0 - 1) - V_-(n_0 - 1)\, {}^t\delta_+(n_0)\, {}^t\delta_-(n_0).$$

By taking the both-hand sides of (i) with $n = n_0$ and noting (7.2_+), we have

$$V_-(n_0 - 1)\, {}^t\delta_+(n_0) = -R_+(0, n_0) - \sum_{k=1}^{n_0-1} R_+(0, k)\, {}^t\gamma_+(n_0 - 1, k - 1).$$

By substituting this into the above second term, we obtain

$$V_-(n_0) = V_-(n_0 - 1) + \left\{ R_+(0, n_0) + \sum_{k=1}^{n_0-1} R_+(0, k)\, {}^t\gamma_+(n_0 - 2, k - 1) \right\} {}^t\delta_-(n_0).$$

By applying (ii) with $n = n_0 - 1$ to the first term in the above equation, we have

$$V_-(n_0) = V + \sum_{k=1}^{n_0-1} R_+(0, k)\, {}^t\gamma_-(n_0 - 1, n_0 - 1 - k) + \left\{ R_+(0, n_0) + \sum_{k=1}^{n_0-1} R_+(0, k)\, {}^t\gamma_+(n_0 - 1, k - 1) \right\} {}^t\delta_-(n_0 - 1).$$

Finally, we can apply (DDT-ii) to the above equation to find that (ii) with $n = n_0$ holds.

Thus, we find that (i) and (ii) hold for each n $(1 \le n \le N)$. □

[Step 3] For each n $(1 \le n \le N)$, we shall consider the following four statements $(A_n), (B_n), (C_n)$ and (D_n):

$$(A_n) \qquad \sum_{k=0}^{n-1} \gamma_+(n, k) R_+(k+1, 0) = \sum_{k=0}^{n-1} R_+(k+1, 0)\, {}^t\gamma_-(n, k)$$

$$(B_n) \qquad V_+(n) = V + \sum_{k=1}^{n} R_+(k, 0)\, {}^t\gamma_+(n, n-k)$$

$$(C_n) \qquad R_+(0,n) = -\delta_-(n)V_+(n-1) - \sum_{k=1}^{n-1} \gamma_-(n-1,k-1)R_+(0,k)$$

$$(D_n) \qquad R_+(0,n) = -\sum_{k=0}^{n-1} \gamma_-(n,k)R_+(0,k).$$

We shall investigate some logical relations among the above statements.

Lemma 7.5.

(i) (A_1), (B_1), (C_1) *and* (D_1) *hold.*
(ii) *If* (A_n) *holds, then* (C_{n+1}) *holds for each* n $(1 \le n \le N-1)$.
(iii) *If* (B_{n-1}) *and* (C_n) *hold, then* (B_n) *and* (D_n) *hold for each* n $(2 \le n \le N)$.

Proof. By Lemma 7.3 (i) and (FDT-iii), we see that (A_1) holds. Immediately from (4.7) and (7.3), we see that (B_1) holds. By (4.7), (7.3), Lemma 7.3 (ii) and (FDT-iii), we find that (C_1) and (D_1) hold. Thus, we have (i).

By taking the transpose of the both-hand sides of Lemma 7.4 (i) with n replaced by $n+1$ and then noting (FDT-iii) and (7.2_+), we see that $R_+(0,n+1) = -\delta_-(n+1)V_+(n) - \sum_{k=0}^{n-1} R_+(0,k+1)\,{}^t\gamma_+(n,k)$. Thus, we obtain (ii).

Finally, we shall show (iii). It follows from (DDT-i) and (B_{n-1}) that the right-hand side of $(B_n) = V_+(n-1) + \{R_+(n,0) + \sum_{k=1}^{n-1} R_+(k,0)\,{}^t\gamma_-(n-1,k-1)\}\,{}^t\delta_+(n)$. By taking the transpose of (C_n) and noting (7.2_+), we find that $R_+(n,0) + \sum_{k=1}^{n-1} R_+(k,0)\,{}^t\gamma_-(n-1,k-1) = -V_+(n-1)\,{}^t\delta_-(n)$. Therefore, we see that the right-hand side of $(B_n) = V_+(n-1) - V_+(n-1)\,{}^t\delta_-(n)$ and so that (B_n) comes from (FDT-i) and (FDT-iii). By substituting the transpose of the both-hand sides of (B_{n-1}) into $V_+(n-1)$ in the first term of (C_n) and using (DDT-ii), we see that $R_+(0,n) = -\delta_-(n)V - \sum_{k=1}^{n-1} \gamma_-(n,k)R_+(0,k)$, which with (3.6_-) implies that (D_n) holds. □

Immediately from Lemma 7.5, we have

Lemma 7.6. *Let* n *be any integer such that* $2 \le n \le N$. *If relation* (A_m) *holds for each* m $(2 \le m \le n-1)$, *then relations* (B_m), (C_m) *and* (D_m) *hold for each* m $(1 \le m \le n)$.

