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Integral representation for eigenfunctions of the Laplacian

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Abstract. We study entire functions of exponential type which are eigenfunctions of the Laplacian. We represent them by an integral on the complex light cone. The integral formula is closely related to the Fourier-Borel transformation for analytic functionals on the complex sphere.

Introduction.

Let $\Delta_z = (\partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2)$ be the complex Laplacian on $\tilde{\boldsymbol{E}} = \boldsymbol{C}^{n+1}$, $n \geq 2$. We denote the Lie norm and the dual Lie norm by $L(z)$ and $L^*(z)$ (for the definition, see Section 1). $\mathcal{O}(\tilde{B}(R))$ denotes the space of holomorphic functions on the open Lie ball $\tilde{B}(R) = \{z \in \tilde{\boldsymbol{E}}; L(z) < R\}$ with the topology of uniform convergence on compact sets. It is an FS (Fréchet-Schwartz) space. Put

$$\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R)) = \{f \in \mathcal{O}(\tilde{B}(R)); \Delta_z f(z) = \lambda^2 f(z)\},$$

$$\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}[R]) = \text{ind} \lim\{\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R')); R' > R\}.$$

Since $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R))$ is a closed subspace of $\mathcal{O}(\tilde{B}(R))$, it is an FS space and $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}[R])$ is a DFS (dual Fréchet-Schwartz) space (for FS spaces and DFS spaces see, for example, [**3**]). We denote the spaces of entire eigenfunctions of exponential type (R) and $[R]$ by

$$\begin{aligned}\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; (R)) = \{f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}); \quad &\text{for all } R' > R \quad \text{we have} \\ &\sup\{|f(z)| \exp(-R'L^*(z)); z \in \tilde{\boldsymbol{E}}\} < \infty\}, \quad |\lambda| \leq R,\end{aligned}$$

$$\begin{aligned}\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R]) = \{f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}); \quad &\text{there is } R' < R \quad \text{such that} \\ &\sup\{|f(z)| \exp(-R'L^*(z)); z \in \tilde{\boldsymbol{E}}\} < \infty\}, \quad |\lambda| < R.\end{aligned}$$

Then $\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; (R))$ is an FS space and $\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R])$ is a DFS space.

Let $\tilde{S}_\lambda = \{w \in \tilde{\boldsymbol{E}}; w_1^2 + \cdots + w_{n+1}^2 = \lambda^2\}$ be the complex sphere. A holomorphic function on $\tilde{S}_\lambda$ is called an entire function on $\tilde{S}_\lambda$. Similarly, we denote by $\text{Exp}(\tilde{S}_\lambda; (R))$ and $\text{Exp}(\tilde{S}_\lambda; [R])$ the spaces of entire functions on $\tilde{S}_\lambda$ of exponential type (R) and $[R]$. Put $\mathcal{O}(\tilde{S}_\lambda(R)) = \mathcal{O}(\tilde{B}(R))|_{\tilde{S}_\lambda}$ and $\mathcal{O}(\tilde{S}_\lambda[R]) = \mathcal{O}(\tilde{B}[R])|_{\tilde{S}_\lambda}$. $\mathcal{O}'(X)$ and $\text{Exp}'(X)$ mean the dual spaces of $\mathcal{O}(X)$ and $\text{Exp}(X)$.

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In [**6**], we gave integral representations for $\mathrm{Exp}(\tilde{S}_0;(R))$ and $\mathrm{Exp}_\Delta(\tilde{\boldsymbol{E}};(R))$, and using the integral kernels, we constructed the mappings $\mathscr{M}_R$ and $\mathscr{E}_R$ (see also [**9**]). $\mathscr{M}_R$ is related to the Poisson transformation $\mathscr{P}_R$ and $\mathscr{E}_R$ to the Cauchy transformation $\mathscr{C}_R$ through the Fourier-Borel transformation $\mathscr{F}_0$ and the conical Fourier-Borel transformation $\mathscr{F}^0$; that is, in [**6**], we proved that the following diagrams are commutative and all the mappings in them are topological linear isomorphisms:

$$\begin{array}{ccc} \mathscr{O}'_\Delta(\tilde{B}[R]) & \xrightarrow{\mathscr{F}^0} & \mathrm{Exp}(\tilde{S}_0;(R)) \\ \uparrow \mathscr{P}_R^{-1} & & \downarrow \mathscr{M}_R^{-1} \\ \mathscr{O}_\Delta(\tilde{B}(R)) & \xleftarrow{\mathscr{F}_0} & \mathrm{Exp}'(\tilde{S}_0;[R]), \end{array} \tag{1}$$

where $\mathscr{P}_R^{-1}$ is given by an integral on $S_R = \tilde{S}_R \cap \boldsymbol{R}^{n+1}$ and $\mathscr{M}_R^{-1}$ is given by an integral on $\tilde{S}_0$; and

$$\begin{array}{ccc} \mathrm{Exp}'_\Delta(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{F}^0} & \mathscr{O}(\tilde{S}_0(R)) \\ \uparrow \mathscr{E}_R^{-1} & & \downarrow \mathscr{C}_R^{-1} \\ \mathrm{Exp}_\Delta(\tilde{\boldsymbol{E}};(R)) & \xleftarrow{\mathscr{F}_0} & \mathscr{O}'(\tilde{S}_0[R]), \end{array} \tag{2}$$

where $\mathscr{C}_R^{-1}$ is given by an integral on the boundary of $\tilde{S}_0(R) = \tilde{S}_0 \cap \tilde{B}(R)$ and $\mathscr{E}_R^{-1}$ is given by an integral on $\boldsymbol{R}^{n+1}$.

In this paper, we shall generalize the above results in [**6**] according to the following plan.

First, we give an integral representation for $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$, $|\lambda| \le R$. Then the integral representation gives the inverse mapping of the restriction mapping

$$\beta_\lambda : \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0;(R)),$$

which was proved to be a topological linear isomorphism in [**10**] (see Theorem 2.2 which is our main theorem in this paper).

By using the integral kernel for $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$, we construct the mapping $\mathscr{E}_{\lambda,R}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{E}_{\lambda,R}} & \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) \\ \uparrow \beta_\lambda^* & & \downarrow \beta_\lambda \\ \mathrm{Exp}'(\tilde{S}_0;[R]) & \xrightarrow{\mathscr{M}_R} & \mathrm{Exp}(\tilde{S}_0;(R)), \end{array}$$

where all the mappings are topological linear isomorphisms and β_λ^* is the adjoint mapping of β_λ (Corollary 2.4).

