J. Math. Soc. Japan
Vol. 51, No. 3, 1999

Remark on Fourier coefficients of modular forms of half integral weight belonging to Kohnen's spaces

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(Received Feb. 25, 1997)
(Revised Nov. 26, 1997)

Abstract. The purpose of this paper is to derive that the square of Fourier coefficients $a(n)$ at a square free positive integer n of modular forms f of half integral weight belonging to Kohnen's spaces of arbitrary odd level and of arbitrary primitive character is essentially equal to the critical value of the zeta function attached to the modular form F of integral weight which is the image of f under the Shimura correspondence. Previously, Kohnen-Zagier had obtained an analogous result in the case of Kohnen's spaces of square free level and of trivial character. Our results give some generalizations of them of Kohnen-Zagier. Our method of the proof is similar to that of Shimura's paper concerning Fourier coefficients of Hilbert modular forms of half integral weight over totally real fields.

Introduction.

In [**7**], Shimura established a correspondence $\Psi = \Psi^{4N, \psi_0}_{(2k+1)/2, t}$ between the space $S_{(2k+1)/2}(4N, \psi_0)$ of modular forms of half integral weight $(2k+1)/2$ and the space $S_{2k}(2N, \psi_0^2)$ of those of even weight $2k$. Using methods and languages of representation theory of adeles of metaplectic groups, Waldspurger [**11**] proved that the square of Fourier coefficients $a(n)$ at a square free integer n of modular forms $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ of half integral weight is essentially proportional to the central value of the zeta function at a certain integer attached to the modular form F if f corresponds to F by Ψ and f is an eigen-function of Hecke operators.

On the other hand, Kohnen-Zagier [**2**] determined explicitly the constant of the proportionality in the case of modular forms of half integral weight belonging to Kohnen's spaces $S_{(2k+1)/2}(N, \chi)$ of square free level N and character χ which is a subspace of $S_{(2k+1)/2}(4N, \chi_1)$. Kohnen [**1**] (resp. [**3**]) treated the case where $N = 1$ (resp. N is an odd square free integer and χ is the trivial character of level N) (cf. Kojima [**4**] and [**5**]).

In [**10**] , Shimura intended to generalize such formulas to the case of Hilbert modular forms f of half integral weight and succeeded in obtaining many general interesting formulas. Among these, some explicit and useful formulas about the proportionality constant were formulated under assumption that f satisfies the multiplicity one theorem, but there he did not treat the problem determining whether such

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1991 *Mathematics Subject Classification*. Primary 11F37; Secondary 11F30, 11F67
Key words and phrases. Fourier coefficients of modular forms, modular forms of half integral weight, the special values of zeta functions
This research was partially supported by Grant-in-Aid for Scientific Research (No. 09640018), Ministry of Education, Science and Culture, Japan

f satisfies this condition. Therefore it is a question to explore results of Shimura's type in the case of modular forms of half integral weight which do not satisfy the multiplicity one theorem. There are interesting subspaces $S_{(2k+1)/2}(N,\chi)$, called Kohnen's subspaces, of $S_{(2k+1)/2}(4N,\chi_1)$, whose elements do not satisfy the multiplicity one theorem general. More precisely, this means that if $f \in S_{(2k+1)/2}(N,\chi)$ satisfies $T^{4N}_{(2k+1)/2,\chi_1}(p^2)f = \omega(p)f$ and $T^{4N}_{(2k+1)/2,\chi_1}(p^2)f' = \omega(p)f'$ for every p $((p,M)=1)$ with an $f' \in S_{(2k+1)/2}(4N,\chi_1)$, then f is not equal to cf' for any constant c, where $T^{4N}_{(2k+1)/2,\chi_1}(p^2)$ is the Hecke operator of $S_{(2k+1)/2}(4N,\chi_1)$ given in [**7**] (cf. [**2**] and Lemma 1.3). For modular forms belonging to Kohnen's spaces $S_{(2k+1)/2}(N,\chi)$, Shimura [**10**] did not obtain the same explicit formula as that of Kohnen and Zagier [**1**], [**3**].

The purpose of this paper is to try to derive such explicit formulas concerning the proportionality constant in some cases of modular forms f of half integral weight with multiplicity two by following Shimura's method and to generalize results of Kohnen and Zagier in [**1**], [**3**] to the modular forms of half integral weight belonging to Kohnen's spaces of an arbitrary odd level N and of an arbitrary primitive character χ modulo N. We shall verify an explicit relation between the square of Fourier coefficients $a(4n)$ at a fundamental discriminant $4n$ of modular forms $f(z) = \sum_{\varepsilon(-1)^k n \equiv 0,1(4), n>0} a(n)e[nz]$ belonging to the Kohnen's space of half integral weight $(2k+1)/2$ and of arbitrary odd level N with primitive character χ and the critical value of the zeta function of the modular form F which is the image of f under the Shimura correspondence Ψ. The assumptions on χ and fundamental discriminant $4n$ are technical conditions for modifications of methods in [**10**]. Our methods of the proof are the same as those of Shimura [**10**]. To obtain our results we need to modify slightly his methods.

Section 0 is a preliminary section. In Section 1, we shall summarize some results about Kohnen's spaces, the Shimura correspondence, theta functions and Hecke operators of Kohnen's spaces. There, using these, we shall determine explicitly the image of modular forms of half integral weight belonging to Kohnen's spaces under the Shimura correspondence. We show that $\Psi^{4N,\chi_1}_{(2k+1)/2,\tau}(f)(w)$ coincides with $a(4\tau)g(2w)$ for every $f(z) = \sum_{\varepsilon(-1)^k n \equiv 0,1(4), n>0} a(n)e[nz] \in S_{(2k+1)/2}(N,\chi)$, where τ is a positive square free integer satisfying $\tau \equiv 2,3 (\mathrm{mod}\, 4)$ and $g(w)$ is an element of $S_{2k}(N,\chi^2)$. To overcome difficulties about multiplicity one property for elements of Kohnen's space, we adapt the operator $U(4)$ and we impose the condition (1-13) which is essential for our arguments. In Section 2, we shall verify an integral formula which indicates that a modular form f of half integral weight is expressed as the inner product of a theta function and the image $\Psi(f)$ of f by the Shimura correspondence Ψ. By the above assumption and a method similar to that of Shimura [**10**], we shall show that $\langle \Theta(z,w;\zeta^{\mathbf{Z}}), g(2w)\rangle = cL_\tau(f)(z) + c'L_\tau(f)|U(4)(z)$ for a modular form $f \in S_{(2k+1)/2}(N,\chi)$ with some constants c and c', where $\Theta(z,w;\zeta^{\mathbf{Z}})$ is a theta function and $L_\tau(f)(z) = f(\tau z)\tau^k$. Moreover, applying the above formula in Section 1, we obtain $c = \overline{a(4\tau)}c''$ and $c' = \overline{a(4\tau)}c'''$ with explicit constants c'' and c'''. These formulas are keys for our later arguments. In Section 3, using the results of Section 1, Section 2 , the computation of the image of a product of theta series and Eisenstein series by Shimura correspondence and the method of the Rankin's convolution, under some assumptions, we shall derive an explicit connection between the square of Fourier coefficients of a modular form f of half integral weight and the central value of zeta functions associated with the image $\Psi(f)$ of f by the Shimura correspondence Ψ.

