J. Math. Soc. Japan
Vol. 51, No. 4, 1999

Positive solutions of a forced nonlinear elliptic boundary value problem

By Kenichiro UMEZU

(Received July 25, 1996)
(Revised Dec. 9, 1997)

Abstract. This paper is a continuation of the previous paper Taira and Umezu [**12**] where we studied the existence and uniqueness of positive solutions of a class of *sublinear* elliptic problems with *degenerate* boundary conditions. We intend here to give a further investigation of the set of positive solutions in the *forced* case.

1. Introduction.

Let D be a bounded domain of Euclidean space $\boldsymbol{R}^N$, $N \geq 2$, with C^∞ boundary ∂D; its closure $\bar{D} = D \cup \partial D$ is an N-dimensional, compact C^∞ manifold with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real C^∞ coefficients on $\bar{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$, $x \in \bar{D}$, $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \bar{D}, \xi \in \boldsymbol{R}^N,$$

(2) $c(x) > 0$ in D.

In the first part of this paper, we consider the following semilinear elliptic boundary value problem:

$$(*)_\lambda \qquad \begin{cases} Au = \lambda u + f(u) & \text{in } D, \\ Bu = a\dfrac{\partial u}{\partial \nu} + (1-a)u = 0 & \text{on } \partial D. \end{cases}$$

Here:

(1) λ is a real parameter.
(2) f is a real-valued C^1-function on $[0, \infty)$.
(3) $a \in C^\infty(\partial D)$ and $0 \leq a(x) \leq 1$ on ∂D.
(4) $\partial/\partial\nu$ is the conormal derivative associated with the operator A: $\partial/\partial\nu = \sum_{i,j=1}^{N} a^{ij}n_j\partial/\partial x_i$, where $\boldsymbol{n} = (n_1, n_2, \dots, n_N)$ is the unit exterior normal to the boundary ∂D.

1991 *Mathematics Subject Classification*. 35J65.

Key Words and Phrases. Forced nonlinear elliptic problem, degenerate boundary condition, asymptotic behavior of positive solution, super-sub-solution.

It seems worth pointing out that we study the case of a first order degenerate boundary condition which is the Dirichlet condition if $a \equiv 0$ on ∂D, and which is the Neumann condition if $a \equiv 1$ on ∂D.

A function u is said to be a *solution* of problem $(*)_\lambda$ if $u \in C^2(\bar{D})$ and satisfies problem $(*)_\lambda$. If the solution u is positive everywhere in D, then it is called a *positive solution* of problem $(*)_\lambda$.

We assume that the nonlinearity f is forced, that is,

$$\text{(H.1)} \qquad f(0) > 0,$$

and in addition assume that

$$\text{(H.2)} \qquad \left(\frac{f(t)}{t}\right)' < 0, \quad t > 0.$$

From the condition (H.2), we can see that $f(t)/t$ is strictly decreasing in $t > 0$. So, we can put

$$k_\infty = \lim_{t \to \infty} \frac{f(t)}{t}.$$

Let λ_1 be the first eigenvalue of the linear eigenvalue problem:

$$\begin{cases} Au = \lambda u & \text{in} \quad D, \\ Bu = 0 & \text{on} \quad \partial D. \end{cases}$$

It is known ([**9**]) that the first eigenvalue λ_1 is positive.

Under the conditions (H.1) and (H.2), Taira and Umezu [**12**, Corollary 1] proved that

(1) problem $(*)_\lambda$ has a positive solution if and only if $\lambda < \lambda_1 - k_\infty$ (k_∞ may be equal to $-\infty$),

(2) the existence of positive solutions of problem $(*)_\lambda$ is unique in space $C^2(\bar{D})$.

The purpose of the first part is to give a further precise behaviour of the positive solution set of problem $(*)_\lambda$.

Now the first result of ours is the following.

Theorem 1. *Let the conditions* (H.1) *and* (H.2) *be satisfied. If $u(\lambda)$ is a unique positive solution of problem $(*)_\lambda$ for each $\lambda < \lambda_1 - k_\infty$ (possibly $k_\infty = -\infty$), then the following assertions hold.*

(1.1) *The mapping $\lambda \mapsto u(\lambda)$ is of class C^1.*

(1.2) *$u(\lambda)$ is monotone increasing, that is, $u(\lambda) < u(\mu)$ in D if $\lambda < \mu$.*

(1.3) $\|u(\lambda)\|_{C^0} \to \infty$ as $\lambda \to \lambda_1 - k_\infty$.

(1.4) *$\|u(\lambda)\|_{C^0} \to 0$ as $\lambda \to -\infty$. More precisely, there exist constants $C > 0$ and $\bar{\lambda} < 0$ such that*

$$C^{-1}|\lambda|^{-1} \leq \|u(\lambda)\|_{C^0} \leq C|\lambda|^{-1}, \quad \lambda < \bar{\lambda}.$$

Here $\|\cdot\|_{C^0}$ denotes the maximum norm of space $C^0(\bar{D}) := C(\bar{D})$. Figure 1 *shows the behaviour of the positive solution set.*

Remark 1.1. In the case that the nonlinear term f is monotone decreasing, Keller [**5**] proved the existence and uniqueness of positive solutions of problem $(*)_\lambda$ under the non-degenerate boundary condition, that is, either under the condition that $a \equiv 0$ on ∂D or under the condition that $0 < a \leq 1$ on ∂D.