[Step 4] After the above preparations, we shall show

Lemma 7.7.

$$\text{(i)} \qquad R_+(j,k) = R_+(j-k,0) \quad \textit{for any } k,j \ (0 \le k \le j \le N)$$

$$\text{(ii)} \qquad R_-(-j,-k) = R_+(k,j) \quad \textit{for any } k,j \ (0 \le k \le j \le N).$$

For that purpose, we shall consider for each n $(1 \le n \le N)$ the following statements (E_n) and (F_n):

$$(E_n) \qquad R_+(j,k) = R_+(j-k,0) \quad \text{for any } k,j \ (0 \le k \le j \le n)$$

$$(F_n) \qquad R_-(-j,-k) = R_+(k,j) \quad \text{for any } k,j \ (0 \le k \le j \le n)$$

and show these together with the statement (A_n) by induction on n. Immediately from Lemmas 7.3 (iv) and 7.5 (i), we have

CLAIM 1. *The statements* (A_1), (E_1) *and* (F_1) *hold.*

[Step 5] Fix any integer n_0 $(2 \le n_0 \le N)$ and assume that $(A_m), (E_m)$ and (F_m) hold for any m $(1 \le m \le n_0 - 1)$. The aim is to show that the statements $(A_{n_0}), (E_{n_0})$ and (F_{n_0}) hold. From the assumption that (E_m) hold for any m $(1 \le m \le n_0 - 1)$ and noting (7.2_+), we can show

CLAIM 2. *For each* n $(1 \le n \le n_0 - 1)$,

(i) $$R_+(j,k) = R_+(n-k, n-j) \qquad (0 \le j, k \le n)$$

(ii) $$R_+(j+1,k) = R_+(j, k-1) \qquad (0 \le j, k-1 \le n_0 - 2)$$

(iii) $$R_+(n-k, j+1) = R_+(n-1-k, j) \qquad (0 \le j \le n_0 - 2, 0 \le k \le n-1)$$

(iv) $$R_+(n-k, n-1) = R_+(0, k-1) \qquad (1 \le k \le n)$$

(v) $$R_+(j,0) = R_+(n-k, n-k-j) \qquad (0 \le j \le n-k \le n)$$

(vi) $$R_+(k,n) = R_+(0, n-k) \qquad (0 \le k \le n)$$

(vii) $$R_+(j+2,0) = R_+(n_0-1, n_0-3-j) \qquad (0 \le j \le n_0 - 3)$$

(viii) $$R_+(j+1,0) = R_+(n_0-2, n_0-3-j) \qquad (0 \le j \le n_0 - 3)$$

(ix) $$R_+(n_0-1-k, 0) = R_+(n_0-1, k) \qquad (0 \le k \le n_0 - 1)$$

(x) $$R_+(k, n_0-3-j) = R_+(j, n_0-3-k) \qquad (0 \le j, k \le n_0 - 3).$$

CLAIM 3. *For each* k $(1 \le k \le n_0 - 1)$,

$$R_+(n_0, k) = R_+(n_0-1, k-1) - \delta_+(n_0) H_-(k; n_0 - 1),$$

where the matrix function $H_-(\star; *)$ *is defined by*

$$H_-(k;n) \equiv R_+(0,k) + \sum_{j=0}^{n-1} \gamma_-(n,j) R_+(n-k, j) \quad (1 \le k \le n). \tag{7.4}$$

PROOF. By using Lemma 7.2 (i) and (DDT-i) and noting (7.2_+) and using Claim 2 (ii), we have

$$R_+(n_0,k) = -\delta_+(n_0)\left\{R_+(0,k) + \sum_{j=0}^{n_0-2} \gamma_-(n_0-1, j) R_+(n_0-1-j, k)\right\}$$
$$+ \sum_{j=1}^{n_0-1} \gamma_+(n_0-1, j-1) R_+(j, k-1).$$

Applying Lemma 7.2 (i) to the above equation, we have Claim 3. □

By using Lemma 7.6, we have

CLAIM 4. $H_-(n;n) = 0$ $(1 \le n \le n_0)$.