Second, we define the integral kernel for $\mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R))$ and using it we define the λ-Cauchy transformation $\mathscr{C}_{\lambda,R}$ for $\mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R])$. Then, in Proposition 3.2, we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) & \xrightarrow{\mathscr{C}_{\lambda,R}} & \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)) \\ \uparrow \beta_\lambda^* & & \downarrow \beta_\lambda \\ \mathscr{O}'(\tilde{S}_0[R]) & \xrightarrow{\mathscr{C}_R} & \mathscr{O}(\tilde{S}_0(R)), \end{array}$$

where all the mappings are topological linear isomorphisms.

Third, we can define the spherical Fourier-Borel transformation $\mathscr{F}^\lambda$ for $T \in \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R])$ by

$$\mathscr{F}^\lambda : T \mapsto \mathscr{F}^\lambda T(w) = \langle T_z, \exp(z \cdot w) \rangle, \quad w \in \tilde{S}_\lambda(R).$$

This is well-defined since $(\Delta_z - \lambda^2)\exp(z \cdot w) = 0$ for $w \in \tilde{S}_\lambda$. By using the diagram (1), we prove that the spherical Fourier-Borel transformation

$$\mathscr{F}^\lambda : \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) \xrightarrow{\sim} \mathscr{O}(\tilde{S}_\lambda(R))$$

is a topological linear isomorphism (Theorem 5.2). A different proof is found in [**8**].

Like (2), a relation between the Fourier-Borel transformation $\mathscr{F}_\lambda$ and the spherical Fourier-Borel transformation $\mathscr{F}^\lambda$ is given by the diagram

$$\begin{array}{ccc} \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{F}^\lambda} & \mathscr{O}(\tilde{S}_\lambda(R)) \\ \uparrow \mathscr{E}_{\lambda,R}^{-1} & & \downarrow (\alpha_\lambda \circ \mathscr{P}_R \circ \alpha_\lambda^*)^{-1} \\ \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) & \xleftarrow{\mathscr{F}_\lambda} & \mathscr{O}'(\tilde{S}_\lambda[R]), \end{array}$$

where α_λ is the restriction mapping $\alpha_\lambda : \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)) \xrightarrow{\sim} \mathscr{O}(\tilde{S}_\lambda(R))$, and α_λ^* is the adjoint mapping of α_λ (Corollary 5.3). Note that this diagram for $\lambda = 0$ is different from (2).

Fourth, by using the diagram (2), we prove that the spherical Fourier-Borel transformation

$$\mathscr{F}^\lambda : \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_\lambda;(R)) \tag{3}$$

is a topological linear isomorphism (Theorem 5.4). A different proof is found in [**7**].

We also have

$$\begin{array}{ccc} \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) & \xrightarrow{\mathscr{F}^\lambda} & \mathrm{Exp}(\tilde{S}_\lambda;(R)) \\ \uparrow \mathscr{C}_{\lambda,R}^{-1} & & \downarrow (\alpha_\lambda \circ \mathscr{E}_R \circ \alpha_\lambda^*)^{-1} \\ \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)) & \xleftarrow{\mathscr{F}_\lambda} & \mathrm{Exp}'(\tilde{S}_\lambda;[R]), \end{array}$$

where all the mappings are topological linear isomorphisms (Corollary 5.5). Note that this diagram for $\lambda = 0$ is different from (1).

In this paper, we mainly treat for FS spaces but all the results also hold for DFS spaces: For example, we also have the topological linear isomorphism like (3);

$$\mathscr{F}^\lambda : \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}(R)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_\lambda;[R]).$$

1. Eigenspaces of the Laplacian.

Let $\|x\|$ be the Euclidean norm on $\boldsymbol{E} = \boldsymbol{R}^{n+1}$, $n \geq 2$. The cross norm $L(z)$ on $\tilde{\boldsymbol{E}} = \boldsymbol{C}^{n+1}$ corresponding to $\|x\|$ is the Lie norm defined by

$$L(z) = \{\|z\|^2 + (\|z\|^4 - |z^2|^2)^{1/2}\}^{1/2},$$

where $z \cdot w = z_1 w_1 + z_2 w_2 + \cdots + z_{n+1} w_{n+1}$, $z^2 = z \cdot z$ and $\|z\|^2 = z \cdot \bar{z}$. The dual Lie norm $L^*(z)$ is given by

$$L^*(z) = \sup\{|z \cdot \zeta|; L(\zeta) \leq 1\} = \{(\|z\|^2 + |z^2|)/2\}^{1/2}.$$

The open and the closed Lie balls of radius R with center at 0 are defined by

$$\tilde{B}(R) = \{z \in \tilde{\boldsymbol{E}}; L(z) < R\}, \quad 0 < R \leq \infty,$$

$$\tilde{B}[R] = \{z \in \tilde{\boldsymbol{E}}; L(z) \leq R\}, \quad 0 \leq R < \infty.$$

Note that $\tilde{B}[0] = \{0\}$ and $\tilde{B}(\infty) = \tilde{\boldsymbol{E}}$. We denote by $\mathcal{O}(\tilde{B}(R))$ the space of holomorphic functions on $\tilde{B}(R)$ with the topology of uniform convergence on compact sets. Let $\lambda \in \boldsymbol{C}$ and put

$$\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R)) = \{f \in \mathcal{O}(\tilde{B}(R)); (\Delta_z - \lambda^2) f(z) = 0\},$$

where Δ is the complex Laplacian;

$$\Delta_z f(z) = (\partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2) f(z).$$

For $0 \leq R < \infty$ we put

$$\mathcal{O}(\tilde{B}[R]) = \text{ind lim}\{\mathcal{O}(\tilde{B}(R')); R' > R\},$$

$$\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}[R]) = \text{ind lim}\{\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R')); R' > R\}.$$

We denote by $\mathcal{O}'_{\Delta-\lambda^2}(\tilde{B}(R))$ (resp. $\mathcal{O}'_{\Delta-\lambda^2}(\tilde{B}[R])$) the dual space of $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R))$ (resp. $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}[R])$).

