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Value sharing of an entire function and its derivatives

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Abstract. In this paper, when an entire function f and the linear combination of its derivatives $L(f)$ with small functions as its coefficients share one value CM and another value IM is studied. We also resolved the question when an entire function f and its derivative f' share two values CM jointly. Some of the results remain to be valid if f is meromorphic and satisfying $N(r,f) = o(T(r,f))$ as $r \to \infty$ and the values a, b are replaced by small functions of $f(z)$.

1. Introduction.

Let f and g be two non-constant meromorphic functions and b be a complex number. We say that f and g share the value b CM (IM) provided that $f(z) - b$ and $g(z) - b$ have the same zeros with the same multiplicities (ignoring multiplicities). In 1929, R. Nevanlinna proved [**1**] that (i) if f and g share five values IM, then $f \equiv g$, and (ii) if f and g share four values CM, then f is a Möbius transformation of g. Particularly, if f and g are entire functions, then $f \equiv g$ provided that f and g share four finite values CM. Recently the studies on sharing values have been extended to the studies of sharing small functions of f and sharing several finite sets or even to one finite set only, see, e.g. [**2**], [**3**], [**4**], [**5**] and [**6**]. For instance, it has been shown in [**7**] that there exists a single set S with 15 elements such that $f^{-1}(S) = g^{-1}(S)$ implies $f \equiv g$. For its improvements, we refer the reader to Yi [**8**] and Mues-Reinders [**9**]. In 1976, it was shown [**10**] that if an entire function f and its derivative f' share two values a, b CM, then $f \equiv f'$. Since then the subject of sharing values between a meromorphic function and its derivatives has been studied by many mathematicians. For example, G. Gundersen [**11**] proved that if f is entire and shares two finite nonzero values IM with f', then $f \equiv f'$. E. Mues and N. Steinmetz [**12**] proved that if f is meromorphic and shares three finite values IM with f', then $f \equiv f'$. This result was improved by Frank and Schwick [**13**] to the case that f shares three finite values IM with $f^{(k)}$. Similar questions on f shares three finite values IM with its differential polynomial $L(f)$ were studied in [**14**], [**15**] and [**16**]. When a meromorphic function f shares two finite values CM with its differential polynomial $L(f)$ whose coefficients are polynomials, P. Russmann [**17**] proves that $f \equiv L(f)$ except for six specific cases.

More recently, Bernstein-Chang-Li [**18**] studied the similar questions about meromorphic functions of several complex variables. As a special case, they proved

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Theorem A. *Let f be a non-constant entire function and*

$$L(f) = b_n f^{(n)} + b_{n-1} f^{(n-1)} + \cdots + b_1 f' + b_0 f$$

with all b_j being small meromorphic functions of f. If f and $L(f)$ share two values CM, then $f \equiv L(f)$.

Note, here and in the sequel, a meromorphic function $a(z)$ is called a small function of $f(z)$ iff $T(r, a(z)) = o(T(r, f))$ as $r \to \infty$ except a set of finite measure of $r \in (0, \infty)$.

In this paper, we have improved the above result and resolved the problem when the condition of Theorem A is replaced by assuming that f (entire) and $L(f)$ share one value a_1 CM and another value a_2 IM. We have also resolved an interesting problem, namely: What happens if an entire function f and its derivative f' share two finite values a_1, a_2 CM jointly, i.e., $(f(z) - a_1)(f(z) - a_2) = 0$ and $(f'(z) - a_1)(f'(z) - a_2) = 0$ have the same zeros counting multiplicities? It is assumed that the reader is familiar with the standard notations and basics of Nevanlinna's value distribution theory (cf. [**19**], [**20**]).

2. Lemmas and main results.

The following lemmas will be used in the proof of our theorems. Lemma 1 is obvious by the Lemma of the logarithmic derivative, i.e., $m(r, f'/f) = S(r, f)$, see e.g. [**19**]. Lemma 2 and Lemma 3 are well-known. Lemma 4 can be deduced easily from Lemma 2.

Lemma 1. *Let f be a transcendental meromorphic function, $P_k(f)$ denote a polynomial in f of degree k, and $a_i, i = 1, 2 \ldots, n$ denote finite distinct constants in C. Let*

$$g = \frac{P_k(f) f'}{(f - a_1) \cdots (f - a_n)}.$$

If $k < n$, then $m(r, g) = S(r, f)$, where and in the sequel $S(r, f)$ will be used to denote any quantity $o(T(r, f)), r \to \infty$, except a set of finite measure of $r \in (0, \infty)$.

Lemma 2 ([**21**]). *Let $P_k(f)$ and $P_l(f)$ be two relatively prime polynomials of degree k and l, respectively. That is*

$$P_k(f) = a_0(z) f^k(z) + a_1(z) f^{k-1}(z) + \cdots + a_k(z),$$

and

$$P_l(f) = b_0(z) f^l(z) + b_1(z) f^{l-1}(z) + \cdots + b_l(z)$$

such that no polynomial in f of degree more than or equal to one can be a common factor of $P_k(f)$ and $P_l(f)$. Let

$$R(f) = \frac{P_k(f)}{P_l(f)}.$$

Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{k, l\}$.

LEMMA 3 ([**21**]). *Let f be a transcendental meromorphic function and b_i, $i = 0, 1, \dots, n$ be small functions of f. If*

$$b_n f^n + b_{n-1} f^{n-1} + \cdots + b_0 \equiv 0,$$

then $b_i \equiv 0, i = 0, 1, \dots n$.

LEMMA 4. *Let*

$$f = \sum_{i=0}^{n} b_i e^{i\alpha},$$

where α is a nonconstant entire function and b_i $(i = 0, 1, \dots, n)$ are meromorphic functions satisfying $T(r, b_i) = S(r, e^{\alpha})$, then

$$T(r, f^{(k)}) = T(r, f) + S(r, f).$$

LEMMA 5. *Let f be a nonconstant entire function and*

$$g = L(f) = b_{-1} + \sum_{i=0}^{n} b_i f^{(i)}, \tag{1}$$

where b_i $(i = -1, 0, 1, \dots, n)$ are small meromorphic functions of f. Let a_1 and a_2 be two distinct constants in C. If f and g share a_1, a_2 IM, *then*

$$T(r, f) = \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f),$$

and

$$T(r, f) \le 2T(r, g) + S(r, f)$$

provided that $f \not\equiv g$.