We shall obtain certain algorithm about the matrix function $H_-(\star; *)$.

CLAIM 5. *For each* n, k $(2 \le n \le n_0, 1 \le k \le n-1)$,

$$H_-(k;n) = H_-(k;n-1) + \delta_-(n)\sum_{j=1}^{k-1}\delta_+(n-j)H_-(k-j;n-1-j).$$

PROOF. By using (DDT-ii) and Claim 2 (iii), we have

$$H_-(k;n) = H_-(k;n-1) + \delta_-(n)\{R_+(n-k,0) + J_+(k;n)\},$$

where $J_+(k;n) = \sum_{j=0}^{n-2}\gamma_+(n-1,n-2-j)R_+(n-k,j+1)$. By using (DDT-i), Claim 2 (iii) and Claim 2 (iv), we see that $J_+(k;n) = \delta_+(n-1)H_-(k-1;n-2) + J_+(k-1;n-1)$. Therefore, we get

$$\begin{aligned} H_-(k;n) = H_-(k;n-1) &+ \delta_-(n)\delta_+(n-1)H_-(k-1;n-2) \\ &+ \delta_-(n)\{R_+(n-k,0) + J_+(k-1;n-1)\}. \end{aligned}$$

By repeating the same procedure, we have

$$\begin{aligned} H_-(k;n) = H_-(k;n-1) &+ \delta_-(n)\sum_{j=1}^{k-1}\delta_+(n-j)H_-(k-j;n-1-j) \\ &+ \delta_-(n)\{R_+(n-k,0) + J_+(1;n-k+1)\}. \end{aligned}$$

On the other hand, it can be seen from Claim 2 (v) and Lemma 7.2 (i) that $R_+(n-k,0) + J_+(1;n-k+1) = 0$. Thus we have proved Claim 5. □

CLAIM 6. $H_-(k;n) = 0$ $(1 \le k \le n \le n_0)$.

PROOF. We shall consider for each n $(1 \le n \le n_0)$ the following statement

$$(*_n^-) \qquad H_-(k;n) = 0 \quad \text{for any } k \ (1 \le k \le n)$$

and prove it by induction on n. The statement $(*_1^-)$ follows from Claim 3. Let n_1 be any fixed integer such that $1 \le n_1 \le n_0 - 1$ and assume that the statement $(*_n^-)$ holds for each n $(1 \le n \le n_1)$. The statement that $H_-(k;n_1+1) = 0$ for any k $(1 \le k \le n_1)$ comes from Claim 3 when $k = n_1+1$ and from Claim 4 when $1 \le k \le n_1$, respectively. □

After the above preparations, we shall prove the statement (E_{n_0}).

CLAIM 7. *The statement* (E_{n_0}) *holds*;

(i) $$R_+(n_0,k) = R_+(n_0-k,0) \quad \textit{for any } k \ (1 \le k \le n_0-1)$$

(ii) $$R_+(n_0,n_0) = R_+(0,0).$$

PROOF. (i) follows from Claim 3 and Claim 6. By Lemma 7.2 (i) and Claim 2 (vi), we have

$$R_+(n_0,n_0) = -\delta_+(n_0)R_+(0,n_0) + V_+(n_0) - \sum_{k=1}^{n_0-1}\gamma_+(n_0,k)R_+(0,n_0-k).$$

By applying the statement (C_{n_0}), which comes from Lemma 7.6, to the above equation, we see that

$$R_+(n_0,n_0) = V_+(n_0) + \delta_+(n_0)\delta_-(n_0)V_+(n_0-1) - \sum_{k=1}^{n_0-1}\{\gamma_+(n_0,k) - \delta_+(n_0)\gamma_-(n_0-1,n_0-k-1)\}R_+(0,n_0-k).$$

Therefore, by using (FDT-i), (FDT-iii) and (DDT-i), we have

$$R_+(n_0,n_0) = V_+(n_0-1) - \sum_{k=1}^{n_0-1}\gamma_+(n_0-1,n_0-1-k)R_+(0,k).$$