For $0 < R \leq \infty$ we denote by

$$\text{Exp}(\tilde{\boldsymbol{E}}; [R]) = \{f \in \mathcal{O}(\tilde{\boldsymbol{E}}); \quad \text{there is } R' < R \text{ such that}$$
$$\sup\{|f(z)| \exp(-R' L^*(z)); z \in \tilde{\boldsymbol{E}}\} < \infty\}$$

the space of entire functions on $\tilde{\boldsymbol{E}}$ of exponential type $[R]$ with respect to the dual Lie norm $L^*(z)$. Similarly, for $0 \leq R < \infty$

$$\text{Exp}(\tilde{\boldsymbol{E}}; (R)) = \{f \in \mathcal{O}(\tilde{\boldsymbol{E}}); \quad \text{for all } R' > R \quad \text{we have}$$
$$\sup\{|f(z)| \exp(-R' L^*(z)); z \in \tilde{\boldsymbol{E}}\} < \infty\}$$

denotes the space of entire functions on $\tilde{\boldsymbol{E}}$ of exponential type (R). Put

$$\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R]) = \text{Exp}(\tilde{\boldsymbol{E}}; [R]) \cap \mathcal{O}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}),$$

$$\text{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; (R)) = \text{Exp}(\tilde{\boldsymbol{E}}; (R)) \cap \mathcal{O}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}).$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$. The coefficient of the highest power of $P_{k,n}(t)$ is given by

$$\gamma_{k,n} \equiv \frac{\Gamma(k+(n+1)/2)2^k}{N(k,n)\Gamma((n+1)/2)k!},$$

where $N(k,n)$ is the dimension of the space of k-homogeneous harmonic polynomials of $n+1$ variables; $N(k,n) = (2k+n-1)(k+n-2)!/(k!(n-1)!) = O(k^{n-1})$.

We put

$$\tilde{P}_{k,n}(z,w) = (\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right).$$

Then $\tilde{P}_{k,n}(z,w)$ is a symmetric homogeneous polynomial of degree k in z and in w, satisfies $\Delta_z\tilde{P}_{k,n}(z,w) = \Delta_w\tilde{P}_{k,n}(z,w) = 0$ and is estimated as $|\tilde{P}_{k,n}(z,w)| \le L(z)^k L(w)^k$. Further we have the following orthogonal relation;

$$N(k,n)\int_S \tilde{P}_{k,n}(z,\omega)\tilde{P}_{l,n}(\omega,w)d\dot{\omega} = \delta_{kl}\tilde{P}_{k,n}(z,w), \tag{4}$$

where $d\dot{\omega}$ is the normalized $O(n+1)$-invariant measure on the n-dimensional unit sphere S.

Let $\mu \in \boldsymbol{C}$ and $\tilde{J}_\mu(t)$ be the entire Bessel function defined by

$$\tilde{J}_\mu(t) = \sum_{l=0}^{\infty}\frac{(-1)^l\Gamma(\mu+1)}{\Gamma(\mu+l+1)l!}\left(\frac{t}{2}\right)^{2l} = \Gamma(\mu+1)\left(\frac{t}{2}\right)^{-\mu}J_\mu(t).$$

Note that

$$\tilde{J}_\mu(0) = 1, \quad \tilde{J}_\mu(t) = \tilde{J}_\mu(-t). \tag{5}$$

For simplicity, we set

$$\tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t).$$

Lemma 1.1 (Lemma 7.3 in [**2**]). *For $z, w \in \tilde{\boldsymbol{E}}$, we have*

$$\exp(z\cdot w) = \sum_{k=0}^{\infty}\frac{1}{k!\gamma_{k,n}}\tilde{j}_k(i\sqrt{z^2}\sqrt{w^2})\tilde{P}_{k,n}(z,w).$$

Put

$$\tilde{S}_0 = \{z \in \tilde{\boldsymbol{E}};\, z^2 = 0\}, \quad \tilde{S}_{0,1} = \{z \in \tilde{S}_0; L(z) = 1\}.$$

Proposition 1.2 (Corollary 2.3 in [**11**]). *Let $\lambda \in \boldsymbol{C}$ and $f \in \mathcal{O}_{\Delta-\lambda^2}(\{0\})$. Define*

$$f_k(z) = 2^k N(k,n)\int_{\tilde{S}_{0,1}} f(\rho w)(z\cdot\overline{w}/\rho)^k d\dot{w}, \quad z \in \tilde{\boldsymbol{E}},$$

where $\rho > 0$ is sufficiently small and $d\dot{w}$ is the normalized $O(n+1)$-invariant measure on $\tilde{S}_{0,1}$. Then f is expanded as follows:

$$f(z) = \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2}) f_k(z),$$

where the convergence is in the sense of $\mathcal{O}_{\Delta-\lambda^2}(\{0\})$.

2. Integral representation.

In this section, we consider an integral representation for $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$, $R \geq |\lambda|$. $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$ is closely related to the space

$$\mathrm{Exp}(\tilde{S}_0;(R)) = \{f \in \mathcal{O}(\tilde{S}_0); \quad \text{for all } R' > R \quad \text{we have} \\ \sup\{|f(z)|\exp(-R'L^*(z)); z \in \tilde{S}_0\} < \infty\}.$$

That is, the integral representation will give the inverse mapping of the following restriction mapping β_λ studied in [**10**];

$$\beta_\lambda : \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_0;(R)), \quad |\lambda| \leq R < \infty.$$

Ii [**1**] defined a measure $d\mu$ on $\tilde{S}_0$ by

$$\int_{\tilde{S}_0} f(z)\,d\mu(z) = \int_0^{\infty}\int_{\tilde{S}_{0,1}} f(rz')d\dot{z}' r^{n-1}\rho_n(r)dr,$$

where $\rho_n(r)$ is a function of exponential type -1 and satisfies

$$\int_0^{\infty} r^{2k+n-1}\rho_n(r)dr = (N(k,n)k!)^2\gamma_{k,n}2^k \equiv C(k,n) \tag{6}$$

for $k = 0,1,2,\dots$ (see [**1**], [**10**] or [**6**]).