PROOF. Let

$$\phi = \frac{f'(f - g)}{(f - a_1)(f - a_2)}. \tag{2}$$

From Lemma 1 one can easily see that $m(r, \phi) = S(r, f)$. Since f and g share a_1 and a_2, we see that $N(r, \phi) = S(r, f)$, thus

$$T(r, \phi) = S(r, f). \tag{3}$$

If $\phi \equiv 0$, then $f \equiv g$. Suppose that $\phi \not\equiv 0$, that is $f \not\equiv g$. From (1) we deduce that

$$\begin{aligned} T(r, f - g) &= T\left(r, \frac{\phi(f - a_1)(f - a_2)}{f'}\right) \\ &= T\left(r, \frac{f'}{(f - a_1)(f - a_2)}\right) + S(r, f) \\ &= N\left(r, \frac{f'}{(f - a_1)(f - a_2)}\right) + S(r, f). \end{aligned}$$

That is

$$T(r, f - g) = \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

From the expression of g, it is clearly that $T(r, f - g) \le T(r, f) + S(r, f)$. Thus

$$\overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) \le T(r, f) + S(r, f).$$

According to Nevanlinna's Second Fundamental Theorem and the above inequality, we have

$$\begin{aligned} T(r, f) &= \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f) \\ &\le T(r, g) + T(r, g) + S(r, f), \end{aligned}$$

since f and g share a_1 and a_2. □

Lemma 6. *Let f and g be as in Lemma 5. Furthermore, if f and g share a_1 CM, a_2 IM, and $N(r, 1/(f - a_2)) = S(r, f)$, then $f \equiv g$.*

Proof. Suppose that $f \not\equiv g$. Then the function ϕ in (2) is not identically zero. Set

$$\beta = \frac{g'}{g - a_2} - \frac{f'}{f - a_2}. \tag{4}$$

By the assumption of Lemma 6, we have $T(r, \beta) = S(r, f)$. From (2), we get

$$\phi \frac{f - a_1}{f'} \equiv 1 - \frac{g - a_2}{f - a_2}.$$

By taking the derivative and using (4), we have

$$\phi' \frac{f - a_1}{f'} + \phi\left(1 - \frac{(f - a_1)f''}{(f')^2}\right) \equiv \beta\left(\phi \frac{f - a_1}{f'} - 1\right).$$

That is

$$(\phi + \beta)\frac{f'}{f - a_1} - \phi \frac{f''}{f'} + \phi' - \beta\phi \equiv 0. \tag{5}$$

Since $N(r, 1/(f - a_2)) = S(r, f)$, from Lemma 5 we have

$$\overline{N}\left(r, \frac{1}{f - a_1}\right) = T(r, f) + S(r, f) \ne S(r, f).$$

Since f, g share a_1 CM, from (2) we see that "almost all" a_1-points of f are simple. And (5) implies that "almost all" simple a_1-points of f are the zeros of $\phi + \beta$. Hence we have $\phi + \beta \equiv 0$, and thus

$$-\frac{f''}{f'} + \frac{\phi'}{\phi} - \beta \equiv 0.$$

That is

$$\phi(f-a_2) \equiv cf'(g-a_2), \tag{6}$$

where $c \neq 0$ is a constant. From (2) and (6) we get

$$f-g \equiv c(f-a_1)(g-a_2).$$

This can be rewritten as

$$-c\left(g-\frac{1+ca_2}{c}\right) \equiv \frac{g-a_1}{f-a_1}.$$

Since f, g share a_1 CM, it follows from the above identity that

$$N\left(r, \frac{1}{g-(1+ca_2)/c}\right) = S(r,f).$$

Hence by Nevanlinna's Second Fundamental Theorem,

$$T(r,g) \leq \overline{N}\left(r, \frac{1}{g-a_2}\right) + \overline{N}\left(r, \frac{1}{g-(1+ca_2)/c}\right) + S(r,g) = S(r,f).$$

Thus from Lemma 5, $T(r,f) \leq 2T(r,g) + S(r,f) = S(r,f)$, a contradiction. □

THEOREM 1. *Let f be a nonconstant entire function and*

$$g = L(f) = b_{-1} + \sum_{i=0}^{n} b_i f^{(i)},$$

where b_i $(i = -1, 0, 1, \dots, n)$ are small meromorphic functions of f. Let a_1 and a_2 be two distinct constants in $\boldsymbol{C}$. If f and $g = L(f)$ share a_1 CM *and a_2* IM, *then $f \equiv g$ or f and g have the following expressions,*

$$f = a_2 + (a_1-a_2)(1-e^{\alpha})^2,$$

and

$$g = 2a_2 - a_1 + (a_1-a_2)e^{\alpha},$$

where α is an entire function.

PROOF. Suppose that $f \not\equiv g$. Set

$$\gamma = \frac{f'}{f-a_1} - \frac{g'}{g-a_1}. \tag{7}$$

Since f and g share a_1 CM, we have $T(r,\gamma) = S(r,f)$. From (2)

$$\phi\frac{f-a_2}{f'} \equiv 1 - \frac{g-a_1}{f-a_1}.$$

By taking the derivative in both sides of the above identity and using it again, we deduce that

$$\phi'\frac{f-a_2}{f'} + \phi\left(1 - \frac{(f-a_2)f''}{(f')^2}\right) \equiv \gamma\frac{g-a_1}{f-a_1} \equiv \gamma\left(1 - \phi\frac{f-a_2}{f'}\right).$$

That is

$$(\phi - \gamma)\frac{f'}{f - a_2} - \phi\frac{f''}{f'} + \phi' + \gamma\phi \equiv 0. \tag{8}$$

If $\phi - \gamma \equiv 0$, then

$$-\frac{f''}{f'} + \frac{\phi'}{\phi} + \frac{f'}{f - a_1} - \frac{g'}{g - a_1} \equiv 0.$$

It follows from (2) and the above equation that

$$\frac{f - g}{(f - a_2)(g - a_1)} \equiv c, \quad \text{(nonzero constant)},$$

which leads to that f and g share a_1, a_2 CM. And thus by using Theorem A, we have $f \equiv g$, a contradiction.