Further, by applying the statement (B_{n_0}), which comes from Lemma 7.6, to the above equation, we get (ii). □

[Step 6] Next, we shall prove the statement (A_{n_0}). For that purpose, similarly as in (6.2), we shall define $d\times d$ matrices $\tilde{A}_n$ $(1\le n\le N)$ by

$$\tilde{A}_n \equiv \sum_{k=0}^{n-1} R(k+1,0)\,{}^t\gamma_-(n,k) - \sum_{k=0}^{n-1}\gamma_+(n,k)R(k+1,0). \tag{7.5}$$

By taking the same procedure as in the proof of Lemma 6.5, we can show

Claim 8. *For each natural number* n $(2\le n\le n_0)$,

$$\tilde{A}_n = \delta_+(n)\tilde{I}_n{}^t\delta_-(n) + \delta_+(n)\widetilde{II}_n + \widetilde{III}_n{}^t\delta_-(n) + \widetilde{IV}_n,$$

where

$$\tilde{I}_n = -V_-(n-1)\,{}^t\delta_+(n-1) + \delta_-(n-1)V_+(n-1)$$

$$\widetilde{II}_n = -R_+(1,0) - \sum_{j=1}^{n-2}\gamma_-(n-1,n-1-j)R_+(j+1,0) + \delta_-(n-1)\left\{\sum_{j=0}^{n-2}R_+(j+1,0)\,{}^t\gamma_-(n-1,j)\right\} - V_-(n-1)\,{}^t\gamma_-(n-1,n-2)$$

$$\widetilde{III}_n = R_+(1,0) + \sum_{j=1}^{n-2}R_+(j+1,0)\,{}^t\gamma_+(n-1,n-1-j) - \left\{\sum_{j=0}^{n-2}\gamma_+(n-1,j)R_+(j+1,0)\right\}{}^t\delta_+(n-1) + \gamma_+(n-1,n-2)V_+(n-1)$$

$$\widetilde{IV}_n = -\sum_{j=0}^{n-3}\gamma_+(n-1,j)R_+(j+2,0) + \sum_{j=0}^{n-3}R_+(j+2,0)\,{}^t\gamma_-(n-1,j) - \left\{\sum_{j=0}^{n-3}\gamma_+(n-1,j)R_+(j+1,0)\right\}{}^t\gamma_-(n-1,n-2) + \gamma_+(n-1,n-2)\left\{\sum_{j=0}^{n-3}R_+(j+1,0)\,{}^t\gamma_-(n-1,j)\right\}.$$

After the above preparations, we shall show the statement (A_{n_0}). Immediately from (FDT-iv), we have

Claim 9. $\tilde{I}_{n_0} = 0$.

Claim 10. $\widetilde{II}_{n_0} = 0$.

Proof. By using (DDT-i), (DDT-ii), (FDT-ii) and the statement (A_{n_0-1}), which comes from the assumption of mathematical induction, we see that

$$\widetilde{II}_{n_0} = -(I - \delta_-(n_0-1)\delta_+(n_0-1))\Bigg\{R_+(1,0) + \sum_{j=1}^{n_0-2} \gamma_-(n_0-2, n_0-2-j)R_+(j+1,0) + V_-(n_0-2)\,{}^t\gamma_-(n_0-1, n_0-2)\Bigg\}. \tag{7.6}$$

By using Lemma 7.4 (i) and (DDT-ii), we get

$$\begin{aligned} R_+(1,0) &+ \sum_{j=1}^{n_0-2} \gamma_-(n_0-2, n_0-2-j)R_+(j+1,0) + V_-(n_0-2)\,{}^t\gamma_-(n_0-1, n_0-2) \\ &= R_+(1,0) + \sum_{j=1}^{n_0-3} \gamma_-(n_0-2, n_0-2-j)R_+(j+1,0) \\ &\quad - \delta_-(n_0-2)\delta_+(n_0-1)V_-(n_0-2) - \delta_-(n_0-2)\sum_{j=0}^{n_0-3} \gamma_+(n_0-2, j)R_+(j+1,0) \\ &\quad + V_-(n_0-2)\,{}^t\gamma_-(n_0-2, n_0-3) + V_-(n_0-2)\,{}^t\delta_+(n_0-2)\,{}^t\delta_-(n_0-1). \end{aligned}$$