We mention that our results give a generalization of some results in Kohnen-Zagier [**1**] and [**3**] and Kojima [**4**] and [**5**].

Finally, the author is indebted to the referee for suggesting some revisions of this paper.

§0. Notation and preliminaries.

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $z \in \boldsymbol{C}$, we put $e[z] = \exp(2\pi i z)$ and we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg z^{1/2} \leqq \pi/2$. Further, we set $z^{k/2} = (\sqrt{z})^k$ for every $k \in \boldsymbol{Z}$. Let $SL(2, \boldsymbol{R})$ denote the group of all real matrices of degree 2 with determinant one and $\mathfrak{H}$ the complex upper half plane, i.e.,

$$SL(2, \boldsymbol{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c \text{ and } d \in \boldsymbol{R} \text{ and } ad - bc = 1 \right\}$$

and

$$\mathfrak{H} = \{ z = x + iy \,|\, x, y \in \boldsymbol{R} \text{ and } y > 0 \}.$$

Define an action of $SL(2, \boldsymbol{R})$ on $\mathfrak{H}$ by

$$z \rightarrow \gamma(z) = \frac{az + b}{cz + d} \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \text{ and for all } z \in \mathfrak{H}.$$

For positive integers M and M', put

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \middle| \, a, b, c \text{ and } d \in \boldsymbol{Z} \text{ and } c \equiv 0 (\operatorname{mod} M) \right\}$$

and

$$\Gamma[M, M'] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M') \middle| \, b \equiv 0 (\operatorname{mod} M) \right\}.$$

We introduce an automorphic factor $j_0(\gamma, z)$ of $\Gamma_0(4)$ determined by

$$j_0(\gamma, z) = \vartheta_0(\gamma(z))/\vartheta_0(z) \text{ for every } = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \text{ and}$$

$$\text{for every } z \in \mathfrak{H} \text{ with } \vartheta_0(z) = \sum_{n=-\infty}^{\infty} e[n^2 z] \text{ and } \vartheta(z) = \vartheta_0(z/2).$$

§1. Shimura correspondences of modular forms of half integral weight and theta functions.

This section is devoted to summarizing several fundamental facts which we need later. Let k be a positive integer. Let N denote a positive integer and ψ_0 a Dirichlet character modulo $4N$. We denote by $S_{(2k+1)/2}(4N,\psi_0)$ the set of all cusp forms f on $\mathfrak{H}$ such that

$$f(\gamma(z))=\psi_0(d)j_0(\gamma,z)^{2k+1}f(z)\quad \text{for every } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4N). \tag{1-1}$$

Throughout the rest of the paper we assume that N is an odd integer. Let χ be a Dirichlet character modulo N such that $\chi(-1)=\varepsilon$. Put $\chi_1=(4\varepsilon/*)\chi$. We introduce a subspace $S_{(2k+1)/2}(N,\chi)$ of $S_{(2k+1)/2}(4N,\chi_1)$ defined by

$$S_{(2k+1)/2}(N,\chi)=\left\{f(z)\in S_{(2k+1)/2}(4N,\chi_1)\;\middle|\; f(z)=\sum_{\varepsilon(-1)^k n\equiv 0,1(4),\, n>0} a(n)e[nz]\right\}. \tag{1-2}$$

We call $S_{(2k+1)/2}(N,\chi)$ the Kohnen's space of weight $(2k+1)/2$ and of level N with character χ. We denote by $S_{2k}(M,\omega)$ the space of all cusp forms f' of weight $2k$ and of level M with character ω satisfying

$$f'(\gamma(z))=\omega(d)(cz+d)^{2k}f'(z)\quad \text{for every } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(M).$$

Here we recall the notation and results in Shimura [**10**]. Let $\mathfrak{b}$ and $\mathfrak{b}'$ denote integral ideals of $\boldsymbol{Q}$ and ψ a Hecke character of $\boldsymbol{Q}$ whose conductor divides $4\mathfrak{b}\mathfrak{b}'$. Let $\mathscr{M}_{(2k+1)/2}(\mathfrak{b},\mathfrak{b}';\psi)$ (resp. $\mathscr{S}_{(2k+1)/2}(\mathfrak{b},\mathfrak{b}';\psi)$) be the space of modular forms (resp. modular cusp forms) of half integral weight $(2k+1)/2$ given in [**10**, p. 507]. Let ψ_0 be a Dirichlet character modulo $4N$ such that $\psi_0(-1)=1$. We choose the Hecke character ψ of $\boldsymbol{Q}$ such that

$$\prod_{p|4N}\psi_p(a)=\left(\frac{-1}{a}\right)^k\psi_0(a)^{-1}\quad \text{for every } a\in(\boldsymbol{Z}/4N\boldsymbol{Z})^\times,\ \psi_p(\boldsymbol{Z}_p^\times)=1 \tag{1-3}$$

for every $p\nmid 4N$ and $\psi_a(x)=(\operatorname{sgn}x)^k$ $(x\in\boldsymbol{R})$.

For $f'\in\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$, put $L(f')(z)=f'(2z)$. Then the following mapping is bijective (cf. [**10**, p. 523]).

$$L:\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi)\to S_{(2k+1)/2}(4N,\psi_0). \tag{1-4}$$

Let t denote a positive square free integer. Define a mapping $\Psi^{4N,\psi_0}_{(2k+1)/2,t}$ of $S_{(2k+1)/2}(4N,\psi_0)$ into $S_{2k}(2N,\psi_0^2)$ by

$$\Psi^{4N,\psi_0}_{(2k+1)/2,t}(f)(z)=\sum_{m=1}^{\infty}\left(\sum_{d|m}\psi_0(d)\left(\frac{-1}{d}\right)^k\left(\frac{t}{d}\right)d^{k-1}a(t(m/d)^2)\right)e[mz] \tag{1-5}$$

for every $f(z)=\sum_{n=1}^{\infty}a(n)e[nz]\in S_{(2k+1)/2}(4N,\psi_0)$. This mapping was first shown by Shimura [**7**] and it was reformulated by Niwa [**6**].