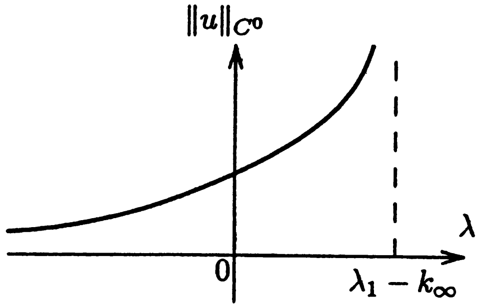

Figure 1

The second part is devoted to the study of the following semilinear elliptic boundary value problem:

$$(**)_\lambda \qquad \begin{cases} Au = \lambda f(u) & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases}$$

where

(1) λ is a positive parameter,
(2) f is a real-valued C^1-function on $[0, \infty)$.

The definitions of solutions and positive solutions of problem $(**)_\lambda$ are given in the same manner as of problem $(*)_\lambda$.

Under the conditions (H.1) and (H.2), Taira and Umezu [**12**, Corollary 2] proved that

(1) problem $(**)_\lambda$ has a positive solution if and only if

$$\begin{cases} 0 < \lambda < \lambda_1/k_\infty & \text{in the case that } k_\infty \geq 0, \\ \lambda > 0 & \text{in the case that } k_\infty < 0 \text{ (possibly } k_\infty = -\infty), \end{cases}$$

(2) the existence of positive solutions of problem $(**)_\lambda$ is unique in space $C^2(\bar{D})$. Here it is understood that $\lambda_1/k_\infty = \infty$ if $k_\infty = 0$.

In this part, we aim to obtain the set of positive solutions of problem $(**)_\lambda$ so precisely as in the previous part.

Now the second result of ours is the following.

Theorem 2. *Suppose that the conditions* (H.1) *and* (H.2) *are satisfied.*

(i) *In the case that $k_\infty \geq 0$, if $u(\lambda)$ is a unique positive solution of problem $(**)_\lambda$ for each $0 < \lambda < \lambda_1/k_\infty$, then in addition to the assertions* (1.1) *and* (1.2), *the following assertions hold.*

$$\|u(\lambda)\|_{C^{2+\alpha}} \to 0, \quad \lambda \to 0+, \tag{1.5}$$

$$\|u(\lambda)\|_{C^0} \to \infty, \quad \lambda \to \lambda_1/k_\infty. \tag{1.6}$$

Here $\|\cdot\|_{C^{2+\alpha}}$ denotes the norm of Hölder space $C^{2+\alpha}(\bar{D})$, $0 < \alpha < 1$.

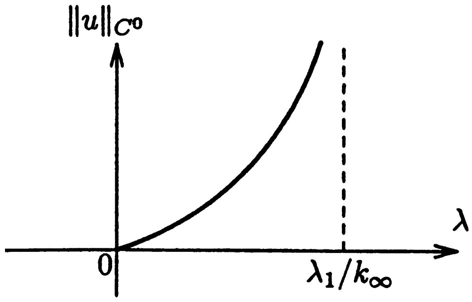

Figure 2

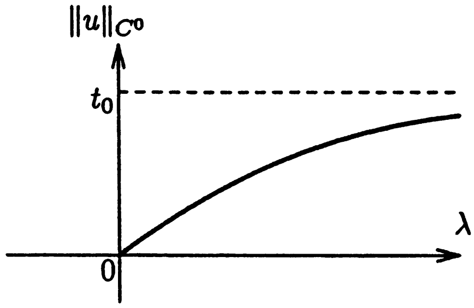

Figure 3

(ii) *In the case that* $k_\infty < 0$ (*possibly* $k_\infty = -\infty$), *if* $u(\lambda)$ *is a unique positive solution of problem* $(**)_\lambda$ *for each* $\lambda > 0$, *then in addition to the assertions* (1.1), (1.2) *and* (1.5), *we have*

$$\|u(\lambda)\|_{C^0} \to t_0, \quad \lambda \to \infty. \tag{1.7}$$

Here t_0 *is the positive constant given by the following.*

$$\begin{cases} f(t_0) = 0, \\ f(t) > 0, \quad 0 \le t < t_0. \end{cases}$$

Figures 2 *and* 3 *exhibit the behaviour of the set of positive solutions in the case that* $k_\infty \ge 0$ *and in the case that* $k_\infty < 0$, *respectively.*

REMARK 1.2. Under the non-degenerate boundary conditions, Ambrosetti and Hess [**2**] studied problem $(**)_\lambda$ in the case that the nonlinearity is asymptotically linear. More extensive classes than in [**2**] are treated by Lions [**7**]. However we find that Theorem 2 characterizes the asymptotic behaviour of the positive solution of $(**)_\lambda$ more exactly by means of the method of super-subsolutions.

EXAMPLE 1.1. We give some examples f satisfying the conditions (H.1) and (H.2).
(1) $f(t) = t + 1/(t+1)$. In this case, $k_\infty = 1$.
(2) $f(t) = e^{-t}$. In this case, $k_\infty = 0$.
(3) $f(t) = \exp[t/(1+t/\alpha)]$ $(0 < \alpha < 4)$. In this case, $k_\infty = 0$. Problem $(**)_\lambda$ for this nonlinearity is called a *modified Gel'fand problem* arising in chemical reactor theory (cf. [**3**], [**13**]).
(4) $f(t) = -t + 2 - 1/(t+1)$. In this case, $k_\infty = -1$.
(5) $f(t) = 1 + \beta t + \alpha t^2$ $(\alpha, \beta \in \boldsymbol{R}, \alpha < 0)$. In this case, $k_\infty = -\infty$.