Since it follows from (FDT-iii) and (FDT-iv) that $\delta_-(n_0-2)\delta_+(n_0-1)V_-(n_0-2) = V_-(n_0-2)\,{}^t\delta_+(n_0-2)\,{}^t\delta_-(n_0-1)$, we find from (A_{n_0-1}) that

$$\begin{aligned} R_+(1,0) &+ \sum_{j=1}^{n_0-2} \gamma_-(n_0-2, n_0-2-j)R_+(j+1,0) + V_-(n_0-2)\,{}^t\gamma_-(n_0-1, n_0-2) \\ &= -\widetilde{II}_{n_0-1}. \end{aligned} \tag{7.7}$$

Thus, it follows from (7.6) and (7.7) that $\widetilde{II}_{n_0} = (I - \delta_-(n_0-1)\delta_+(n_0-1))\widetilde{II}_{n_0-1}$ and so that $\widetilde{II}_{n_0} = (I - \delta_-(n_0-1)\delta_+(n_0-1))\cdots(I - \delta_-(2)\delta_+(2))\widetilde{II}_2$. Since a direct calculation gives us that $\widetilde{II}_2 = 0$, we can conclude that $\widetilde{II}_{n_0} = 0$. □

By taking the same consideration as in Claim 10, we have

Claim 11. $\widetilde{III}_{n_0} = 0$.

[Step 7] For completing the proof of the statement (A_{n_0}), we have to show that $\widetilde{IV}_{n_0} = 0$. For that purpose, we shall prove the statement (F_{n_0}). We have never used the statements (F_m) $(1 \le m \le n_0 - 1)$ until now. By using the same procedure as in Claim 3, we can show

CLAIM 12. *For each* k $(1 \le k \le n_0 - 1)$,

$$R_-(-k, -n_0) = R_-(-(k-1), -(n_0-1)) - H_+(k; n_0 - 1)\, {}^t\delta_-(n_0),$$

where the matrix function $H_+(\star; *)$ *is defined by*

$$H_+(k; n) \equiv R_+(0, k) + \sum_{j=0}^{n-1} R_+(n-k, j)\, {}^t\gamma_+(n, j) \quad (1 \le k \le n). \tag{7.8}$$

Relating to Claim 4, we shall show

CLAIM 13. $H_+(n; n) = 0$ $(1 \le n \le n_0)$.

PROOF. Let n be any integer such that $1 \le n \le n_0$. By (7.8), we get

$$H_+(n; n) = R_+(0, n) + \sum_{j=0}^{n-1} R_+(0, j)\, {}^t\gamma_+(n, j).$$

By taking the transpose of the both-hand sides for $\ell = 0$ in Lemma 7.2 (i), we find that the right-hand side of the above equation is equal to 0. □

We shall obtain an algorithm for the matrix function $H_+(\star; *)$ that corresponds to Claim 5. By using the same procedure as in Claim 5, we can prove

CLAIM 14. *For each* k, n $(2 \le n \le n_0, 1 \le k \le n - 1)$,

$$H_+(k; n) = H_+(k; n-1) + \left\{ \sum_{j=1}^{k-1} H_+(k-j; n-1-j)\, {}^t\delta_+(n-j) \right\} {}^t\delta_+(n).$$

By taking the same consideration as in the proof of Claim 6, we can see from Claim 13 and Claim 14 that

CLAIM 15. $H_+(k; n) = 0$ $(1 \le k \le n \le n_0)$.

After the above preparations, we shall prove the statement (F_{n_0}).

CLAIM 16. *The statement* (F_{n_0}) *holds, i.e.,*

(i) $$R_-(-n_0, -k) = R_+(k, n_0) \quad \textit{for any } k \ (1 \le k \le n_0 - 1)$$

(ii) $$R_-(-n_0, -n_0) = R_+(n_0, n_0).$$

PROOF. Let k be any integer such that $1 \le k \le n_0 - 1$. By applying Claim 15 to Claim 12, we see that $R_-(-n_0, -k) = R_-(-(n_0 - 1), -(k-1))$. Hence, by using the statement (F_{n_0-1}), we have

$$R_-(-n_0, -k) = R_+(k-1, n_0-1). \tag{7.9}$$

By using the statement (E_{n_0}), which has been proved in Claim 7, we find that (i) holds.