Next we recall the definition of theta functions given in [**10**]. Put

$$V=\left\{\xi=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\boldsymbol{Q})\;\middle|\;\operatorname{tr}\xi=0\right\}\quad\text{and}$$

$$\mathscr{S}(V)=\{\eta:V\to\boldsymbol{C}\,|\,\eta \text{ is a locally constant function in the sence of [10]}\}.$$

Let η be an element of $\mathscr{S}(V)$. Define a theta function $\Theta(z,w;\eta)$ on $\mathfrak{H}\times\mathfrak{H}$ by

$$\Theta(z,w;\eta)=y^{1/2}(\Im w)^{-2k}\sum_{\xi\in V}\eta(\xi)[\xi,\overline{w}]^{k}e[2^{-1}R[\xi,z,w]] \tag{1-6}$$

for every $(z,w)\in\mathfrak{H}\times\mathfrak{H}$, where

$$[\xi,w]=[\xi,w,w],\quad [\xi,w,w']=(cww'+dw-aw'-b)\text{ and}$$

$$R[\xi,z,w]=(\det\xi)z+\frac{iy}{2}\Im w^{-2}|[\xi,w]|^{2}\quad\left(\xi=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in V,\,y=\Im z\right).$$

Throughout the rest of the paper, we take a Dirichlet character ψ_0 with $\psi_0(-1)=1$ and a positive square free integer τ such that

$$\text{The conductor of }\psi_0\text{ is }N,(\tau,N)=1\text{ and }\tau\equiv 2,3(\operatorname{mod}4). \tag{1-7}$$

Put $\varphi=\psi\varepsilon_\tau$ with the Hecke character ε_τ associated with the quadratic field $\boldsymbol{Q}(\sqrt{\tau})$. Observe that the conductor of φ is $4N\tau$. We put $e=2N$ and

$$\mathfrak{o}[e^{-1},e]=\left\{x=\begin{pmatrix}a_x&b_x\\c_x&d_x\end{pmatrix}\in M_2(\boldsymbol{Q})\,\middle|\,a_x\in\boldsymbol{Z},b_x\in e^{-1}\boldsymbol{Z},c_x\in e\boldsymbol{Z}\text{ and }d_x\in\boldsymbol{Z}\right\}.$$

We consider an element $\eta\in\mathscr{S}(V)$ determined by

$$\eta(x)=\begin{cases}0 & \text{if } x\notin\mathfrak{o}[e^{-1},e],\\ \sum_{t\in(1/2\tau)\boldsymbol{Z}/2N\boldsymbol{Z}}\varphi_a(t)\varphi^{*}((2t\tau))e[-b_xt] & \text{otherwise,}\end{cases} \tag{1-8}$$

where φ^* is the ideal character associated with φ (cf. [**10**, p. 505]). Introduce a mapping L_τ of $\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$ into $\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\tau\boldsymbol{Z};\psi\varepsilon_\tau)$ defined by

$$L_\tau(f)(z)=f(\tau z)\tau^{k}\quad\text{for every } f(z)\in\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi). \tag{1-9}$$

We put $h(z)=L_\tau(f)(z)$. The following lemma is proved by Shimura [**10**, Prop. 1.4 and 1.5].

Lemma 1.1. *The notation being as above, the mapping L_τ gives a bijection of $\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$ onto the set*

$$\left\{h(z)=\sum_{n=1}^{\infty}a'(n)e[nz/2]\in\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\tau\boldsymbol{Z},\psi\varepsilon_\tau)\middle|a'(n)=0\text{ if }\tau\nmid n\right\}.$$

Moreover,

$$\langle f,f\rangle=\tau^{-(2k+1)/2}\tau\langle h,h\rangle,$$

where for given two cusp forms f,g of weight l with respect to Γ, their inner product $\langle f,g\rangle$ means

$$\langle f,g\rangle=\operatorname{vol}(\Gamma\backslash\mathfrak{H})^{-1}\int_{\Gamma\backslash\mathfrak{H}}\overline{f(z)}g(z)\Im z^{l}d_{\mathfrak{H}}z\text{ with }d_{\mathfrak{H}}z=\frac{dxdy}{y^{2}}\quad(x=\Re z,\,y=\Im z).$$

The following proposition is confirmed by Shimura [**8**] and [**10**] (cf. Niwa [**6**]).

Proposition 1.1. *For* $f(z)=\sum_{n=1}^{\infty}a(n)e[nz/2]\in\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z},N\boldsymbol{Z};\psi)$, *put* $h(z)=L_{\tau}(f)(z)$. *Suppose that* ψ_0,τ *and* ψ *satisfy the condition* (1–7) *and* η *is the function on* V *determined by* (1–8). *Then there exists the element*

$$g_{\tau}(w)=\sum_{m=1}^{\infty}\left(\sum_{d|m}\psi_0(d)\left(\frac{-1}{d}\right)^k\left(\frac{\tau}{d}\right)d^{k-1}a(\tau(m/d)^2)\right)e[mw]$$

belonging to $S_{2k}(2N,\psi_0^2)$ *such that*

$$C'g_{\tau}(w)=\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}h(z)\Theta(z,w;\eta)y^{(2k+1)/2}d_{\mathfrak{H}}z \text{ with } C'=2^{2+k}i^k\tau 4N.$$

Next we describe basic results of Hecke operators of the Kohnen's space. Let χ denote a Dirichlet character modulo N. We denote by $T^{4N}_{(2k+1)/2,\chi_1}(p^2)$ the Hecke operator on $S_{(2k+1)/2}(4N,\chi_1)$ given in [**7**]. For a positive integer m, we define a function $f|U(m)$ on $\mathfrak{H}$ by

$$f|U(m)(z)=\sum_{n=0}^{\infty}a(mn)e[nz]$$

for every function $f(z)=\sum_{n=0}^{\infty}a(n)e[nz]$ on $\mathfrak{H}$. We denote by $T_{(2k+1)/2,N,\chi}(p)$ $(p\nmid N)$ the Hecke operator on $S_{(2k+1)/2}(N,\chi)$ defined in [**2**, p. 42] (see also Remark in [**2**, p. 46]). The following lemma was verified by Kohnen [**2**].

Lemma 1.2. *The notation being as above, for* $f(z)=\sum_{n\geqq 1,\varepsilon(-1)^kn\equiv 0,1(4)}a(n)e[nz]\in S_{(2k+1)/2}(N,\chi)$, *Fourier expansions of* $f|T_{(2k+1)/2,N,\chi}(p)(z)$ $(p\nmid N)$ *and* $f|U(p^2)(z)$ $(p|N)$ *are given as follows*:

$$f|T_{(2k+1)/2,N,\chi}(p)(z)=\sum_{n\geqq 1,\varepsilon(-1)^kn\equiv 0,1(4)}\left(a(p^2n)+\chi(p)\left(\frac{\varepsilon(-1)^kn}{p}\right)p^{k-1}a(n)\right.$$
$$\left.+\chi^2(p)p^{2k-1}a(n/p^2)\right)e[nz]\quad(p\nmid N)$$

and

$$f|U(p^2)(z)=\sum_{n\geqq 1,\varepsilon(-1)^kn\equiv 0,1(4)}a(p^2n)e[nz]\quad(p|N),$$

where $a(n/p^2)$ *means* 0 *if* $p^2\nmid n$.