In the following, we assume that $\phi - \gamma \not\equiv 0$. Denote by $N_{k)}(r, 1/(f - a))$ the counting function of those a-points of f whose multiplicities are less than or equal to k and by $N_{(k+1}(r, 1/(f - a))$ the counting function of those a-points of f whose multiplicities are greater than k.

Let z_0 be an a_2-point of f of multiplicity $k \geq 1$ but not the zero of $\phi - \gamma$ and the pole of $\phi' + \gamma\phi$. Then the formula (8) implies that $\phi(z_0) - k\gamma(z_0) = 0$. If $\phi - k\gamma \not\equiv 0$ for any $k \geq 1$, then

$$N_{k)}\left(r, \frac{1}{f - a_2}\right) = S(r, f).$$

Let z_1 be an a_2-point of f of multiplicity $k \geq n + 2$, but not the zero of $\phi - \gamma$ and not the pole of $\phi' + \gamma\phi$ and b_i $(i = -1, 0, 1, \ldots)$. Then from (1), we have $b_{-1}(z_1) + b_0(z_1)a_2 = a_2$. If $b_{-1} + b_0 a_2 \not\equiv a_2$, then we get $N_{(n+2}(r, 1/(f - a_2)) = S(r, f)$. If $b_{-1} + b_0 a_2 \equiv a_2$, then it follows from (1) that

$$g - f \equiv (b_0 - 1)(f - a_2) + \sum_{i=1}^{n} b_i f^{(i)}.$$

Hence z_1 is a multiple zero of $g - f$ and thus a zero of ϕ. Hence $N_{(n+2}(r, 1/(f - a_2)) = S(r, f)$ still holds. In any case, we can deduce that $N(r, 1/(f - a_2)) = S(r, f)$. Hence $f \equiv g$ by Lemma 6.

Now we suppose that there exist an integer $k \geq 1$ such that $\phi - k\gamma \equiv 0$ and $\phi \not\equiv 0$. Then it follows from (8) that

$$\left(1 - \frac{1}{k}\right)\frac{f'}{f - a_2} - \frac{f''}{f'} + \frac{\phi'}{\phi} + \gamma \equiv 0. \tag{9}$$

By integrating, we obtain that

$$(f - a_2)^{k-1} \equiv c\left[\frac{f'(g - a_1)}{\phi(f - a_1)}\right]^k,$$

where $c \neq 0$ is a constant. From this and (2), by eliminating ϕ, we have

$$f \equiv a_2 + \frac{1}{c}(h-1)^k, \tag{10}$$

where

$$h \equiv \frac{f-a_1}{g-a_1}. \tag{11}$$

Clearly, $h' \equiv \gamma h$, from (1) and (10) we see that there exist small functions d_i $(i = 0, 1, \dots, k)$ of f such that

$$g \equiv \sum_{i=0}^{k} d_i h^i. \tag{12}$$

From (10), (11) and (12), we have

$$\begin{aligned} d_k h^{k+1} &+ \sum_{i=2}^{k}\left[d_{i-1} - \frac{(-1)^{k-i}}{c}\binom{k}{i}\right]h^i \\ &+ \left[d_0 - a_1 - (-1)^{k-1}\frac{k}{c}\right]h + a_1 - a_2 - \frac{(-1)^k}{c} \equiv 0. \end{aligned} \tag{13}$$

From this and Lemma 3, we get

$$\begin{aligned} c &\equiv \frac{(-1)^k}{a_1 - a_2}, \\ d_0 &\equiv a_1 - k(a_1 - a_2), \\ d_{i-1} &\equiv (-1)^i\binom{k}{i}(a_1 - a_2), \quad i = 2, \dots, k, \\ d_k &\equiv 0. \end{aligned}$$

Thus it follows from (10), (11) and (12) that

$$\begin{aligned} f &\equiv a_2 + (a_1 - a_2)(1-h)^k, \\ g &\equiv a_1 + \frac{(a_1 - a_2)[(1-h)^k - 1]}{h}. \end{aligned}$$

These two identities can be rewritten as

$$f - a_2 \equiv (a_1 - a_2)(1-h)^k, \tag{14}$$

$$g - a_2 \equiv (a_1 - a_2)\frac{h-1}{h}[1 - (1-h)^{k-1}]. \tag{15}$$

Since f and g share a_1 CM, we have $N(r,h) = S(r,f)$ and $N(r,1/h) = S(r,f)$. On the other hand, from (10) and by Lemma 2, we have

$$T(r,h) = \frac{1}{k}T(r,f) + S(r,f) \neq S(r,f).$$

Hence h can take any finite value $b \neq 0, 1$. Thus when $k > 2$, there exists a value $b \neq 0, 1$ such that $(1 - b)^{k-1} = 1$. Noting that f and g share a_2, from (14) and (15) we can conclude that $k = 2$. Thus $g \equiv 2a_2 - a_1 + (a_1 - a_2)h$ is an entire function. Hence $h = (f - a_1)/(g - a_1) = e^{\alpha}$, where α is an entire function. Finally from this, (14) and (15), we obtain that

$$f \equiv a_2 + (a_1 - a_2)(1 - e^{\alpha})^2,$$

and

$$g \equiv 2a_2 - a_1 + (a_1 - a_2)e^{\alpha},$$

which completes the proof of Theorem 1. □

COROLLARY 1. *Let f be an entire function, and a_1, a_2 be two distinct numbers in C. If f and $f^{(k)}$ share a_1* CM *and a_2* IM, *then $f \equiv f^{(k)}$.*

PROOF. If $f \equiv a_2 + (a_1 - a_2)(1 - e^{\alpha})^2$, then, by Lemma 4, $f^{(k)}$ can not be $2a_2 - a_1 + (a_1 - a_2)e^{\alpha}$. Hence Corollary 1 follows from Theorem 1. □

REMARK 1. (i) There are examples to show that the word "entire function" in Theorem 1 can not be replaced by "meromorphic function". (ii) The assumption "f and $L(f)$ share a_1 CM" in Theorem 1 can not be replaced by "f and $L(f)$ share a_1 IM".