The rest of this paper is organized as follows. For proofs of Theorems 1 and 2 we prepare, in Section 2, some results from the linear theory of elliptic boundary value problems with the degenerate boundary condition. Our problems are reduced to operator equations of the resolvent associated with the linear boundary value problem. More precisely, since the resolvent has positivity and compactness in some Banach spaces, we can apply to our case the theory of positive mappings in ordered Banach spaces ([**1**]) and the positivity lemma for strongly positive compact linear operators ([**6**]), by establishing existence and uniqueness theorems both in Sobolev spaces ([**14**, Theorem 1]) and in Hölder spaces ([**11**, Theorem 1.1]) for the degenerate linear elliptic boundary value problem. We remark that it seems to be difficult to use the variational method in the degenerate case (cf. Remark 4.1).

Section 3 is devoted to the proof of Theorem 1. Our main tools are the implicit function theorem and the method of super-subsolutions.

In Section 4, we prove Theorem 2 where the *a priori* upper bounds for positive solutions plays an important role. The proof of Theorem 2 is given essentially in the same manner as in Section 3.

2. Linear theory.

In this section, we prepare the linear theory of degenerate elliptic boundary value problems to prove Theorems 1 and 2. We state the following existence and uniqueness theorems both in the framework of Sobolev spaces and in the framework of Hölder spaces.

Theorem 2.1 ([**14**, Theorem 1]). *The mapping*

$$(A, B) \colon W^{s,p}(D) \to W^{s-2,p}(D) \oplus B_{(a)}^{s-1-1/p,p}(\partial D),$$

$$u \mapsto (Au, Bu)$$

is isomorphic for $1 < p < \infty$ *and* $s \geq 2$ *where*

$$B^{\sigma-1/p,p}(\partial D) = \{\varphi : \varphi = u|_{\partial D}, u \in W^{\sigma,p}(D)\}, \quad \sigma \geq 1,$$

with norm

$$\|\varphi\|_{B^{\sigma-1/p,p}(\partial D)} = \inf\Big\{\|u\|_{W^{\sigma,p}(D)} : \varphi = u|_{\partial D}, u \in W^{\sigma,p}(D)\Big\},$$

and

$$B_{(a)}^{s-1-1/p,p}(\partial D) = \Big\{\varphi = a\varphi_1 + (1-a)\varphi_0 : \varphi_i \in B^{s-i-1/p,p}(\partial D), \ \ i = 0, 1\Big\},$$

with norm

$$\|\varphi\|_{B_{(a)}^{s-1-1/p,p}(\partial D)} = \inf\{\|\varphi_1\|_{B^{s-1-1/p,p}(\partial D)} + \|\varphi_0\|_{B^{s-1/p,p}(\partial D)} :$$

$$\varphi = a\varphi_1 + (1-a)\varphi_0, \varphi_i \in B^{s-i-1/p,p}(\partial D), \ \ i = 0, 1\}.$$

Theorem 2.2 ([**11**, Theorem 1.1]). *The mapping*

$$(A,B)\colon\ C^{2+\alpha}(\bar{D}) \to C^{\alpha}(\bar{D}) \oplus C^{1+\alpha}_{(a)}(\partial D),$$

$$u \mapsto (Au, Bu)$$

is isomorphic for $0 < \alpha < 1$ *where*

$$C^{1+\alpha}_{(a)}(\partial D) = \{\varphi = a\varphi_1 + (1-a)\varphi_0 : \varphi_i \in C^{2-i+\alpha}(\partial D),\ \ i = 0,1\},$$

with norm

$$\|\varphi\|_{C^{1+\alpha}_{(a)}(\partial D)} = \inf\{\|\varphi_1\|_{C^{1+\alpha}(\partial D)} + \|\varphi_0\|_{C^{2+\alpha}(\partial D)} :$$
$$\varphi = a\varphi_1 + (1-a)\varphi_0, \varphi_i \in C^{2-i+\alpha}(\partial D), i = 0,1\}.$$

We remark that interpolation spaces $B^{s-1-1/p,p}_{(a)}(\partial D)$ and $C^{1+\alpha}_{(a)}(\partial D)$ are Banach spaces with respect to norms $\|\cdot\|_{B^{s-1-1/p,p}_{(a)}(\partial D)}$ and $\|\cdot\|_{C^{1+\alpha}_{(a)}(\partial D)}$, respectively.

First we consider the linear boundary value problem

$$\begin{cases} Au = v & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Then we find from Theorem 2.2 that there exists a resolvent K which maps $C^{\alpha}(\bar{D})$ isomorphically onto $C^{2+\alpha}_B(\bar{D})$, where

$$C^{2+\alpha}_B(\bar{D}) = \{u \in C^{2+\alpha}(\bar{D}) : Bu = 0 \ \text{ on } \ \partial D\}.$$

This implies that for any $v \in C^{\alpha}(\bar{D})$, Kv is a unique solution in $C^{2+\alpha}(\bar{D})$ of the linear problem. Hence we can find that a function u is a solution of $(*)_\lambda$ if and only if

$$u = K(\lambda u + f(u)) \quad \text{in } \ C^{\alpha}(\bar{D}),$$

because $f \in C^1([0,\infty))$.