Observe that $T_{(2k+1)/2,N,\chi}(p)$ coincides with the restriction of $T^{4N}_{(2k+1)/2,\chi_1}(p^2)$ to $S_{(2k+1)/2}(N,\chi)$ for every odd prime p $(p\nmid N)$. Now we impose the following assumption.

(1-10) $\varepsilon=\chi(-1)$ satisfies $(-1)^k\varepsilon>0,\chi$ is a primitive Dirichlet character modulo N, $f\in S_{(2k+1)/2}(N,\chi)$ is an eigenfunction of all Hecke operators

$$T_{(2k+1)/2,N,\chi}(p)\ (p \nmid N) \quad \text{and} \quad U(p^2)\ (p|N), \text{ i.e.,}$$

$$f|T_{(2k+1)/2,N,\chi}(p)(z) = \omega(p)f(z)\ (p \nmid N) \quad \text{and} \quad f|U(p^2)(z) = \omega(p)f(z)\ (p|N)$$

and $a(4\tau_0) \neq 0$ for some square free positive integer τ_0 such that $\tau_0 \equiv 2, 3 (\mathrm{mod}\, 4)$.

Since $S_{(2k+1)/2}(N,\chi)$ is contained in $S_{(2k+1)/2}(4N,(4\varepsilon/*)\chi)$, we have the following diagram.

$$\text{(1-11)} \quad S_{(2k+1)/2}(N,\chi) \subset S_{(2k+1)/2}(4N,\psi_0) \xrightarrow{L^{-1}} \mathscr{S}_{(2k+1)/2}(\boldsymbol{Z}, N\boldsymbol{Z};\psi) \text{ with } \psi_0 = \left(\frac{4\varepsilon}{*}\right)\chi.$$

By this relation, we may identify elements $f(z)$ of $S_{(2k+1)/2}(N,\chi)$ with those of $\mathscr{S}_{(2k+1)/2}(\boldsymbol{Z}, N\boldsymbol{Z};\psi)$. Now we may derive the following lemma.

Lemma 1.3. *Let* $f(z) = \sum_{n \geqq 1, \varepsilon(-1)^k n \equiv 0,1(4)} a(n)e[nz]$ *be an element of* $S_{(2k+1)/2}(N,\chi)$ *satisfying the condition* (1–10) *and let* τ *be an integer satisfying* (1-7). *Then*

$$\Psi^{4N,\chi(4\varepsilon/*)}_{(2k+1)/2,\tau}(f)(w) = \sum_{m=1}^{\infty}\left(\sum_{d|m}\left(\frac{4\tau}{d}\right)\chi(d)d^{k-1}a(\tau(m/d)^2)\right)e[mw]$$

coincides with an element $a(4\tau)g(2w)$ *of* $S_{2k}(2N,\chi^2)$, *where* $g(w) = \sum_{n=1}^{\infty} c(n)e[nw]$ *belongs to* $S_{2k}(N,\chi^2)$ *and*

$$\sum_{n=1}^{\infty} c(n)n^{-s} = \prod_p (1 - \omega(p)p^{-s} + \chi^2(p)p^{2k-1-2s})^{-1}.$$

Proof. Putting $F(w) = \Psi^{4N,\chi(4\varepsilon/*)}_{(2k+1)/2,\tau}(f)(w) = \sum_{n=1}^{\infty} b(n)e[nw]$, by (1.4) and (1.5), we see that $F(w)$ is an element of $S_{2k}(2N,\chi^2)$. Put $l(w) = F(w/2)$. Since $b(m) = 0$ if m is odd, $l(w+1) = l(w)$. Using the fact that $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\Gamma[2,N]\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} = \Gamma_0(2N)$ and $F(w)$ belongs to $S_{2k}(2N,\chi^2)$, we can easily check that

$$l(\gamma(w)) = \chi^2(d)(cw+d)^{2k}l(w) \quad \text{for every } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma[2,N].$$

Observing that $\Gamma_0(N)$ is generated by elements of two groups $\Gamma[2,N]$ and $\left\{\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \boldsymbol{Z}\right\}$, $l(w)$ is contained in $S_{2k}(N,\chi^2)$. Now

$$l(w) = \sum_{n=1}^{\infty} b(2n)e[nw] \text{ and } b(2n) = \sum_{d|2n}\left(\frac{4\tau}{d}\right)\chi(d)d^{k-1}a(4\tau(n/d)^2)$$

$$= \sum_{d|n}\left(\frac{4\tau}{d}\right)\chi(d)d^{k-1}a(4\tau(n/d)^2).$$

Hence, we have

$$\sum_{n=1}^{\infty} b(2n)n^{-s} = L\left(s-k+1, \left(\frac{4\tau}{*}\right)\chi\right)\sum_{n=1}^{\infty} a(4\tau n^2)n^{-s}.$$

By Lemma 1.2, we see that

$$L\left(s-k+1,\left(\frac{4\tau}{*}\right)\chi\right)\sum_{n=1}^{\infty}a(4\tau n^2)n^{-s}=a(4\tau)\prod_p(1-\omega(p)p^{-s}+\chi^2(p)p^{2k-1-2s})^{-1}$$

(cf. [**7**, p. 452 and p. 453] and [**2**, p. 69]). This completes the proof.

It is well known that $f|U(4)(z)$ belongs to $S_{(2k+1)/2}(4N,\chi(4\varepsilon/*))$ for every $f(z)\in S_{(2k+1)/2}(4N,\chi(4\varepsilon/*))$. Here we impose the further condition on f in (1-10).

$$\text{(1-12)}\qquad \text{If } f'\in S_{(2k+1)/2}\left(4N,\chi\left(\frac{4\varepsilon}{*}\right)\right)\quad\text{and}\quad f'|T^{4N}_{(2k+1)/2,\chi(4\varepsilon/*)}(p^2)=\omega(p)f'$$

for every prime p ($p\nmid 4\tau N$), then $f'(z)$ equals

$$cf(z)+c'f|U(4)(z)$$

for some constants c and c', where $\omega(p)$ is the eigenvalue given in Lemma 1.3. Moreover, we impose the condition that

(1-13) $g(w)$ is a primitive form of $S_{2k}(N,\chi^2)$ and $f|U(4)(z)$ is not a constant times $f(z)$,

where $f(z)$ and $g(w)$ are the same elements given in Lemma 1.3. Furthermore, we consider the condition that

(1-14) if $2|\tau$, then the conductor of φ is $4N\tau$ and

$$\varphi_2(1+4x)=\varphi_2(1+4x^2)\text{ for every }x\in\boldsymbol{Z}_2,$$

where φ_2 is the restriction of φ to $\boldsymbol{Q}_2^{\times}$ and $\boldsymbol{Z}_2$ (*resp.* $\boldsymbol{Q}_2$) means the ring of all 2-adic integers (numbers).

§2. Some formulas of theta integrals.