EXAMPLE 1. Let $a_1, a_2 \in C$, $a_1 - a_2 = \sqrt{2}i$, w be a nonconstant solution of the following Riccati equation

$$w' = (w - a_1)(w - a_2).$$

Let

$$f = (w - a_1)(w - a_2) - \frac{1}{3}.$$

Then w and f are transcendental meromorphic functions and $w' \neq 0$. It is easy to verify that

$$f'' = 6w'f, \quad f'' + \frac{1}{6} = 6\left(f + \frac{1}{6}\right)^2.$$

Hence f and f'' share 0 CM and $-(1/6)$ IM. However, neither $f \equiv f''$ nor f has the form $a_2 + (a_1 - a_2)(1 - e^{\alpha})^2$.

EXAMPLE 2. Let $f = (1/2)e^z + (1/2)a^2e^{-z}$ and $L(f) = f'' + f' = e^z$, where a is a nonzero constant. It is obviously that

$$(L(f) - f)^2 = (f - a)(f + a).$$

Hence f and $L(f)$ share $-a, a$ IM and not CM. Again neither $f \equiv L(f)$ nor f assumes the form $a_2 + (a_1 - a_2)(1 - e^{\alpha})^2$.

Now we state a slight generalization of Theorem 1. First of all, we generalise the definitions of CM and IM to CM* and IM*.

Let f and g be two meromorphic functions. Denote by $N_c(r, 1/(f-a))$ the counting function of those a-points of f where a is taken by f and g with the same multiplicity, counted only once regardless of the multiplicity, and $N_i(r, 1/(f-a))$ the counting function of those a-points of f where a is taken by f and g regardless of the multiplicity, counted only once. We say that f and g share the value a CM*, if

$$\overline{N}\left(r, \frac{1}{f-a}\right) - N_c\left(r, \frac{1}{f-a}\right) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g-a}\right) - N_c\left(r, \frac{1}{g-a}\right) = S(r, f).$$

Similarly, we say that f and g share the value a IM*, if

$$\overline{N}\left(r, \frac{1}{f-a}\right) - N_i\left(r, \frac{1}{f-a}\right) = S(r, f),$$

and

$$\overline{N}\left(r, \frac{1}{g-a}\right) - N_i\left(r, \frac{1}{g-a}\right) = S(r, f).$$

Remark 2. From the proofs of Lemma 5, Lemma 6 and Theorem 1, one can easily deduce that the result in Theorem 1 is still valid for a nonconstant meromorphic function f satisfying $N(r, f) = S(r, f)$ and sharing a_1 CM* and a_2 IM* with $g = L(f)$.

When a_1, a_2 are two small functions of f, we have the following

Theorem 2. *Let f be a nonconstant meromorphic function satisfying $N(r, f) = S(r, f)$, and*

$$g = L(f) = b_{-1} + \sum_{i=0}^{n} b_i f^{(i)},$$

where b_i $(i = -1, 0, 1, \dots, n)$ are small meromorphic functions of f. Let a_1 and a_2 be two distinct small meromorphic functions of f. If f and g share a_1 CM* and a_2 IM*, then $f \equiv g$ or

$$f \equiv a_2 + (a_1 - a_2)(1 - e^{\alpha})^2,$$

and

$$g \equiv 2a_2 - a_1 + (a_1 - a_2)e^{\alpha},$$

where α is an entire function.

Proof. Let

$$F = \frac{f - a_1}{a_2 - a_1}, \quad \text{and} \quad G = \frac{g - a_1}{a_2 - a_1}.$$

Then F and G share 0 CM* and 1 IM*. Obviously, G still has the form $B_{-1} +$

$\sum_{i=0}^{n} B_i F^{(i)}$, where B_i $(i = -1, 0, 1, \ldots, n)$ are small functions of F. According to Remark 2, we can deduce that $F \equiv G$ or

$$F \equiv 1 - (1 - e^{\alpha})^2,$$

and

$$G \equiv 2 - e^{\alpha},$$

where α is an entire function. Hence we get $f \equiv g$ or

$$f \equiv a_2 + (a_1 - a_2)(1 - e^{\alpha})^2,$$

and

$$g \equiv 2a_2 - a_1 + (a_1 - a_2)e^{\alpha}. \qquad \square$$

Corollary 2. *Let f be a meromorphic function satisfying $N(r, f) = S(r, f)$ and a_1, a_2 be two distinct small meromorphic functions of f. If f and $f^{(k)}$ share a_1* CM* *and share a_2* IM*, *then $f \equiv f^{(k)}$.*

Thus we have completely resolved the question: What happens when an entire function f and the linear combination of its derivatives $L(f)$ share a small function a_1 CM and another small function a_2 IM? Next we propose to solve a new interesting question, namely: What happens when an entire function f and its derivative f' share two finite values a_1, a_2 CM jointly, that is $f^{-1}\{a_1, a_2\} = (f')^{-1}\{a_1, b_2\}$ counting multiplicities? Firstly, we prove two lemmas which will be needed in the proof of the theorem.

Lemma 7. *Let f be a nonconstant entire function and a_1, a_2 be two nonzero distinct finite values. If f and f' share the set $\{a_1, a_2\}$* IM *and $T(r, h) \neq S(r, f)$, where*

$$h \equiv \frac{(f' - a_1)(f' - a_2)}{(f - a_1)(f - a_2)}, \tag{16}$$

then following conclusions hold.