Let $\lambda \in \boldsymbol{C}$. For $z \in \tilde{\boldsymbol{E}}$ and $w \in \tilde{S}_0$, define

$$\begin{aligned} E^\lambda(z,w) &= \sum_{k=0}^{\infty} \frac{2^k N(k,n)}{\gamma_{k,n}C(k,n)}\tilde{j}_k(i\lambda\sqrt{z^2})\tilde{P}_{k,n}(z,w) \\ &= \sum_{k=0}^{\infty} \frac{\tilde{j}_k(i\lambda\sqrt{z^2})}{C(k,n)} N(k,n)(2z\cdot w)^k. \end{aligned} \tag{7}$$

Note that

$$E^\lambda(z,\cdot\,) \in \mathrm{Exp}(\tilde{S}_0;(0)), \quad E^\lambda(\,\cdot\,,w) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(|\lambda|)).$$

Lemma 2.1. *For* $\lambda \neq 0$, $E^\lambda(z,w)$ *has the following integral representation*:

$$E^\lambda(z,w) = \int_S \exp(\lambda\omega\cdot z)\exp(w\cdot\omega/\lambda)d\dot{\omega}, \quad z \in \tilde{\boldsymbol{E}}, \quad w \in \tilde{S}_0. \tag{8}$$

Proof. By Lemma 1.1, (4), (5), (6) and (7), we have (8). □

For $z,w \in \tilde{S}_0$, $E^\lambda(z,w)$ does not depend on λ, thus for $z,w \in \tilde{S}_0$ we denote $E^\lambda(z,w)$ by $E_1(z,w)$. Note that $E_1(z,w) = E^0(z,w)$. $E_1(z,w)$ is the integral kernel for $\mathrm{Exp}(\tilde{S}_0;(R))$ in [**6**]; that is, for $f \in \mathrm{Exp}(\tilde{S}_0;(R))$ and $s > R/2$, we have the following integral representation:

$$f(z) = \int_{\tilde{S}_0} f(w/s) E_1(z, s\overline{w})\, d\mu(w), \quad z \in \tilde{S}_0 \tag{9}$$

(Theorem 16 in [**6**]).

Using the kernel $E^{\lambda}(z,w)$, we shall give an integral representation for $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)), R \geq |\lambda|$.

THEOREM 2.2. *Let $f \in \mathrm{Exp}(\tilde{S}_0;(R))$. For $s > R/2$, define $F(z)$ by*

$$F(z) = \int_{\tilde{S}_0} f(w/s) E^{\lambda}(z, s\overline{w})\, d\mu(w), \quad z \in \tilde{\boldsymbol{E}}. \tag{10}$$

Then $(\Delta_z - \lambda^2)F(z) = 0$ and $F|_{\tilde{S}_0} = \beta_\lambda F = f$. Further, if $R \geq |\lambda|$, then $F(z) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$.

Conversely, let $F(z) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)), R \geq |\lambda|$. Then for $s > R/2$ we have

$$F(z) = \int_{\tilde{S}_0} F(w/s) E^{\lambda}(z, s\overline{w})\, d\mu(w), \quad z \in \tilde{\boldsymbol{E}}; \tag{11}$$

that is, the mapping $f \mapsto F$ in (10) *is the inverse mapping of the restriction mapping β_λ and* (11) *is an integral representation for $F \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$, $R \geq |\lambda|$.*

PROOF. Since $(\Delta_z - \lambda^2)E^{\lambda}(z,w) = 0$, the first half is clear by (9). Further, noting that $\lim_{k\to\infty} |\tilde{j}_k(t)| = 1$ for $t \in \boldsymbol{C}$, we get $F(z) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$ by the growth conditions of $\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$ and $\mathrm{Exp}(\tilde{S}_0;(R))$ (Theorem 12 in [**5**] and Theorem 16 in [**4**]).

Because $E^{\lambda}(z,\cdot) \in \mathrm{Exp}(\tilde{S}_0;(0))$ and ρ_n is of exponential type -1, the right-hand side in (11) is finite, and by Proposition 1.2, (6) and (7) we have

$$\begin{aligned}
&\int_{\tilde{S}_0} F(w/s) E^{\lambda}(z, s\overline{w})\, d\mu(w) \\
&= \int_{\tilde{S}_0} \sum_{l=0}^{\infty} F_l(w/s) \sum_{k=0}^{\infty} \frac{2^k N(k,n)}{C(k,n)} \tilde{j}_k(i\lambda\sqrt{z^2})(sz\cdot\overline{w})^k d\mu(w) \\
&= \sum_{k=0}^{\infty} \frac{2^k N(k,n)}{C(k,n)} \tilde{j}_k(i\lambda\sqrt{z^2}) \int_{\tilde{S}_0} F_k(w/s)(sz\cdot\overline{w})^k d\mu(w) \\
&= \sum_{k=0}^{\infty} \frac{2^k N(k,n)}{C(k,n)} \tilde{j}_k(i\lambda\sqrt{z^2}) C(k,n) \frac{F_k(z)}{2^k N(k,n)} \\
&= \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2}) F_k(z) = F(z), \quad z \in \tilde{\boldsymbol{E}}.
\end{aligned}$$

□

Now, define a measure $d\mu_R$ on $\tilde{S}_0$ by

$$\int_{\tilde{S}_0} f(w)\, d\mu_R(w) = \int_0^{\infty} \int_{\tilde{S}_{0,R}} f(rw')\, d\dot{w}' r^{n-1} \rho_n(r)\, dr,$$

where $d\dot{w}'$ is the normalized $O(n+1)$-invariant measure on

$$\tilde{S}_{0,R} = \{z \in \tilde{S}_0; L(z) = R\}.$$

Put

$$\begin{aligned}\operatorname{Exp}(\tilde{S}_0;[R]) = \{f \in \mathcal{O}(\tilde{S}_0); \quad &\text{there is } R' < R \quad \text{such that} \\ &\sup\{|f(z)|\exp(-R'L^*(z)); z \in \tilde{S}_0\} < \infty\}.\end{aligned}$$

For $f \in \operatorname{Exp}(\tilde{S}_0;(R))$ and $g \in \operatorname{Exp}(\tilde{S}_0;[R])$, we set

$$\langle\!\langle f, g\rangle\!\rangle_{\tilde{S}_0,1/R} = \int_{\tilde{S}_0} f(w)g(\overline{w})\,d\mu_{1/R}(w).$$

Put

$$E_R^\lambda(z,w) = E^\lambda(z, R^2 w), \quad z \in \tilde{\boldsymbol{E}}, \quad w \in \tilde{S}_0.$$

If $z, w \in \tilde{S}_0$, $E^\lambda(z,w)$ does not depend on λ. Thus we denote

$$E_R(z,w) = E^\lambda(z, R^2 w) = E^\lambda(Rz, Rw), \quad z, w \in \tilde{S}_0.$$