This section is devoted to confirming a key proposition concerning theta integrals. Let $\zeta^{\boldsymbol{Z}}$ and $\zeta_{\boldsymbol{Z}}$ denote two elements in $\mathscr{S}(V)$ determined by

$$\text{(2-1)}\qquad \zeta^{\boldsymbol{Z}}(x)=\begin{cases}\bar\varphi_a(b_x)\bar\varphi^*((b_xe)) & \text{if } x\in\mathfrak{o}[e^{-1},e],\\ 0 & \text{otherwise}\end{cases}$$

and

$$\zeta_{\boldsymbol{Z}}(x)=\begin{cases}\bar\varphi_a(b_x)\bar\varphi^*((b_xe)) & \text{if } x\in\mathfrak{o}[e^{-1},e]\text{ and }(b_xe,4N\tau)=1,\\ 0 & \text{otherwise.}\end{cases}$$

By the same method as that of Shimura [**10**, Lemma 5.7], we may derive the following lemma (cf. [**9**]).

Lemma 2.1. *Suppose that N is odd, τ is a positive square free integer satisfying $\tau\equiv 2,3 \pmod 4$, the conductor of $\psi\varepsilon_\tau$ is $4N\tau$ with ψ in* (1-11) *and the conditions* (1-7), (1-10), (1-12), (1-13) *and* (1-14) *are satisfied. Then there are constants* M *and* M' *such that*

$$\langle \Theta(z, w; \zeta^{\boldsymbol{Z}}), g(2w) \rangle = Mh(z) + M'h|U(4)(z) \text{ with } h(z) = L_\tau(f)(z),$$

where $g(w)$ *is the same function given in Lemma* 1.3.

By virtue of Proposition 1.1, Lemma 1.3 and Lemma 2.1, we may conclude the following proposition.

Proposition 2.1. *Suppose that the assumption in Lemma* 2.1 *is satisfied. Then we have*

$$Ah(z) + Bh|U(4)(z) = \langle \Theta(z, w; \eta), g(2w) \rangle$$

and

$$Ch(z) + Dh|U(4)(z) = \langle \Theta(z, w; \zeta_{\boldsymbol{Z}}), g(2w) \rangle$$

with

$$A = \overline{a(4\tau)}\,\overline{C'}\,\mathrm{vol}(\Gamma[2, 2N\tau]\backslash\mathfrak{H})^{-1} \frac{\langle g(2w), g(2w)\rangle\langle h', h'\rangle - \langle g(w), g(2w)\rangle\langle h, h'\rangle}{\langle h, h\rangle\langle h', h'\rangle - \langle h, h'\rangle\langle h', h\rangle},$$

$$B = \overline{a(4\tau)}\,\overline{C'}\,\mathrm{vol}(\Gamma[2, 2N\tau]\backslash\mathfrak{H})^{-1} \frac{\langle g(w), g(2w)\rangle\langle h, h\rangle - \langle g(2w), g(2w)\rangle\langle h', h\rangle}{\langle h, h\rangle\langle h', h'\rangle - \langle h, h'\rangle\langle h', h\rangle},$$

$$A = \varphi_a(-1)\overline{\gamma(\varphi)}C, \quad B = \varphi_a(-1)\overline{\gamma(\varphi)}D,$$

where η *is the same element given in* (1-8), $h'(z) = h|U(4)(z)$ *and* $\gamma(\varphi)$ *means the Gauss sum of* φ.

Proof. By Shimura [**10**, p. 539], we see that

$$\langle \Theta(z, w; \eta), g(2w) \rangle = \varphi_a(-1)\overline{\gamma(\varphi)}\langle \Theta(z, w; \zeta^{\boldsymbol{Z}}), g(2w) \rangle \tag{2-2}$$

and

$$\langle \Theta(z, w; \zeta_{\boldsymbol{Z}}), g(2w) \rangle = \langle \Theta(z, w; \zeta^{\boldsymbol{Z}}), g(2w) \rangle.$$

Hence, by Lemma 2.1, we have

$$\langle \Theta(z, w; \eta), g(2w) \rangle = \varphi_a(-1)\overline{\gamma(\varphi)}(Mh(z) + M'h'(z)), \tag{2-3}$$

which deduces that

$$\begin{aligned} \text{(2-4)}\quad & \langle h(z), \langle \Theta(z, w; \eta), g(2w) \rangle \rangle \\ &= \varphi_a(-1)\overline{\gamma(\varphi)}(M\langle h, h\rangle + M'\langle h, h'\rangle) \\ &= \mathrm{vol}(\Gamma[2\tau, 4N]\backslash\mathfrak{H})^{-1}\mathrm{vol}(\Gamma[2, 2\tau N]\backslash\mathfrak{H})^{-1} \\ &\quad \cdot \int_{\Gamma[2, 2\tau N]\backslash\mathfrak{H}} \overline{h(z)} \left\{ \int_{\Gamma[2\tau, 4N]\backslash\mathfrak{H}} \overline{\Theta(z, w; \eta)} g(2w) \Im w^{2k} d_{\mathfrak{H}} w \right\} \Im z^{(2k+1)/2} d_{\mathfrak{H}} z \end{aligned}$$

$$= \mathrm{vol}(\Gamma[2\tau, 4N]\backslash\mathfrak{H})^{-1}\mathrm{vol}(\Gamma[2, 2\tau N]\backslash\mathfrak{H})^{-1}$$

$$\cdot \int_{\Gamma[2\tau, 4N]\backslash\mathfrak{H}} \left\{ \overline{\int_{\Gamma[2, 2\tau N]\backslash\mathfrak{H}} h(z)\Theta(z, w; \eta)\Im z^{(2k+1)/2} d_{\mathfrak{H}} z} \right\} g(2w)\Im w^{2k} d_{\mathfrak{H}} w.$$

On the other hand, by Lemma 1.2 and Lemma 1.3, we can easily check

$$\text{(2-5)} \qquad \sum_{m=1}^{\infty}\left(\sum_{d|m}\psi_0(d)\left(\frac{4\tau\varepsilon}{d}\right)d^{k-1}a(\tau(m/d)^2)\right)e[mw]$$

$$= \sum_{m=1}^{\infty}\left(\sum_{d|2m}\psi_0(d)\left(\frac{4\tau\varepsilon}{d}\right)d^{k-1}a(4\tau(m/d)^2)\right)e[2mw]$$

$$= a(4\tau)g(2w).$$

Similarly we may justify

$$\text{(2-6)} \qquad \sum_{m=1}^{\infty}\left(\sum_{d|m}\psi_0(d)\left(\frac{4\tau\varepsilon}{d}\right)d^{k-1}a(4\tau(m/d)^2)\right)e[mw] = a(4\tau)g(w).$$