(i) $T(r, \psi) = S(r, f)$, *where*

$$\psi \equiv \frac{(f'h - f'')(f'h + f'')}{(f' - a_1)(f' - a_2)}. \tag{17}$$

(ii) $T(r, f') = N(r, 1/(f' - a_i)) + S(r, f),\ i = 1, 2.$

(iii) $m(r, 1/(f - c)) = S(r, f)$, *where $c \neq a_1, a_2$ is a constant.*

(iv) $T(r, h) = m(r, 1/(f - a_1)) + m(r, 1/(f - a_2)) + S(r, f) = m(r, 1/f') + S(r, f).$

(v) $2T(r, f) - 2T(r, f') = m(r, 1/h) + S(r, f).$

Proof. (i) Since f, f' share $a_i (i = 1, 2)$, any a_i-point of f is simple and thus h is an entire function. By assumption, $T(r, h) \neq S(r, f)$, hence $\psi \not\equiv 0$. Rewrite (16) as

$$(f' - a_1)(f' - a_2) \equiv (f - a_1)(f - a_2)h, \tag{18}$$

and then by taking the derivative in both sides of (18), we have

$$(2f' - a_1 - a_2)f'' \equiv [(2f - a_1 - a_2)f'h + (f - a_1)(f - a_2)h']. \tag{19}$$

When, say at $z = z_0$, $(f'(z_0) - a_1)(f'(z_0) - a_2) = 0$, and thus $(f(z_0) - a_1)(f(z_0) - a_2) = 0$, we have

$$\frac{2f'(z_0) - a_1 - a_2}{2f(z_0) - a_1 - a_2} = \pm 1.$$

It follows that

$$(f'(z_0)h(z_0) - f''(z_0))(f'(z_0)h(z_0) + f''(z_0)) = 0.$$

Hence we see that the simple a_i-points of f' are not the poles of ψ. If z_0 is an a_i-point of f' of multiplicity $m \geq 2$, thus a zero of f'' of multiplicity $m-1$, then from (16), z_0 is also a zero of h of multiplicity $m-1$. Hence z_0 is not the pole of ψ. We conclude that ψ is an entire function. Furthermore, since

$$\frac{f'h - f''}{f' - a_1} \equiv \frac{(f')^2 - a_2 f'}{(f - a_1)(f - a_2)} - \frac{f''}{f' - a_1}, \tag{20}$$

by using Lemma 1, we have $m(r, (f'h - f'')/(f' - a_1)) = S(r, f)$. Similarly, we have $m(r, (f'h + f'')/(f' - a_2)) = S(r, f)$. Hence $m(r, \psi) = S(r, f)$, and thus $T(r, \psi) = S(r, f)$.

(ii) By rewriting (17) as

$$\frac{\psi}{f'h - f''} \equiv \frac{f'}{(f - a_1)(f - a_2)} + \frac{f''}{(f' - a_1)(f' - a_2)},$$

and then by Lemma 1, we can deduce that $m(r, 1/(f'h - f'')) = S(r, f)$. Similarly, we have $m(r, 1/(f'h + f'')) = S(r, f)$. Hence it follows from (17) that $m(r, 1/((f' - a_1)(f' - a_2))) = S(r, f)$, which implies that $T(r, f') = N(r, 1/(f' - a_i)) + S(r, f), i = 1, 2$.

(iii) From (17) and (20), we have

$$\frac{\psi}{f - c} \equiv \left[\frac{(f')^2 - a_2 f'}{(f - c)(f - a_1)(f - a_2)} - \frac{f'}{f - c}\,\frac{f''}{f'(f' - a_1)}\right]\frac{f'h + f''}{f' - a_2}.$$

Hence by Lemma 1, we get $m(r, 1/(f - c)) = S(r, f)$, for $c \neq a_1, a_2$.

(iv) Since the function h in (16) is entire and

$$h = \frac{f'}{f - a_1}\,\frac{f'}{f - a_2} - \frac{(a_1 + a_2)f'}{(f - a_1)(f - a_2)} + \frac{a_1 a_2}{(f - a_1)(f - a_2)},$$

by using Lemma 1, it is not difficult to get

$$\begin{aligned} T(r, h) &= m\left(r, \frac{1}{(f - a_1)(f - a_2)}\right) + S(r, f) \\ &= m\left(r, \frac{1}{f - a_1}\right) + m\left(r, \frac{1}{f - a_2}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{f'}\right) + S(r, f). \end{aligned}$$

On the other hand, from (16) and (17) by eliminating h, we have

$$\frac{\psi}{f'} \equiv \frac{(f')^3-(a_1+a_2)(f')^2}{(f'-a_1)^2(f'-a_2)^2} - \frac{(f'')^2}{f'(f'-a_1)(f'-a_2)} + \frac{a_1a_2f'}{(f-a_1)(f-a_2)}\frac{1}{(f-a_1)(f-a_2)},$$

thus by Lemma 1, we get

$$m\left(r,\frac{1}{f'}\right) \le m\left(r,\frac{1}{f-a_1}\right) + m\left(r,\frac{1}{f-a_2}\right) + S(r,f).$$

Hence we obtain that

$$T(r,h) = m\left(r,\frac{1}{f-a_1}\right) + m\left(r,\frac{1}{f-a_2}\right) + S(r,f) = m\left(r,\frac{1}{f'}\right) + S(r,f).$$

(v) By using the conclusion in (ii), we have

$$2T(r,f') = N\left(r,\frac{1}{(f'-a_1)(f'-a_2)}\right) + S(r,f).$$

It follows from (18) and the conclusion in (iv) that

$$\begin{aligned}
2T(r,f') &= N\left(r,\frac{1}{(f-a_1)(f-a_2)h}\right) + S(r,f)\\
&= N\left(r,\frac{1}{(f-a_1)(f-a_2)}\right) + N\left(r,\frac{1}{h}\right) + S(r,f)\\
&= 2T(r,f) - m\left(r,\frac{1}{f-a_1}\right) - m\left(r,\frac{1}{f-a_2}\right) + N\left(r,\frac{1}{h}\right) + S(r,f)\\
&= 2T(r,f) - T(r,h) + N\left(r,\frac{1}{h}\right) + S(r,f).
\end{aligned}$$

That is $2T(r,f) - 2T(r,f') = m(r,1/h) + S(r,f)$, which completes the proof of Lemma 7. □

Lemma 8. *Let f be a nonconstant entire function and a_1, a_2 be two distinct finite values. If f and f' share the set $\{a_1,a_2\}$* CM, *then $T(r,h) = S(r,f)$, where h is the same as in Lemma* 7.