In [**6**], we defined a transformation $\mathcal{M}_R$ for $T \in \operatorname{Exp}'(\tilde{S}_0;[R])$ by

$$\mathcal{M}_R : T \mapsto \mathcal{M}_R T(w) = \langle T_z, E_R(w,z)\rangle$$

and proved that the transformation $\mathcal{M}_R$ establishes the following topological linear isomorphism:

$$\mathcal{M}_R : \operatorname{Exp}'(\tilde{S}_0;[R]) \xrightarrow{\sim} \operatorname{Exp}(\tilde{S}_0;(R)).$$

Further, for $T \in \operatorname{Exp}'(\tilde{S}_0;[R])$ and $f \in \operatorname{Exp}(\tilde{S}_0;[R])$, we have

$$\langle T, f\rangle = \langle\!\langle f, \mathcal{M}_R T\rangle\!\rangle_{\tilde{S}_0,1/R},$$

which gives $\mathcal{M}_R^{-1}$.

Now, we define a transformation $\mathcal{M}_R^\lambda$ for $\operatorname{Exp}'(\tilde{S}_0;[R])$ and a transformation $\mathcal{E}_R^\lambda$ for $\operatorname{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R])$ by

$$\mathcal{M}_R^\lambda : T \mapsto \mathcal{M}_R^\lambda T(z) = \langle T_w, E_R^\lambda(z,w)\rangle, \quad T \in \operatorname{Exp}'(\tilde{S}_0;[R]),$$

$$\mathcal{E}_R^\lambda : T \mapsto \mathcal{E}_R^\lambda T(w) = \langle T_z, E_R^\lambda(z,w)\rangle, \quad T \in \operatorname{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]).$$

Then we have the following proposition:

Proposition 2.3. *The following diagram is commutative and all the mappings in it are topological linear isomorphisms*:

$$\begin{array}{ccc}
\operatorname{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) & \xleftarrow{\ \mathcal{M}_R^\lambda\ } & \operatorname{Exp}'(\tilde{S}_0;[R]) \\
\beta_\lambda \downarrow\uparrow \beta_\lambda^{-1} & \mathcal{M}_R \swarrow\!\nearrow \mathcal{M}_R^{-1} & \beta_\lambda^* \downarrow\uparrow (\beta_\lambda^*)^{-1} \\
\operatorname{Exp}(\tilde{S}_0;(R)) & \xleftarrow{\ \mathcal{E}_R^\lambda\ } & \operatorname{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]),
\end{array}$$

where β_λ^* is the adjoint mapping of β_λ.

Since we have $\mathscr{M}_R^{-1}$ and β_λ^{-1} explicitly, we also have $(\mathscr{M}_R^\lambda)^{-1}$ and $(\mathscr{E}_R^\lambda)^{-1}$ explicitly.

Proof. We have only to show that the mappings are given by

$$\mathscr{M}_R^\lambda = \beta_\lambda^{-1} \circ \mathscr{M}_R, \quad \mathscr{E}_R^\lambda = \mathscr{M}_R \circ (\beta_\lambda^*)^{-1}. \tag{12}$$

Since the integral representation (11) for $F \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R))$ is rewritten by

$$F(z) = \int_{\tilde{S}_0} F(w) E_R^\lambda(z, \overline{w}) d\mu_{1/R}(w), \quad z \in \tilde{\boldsymbol{E}},$$

and

$$\int_{\tilde{S}_0} E_R(\zeta, w) E_R^\lambda(z, \bar{\zeta})\, d\mu_{1/R}(\zeta) = \int_{\tilde{S}_0} E_R^\lambda(\zeta, w) E_R^\lambda(z, \bar{\zeta})\, d\mu_{1/R}(\zeta)$$
$$= E_R^\lambda(z, w), \quad z \in \tilde{\boldsymbol{E}}, w \in \tilde{S}_0,$$

we have (12). □

Furthermore, by composing β_λ^{-1} and $\mathscr{E}_R^\lambda$, there is another topological linear isomorphism

$$\mathscr{E}_{\lambda,R} \equiv \beta_\lambda^{-1} \circ \mathscr{E}_R^\lambda : \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)).$$

Then for $T \in \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R])$, $\mathscr{E}_{\lambda,R}$ is given by

$$\mathscr{E}_{\lambda,R} : T \mapsto \mathscr{E}_{\lambda,R} T(z) = \langle T_w, E_{\lambda,R}(z, w) \rangle,$$

where

$$E_{\lambda,R}(z, w) = \int_{\tilde{S}_0} E_R^\lambda(z, \zeta) E_R^\lambda(w, \bar{\zeta})\, d\mu_{1/R}(\zeta), \quad z, w \in \tilde{\boldsymbol{E}}.$$

Therefore, we have the following corollary:

Corollary 2.4. *The following diagram is commutative and all the mappings in it are topological linear isomorphisms*:

$$\begin{array}{ccc} \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{E}_{\lambda,R}} & \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) \\ \uparrow{\scriptstyle \beta_\lambda^*} & & \downarrow{\scriptstyle \beta_\lambda} \\ \mathrm{Exp}'(\tilde{S}_0;[R]) & \xrightarrow{\mathscr{M}_R} & \mathrm{Exp}(\tilde{S}_0;(R)). \end{array}$$

Since we have $\mathscr{M}_R^{-1}$ explicitly, we also have $\mathscr{E}_{\lambda,R}^{-1}$ explicitly.

3. λ-Cauchy transformation.

In this section, first, we consider an integral representation for $\mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R))$. The integral representation will give the inverse mapping of the following restriction mapping β_λ;

$$\beta_\lambda : \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)) \xrightarrow{\sim} \mathscr{O}(\tilde{S}_0(R)), \quad 0 < R \leq \infty$$

(see [**11**]). For $z, w \in \tilde{\boldsymbol{E}}$ with $L(z)L(w) < R^2$ define

$$K_{\lambda,R}(z,w) = \sum_{k=0}^{\infty} 2^k/\gamma_{k,n} N(k,n)\tilde{j}_k(i\lambda\sqrt{z^2})\tilde{j}_k(i\lambda\sqrt{w^2})\tilde{P}_{k,n}(z/R, w/R).$$

Note that $(\Delta_z - \lambda^2)K_{\lambda,R}(z,w) = (\Delta_w - \lambda^2)K_{\lambda,R}(z,w) = 0$.