Secondly since $C(\bar{D}) \subset L^p(D)$ for any $1 < p < \infty$, Theorem 2.1 with $s = 2$ tells us that there exists a unique extension of K, again denoted by K, to $C(\bar{D})$ with its image contained $W^{2,p}_B(D)$ for all $1 < p < \infty$ where

$$W^{2,p}_B(D) = \{u \in W^{2,p}(D) : Bu = 0 \ \text{ on } \ \partial D\}.$$

If we take the exponent p such that $p > N$, then we obtain by Sobolev's imbedding theorem that K is a continuous mapping of $C(\bar{D})$ into $C^1(\bar{D})$, and obtain by Ascoli-Arzelà's theorem that K maps $C(\bar{D})$ compactly into itself. Hence, it is easy to see that a function u is a solution of $(*)_\lambda$ if and only if

$$u = K(\lambda u + f(u)) \quad \text{in } \ C(\bar{D}).$$

Similarly we see that a function u is a solution of $(**)_\lambda$ if and only if

$$u = K(\lambda f(u)) \quad \text{in } \ C(\bar{D}).$$

Now, we know ([**10**, Lemma 2.1]) that K is *strictly positive* in $C(\bar{D})$, meaning that for any non-negative function $v \in C(\bar{D})$ such that $v \not\equiv 0$, Kv is positive everywhere in D.

More precisely, the function u which denotes Kv, has the following property.

$$\begin{cases} u > 0 & \text{in } \bar{D}\backslash\{x \in \partial D : a(x) = 0\}, \\ \dfrac{\partial u}{\partial \nu} < 0 & \text{on } \partial D\backslash\{x \in \partial D : a(x) = 1\}. \end{cases}$$

Furthermore K is *strongly positive* in the following sense. Let $e = K1$. The space $C_e(\bar{D})$ denotes

$$C_e(\bar{D}) = \{u \in C(\bar{D}) : \exists c > 0\,; -ce \le u \le ce\}.$$

It is known (cf. [**1**]) that the space $C_e(\bar{D})$ is a Banach space with respect to norm $\|\cdot\|_e$ given by

$$\|u\|_e = \inf\{c > 0 : -ce \le u \le ce\}.$$

According to [**10**, Proposition 2.2], $K : C_e(\bar{D}) \to C_e(\bar{D})$ is compact and strongly positive, meaning that for any v in $P_e\backslash\{0\}$, Kv is an interior point in P_e where $P_e = \{u \in C_e(\bar{D}) : u \ge 0\}$. We can check easily that a function u is a solution of $(*)_\lambda$ if and only if

$$u = K(\lambda u + f(u)) \quad \text{in } C_e(\bar{D}).$$

Similarly, we can see that a function u is a solution of $(**)_\lambda$ if and only if

$$u = K(\lambda f(u)) \quad \text{in } C_e(\bar{D}).$$

Finally we state the following result which guarantees the existence and uniqueness of positive solutions to equations for strongly positive, compact linear operators.

Theorem 2.3 ([**6**, Theorem 2.16]). *Let T be a strongly positive, compact linear operator of $C_e(\bar{D})$ and λ_0 the largest eigenvalue of T. Then for any non-negative function $g \in C_e(\bar{D})$ such that $g \not\equiv 0$, the equation*

$$\lambda v - Tv = g$$

has exactly one positive solution $v \in C_e(\bar{D})$ for each $\lambda > \lambda_0$.

3. Proof of Theorem 1.

In this section, we prove Theorem 1. First we shall show the assertion (1.1). We define a nonlinear mapping F by

$$F\colon\ \boldsymbol{R} \times C_B^{2+\alpha}(\bar{D}) \to C^\alpha(\bar{D})$$

$$(\lambda, u) \mapsto Au - \lambda u - f(u).$$

From Theorem 2.1 with $s = 3$, it is easy to see that a function u is a solution of problem $(*)_\lambda$ if and only if $F(\lambda, u) = 0$. Let $u(\lambda)$ be the unique positive solution of $(*)_\lambda$ for $\lambda < \lambda_1 - k_\infty$, that is, $u(\lambda)$ satisfies that

$$\begin{cases} Au(\lambda) - \lambda u(\lambda) - f(u(\lambda)) = 0 & \text{in } D, \\ u(\lambda) > 0 & \text{in } D, \\ Bu(\lambda) = 0 & \text{on } \partial D. \end{cases} \tag{3.1}$$

The Fréchet derivative $F_u(\lambda, u(\lambda))$ of F at $(\lambda, u(\lambda))$ is given by

$$F_u(\lambda, u(\lambda)) : C_B^{2+\alpha}(\bar{D}) \to C^\alpha(\bar{D})$$

$$\varphi \mapsto A\varphi - \lambda\varphi - f'(u(\lambda))\varphi.$$

Let $\delta(\lambda)$ be the first eigenvalue of $F_u(\lambda, u(\lambda))$ and let $\varphi(\lambda)$ a corresponding eigenfunction to $\delta(\lambda)$:

$$\begin{cases} A\varphi(\lambda) - \lambda\varphi(\lambda) - f'(u(\lambda))\varphi(\lambda) = \delta(\lambda)\varphi(\lambda) & \text{in } D, \\ B\varphi(\lambda) = 0 & \text{on } \partial D. \end{cases} \tag{3.2}$$

It is known ([**9**, Theorem 7.4]) that $\varphi(\lambda)$ can be chosen as positive everywhere in D.