Proof. For the sake of convenience, we write $f_1 = f'$, $f_2 = f''$, and $f_3 = f'''$. Because f and f_1 share the set $\{a_1,a_2\}$ CM, there exists an entire function α such that $h \equiv e^{\alpha}$. If $a_1a_2 = 0$, then from (16)

$$h \equiv \frac{f_1^2}{(f-a_1)(f-a_2)} - \frac{(a_1+a_2)f_1}{(f-a_1)(f-a_2)}.$$

Hence by Lemma 1 we have $T(r,h) = S(r,f)$. Without loss of generality, we may assume that $a_1a_2 \ne 0$. Suppose $T(r,h) \ne S(r,f)$. From (17), (18) and (19) by eliminating h, we have

$$\frac{[(f-a_1)+(f-a_2)]f_1}{(f-a_1)(f-a_2)} \equiv \frac{(2f-a_1-a_2)f_1}{(f-a_1)(f-a_2)} \equiv \frac{(2f_1-a_1-a_2)f_2}{(f_1-a_1)(f_1-a_2)} - \beta, \tag{21}$$

where, and in the sequel $\beta \equiv \alpha'$, and

$$\frac{f_1^2}{(f-a_1)^2(f-a_2)^2} \equiv \frac{f_2^2}{(f_1-a_1)^2(f_1-a_2)^2} + \frac{\psi}{(f_1-a_1)(f_1-a_2)}. \tag{22}$$

By squaring all sides of (21), we get

$$\frac{f_1^2}{(f-a_1)^2} + \frac{2f_1^2}{(f-a_1)(f-a_2)} + \frac{f_1^2}{(f-a_2)^2}$$
$$\equiv \frac{(2f_1-a_1-a_2)^2 f_2^2}{(f_1-a_1)^2(f_1-a_2)^2} - \frac{2\beta(2f_1-a_1-a_2)f_2}{(f_1-a_1)(f_1-a_2)} + \beta^2. \tag{23}$$

Now (22) can be written as

$$\left[\frac{f_1^2}{(f-a_1)^2} - \frac{2f_1^2}{(f-a_1)(f-a_2)} + \frac{f_1^2}{(f-a_2)^2}\right]$$
$$\equiv \frac{(a_1-a_2)^2\psi(f_1-a_1)(f_1-a_2) + (a_1-a_2)^2 f_2^2}{(f_1-a_1)^2(f_1-a_2)^2}. \tag{24}$$

By taking the difference of (23) and (24), we get

$$\frac{4f_1^2}{(f-a_1)(f-a_2)} \equiv \frac{4f_2^2 - 2\beta(2f_1-a_1-a_2)f_2}{(f_1-a_1)(f_1-a_2)} + \frac{\beta^2(f_1-a_1)(f_1-a_2) - (a_1-a_2)^2\psi}{(f_1-a_1)(f_1-a_2)}. \tag{25}$$

By eliminating f from (17), (22) and (24), we have

$$\frac{16\psi}{(f_1-a_1)(f_1-a_2)} + \frac{16f_2^2}{(f_1-a_1)^2(f_1-a_2)^2} \equiv \left[\frac{4f_2^2 - 2\beta(2f_1-a_1-a_2)f_2}{f_1(f_1-a_1)(f_1-a_2)} + H\right]^2, \tag{26}$$

where

$$H = \frac{\beta^2(f_1-a_1)(f_1-a_2) - (a_1-a_2)^2\psi}{f_1(f_1-a_1)(f_1-a_2)}.$$

From Lemma 7, $m(r, 1/(f_1-a_1)) + m(r, 1/(f_1-a_2)) = S(r,f)$. Hence from (26) and by using Lemma 1, we get

$$m(r,H) = S(r,f). \tag{27}$$

We shall treat two cases: $a_1a_2\beta^2 - (a_1-a_2)^2\psi \not\equiv 0$ and $a_1a_2\beta^2 - (a_1-a_2)^2\psi \equiv 0$, separately.

If $a_1a_2\beta^2 - (a_1 - a_2)^2\psi \not\equiv 0$, then from (27) and Lemma 2, we can deduce that

$$3T(r, f_1) = N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_1 - a_1}\right) + N\left(r, \frac{1}{f_1 - a_2}\right) + S(r, f).$$

By (ii) of Lemma 7 and above formula, we get

$$m\left(r, \frac{1}{f_1}\right) = S(r, f). \tag{28}$$

Hence by (iv) of Lemma 7, we have $T(r, h) = S(r, f)$.

Now we consider the case

$$a_1a_2\beta^2 - (a_1 - a_2)^2\psi \equiv 0, \tag{29}$$

and rewrite (17) as

$$\psi(f_1 - a_1)(f_1 - a_2) \equiv f_1^2 e^{2\alpha} - f_2^2. \tag{30}$$

By taking the derivative on both sides of (30), we get

$$\begin{aligned}\psi'(f_1 - a_1)(f_1 - a_2) + \psi(2f_1 - a_1 - a_2)f_2 \\ \equiv 2\alpha' f_1^2 e^{2\alpha} + 2f_1f_2e^{2\alpha} - 2f_2f_3.\end{aligned} \tag{31}$$

Let z_0 be a zero of f_1. From (17), (18), (19) and (31), we can see that

$$\psi(z_0) = -\frac{f_2^2(z_0)}{a_1a_2}, \quad \beta(z_0) = -\frac{(a_1 + a_2)f_2(z_0)}{a_1a_2},$$

and

$$a_1a_2\psi'(z_0) - (a_1 + a_2)\psi(z_0)f_2(z_0) = -2f_2(z_0)f_3(z_0).$$

Thus by using (29), we have

$$\left(\frac{\beta'(z_0)}{\beta(z_0)} + \frac{\beta(z_0)}{2}\right)f_2(z_0) - f_3(z_0) = 0.$$

Again from (17) we see that any zero of f_1 and f_2 must be the zero of ψ, thus "almost all" zeros of f_1 are simple. Let

$$\gamma \equiv \left(\frac{\beta'}{\beta} + \frac{\beta}{2}\right)\frac{f_2}{f_1} - \frac{f_3}{f_1}. \tag{32}$$

Then we have $T(r, \gamma) = S(r, f)$, which also holds when f_1 is zero free.