If $z, w \in \tilde{S}_0$ with $L(z)L(w) < R^2$, then $K_{\lambda,R}(z,w)$ does not depend on λ and reduces to the Cauchy kernel $K_R^0(z,w)$ on $\tilde{S}_0$ introduced in [**5**];

$$\begin{aligned} K_R^0(z,w) &= \sum_{k=0}^{\infty} N(k,n)(2z\cdot w/R^2)^k = (1+2z\cdot w/R^2)/(1-2z\cdot w/R^2)^n \\ &= \int_0^{\infty}\int_{S_{1/R}} \exp(z\cdot r\zeta)\exp(r\zeta\cdot w)\,d\dot{\zeta} r^{n-1}\rho_n(r)\,dr, \quad z, w \in \tilde{S}_0 \\ &\equiv \int_{\boldsymbol{E}} \exp(z\cdot x)\exp(x\cdot w)\,dE_{1/R}(x), \end{aligned}$$

where $d\dot{\zeta}$ is the normalized $O(n+1)$-invariant measure on $S_{1/R}$. Note that $K_{0,R}(z,w) = K_R^0(z,w)$ if z or $w \in \tilde{S}_0$.

From the point of view of integral representations, Theorem 2.4 in [**11**] may be restated as follows;

Proposition 3.1. *Let $\lambda \in \boldsymbol{C}$ and $f \in \mathcal{O}(\tilde{S}_0(R))$. For $0 < \rho < R$, define $F(z)$ by*

$$F(z) = \int_{\tilde{S}_{0,1}} f(\rho w)K_{\lambda,1}(z, \overline{w}/\rho)\,d\dot{w}, \quad z \in \tilde{B}(\rho). \tag{13}$$

Then $F(z) \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R))$ and $F|_{\tilde{S}_0} = \beta_\lambda F = f$.

Conversely, for $F \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R))$ we have

$$F(z) = \int_{\tilde{S}_{0,1}} F(\rho w)K_{\lambda,1}(z, \overline{w}/\rho)d\dot{w}, \quad z \in \tilde{B}(\rho); \tag{14}$$

that is, the mapping $f \mapsto F$ in (13) *is the inverse mapping of the restriction mapping β_λ and* (14) *is an integral representation for $F \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R))$.*

In [**5**], we defined the Cauchy transformation $\mathscr{C}_R$ for $T \in \mathcal{O}'(\tilde{S}_0[R])$ by

$$\mathscr{C}_R : T \mapsto \mathscr{C}_R T(z) = \langle T_w, K_R^0(z,w)\rangle.$$

For $f \in \mathcal{O}(\tilde{S}_0[R])$ and $g \in \mathcal{O}(\tilde{S}_0(R))$ we set

$$\langle f, g\rangle_{\tilde{S}_{0,R}} = \int_{\tilde{S}_{0,R}} f(\rho\zeta)g(\overline{\zeta}/\rho)\,d\dot{\zeta},$$

where $\rho > 1$ is sufficiently close to 1. Then we proved that the Cauchy transformation $\mathscr{C}_R$ establishes the following topological linear isomorphism (Theorem 9 in [**5**]):

$$\mathscr{C}_R : \mathcal{O}'(\tilde{S}_0[R]) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_0(R)).$$

Moreover, for $T \in \mathcal{O}'(\tilde{S}_0[R])$ and $f \in \mathcal{O}(\tilde{S}_0[R])$ we have

$$\langle T, f\rangle = \langle f, \mathscr{C}_R T\rangle_{\tilde{S}_{0,R}} \tag{15}$$

which gives $\mathscr{C}_R^{-1}$ (Theorem 11 in [**6**]).

Using the kernel $K_{\lambda,R}(z,w)$, we define the λ-Cauchy transformation for $T \in \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R])$ by

$$\mathscr{C}_{\lambda,R} : T \mapsto \mathscr{C}_{\lambda,R}T(z) = \langle T_w, K_{\lambda,R}(z,w)\rangle, \quad z \in \tilde{B}(R).$$

For the λ-Cauchy transformation, we have

Proposition 3.2. *Let $T \in \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R])$. Then the λ-Cauchy transformation $\mathscr{C}_{\lambda,R}$ establishes the following topological linear isomorphism*:

$$\mathscr{C}_{\lambda,R} : \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) \xrightarrow{\sim} \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)).$$

Further, for $T \in \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R])$ and $f \in \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}[R])$ we have

$$\langle T, f\rangle = \langle f, \mathscr{C}_{\lambda,R} T\rangle_{\tilde{S}_{0,R}}. \tag{16}$$

Proof. Since $K_{\lambda,R}(z,w) = K_R^0(z,w)$ for $z, w \in \tilde{S}_0$ and

$$\int_{\tilde{S}_{0,R}} K_{\lambda,R}(z,\zeta) K_{\lambda,R}(w,\bar{\zeta})\, d\dot{\zeta} = K_{\lambda,R}(z,w), \quad z, w \in \tilde{\boldsymbol{E}},$$

we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) & \xrightarrow{\mathscr{C}_{\lambda,R}} & \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)) \\ \uparrow \beta_\lambda^* & & \downarrow \beta_\lambda \\ \mathscr{O}'(\tilde{S}_0[R]) & \xrightarrow{\mathscr{C}_R} & \mathscr{O}(\tilde{S}_0(R)). \end{array} \tag{17}$$

Because β_λ and $\mathscr{C}_R$ in (17) are topological linear isomorphisms, $\mathscr{C}_{\lambda,R}$ is also topological linear isomorphism. (16) is clear by (17) and (15). □

4. Poisson transformation.

Let

$$\tilde{S}_\lambda = \{z \in \tilde{\boldsymbol{E}}; z^2 = \lambda^2\}$$

be the complex sphere of radius $\lambda \in \boldsymbol{C}$. Put $\tilde{S}_\lambda(R) = \tilde{B}(R) \cap \tilde{S}_\lambda, \tilde{S}_\lambda[R] = \tilde{B}[R] \cap \tilde{S}_\lambda$ and $\tilde{S}_{\lambda,R} = \partial\tilde{S}_\lambda[R] = \{z \in \tilde{S}_\lambda; L(z) = R\}$. Further we put

$$S_\lambda = \tilde{S}_{\lambda,|\lambda|} = \partial\tilde{S}_\lambda[|\lambda|] = \lambda S = \{\lambda x, x \in S\}.$$