Since it follows from $(7.2_\pm)$, (7.9) and Claim 2 (ix) that $R_-(-k,-n_0) = R_+(n_0-k,0)$ $(0 \le k \le n_0-1)$, we see from Lemma 7.2 (ii) that

$$R_-(-n_0,-n_0) = -\delta_-(n_0)R_+(n_0,0) + V_-(n_0) - \sum_{k=1}^{n_0-1} \gamma_-(n_0,k)R_+(n_0-k,0).$$

By applying Lemma 7.4 (i) to the above equation and then using (DDT-ii) and Claim 2 (ix), we obtain

$$R_-(-n_0,-n_0) = \delta_-(n_0)\delta_+(n_0)V_-(n_0-1) + V_-(n_0) - \sum_{k=1}^{n_0-2} \gamma_-(n_0-1,k)R_+(n_0-1,k).$$

Therefore, by applying the transpose of the both-hand sides of Lemma 7.4 (ii), we find that $R_+(n_0,n_0) = V$, which with Claim 7 implies that (ii) holds. □

[Step 8] Next, we shall complete the proof of the statement (A_{n_0}).

Claim 17. $\widetilde{IV}_{n_0} = 0$.

Proof. By Claim 2 (vii) and Lemma 7.2 (i), we find that for each j $(0 \le j \le n_0-3)$,

$$\begin{aligned} R_+(j+2,0) &= R_+(n_0-1,n_0-3-j) \\ &= -\gamma_+(n_0-1,n_0-2)R_+(n_0-2,n_0-3-j) \\ &\quad - \sum_{k=0}^{n_0-3} \gamma_+(n_0-1,k)R_+(k,n_0-3-j). \end{aligned}$$

Hence, by using Claim 2 (viii), we have

$$\begin{aligned} (7.10) \qquad & \sum_{j=0}^{n_0-3} R_+(j+2,0)\, {}^t\gamma_-(n_0-1,j) \\ &= -\gamma_+(n_0-1,n_0-2)\left\{\sum_{j=0}^{n_0-3} R_+(j+1,0)\, {}^t\gamma_-(n_0-1,j)\right\} \\ &\quad - \sum_{k=0}^{n_0-3} \gamma_+(n_0-1,k)\left\{\sum_{j=0}^{n_0-3} R_+(k,n_0-3-j)\, {}^t\gamma_-(n_0-1,j)\right\}. \end{aligned}$$

On the other hand, by using Claim 2 (ii) and (F_{n_0-1}), we have

$$R_+(j+2,0) = R_+(n_0-1,n_0-3-j) = R_-(-(n_0-3-j),-(n_0-1)).$$

Hence, by (7.2_-) and Lemma 7.2 (i), we find that for each j $(0 \le j \le n_0-3)$,

$$\begin{aligned} R_+(j+2,0) &= -{}^tR_-(-(n_0-2),-(n_0-3-j))\, {}^t\gamma_-(n_0-1,n_0-2) \\ &\quad - \sum_{k=0}^{n_0-3} {}^tR_-(-k,-(n_0-3-j))\, {}^t\gamma_-(n_0-1,k). \end{aligned}$$

By using $(7.2_{\pm})$ and (F_{n_0-1}) again, we see that ${}^tR_-(-k,-(n_0-3-j)) = R_+(k,n_0-3-j)$ $(0 \le k \le n_0-2)$. Hence, it follows from Claim 2 (viii) and Claim 2 (x) that

$$\text{(7.11)} \quad \sum_{j=0}^{m-3}\gamma_+(m-1,j)R_+(j+2,0)$$
$$= -\left\{\sum_{j=0}^{m-3}\gamma_+(n_0-1,j)R_+(j+1,0)\right\}{}^t\gamma_-(n_0-1,n_0-2)$$
$$-\sum_{k=0}^{n_0-3}\gamma_+(n_0-1,k)\left\{\sum_{j=0}^{n_0-3}R_+(k,n_0-3-j))\,{}^t\gamma_-(n_0-1,j)\right\}.$$

Hence, we can see from (7.10) and (7.11) that $\widetilde{IV}_{n_0} = 0$. □

By substituting Claim 9, Claim 10, Claim 11 and Claim 17 into Claim 8, we can show

Claim 18. *The statement* (A_{n_0}) *holds.*

[Step 9] Consequently, we have arrived at the final position to show Proposition 7.1. From Claim 1, Claim 7, Claim 16 and Claim 18, we have Lemma 7.7. Thus, we can conclude that Proposition 7.1 holds.

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