The boundary conditions imply that

$$\begin{pmatrix} \dfrac{\partial u(\lambda)}{\partial \nu} & u(\lambda) \\ \dfrac{\partial \varphi(\lambda)}{\partial \nu} & \varphi(\lambda) \end{pmatrix} \begin{pmatrix} a \\ 1-a \end{pmatrix} = 0 \quad \text{on } \partial D,$$

and hence

$$\frac{\partial u(\lambda)}{\partial \nu}\varphi(\lambda) - u(\lambda)\frac{\partial \varphi(\lambda)}{\partial \nu} = 0 \quad \text{on } \partial D,$$

because $(a, 1-a) \neq (0,0)$. In view of this assertion, we have from (3.1) and (3.2)

$$\delta(\lambda) = \frac{\int_D (f(u(\lambda)) - f'(u(\lambda))u(\lambda))\varphi(\lambda)\,dx}{\int_D u(\lambda)\varphi(\lambda)\,dx},$$

by integration by parts. Since it follows from the condition (H.2) that

$$f(t) - f'(t)t > 0, \quad t > 0,$$

we obtain that $\delta(\lambda) > 0$. This implies that $F_u(\lambda, u(\lambda))$ is injective for all $\lambda < \lambda_1 - k_\infty$. Moreover the bijectivity of $F_u(\lambda, u(\lambda))$ follows from the combination of Theorem 2.2 and the index theory of Fredholm operators. So, we can apply the implicit function theorem to obtain that the mapping $\lambda \mapsto u(\lambda)$ is of class C^1 in $\lambda < \lambda_1 - k_\infty$. The proof of the assertion (1.1) is complete.

We shall prove the assertion (1.2). For our purpose, we use the method of super-subsolutions. For a real-valued C^1-function g on $[0, \infty)$ we consider

$$\begin{cases} Au = g(u) & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases} \tag{3.3}$$

A non-negative function $\psi \in C^2(\bar{D})$ is said to be a *supersolution* (resp. *subsolution*) of (3.3) if

$$\begin{cases} A\psi \geq (\text{resp.} \leq) g(\psi) & \text{in } D, \\ B\psi \geq (\text{resp.} \leq) 0 & \text{in } \partial D. \end{cases}$$

Now we give the following existence theorem of solutions for (3.3) relying on the super-subsolution method.

Theorem 3.1 ([**11**, Theorem 1]). *Let ϕ be a subsolution and let ψ a supersolution of* (3.3) *such that $\phi \le \psi$ in $\bar{D}$. Then there exists a function $u \in C^2(\bar{D})$ such that u satisfies* (3.3) *and $\phi \le u \le \psi$ in $\bar{D}$.*

Let $\lambda < \mu < \lambda_1 - k_\infty$. It is clear that the positive solution $u(\mu)$ of $(*)_\mu$ is a supersolution of $(*)_\lambda$, and that $u \equiv 0$ is a subsolution but not a solution of $(*)_\lambda$ because of the condition (H.1). By Theorem 3.1, there exists a solution v of $(*)_\lambda$ such that

$$0 \le v \le u(\mu) \quad \text{in } \bar{D}.$$

Since $f \in C^1([0,\infty))$, there exists a constant $\xi > 0$ such that $(\lambda+\xi)t + f(t)$ is monotone increasing in the closed interval $[0, \max\{\|u(\lambda)\|_{C^0}, \|u(\mu)\|_{C^0}\}]$. Hence, we find that

$$\begin{cases} (A+\xi)v = (\lambda+\xi)v + f(v) \ge f(0) > 0 & \text{in } D, \\ Bv = 0 & \text{on } \partial D. \end{cases}$$

By the strong maximum principle and boundary point lemma (see [**8**]), the solution v is positive everywhere in D. From the uniqueness of positive solutions of $(*)_\lambda$ it follows that $v \equiv u(\lambda)$.

Next, we see that

$$(A+\xi)u(\mu) = (\mu+\xi)u(\mu) + f(u(\mu)) > (\lambda+\xi)u(\mu) + f(u(\mu)) \quad \text{in } D,$$

$$(A+\xi)u(\lambda) = (\lambda+\xi)u(\lambda) + f(u(\lambda)) \quad \text{in } D,$$

and hence that

$$\begin{cases} (A+\xi)(u(\mu)-u(\lambda)) & \\ > (\lambda+\xi)u(\mu) + f(u(\mu)) - \{(\lambda+\xi)u(\lambda) + f(u(\lambda))\} \ge 0 & \text{in } D, \\ B(u(\mu)-u(\lambda)) = 0 & \text{on } \partial D. \end{cases}$$

Therefore we have the assertion (1.2) by the strong maximum principle and the boundary point lemma. The proof of the assertion (1.2) is complete.

We shall verify the assertion (1.3). First we discuss the case that k_∞ is finite. Assume to the contrary that $\|u(\lambda)\|_{C^0}$ is bounded near $\lambda = \lambda_1 - k_\infty$. We recall that $u(\lambda)$ satisfies

$$u(\lambda) = K(\lambda u(\lambda) + f(u(\lambda))).$$

Since $\|\lambda u(\lambda) + f(u(\lambda))\|_{C^0}$ is bounded near $\lambda = \lambda_1 - k_\infty$, $\|u(\lambda)\|_{C^1}$ is bounded near $\lambda = \lambda_1 - k_\infty$ where $\|\cdot\|_{C^1}$ denotes the norm of $C^1(\bar{D})$. Hence we obtain that $\|u(\lambda)\|_{C^{2+\alpha}}$ is bounded near $\lambda = \lambda_1 - k_\infty$ because of Theorem 2.2. Therefore, without loss of generality, we may obtain that there exists a function $\bar{u} \in C^2(\bar{D})$ such that

$$u(\lambda) \to \bar{u} \quad \text{in } C^2(\bar{D}), \quad \lambda \to \lambda_1 - k_\infty.$$

With the aid of the assertion (1.2) we have

$$\begin{cases} A\bar{u} = (\lambda_1 - k_\infty)\bar{u} + f(\bar{u}) & \text{in } D, \\ \bar{u} > 0 & \text{in } D, \\ B\bar{u} = 0 & \text{on } \partial D. \end{cases}$$

This is a contradiction.