We denote by $\mathscr{O}(\tilde{S}_\lambda(R))$ the space of holomorphic functions on $\tilde{S}_\lambda(R)$ with the topology of uniform convergence on compact sets and put

$$\begin{aligned} \mathrm{Exp}(\tilde{S}_\lambda; (R)) = \{f \in \mathscr{O}(\tilde{S}_\lambda); \quad & \text{for all } R' > R \quad \text{we have} \\ & \sup\{|f(z)| \exp(-R' L^*(z)); z \in \tilde{S}_\lambda\} < \infty\}. \end{aligned}$$

We know the following theorem (Theorem 5.3 in [**2**] and Theorem 3.1 in [**12**]):

THEOREM 4.1. *The restriction mapping α_λ establishes the following topological linear isomorphisms*:

(i) $$\alpha_\lambda : \mathcal{O}_\Delta(\tilde{B}(R)) \xrightarrow{\sim} \mathcal{O}(\tilde{S}_\lambda(R)), \quad |\lambda| < R \leq \infty,$$

(ii) $$\alpha_\lambda : \mathrm{Exp}_\Delta(\tilde{\boldsymbol{E}}; (R)) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_\lambda; (R)), \quad 0 \leq R < \infty.$$

Note that $\alpha_0 = \beta_0$ and the inverse mappings of α_0 are given in Theorem 2.2 or in Proposition 3.1.

Here, we review the Poisson transformation. For $z, w \in \tilde{\boldsymbol{E}}$ with $L(z)L(w) < 1$,

$$K_1(z, w) = \sum_{k=0}^{\infty} N(k, n)\tilde{P}_{k,n}(z, w) = \frac{1 - z^2 w^2}{(1 + z^2 w^2 - 2z \cdot w)^{(n+1)/2}}$$

is the well-known Poisson kernel. Set $K_R(z, w) = K_1(z/R, w/R)$. $K_R(z, w)$ has the following integral representation (Lemma 14 in [**6**]):

$$K_R(z, w) = \int_{\tilde{S}_0} \exp(z \cdot \zeta)\exp(\bar{\zeta} \cdot w)\, d\mu_{1/R}(\zeta).$$

For $T \in \mathcal{O}'_\Delta(\tilde{B}[R])$, the Poisson transformation $\mathscr{P}_R$ is defined by

$$\mathscr{P}_R : T \mapsto \mathscr{P}_R T(z) = \langle T_w, K_R(z, w) \rangle, \quad z \in \tilde{B}(R)$$

and the following topological linear isomorphism is known;

$$\mathscr{P}_R : \mathcal{O}'_\Delta(\tilde{B}[R]) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(R)).$$

Further, for $T \in \mathcal{O}'_\Delta(\tilde{B}[R])$ and $f \in \mathcal{O}_\Delta(\tilde{B}[R])$, we have

$$\langle T, f \rangle = \int_{S_R} f(\rho\omega)\mathscr{P}_R T(\omega/\rho)\, d\dot{\omega} \equiv \langle f, \mathscr{P}_R T \rangle_{S_R},$$

where $\rho > 1$ is sufficiently close to 1 and $d\dot{\omega}$ is the normalized $O(n+1)$-invariant measure on S_R, which gives $\mathscr{P}_R^{-1}$ (Theorem 5 in [**6**]).

Moreover, for $f \in \mathcal{O}_\Delta(\tilde{B}(R))$ we have the following integral representation:

$$f(z) = \langle f, K_R(z, \cdot) \rangle_{S_R}.$$

5. Spherical Fourier-Borel transformation.

We denote by $\mathcal{O}'(\tilde{S}_\lambda[R])$ the dual space of

$$\mathcal{O}(\tilde{S}_\lambda[R]) = \mathrm{ind}\lim\{\mathcal{O}(\tilde{S}_\lambda(R')); R' > R\}.$$

Put

$$\mathrm{Exp}(\tilde{S}_\lambda; [R]) = \{f \in \mathcal{O}(\tilde{S}_\lambda); \quad \text{there is } R' < R \quad \text{such that} \\ \sup\{|f(z)| \exp(-R' L^*(z)); z \in \tilde{S}_\lambda\} < \infty\}.$$

We denote its dual space by $\mathrm{Exp}'(\tilde{S}_\lambda; [R])$. For $T \in \mathrm{Exp}'(\tilde{S}_\lambda; [R])$ the Fourier-Borel transformation $\mathscr{F}_\lambda$ is defined by

$$\mathscr{F}_\lambda : T \mapsto \mathscr{F}_\lambda T(\zeta) = \langle T_z, \exp(\zeta \cdot z)\rangle, \quad \zeta \in \tilde{B}(R)$$

and we have the following theorem (Theorems 18, 19 in [**5**], Theorem 3.1 in [**11**] and Theorem 3.1 in [**10**]):

Theorem 5.1. *The Fourier-Borel transformation $\mathscr{F}_\lambda$ establishes the following topological linear isomorphisms*:

(i) $$\mathscr{F}_\lambda : \mathrm{Exp}'(\tilde{S}_\lambda; [R]) \xrightarrow{\sim} \mathscr{O}_{\Delta-\lambda^2}(\tilde{B}(R)), \quad |\lambda| < R \le \infty,$$

(ii) $$\mathscr{F}_\lambda : \mathscr{O}'(\tilde{S}_\lambda[R]) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; (R)), \quad |\lambda| \le R < \infty.$$

If $w \in \tilde{S}_\lambda(R)$, then $\exp(z \cdot w) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R])$. Therefore, we can define the *spherical Fourier-Borel transformation* $\mathscr{F}^\lambda$ for $T \in \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R])$ by

$$\mathscr{F}^\lambda : T \mapsto \mathscr{F}^\lambda T(w) = \langle T_z, \exp(w \cdot z)\rangle, \quad w \in \tilde{S}_\lambda(R).$$

In [**6**], we denote $\mathscr{F}^0$ by $\mathscr{F}^\Delta$ and call it the conical Fourier-Borel transformation, and we proved that the diagrams (1) and (2) are commutative and all the mappings in them are topological linear isomorphisms. In (1), $\mathscr{P}_R$ is defined in Section 4 and $\mathscr{M}_R$ in Section 2. In (2), $\mathscr{C}_R$ is defined in Section 3 and $\mathscr{E}_R$ is given as follows;