Combining Proposition 1.1 with (2-4), (2-5) and (2-6), we have

$$\text{(2-7)} \qquad (M\langle h, h\rangle + M'\langle h, h'\rangle)\varphi_a(-1)\overline{\gamma(\varphi)}$$

$$= \mathrm{vol}(\Gamma[2\tau, 4N]\backslash\mathfrak{H})^{-1}\mathrm{vol}(\Gamma[2, 2\tau N]\backslash\mathfrak{H})^{-1}$$

$$\cdot \int_{\Gamma[2\tau, 4N]\backslash\mathfrak{H}} \overline{a(4\tau)g(2w)C'}g(2w)\Im w^{2k} d_{\mathfrak{H}} w$$

$$= \mathrm{vol}(\Gamma[2, 2\tau N]\backslash\mathfrak{H})^{-1}\overline{a(4\tau)C'}\langle g(2w), g(2w)\rangle.$$

Also we may deduce

$$\text{(2-8)} \qquad \langle h'(z), \langle\Theta(z, w; \eta), g(2w)\rangle\rangle = \varphi_a(-1)\overline{\gamma(\varphi)}(M\langle h', h\rangle + M'\langle h', h'\rangle)$$

$$= \overline{a(4\tau)C'}\mathrm{vol}(\Gamma[2, 2\tau N]\backslash\mathfrak{H})^{-1}\langle g(w), g(2w)\rangle.$$

The assumption (1-13) shows that

$$\langle h, h\rangle\langle h', h'\rangle - \langle h, h'\rangle\langle h', h\rangle = ||h||^2||h'||^2 - |\langle h, h'\rangle|^2 > 0.$$

Consequently, by (2-7) and (2-8), we conclude that

$$M = \overline{a(4\tau)}\,\overline{C'}\frac{\mathrm{vol}(\Gamma[2, 2N\tau]\backslash\mathfrak{H})^{-1}}{\varphi_a(-1)\overline{\gamma(\varphi)}}\frac{\langle g(2w), g(2w)\rangle\langle h', h'\rangle - \langle g(w), g(2w)\rangle\langle h, h'\rangle}{\langle h, h\rangle\langle h', h'\rangle - \langle h, h'\rangle\langle h', h\rangle},$$

$$M' = \overline{a(4\tau)}\,\overline{C'}\frac{\mathrm{vol}(\Gamma[2, 2N\tau]\backslash\mathfrak{H})^{-1}}{\varphi_a(-1)\overline{\gamma(\varphi)}}\frac{\langle g(w), g(2w)\rangle\langle h, h\rangle - \langle g(2w), g(2w)\rangle\langle h', h\rangle}{\langle h, h\rangle\langle h', h'\rangle - \langle h, h'\rangle\langle h', h\rangle}.$$

By (2-2), we can prove the remainders of the assertions, which completes our proof.

§3. Rankin's convolution of theta series, Eisenstein series and final calculation.

Put

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e[n^2 z/2] \quad \text{and} \quad L_M(s,\omega) = \sum_{n=1}^{\infty} \omega^*(n\boldsymbol{Z})n^{-s}$$

for each Hecke character ω of $\boldsymbol{Q}$ and for each positive integer M, where n runs over all positive integers such that $(n, M) = 1$ and ω^* is the ideal character associated with ω (cf. [**10**, p. 505]). Let $g(w)$ and $h(z)$ be the same functions in Lemma 2.1.

We consider an integral

$$\text{(3-1)} \qquad \int_{\Gamma\backslash\mathfrak{H}} h(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}d_{\mathfrak{H}}z \quad (z = x+iy),$$

where $\Gamma = \Gamma[2, 2\tau N]$, $C(z,s:k,\bar{\varphi},\Gamma) = L_{4N\tau}(2s,\bar{\varphi})E(z,s:k,\bar{\varphi},\Gamma)$, $C(z,s:k,\bar{\varphi},\Gamma)$ and $E(z,s:k,\bar{\varphi},\Gamma)$ means functions given in [**10**, (4-6) and (4-11)]. By the same method as that of Shimura [**10**, p. 542], we may reduced (3-1) to the form

$$\text{(3-2)} \qquad L_{4N\tau}(2s+1,\varphi)\int_{\Gamma\backslash\mathfrak{H}} h(z)\overline{\vartheta(z)E(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}d_{\mathfrak{H}}z$$

$$= L_{4N\tau}(2s+1,\varphi)\int_{\Psi} h(z)\overline{\vartheta(z)}\,y^{s+1+k/2}d_{\mathfrak{H}}z$$

with $\Psi = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma[2,2\tau N] \,\middle|\, c = 0\right\}\backslash\mathfrak{H}$, which implies that

$$\text{(3-3)} \qquad L_{4N\tau}(2s+1,\varphi)\tau^k \int_0^2\int_0^\infty \sum_{m=1}^{\infty} a(m)\exp(2\pi i m\tau(x+iy)/2)$$

$$\cdot \sum_{n=-\infty}^{\infty} \overline{\exp(2\pi i n^2(x+iy)/2)}\,y^{s+1+k/2}d_{\mathfrak{H}}z$$

$$= 4\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} L_{4N\tau}((2s+k)-k+1,\varphi)\sum_{l=1}^{\infty} a(\tau l^2)l^{-(2s+k)}.$$

Therefore, the integral (3-1) is equal to

$$\text{(3-4)} \qquad 4a(4\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} 2^{-(2s+k)}L(2s+k,g)$$

with

$$L(s,g) = \sum_{n=1}^{\infty} c(n)n^{-s} \quad \text{and} \quad g(w) = \sum_{n=1}^{\infty} c(n)e[nw].$$

As we did for (3-2), (3-3) and (3-4), we obtain

$$\text{(3-5)} \qquad \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}} h'(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma)}\,y^{(2k+1)/2}d_{\mathfrak{H}}z$$

$$= 4a(4\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}} L(2s+k,g).$$

Next we calculate an integral

$$(3\text{-}6)\qquad \int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})E(w,\bar{t}+1/2:\bar{\varphi})}\Im w^{2k}d_{\mathfrak{H}}w,$$

where $C(w,s:\bar{\varphi})=C(w,s:k,\bar{\varphi},\Gamma[2\tau,4N])$ and $E(w,t:\bar{\varphi})=E(w,t:k,\bar{\varphi},\Gamma[2\tau,4N])$ are given in [**10**, (4-6) and (4-11)]. By a method similar to that of [**10**, p. 550], the integral (3-6) may be reduced to the form:

$$(3\text{-}7)\qquad \int_{\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma[2\tau,4N]|c=0\right\}\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})}\Im w^{t+((1+3k)/2)}d_{\mathfrak{H}}w$$

$$=\int_0^{2\tau}\int_0^{\infty} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})}\Im w^{t+((1+3k)/2)}d_{\mathfrak{H}}w.$$

We recall the following formula (cf. [**10**, p. 531]).