If $\gamma \equiv 0$, then we can deduce that $f_2'/f_2 \equiv (\beta'/\beta) + (\alpha'/2)$, and thus by integrating, we have $f_2 \equiv c(\beta/2)\exp(\alpha/2)$, and thus $f_1 \equiv c\{\exp(\alpha/2) + d\}$, where $c \neq 0$ and d are constants. This implies

$$m\left(r, \frac{1}{f_1}\right) = m\left(r, \frac{1}{\exp(\alpha/2) + d}\right) \le \frac{1}{2}T(r, h) + S(r, f),$$

which leads to $T(r, h) = S(r, f)$, by Lemma 7.

In the following, we assume that $\gamma \not\equiv 0$. From (30), (31), by eliminating $e^{2\alpha}$, we have

$$(\psi' - 2\alpha'\psi)f_1(f_1 - a_1)(f_1 - a_2) + \psi(2f_1 - a_1 - a_2)f_1 f_2$$
$$\equiv 2\alpha' f_1 f_2^2 + 2\psi(f_1 - a_1)(f_1 - a_2)f_2 + 2f_2^3 - 2f_1 f_2 f_3. \tag{33}$$

If $\psi' - 2\alpha'\psi \equiv 0$, then we can get $e^{2\alpha} \equiv c\psi$, where c is a constant. Hence $T(r,h) = T(r,e^{\alpha}) = S(r,f)$. Without loss of generality, we may assume that $\psi' - 2\alpha'\psi \not\equiv 0$. Since any a_1-point and any a_2-point of f_1 are simple, from (33) any zero of f_2 but not a zero of f_1 must be also a zero of $\psi' - 2\alpha'\psi$. Hence we can conclude that $T(r, f_3/f_2) = S(r,f)$. From (32), we have

$$\gamma \equiv \left(\frac{\beta'}{\beta} + \frac{\beta}{2} - \frac{f_3}{f_2}\right)\frac{f_2}{f_1}.$$

Thus

$$T\left(r, \frac{f_2}{f_1}\right) = S(r,f). \tag{34}$$

Now since (29) holds, (26) can be rewritten as

$$b_0 f_1^2 + b_1 f_1 + b_2 \equiv 0, \tag{35}$$

where

$$b_0 \equiv (16 - 12\beta^2)\left(\frac{f_2}{f_1}\right)^2 + 4\beta^3\left(\frac{f_2}{f_1}\right) + 16\psi + 16\beta - 16 - \beta^4,$$

$$b_1 \equiv -16(a_1 + a_2)\beta\left(\frac{f_2}{f_1}\right)^3 + 16(a_1 + a_2)\beta^2\left(\frac{f_2}{f_1}\right)^2 - 8(a_1 + a_2)\beta^3\left(\frac{f_2}{f_1}\right)$$
$$+ 2(a_1 + a_2)\beta^4 - 16(a_1 + a_2)\psi,$$

$$b_2 \equiv -4(a_1 + a_2)^2\beta^2\left(\frac{f_2}{f_1}\right)^2 + 4(a_1 + a - 2)^2\beta^3\left(\frac{f_2}{f_1}\right) + 16a_1 a - 2\psi - (a_1 + a_2)^2\beta^4.$$

It is obviously that $T(r,b_i) = S(r,f), i = 0,1,2$. Since

$$T(r,f) < N\left(r, \frac{1}{f - a_1}\right) + N\left(r, \frac{1}{f - a_2}\right) + S(r,f)$$
$$= N\left(r, \frac{1}{f_1 - a_1}\right) + N\left(r, \frac{1}{f_1 - a_2}\right) + S(r,f)$$
$$\leq 2T(r,f_1) + S(r,f),$$

we have $T(r,b_i) = S(r,f_1)$, $i = 0,1,2$. Thus by Lemma 3, we have

$$b_i \equiv 0, \quad i = 0,1,2. \tag{36}$$

From this, (29), and (36), it is not difficult to show that f_2/f_1 is a constant. Hence

$$f' \equiv c_1(f - c_2), \tag{37}$$

where $c_1 \neq 0$, and $c_2 \neq a_1, a_2$ are constants. From (30) and (37), we have $N(r, 1/(f - c_2)) = S(r, f)$. On the other hand, from (21), Lemma 7 and Lemma 1, we can conclude that $m(r, \beta/(f - c_2)) = S(r, f)$. Thus $m(r, 1/(f - c_2)) = S(r, f)$. Hence $T(r, f) = S(r, f)$, a contradiction. □

Theorem 3. *Let f be a nonconstant entire function and a_1, a_2 be two distinct complex numbers. If f and f' share the set $\{a_1, a_2\}$* CM, *then one and only one of the following conclusions holds*:

(i) $f \equiv f'$.

(ii) $f + f' \equiv a_1 + a_2$.

(iii) $f \equiv c_1 e^{cz} + c_2 e^{-cz}$, *with* $a_1 + a_2 = 0$, *where* c, c_1 *and* c_2 *are nonzero constants which satisfy* $c^2 \neq 1$ *and* $c_1 c_2 = (1/4) a_1^2 (1 - c^{-2})$.