Next, we treat the case that $k_\infty = -\infty$. For any $\lambda > \lambda_1$, there exists a unique $t_\lambda > 0$ such that

$$-(\lambda - \lambda_1) = \frac{f(t_\lambda)}{t_\lambda},$$

and we can see

$$t_\lambda \to \infty \quad \text{as } \lambda \to \infty.$$

By the condition (H.2) we have for $0 < t \le t_\lambda$,

$$\frac{\lambda t + f(t)}{t} = \lambda + \frac{f(t)}{t} \ge \lambda + \frac{f(t_\lambda)}{t_\lambda} = \lambda_1.$$

Hence

$$\lambda t + f(t) \ge \lambda_1 t, \quad 0 \le t \le t_\lambda.$$

Let φ_1 be a corresponding positive eigenfunction to the first eigenvalue λ_1 such that $\|\varphi_1\|_{C^0} = 1$. Then we see that

$$A\left(\frac{t_\lambda}{2}\varphi_1\right) - \lambda\frac{t_\lambda}{2}\varphi_1 - f\left(\frac{t_\lambda}{2}\varphi_1\right) \le \frac{t_\lambda}{2}\lambda_1\varphi_1 - \lambda_1\frac{t_\lambda}{2}\varphi_1 = 0 \quad \text{in } D.$$

This implies that $t_\lambda\varphi_1/2$ is a subsolution of $(*)_\lambda$. On the other hand, if we take a constant $M_\lambda > 0$ sufficiently large, then M_λ is a supersolution of $(*)_\lambda$. Combining Theorem 3.1 and the uniqueness result for positive solutions of $(*)_\lambda$, we obtain that for $\lambda > \lambda_1$,

$$\frac{t_\lambda}{2}\varphi_1 \le u(\lambda) \le M_\lambda \quad \text{in } \bar{D}.$$

Hence,

$$\frac{t_\lambda}{2} \le \|u(\lambda)\|_{C^0}, \quad \lambda > \lambda_1,$$

which implies that

$$\|u(\lambda)\|_{C^0} \to \infty, \quad \lambda \to \infty.$$

The proof of the assertion (1.3) is complete.

Finally we shall prove the assertion (1.4). For any constant $\varepsilon > 0$, there exists a constant $\bar{\lambda}_\varepsilon < 0$ such that if $\lambda < \bar{\lambda}_\varepsilon$, then

$$\begin{cases} A\left(-\dfrac{f(0)+\varepsilon}{\lambda}\right) - \lambda\left(-\dfrac{f(0)+\varepsilon}{\lambda}\right) - f\left(-\dfrac{f(0)+\varepsilon}{\lambda}\right) \\ > f(0) + \varepsilon - f\left(-\dfrac{f(0)+\varepsilon}{\lambda}\right) > 0 \quad \text{in } D, \end{cases}$$

because

$$f\left(-\frac{f(0)+\varepsilon}{\lambda}\right) \to f(0), \quad \lambda \to -\infty.$$

This shows that $-(f(0)+\varepsilon)/\lambda$ is a supersolution of $(*)_\lambda$ for $\lambda < \bar{\lambda}_\varepsilon$. If we put

$$\varphi_f = \frac{f(0)}{2}\varphi_1,$$

then there exists a constant $\bar{\lambda}_0 < 0$ such that, for $\lambda < \bar{\lambda}_0$,

$$A\left(-\frac{\varphi_f}{\lambda}\right) - \lambda\left(-\frac{\varphi_f}{\lambda}\right) - f\left(-\frac{\varphi_f}{\lambda}\right) = -\frac{\lambda_1}{\lambda}\varphi_f + \varphi_f - f\left(-\frac{\varphi_f}{\lambda}\right) < 0 \quad \text{in } D,$$

which implies that $-\varphi_f/\lambda$ is a subsolution of $(*)_\lambda$. Combining Theorem 3.1 with the uniqueness of positive solutions and letting $\bar{\lambda} = \min\{\bar{\lambda}_\varepsilon, \bar{\lambda}_0\}$, we have

$$-\frac{\varphi_f}{\lambda} \le u(\lambda) \le -\frac{f(0)+\varepsilon}{\lambda}, \quad \lambda < \bar{\lambda}.$$

This shows the assertion (1.4). The proof of the assertion (1.4) is complete.

The proof of Theorem 1 is now complete. □

4. Proof of Theorem 2.

4.1. Proof of part (i) of Theorem 2.

The proof of the assertion (1.1) is the same as in Theorem 1, and we can prove the assertion (1.2) similarly since if $k_\infty \ge 0$ then $f(t) > 0$ for $t > 0$.