$$(3\text{-}8)\qquad v^{-s+k/2}C(w,s:\bar{\varphi})$$

$$=L_{8N\tau}(2s,\bar{\varphi})+\gamma(\bar{\varphi})(4N\tau)^{-1}\sum_{h'\in\boldsymbol{Z},\,c'\in\boldsymbol{Z}(c'>0)}(4Nc')^{1-2s}\varphi^*((h'))$$

$$\cdot\,\varphi_a(h'/4N\tau)e[(c'h'/\tau)u]\xi(v,h'c'/\tau;s+k/2,s-k/2),$$

where $w=u+iv$ and

$$\xi(y,h;\alpha,\beta)=i^{\beta-\alpha}(2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}h^{\alpha+\beta-1}e^{-2\pi hy}\int_0^{\infty}e^{-4\pi hyt}(t+1)^{\alpha-1}t^{\beta-1}dt$$

$(y>0,h>0,(\alpha,\beta)\in\boldsymbol{C}^2,\Re(\beta)>0)$. Hence, the above integral (3-7) can be rewritten as follows:

$$(3\text{-}9)\qquad 2\tau\int_0^{\infty}\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}\sum_{h'\in\boldsymbol{Z},\,c'\in\boldsymbol{Z}(c'>0)}(4Nc')^{-2s}\overline{\varphi_a(h')\varphi^*((h'))}\sum_{n=1}^{\infty}c(n)$$

$$\cdot\,\exp(-4\pi nv)\overline{\xi(v,h'c'/\tau;\bar{s}+(1+k)/2,\bar{s}+(1-k)/2)}$$

$$\cdot\,v^{t+((1+3k)/2)}v^{(s+(1/2))-(k/2)-2}dv$$

with $g(w)=\sum_{n=1}^{\infty}c(n)e[nw]$. Observe that

$$(3\text{-}10)\qquad \int_0^{\infty}\exp(-4\pi nv)\xi(v,h'c'/\tau;s+(1+k)/2,s+(1-k)/2)v^{t+s+k-1}dv$$

$$=i^{-k}2^{-(t+s+k)}(2\pi)^{s-t-k+1}(h'c'/\tau)^{s-t-k}\frac{\Gamma(t+s+k)\Gamma(t-s+k)}{\Gamma(s+((k+1)/2))\Gamma(t+((k+1)/2))},$$

if $n=c'h'/2\tau$. We can check that

$$(3\text{-}11)\qquad \sum_{h'=1}^{\infty}\sum_{c'=1,\,2\tau|c'h'}^{\infty}(c')^{-2s}\overline{\varphi_a(h')\varphi^*((h'))}c(c'h'/2\tau)(h'c'/\tau)^{s-t-k}$$

$$=(\tau)^{-2s}2^{-s-t-k}\sum_{h'=1}^{\infty}\sum_{c''=1}^{\infty}\bar{\varphi}^*((h'))c(h'c'')(c'')^{-s-t-k}(h')^{-t+s-k}.$$

Putting $s = t$, the sum (3-11) equals

$$(3\text{-}12)\qquad (2\tau)^{-2s}2^{-k}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\varphi}^*((m))c(mn)m^{-k}n^{-2s-k}$$

$$= (2\tau)^{-2s}2^{-k}L_{4N}(2s+1,\varphi)^{-1}L(2s+k,g)L(k,g,\bar{\varphi}),$$

where

$$L(s,g,\bar{\varphi}) = \sum_{n=1}^{\infty}\bar{\varphi}^*((n))c(n)n^{-s} \quad \text{and} \quad g(w) = \sum_{n=1}^{\infty}c(n)e[nw].$$

Employing (3-9), (3-10) and (3-12), we find that

$$(3\text{-}13)\qquad \int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}} g(2w)\overline{C(w,\bar{s}+1/2:\bar{\varphi})E(w,\bar{s}+1/2,\bar{\varphi})}\Im w^{2k}d_{\mathfrak{H}}w$$

$$= A(s)L_{4N}(2s+1,\varphi)^{-1}L(2s+k,g)L(k,g,\bar{\varphi}),$$

where

$$A(s) = 2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}(4N)^{-2s}i^{-k}2^{-(2s+k)}(2\pi)^{-k+1}$$

$$\cdot\,(2\pi)^{-(1/2)}2^{2s+k-(1/2)}(2\tau)^{-2s}2^{-k}\frac{\Gamma(s+(k/2))\Gamma(k)}{\Gamma(s+(k/2)+(1/2))}.$$

Exchanging the order of integration, we have

$$(3\text{-}14)\qquad \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\langle\Theta(z,w;\zeta_{\boldsymbol{Z}}),g(2w)\rangle\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}y^{(2k+1)/2}d_{\mathfrak{H}}z$$

$$= \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}\left\{\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}}\overline{\Theta(z,w;\zeta_{\boldsymbol{Z}})}g(2w)\Im w^{2k}d_{\mathfrak{H}}w\right\}$$

$$\cdot\,\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}y^{(2k+1)/2}d_{\mathfrak{H}}z$$

$$= \mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1}\int_{\Gamma[2\tau,4N]\backslash\mathfrak{H}}\left\{\int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\overline{\vartheta(z)\Theta(z,w;\zeta_{\boldsymbol{Z}})}\right.$$

$$\left.\cdot\,\overline{C(z,\bar{s}+1/2:k,\bar{\varphi})y^{(2k+1)/2}d_{\mathfrak{H}}z}\right\}g(2w)\Im w^{2k}d_{\mathfrak{H}}w$$

$$= \langle M'(w,\bar{s}),g(2w)\rangle,$$

where

$$M'(w,s) = \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\vartheta(z)\Theta(z,w;\zeta_{\boldsymbol{Z}})C(z,s+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])y^{(2k+1)/2}d_{\mathfrak{H}}z.$$

For a function $p(z)$ on $\mathfrak{H}$ and an $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\boldsymbol{R})$, $p\|_l\,\alpha(z)$ means

$p(\alpha(z))(cz+d)^{-l}$. Observing that

$$(3\text{-}15)\qquad C(z,s:k,\bar{\varphi},\Gamma[2,2\tau N]) = L_{4\tau N}(2s,\bar{\varphi})\sum_{\alpha\in\Gamma[2,2\tau N]_\infty\backslash\Gamma[2,2\tau N]}\overline{\varphi}_a(d)\bar{\varphi}^*((d))\Im z^{s-(k/2)}||_k\alpha(z),$$

we obtain

$$(3\text{-}16)\qquad M'(w,s) = L_{4N\tau}(2s+1,\bar{\varphi})\int_{\alpha\in\Gamma[2,2\tau N]_\infty\backslash\mathfrak{H}}\vartheta(z)\Theta(z,w;\zeta_{\boldsymbol{Z}})y^{s+1+(k/2)}d_{\mathfrak{H}}z = B(s)L_{4N\tau}(2s+1,\bar{\varphi})S'(w,s),$$

where

$$\Gamma[2,2\tau N]_\infty = \left\{\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma[2,2\tau N]\,\middle|\, c=0\right\},$$

$$B(s) = 2\frac{2^{s+(k/2)+(1/2)}}{\pi^{s+(k/2)+(1/2)}}\Gamma(s+(k+1)/2),$$

$$S'(w,s) = \sum_{(\alpha,b)\in X}\zeta_{\boldsymbol{Z}}(\alpha)\mu(b)[\alpha,w]^{-k}\left|\frac{[\alpha,w]}{\Im w}\right|^{2k-2t'},$$

$$X = \left\{(\alpha,b)\in V\times\boldsymbol{Q}\,\middle|\,\alpha\neq 0, -\det\alpha = b^2\right\},$$

$$\mu(b) = \begin{cases} 1 & \text{if } b\in\boldsymbol{Z}, \\ 0 & \text{if } b\in\boldsymbol{Q}-\boldsymbol{Z}\end{cases}$$

and $t' = s+(1+k)/2$.