Proof. Under the assumption of Theorem 3, there exists an entire function α satisfying $T(r, e^\alpha) = S(r, f)$ such that $(f' - a_1)(f' - a_2) \equiv (f - a_1)(f - a_2)e^\alpha$, which can be expressed as

$$\left(e^{\alpha/2} f - \frac{a_1 + a_2}{2} e^{\alpha/2} + f' - \frac{a_1 + a_2}{2}\right)\left(e^{\alpha/2} f - \frac{a_1 + a_2}{2} e^{\alpha/2} - f' + \frac{a_1 + a_2}{2}\right) \equiv \left(\frac{a_1 - a_2}{2}\right)^2 (e^\alpha - 1). \tag{38}$$

Set

$$G \equiv e^{\alpha/2} f - \frac{a_1 + a_2}{2} e^{\alpha/2} + f' - \frac{a_1 + a_2}{2}, \tag{39}$$

and

$$H \equiv e^{\alpha/2} f - \frac{a_1 + a_2}{2} e^{\alpha/2} - f' + \frac{a_1 + a_2}{2}. \tag{40}$$

Then G and H are entire functions and, if $G \cdot H \not\equiv 0$,

$$N\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{H}\right) = S(r, f). \tag{41}$$

Thus

$$T\left(r, \frac{G'}{G}\right) + T\left(r, \frac{H'}{H}\right) = S(r, f). \tag{42}$$

From (38), (39) and (40), we have

$$G + H \equiv e^{\alpha/2}(2f - a_1 - a_2), \tag{43}$$

$$G - H \equiv 2f' - a_1 - a_2, \tag{44}$$

$$GH \equiv \left(\frac{a_1 - a_2}{2}\right)^2 (e^\alpha - 1). \tag{45}$$

We deduce easily from above three equations that

$$\left(\frac{\alpha'}{2}+e^{\alpha/2}-\frac{G'}{G}\right)G+\left(\frac{\alpha'}{2}-e^{\alpha/2}-\frac{H'}{H}\right)H+(a_1+a_2)e^{\alpha/2}\equiv 0. \tag{46}$$

By multiplying G on both sides of (46), we get

$$\phi_1 G^2+\phi_2 G+\phi_3\equiv 0, \tag{47}$$

where

$$\phi_1\equiv\frac{\alpha'}{2}+e^{\alpha/2}-\frac{G'}{G},$$

$$\phi_2\equiv(a_1+a_2)e^{\alpha/2},$$

$$\phi_3\equiv\left(\frac{a_1-a_2}{2}\right)^2(e^{\alpha}-1)\left(\frac{\alpha'}{2}-e^{\alpha/2}-\frac{H'}{H}\right).$$

From (42), we see that

$$T(r,\phi_i)=S(r,f),\quad i=1,2,3. \tag{48}$$

When $e^{\alpha}=h\equiv 1$, we can easily get from (16) that

$$\text{either}\quad f\equiv f'\quad\text{or}\quad f+f'\equiv a_1+a_2.$$

Now we assume that $e^{\alpha}\not\equiv 1$. If $T(r,G)=S(r,f)$, then from (45) we have $T(r,H)=S(r,f)$. Thus $T(r,f)=S(r,f)$ from (43). This is impossible. Hence $T(r,G)\neq S(r,f)$. If $\phi_1\not\equiv 0$, then from (47) and (48), we get

$$2T(r,G)=T\left(r,\frac{\phi_2}{\phi_1}G+\frac{\phi_3}{\phi_1}\right)\le T(r,G)+S(r,f),$$

and thus $T(r,G)=S(r,f)$, a contradiction. Hence $\phi_1\equiv 0$. Similarly we have $\phi_i\equiv 0$, $i=2,3$. That is

$$\frac{\alpha'}{2}+e^{\alpha/2}-\frac{G'}{G}\equiv 0, \tag{49}$$

$$\frac{\alpha'}{2}-e^{\alpha/2}-\frac{H'}{H}\equiv 0, \tag{50}$$

$$a_1+a_2=0. \tag{51}$$

Formulas (49) and (50) lead to $(G'/G)+(H'/H)\equiv\alpha'$. Thus

$$GH\equiv c_0e^{\alpha}, \tag{52}$$

where c_0 is a nonzero constant. By combining (45), (51), and (52) we can see that e^{α} and thus α is a constant. Hence (49) and (50) become $G'\equiv e^{\alpha/2}G$ and $H'\equiv -e^{\alpha/2}H$, respectively. This and (45) lead to

$$G\equiv c_1e^{cz},\quad H\equiv c_2e^{-cz}, \tag{53}$$

where $c = e^{\alpha/2} \neq \pm 1$, with c_1, c_2 are constants satisfying

$$c_1 c_2 = \left(\frac{a_1 - a_2}{2}\right)^2 (e^\alpha - 1) = \left(\frac{a_1 - a_2}{2}\right)^2 (c^2 - 1). \tag{54}$$

Hence from (43), (53) and (54), we have

$$f \equiv \frac{c_1}{2} e^{-(\alpha/2)} e^{cz} + \frac{c_2}{2} e^{-(\alpha/2)} e^{-cz}.$$

The above expression can also be rewritten as

$$f \equiv \tilde{c}_1 e^{cz} + \tilde{c}_2 e^{-cz},$$

where $\tilde{c}_1 = (c_1/2) e^{-(\alpha/2)}$, and $\tilde{c}_2 = (c_2/2) e^{-(\alpha/2)}$ satisfy

$$\tilde{c}_1 \tilde{c}_2 = \frac{1}{4} \left(\frac{a_1 - a_2}{2}\right)^2 (1 - c^{-2}),$$

which completes the proof of Theorem 3. □

Remark 3. We suspect that the condition 'f and f' share the set $\{a_1, a_2\}$ CM' in Theorem 3 can be replaced by 'f and f' share the set $\{a_1, a_2\}$ IM'. But it can be shown that for a meromorphic function f, the word 'CM' in Theorem 3 can not be replaced by 'IM'. For example, if $f = (e^{2z} - 1)/(e^{2z} + 1)$, then f and f' share $0, 1$ IM jointly. The following is a more complicated example.

Example 3. Taking a constant $a, a \neq 0, -(27/32)$. Then the equation $z^3 - az^2 - a^2 z + a^3 + a^2 = 0$ has no multiple root. Let f be the elliptic function satisfying

$$(f')^2 = f^3 - af^2 - a^2 f + a^3 + a^2.$$

Then

$$(f' - a)(f' + a) = (f - a)^2 (f + a),$$

and f, f' share $a, -a$ IM jointly.

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