We shall show the assertion (1.5). From the assertion (1.2) it follows that $\|u(\lambda)\|_{C^0}$ is bounded near $\lambda = 0$. By Theorem 2.1 with $s = 2$, we get

$$\|u(\lambda)\|_{C^1} \le C, \quad 0 < \lambda < \bar{\lambda} \tag{4.1}$$

with some constants $C > 0$ and $\bar{\lambda} > 0$. On the other hand, from Theorem 2.2 we have

$$\|u(\lambda)\|_{C^{2+\alpha}} \le C'\lambda\|f(u(\lambda))\|_{C^\alpha} \tag{4.2}$$

with some constant $C' > 0$ independent of λ. From the assertions (4.1) and (4.2) we obtain the assertion (1.5). The proof of the assertion (1.5) is complete.

We shall verify the assertion (1.6). In the case that $k_\infty > 0$, we can prove it similarly as the assertion (1.3) of Theorem 1. Next we deal with the case that $k_\infty = 0$. Assume to the contrary that there exist constants $C > 0$ and $\bar{\lambda} > 0$ such that

$$\|u(\lambda)\|_{C^0} \le C, \quad \lambda > \bar{\lambda}.$$

From the two equations:

$$\begin{cases} A\varphi_1 = \lambda_1\varphi_1 & \text{in } D, \\ B\varphi_1 = 0 & \text{on } \partial D, \end{cases}$$

$$\begin{cases} Au(\lambda) = \lambda f(u(\lambda)) & \text{in } D, \\ Bu(\lambda) = 0 & \text{on } \partial D, \end{cases}$$

we derive by integration by parts

$$\lambda = \frac{\lambda_1 \int_D u(\lambda)\varphi_1\,dx}{\int_D f(u(\lambda))\varphi_1 dx}.$$

We put

$$C_0 = \min_{0 \le t \le C} f(t) > 0.$$

Then we have

$$\lambda \le \frac{\lambda_1 C}{C_0}.$$

On letting $\lambda \to \infty$, this leads to a contradiction. Hence, with the aid of the assertion (1.2), we obtain that

$$\|u(\lambda)\|_{C^0} \to \infty, \quad \lambda \to \infty.$$

The proof of the assertion (1.6) is complete.

The proof of part (i) of Theorem 2 is complete. □

4.2. Proof of part (ii) of Theorem 2.

The assertion (1.1) can be proved similarly as in Theorem 1.

First we shall verify the assertion (1.2). To do so, we need the following lemma which gives the *a priori* bounds for positive solutions of a class of sublinear elliptic problems.

LEMMA 4.1. *For the nonlinearity g of problem* (3.3), *we assume that there exists a $t_0 > 0$ such that*

$$g(t_0) = 0,$$

$$g(t) < 0, \quad t > t_0.$$

Then any positive solution u of problem (3.3) *satisfies that*

$$0 \le u \le t_0 \quad \text{in } \bar{D}.$$

PROOF. Let u be a positive solution of (3.3). We take a constant $\sigma > 0$ such that

$$u(x) \le \sigma \quad \text{in } \bar{D},$$

$$t_0 \le \sigma.$$

Since $g \in C^1([0, \infty))$, there exists a constant $\xi > 0$ such that the mapping $t \mapsto \xi t + g(t)$ is increasing in the closed interval $[0, \sigma]$. As seen in Section 2, if K_ξ denotes the inverse of $(A + \xi) : C_B^{2+\alpha}(\bar{D}) \to C^\alpha(\bar{D}), 0 < \alpha < 1,$ then we have

$$u = K_\xi(\xi u + g(u)).$$

Now, assume to the contrary that there exists $x_0 \in \bar{D}$ such that

$$u(x_0) > t_0. \tag{4.3}$$

Let

$$\omega(x) = \min\{u(x), t_0\}, \quad x \in \bar{D}. \tag{4.4}$$

First we claim that

$$\omega \geq K_\xi(\xi\omega + g(\omega)). \tag{4.5}$$

From the definition of ω, it follows that

$$\xi\omega + g(\omega) \leq \xi u + g(u).$$

By the positivity of K_ξ,

$$K_\xi(\xi\omega + g(\omega)) \leq K_\xi(\xi u + g(u)) = u. \tag{4.6}$$

Similarly

$$\xi\omega + g(\omega) \leq \xi t_0 + g(t_0) = \xi t_0,$$

and hence

$$K_\xi(\xi\omega + g(\omega)) \leq \xi t_0 K_\xi 1. \tag{4.7}$$

Putting $v = \xi t_0 K_\xi 1$, we obtain that

$$\begin{cases} (A + \xi)(t_0 - v) = ct_0 > 0 & \text{in } D, \\ B(t_0 - v) = (1 - a)t_0 \geq 0 & \text{on } \partial D. \end{cases}$$

Using the strong maximum principle and boundary point lemma, we obtain that

$$v(x) < t_0 \quad \text{in } D. \tag{4.8}$$

Therefore, the claim (4.5) follows from the assertions (4.6)–(4.8).

Now, using the assertion (4.5), we get

$$(\omega - u) - \xi K_\xi(\omega - u) \geq K_\xi(g(\omega) - g(u)).$$

By (4.3) and the assumption of the function g, we have

$$g(\omega) - g(u) \not\equiv 0 \text{ and } \geq 0 \quad \text{in } \bar{D}.$$

So, it follows from the strict positivity of K_ξ that

$$K_\xi(g(\omega) - g(u)) > 0 \quad \text{in } D.$$

Since the largest eigenvalue of the operator ξK_ξ is less than 1, the function $\omega - u$ must be positive everywhere in D because of Theorem 2.3. This statement contradicts the definition (4.4).

The proof of Lemma 4.1 is complete. □

Remark 4.1. Figueiredo [**4**, Lemma 2.9] proved Lemma 4.1 in the Dirichlet condition case. However it seems to be difficult to apply his method to the degenerate case since it relies on the variational method.