Combining Proposition 2.1 with (3-4), (3-5), (3-14) and (3-16), we may derive that

$$(3\text{-}17)\quad \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}Ch(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}y^{(2k+1)/2}d_{\mathfrak{H}}z + \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}Dh'(z)\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}y^{(2k+1)/2}d_{\mathfrak{H}}z$$
$$= 4a(4\tau)\tau^k\frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+k/2}}(C2^{-(2s+k)}+D)L(2s+k,g)$$
$$= \int_{\Gamma[2,2\tau N]\backslash\mathfrak{H}}\langle\Theta(z,w;\zeta_{\boldsymbol{Z}}),g(2w)\rangle\overline{\vartheta(z)C(z,\bar{s}+1/2:k,\bar{\varphi},\Gamma[2,2\tau N])}y^{(2k+1)/2}d_{\mathfrak{H}}z$$
$$= \overline{B(\bar{s})}L_{4N\tau}(2s+1,\varphi)\langle S'(w,\bar{s}),g(2w)\rangle.$$

For further computation, we need the following formula in [**10**, (7.9a) and (7.13)]

(3-18) $$S'(w,s) = (-1)^k \sum_{q \in 2N\mathbf{Z}/4N\mathbf{Z}} T'(w,s)||_{2k}\tau_q \quad \text{and}$$

$$T'\left(w, s-\frac{1}{2}\right) = (2N)^{2s} C(w,s;\bar{\varphi})E(w,s;\bar{\varphi}) \quad \text{with } \tau_q = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}.$$

Employing (3-13) and (3-18), we obtain

(3-19) $$\begin{aligned} \langle M'(w,\bar{s}), g(2w)\rangle &= \overline{B(\bar{s})} L_{4N\tau}(2s+1,\varphi)(-1)^k (2N)^{2s+1} \\ &\quad \cdot \left\langle \sum_{q \in 2N\mathbf{Z}/4N\mathbf{Z}} C(w,\bar{s}+1/2;\bar{\varphi})E(w,\bar{s}+1/2;\bar{\varphi})||_{2k}\tau_q, g(2w) \right\rangle \\ &= (-1)^k \overline{B(\bar{s})} L_{4N\tau}(2s+1,\varphi) 2(2N)^{2s+1} \\ &\quad \cdot \langle C(w,\bar{s}+1/2;\bar{\varphi})E(w,\bar{s}+1/2;\bar{\varphi}), g(2w)\rangle \\ &= (-1)^k \overline{B(\bar{s})} 2(2N)^{2s+1} A(s)\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1} \\ &\quad \cdot L(2s+k,g)L(k,g,\bar{\varphi}). \end{aligned}$$

Combining this with (3-17), we have

(3-20) $$\begin{aligned} &4a(4\tau)\tau^k \frac{\Gamma(s+k/2)}{(2\pi\tau^2)^{s+(k/2)}} (C2^{-(2s+k)}+D)L(2s+k,g) \\ &= (-1)^k 2(2N)^{2s+1} 2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}(4N)^{-2s} i^{-k} 2^{-(2s+k)} \\ &\quad \cdot (2\pi)^{-k+1}(2\pi)^{-(1/2)} 2^{2s+k-(1/2)}(2\tau)^{-2s} 2^{-k} \frac{\Gamma(s+k/2)\Gamma(k)}{\Gamma(s+(k+1)/2)} \\ &\quad \cdot 2\overline{\left(\frac{2^{\bar{s}+(k+1)/2}}{\pi^{\bar{s}+(k+1)/2}}\right)} \overline{\Gamma(\bar{s}+(k+1)/2)} \mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1} \\ &\quad \cdot L(2s+k,g)L(k,g,\bar{\varphi}), \end{aligned}$$

which yields that

(3-21) $$\begin{aligned} &a(4\tau)4\tau^k \frac{1}{(2\pi\tau^2)^{s+k/2}} (C2^{-(2s+k)}+D) \\ &= (-1)^k 2(2N)^{2s+1} 2\tau\overline{\gamma(\bar{\varphi})}(4N\tau)^{-1}(4N)^{-2s} i^{-k} \\ &\quad \cdot 2^{-(2s+k)}(2\pi)^{-k+1}(2\pi)^{-(1/2)} 2^{2s+k-(1/2)}(2\tau)^{-2s} 2^{-k}\Gamma(k) \\ &\quad \cdot 2\overline{\left(\frac{2^{\bar{s}+(k+1)/2}}{\pi^{\bar{s}+(k+1)/2}}\right)} \mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1} L(k,g,\bar{\varphi}). \end{aligned}$$

Putting $2s+k=0$, we may deduce that

(3-22) $$a(4\tau)(C+D) = i^k \overline{\gamma(\bar{\varphi})} 2^{1/2}\pi^{-k}(k-1)!\mathrm{vol}(\Gamma[2\tau,4N]\backslash\mathfrak{H})^{-1} L(k,g,\bar{\varphi}).$$

By Proposition 2.1, we see that

$$C+D=\overline{a(4\tau)}\frac{\mathrm{vol}(\Gamma[2,2N\tau]\backslash\mathfrak{H})^{-1}}{\varphi_a(-1)\overline{\gamma(\varphi)}}\overline{C'}E\tau^{-((2k+1)/2)+1} \tag{3-23}$$

with

$$E=\frac{\langle g(2w),g(2w)\rangle(\langle f',f'\rangle-\langle f',f\rangle)+\langle g(w),g(2w)\rangle(\langle f,f\rangle-\langle f,f'\rangle)}{\langle f,f\rangle\langle f',f'\rangle-\langle f,f'\rangle\langle f',f\rangle},$$

where $f'(z)=f|U(4)(z)$.

Consequently, by (3-22) and (3-23), we conclude the following theorem.

Theorem. *Let the notation be as above. Suppose the assumption in Lemma* 2.1. *Then*

$$|a(4\tau)|^2E=2^{-(5/2)-k}\pi^{-k}(k-1)!\overline{\gamma(\varphi)\gamma(\bar\varphi)}(4N\tau)^{-1}\tau^{((2k+1)/2)-1}(-1)^k\varphi_a(-1)L(k,g,\bar\varphi).$$

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