Now, in the case that $k_\infty < 0$ (possibly $-\infty$), there exists a $t_0 > 0$ such that

$$\begin{cases} f(t_0) = 0, \\ f(t) > 0, & 0 \leq t < t_0, \\ f(t) < 0, & t > t_0. \end{cases} \tag{4.9}$$

From Lemma 4.1 it follows that for all $\lambda > 0$,

$$0 \le u(\lambda) \le t_0 \quad \text{in } \bar{D},$$

and hence that for all $\lambda > 0$,

$$f(u(\lambda)) \ge 0 \quad \text{in } \bar{D}.$$

Having this statement in mind, we can prove the assertion (1.2) similarly as in Theorem 1. By virtue of the assertion (1.2), we can show the assertion (1.5) in the same manner as in part (i).

Finally we shall prove the assertion (1.7). Lemma 4.1 and (4.9) assert the maximum norm $\|u(\lambda)\|_{C^0}$ is bounded above by the value t_0. In addition, from the assertion (1.2) it follows that $\|u(\lambda)\|_{C^0}$ is increasing in λ. Hence we can put

$$\bar{t} = \lim_{\lambda\to\infty} \|u(\lambda)\|_{C^0} (\le t_0).$$

We use the super-subsolution method to prove that $\bar{t} = t_0$. It is obvious that $u \equiv t_0$ is a supersolution of $(**)_\lambda$. Let $0 < \varepsilon < 1$. From the condition (H.2) we have for $\lambda > 1/t_0$,

$$\begin{aligned} A&\left(\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1\right) - \lambda f\left(\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1\right) \\ &= \lambda_1\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1 - \lambda f\left(\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1\right) \\ &= \left\{\lambda_1 - \lambda\frac{f((t_0 - (1/\lambda))\varepsilon\varphi_1)}{(t_0 - (1/\lambda))\varepsilon\varphi_1}\right\}\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1 \\ &\le \left(\lambda_1 - \lambda\frac{f(t_0\varepsilon)}{t_0\varepsilon}\right)\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1 \quad \text{in } D. \end{aligned}$$

This shows that $(t_0 - 1/\lambda)\varepsilon\varphi_1$ is a subsolution of $(**)_\lambda$ if $\lambda > \bar{\lambda}_\varepsilon$ where $\bar{\lambda}_\varepsilon = \max\{1/t_0, \lambda_1 t_0\varepsilon/f(t_0\varepsilon)\}$. Combining Theorem 3.1 with the uniqueness, we obtain that for $\lambda > \bar{\lambda}_\varepsilon$,

$$\left(t_0 - \frac{1}{\lambda}\right)\varepsilon\varphi_1 \le u(\lambda) \le t_0 \quad \text{in } \bar{D}.$$

Hence

$$\left(t_0 - \frac{1}{\lambda}\right)\varepsilon \le \|u(\lambda)\|_{C^0} \le t_0, \quad \lambda > \bar{\lambda}_\varepsilon,$$

and hence as $\lambda \to \infty$

$$t_0\varepsilon \le \bar{t} = \lim_{\lambda\to\infty} \|u(\lambda)\|_{C^0} \le t_0.$$

Letting $\varepsilon \to 1$, we have

$$\lim_{\lambda\to\infty} \|u(\lambda)\|_{C^0} = t_0.$$

The proof of the assertion (1.7) is complete.

We have finished the proof of part (ii) of Theorem 2. □

The proof of Theorem 2 is now complete. □

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., **18** (1976), 620–709.

[2] A. Ambrosetti and P. Hess, Positive solutions of asymptotically linear elliptic eigenvalue problems, J. Math. Appl., **73** (1980), 411–422.

[3] T. Boddington, P. Gray and G. C. Wake, Criteria for thermal explosions with and without reactant consumption, Proc. R. Soc. London A., **357** (1977), 403–422.

[4] D. G. de Figueiredo, Positive solutions of semilinear elliptic problems, in Lecture Notes in Mathematics, No. 957, Springer-Verlag Berlin Heidelberg New York, (1982), 34–87.

[5] H. B. Keller, Positive solutions of some nonlinear eigenvalue problems, J. Math. Mech., **19** (1969), 279–295.

[6] M. A. Krasnosel'skii, Positive solutions of operator equations, P. Noordhoff, Groningen (1964).

[7] P. L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev., **24** (1982), 441–467.

[8] M. H. Protter and H. F. Weinberger, Maximum principles in differential equations, Prentice-Hall, Englewood Cliffs New Jersey (1967).

[9] K. Taira, Bifurcation for nonlinear elliptic boundary value problems I, Collect. Math., **47** (1996), 207–229.

[10] K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems II, Tokyo J. Math., **19** (1996), 387–396.

[11] K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems III, Adv. Differential Equations, **1** (1996), 709–727.

[12] K. Taira and K. Umezu, Positive solutions of sublinear elliptic boundary value problems, Nonlinear Analysis, TMA, **29** (1997), 761–771.

[13] K. Taira and K. Umezu, Semilinear elliptic boundary value problems in chemical reactor theory, J. Differential Equations, **142** (1997), 434–454.

[14] K. Umezu, L^p-approach to mixed boundary value problems for second-order elliptic operators, Tokyo J. Math., **17** (1994), 101–123.

Kenichiro Umezu

Maebashi Institute of Technology
Maebashi 371-0816, Japan