For $T \in \mathrm{Exp}'_\Delta(\tilde{\boldsymbol{E}}; [R])$ the mapping $\mathscr{E}_R$ is defined by

$$\mathscr{E}_R : T \mapsto \mathscr{E}_R T(w) = \langle T_z, \tilde{E}_R(w, z)\rangle,$$

where

$$\tilde{E}_R(z, w) = \int_{\tilde{S}_{0,R}} \exp(z \cdot \zeta)\exp(\bar{\zeta} \cdot w)\, d\dot{\zeta} = \sum_{k=0}^{\infty} N(k, n)/C(k, n) R^{2k} \tilde{P}_{k,n}(z, w).$$

Further, for $T \in \mathrm{Exp}'_\Delta(\tilde{\boldsymbol{E}}; [R])$ and $f \in \mathrm{Exp}_\Delta(\tilde{\boldsymbol{E}}; [R])$ we have

$$\langle T, f\rangle = \int_{\boldsymbol{E}} f(x)\mathscr{E}_R T(x) dE_{1/R}(x).$$

This duality formula gives $\mathscr{E}_R^{-1}$.

Moreover, for $f \in \mathrm{Exp}_\Delta(\tilde{\boldsymbol{E}}; (R))$, beside the integral representation (11), we can represent it as follows (see Theorem 24 in [**6**]):

$$f(z) = \int_{\boldsymbol{E}} f(x)\tilde{E}_R(z, x) dE_{1/R}(x).$$

First, we extend the topological linear isomorphism under $\mathscr{F}^0$ in (2) to $\mathscr{F}^\lambda$.

Theorem 5.2. *The spherical Fourier-Borel transformation $\mathscr{F}^\lambda$ establishes the following topological linear isomorphism*:

$$\mathscr{F}^\lambda : \mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}}; [R]) \xrightarrow{\sim} \mathscr{O}(\tilde{S}_\lambda(R)), \quad |\lambda| < R < \infty.$$

Proof. By the definition we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{F}^{\lambda}} & \mathcal{O}(\tilde{S}_{\lambda}(R)) \\
\Big\uparrow {\scriptstyle \beta_{\lambda}^{*}} & & \Big\uparrow {\scriptstyle \alpha_{\lambda}} \\
\mathrm{Exp}'(\tilde{S}_0;[R]) & \xrightarrow{\mathscr{F}_0} & \mathcal{O}_{\Delta}(\tilde{B}(R)).
\end{array}$$

Because $\beta_{\lambda}^{*}, \mathscr{F}_0$ and α_{λ} are topological linear isomorphisms, $\mathscr{F}^{\lambda}$ is also a topological linear isomorphism. □

From (1) and Corollary 2.4, we have the following corollary:

Corollary 5.3. *Let* $|\lambda| < R < \infty$. *We have the following commutative diagram*:

$$\begin{array}{ccc}
\mathrm{Exp}'_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{F}^{\lambda}} & \mathcal{O}(\tilde{S}_{\lambda}(R)) \\
\Big\uparrow {\scriptstyle \mathscr{E}_{\lambda,R}^{-1}} & & \Big\downarrow {\scriptstyle (\alpha_{\lambda}\circ\mathscr{P}_R\circ\alpha_{\lambda}^{*})^{-1}} \\
\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\boldsymbol{E}};(R)) & \xleftarrow{\mathscr{F}_{\lambda}} & \mathcal{O}'(\tilde{S}_{\lambda}[R]),
\end{array}$$

where all the mappings in it are topological linear isomorphisms.

Especially for $\lambda = 0$ we have

$$\begin{array}{ccc}
\mathrm{Exp}'_{\Delta}(\tilde{\boldsymbol{E}};[R]) & \xrightarrow{\mathscr{F}^{0}} & \mathcal{O}(\tilde{S}_{0}(R)) \\
\Big\uparrow {\scriptstyle \mathscr{E}_{0,R}^{-1}} & & \Big\downarrow {\scriptstyle (\alpha_{0}\circ\mathscr{P}_R\circ\alpha_{0}^{*})^{-1}} \\
\mathrm{Exp}_{\Delta}(\tilde{\boldsymbol{E}};(R)) & \xleftarrow{\mathscr{F}_{0}} & \mathcal{O}'(\tilde{S}_{0}[R]).
\end{array}$$

Because $\alpha_0 \circ \mathscr{P}_R \circ \alpha_0^{*}$ and $\mathscr{C}_R$ are different mappings, this diagram is different from (2).

At last, we extend the topological linear isomorphism under $\mathscr{F}^0$ in (1) to $\mathscr{F}^{\lambda}$.

Theorem 5.4. *Let* $R > 0$. *Then the spherical Fourier-Borel transformation* $\mathscr{F}^{\lambda}$ *establishes the following topological linear isomorphism*:

$$\mathscr{F}^{\lambda} : \mathcal{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) \xrightarrow{\sim} \mathrm{Exp}(\tilde{S}_{\lambda};(R)).$$

Proof. By the definition we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) & \xrightarrow{\mathscr{F}^{\lambda}} & \mathrm{Exp}(\tilde{S}_{\lambda};(R)) \\
\Big\uparrow {\scriptstyle \beta_{\lambda}^{*}} & & \Big\uparrow {\scriptstyle \alpha_{\lambda}} \\
\mathcal{O}'(\tilde{S}_0[R]) & \xrightarrow{\mathscr{F}_0} & \mathrm{Exp}_{\Delta}(\tilde{\boldsymbol{E}};(R)).
\end{array}$$

Because $\beta_{\lambda}^{*}, \mathscr{F}_0$ and α_{λ} are topological linear isomorphisms, $\mathscr{F}^{\lambda}$ is also a topological linear isomorphism. □

By (2) and Proposition 3.2 we have the following corollary:

Corollary 5.5. *Let $R > 0$. Then we have the following commutative diagram and all the mappings in it are topological linear isomorphisms*:

$$\begin{array}{ccc} \mathcal{O}'_{\Delta-\lambda^2}(\tilde{B}[R]) & \xrightarrow{\mathcal{F}^\lambda} & \mathrm{Exp}(\tilde{S}_\lambda;(R)) \\ \uparrow \mathscr{C}_{\lambda,R}^{-1} & & \downarrow (\alpha_\lambda \circ \mathscr{E}_R \circ \alpha_\lambda^*)^{-1} \\ \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(R)) & \xleftarrow{\mathcal{F}_\lambda} & \mathrm{Exp}'(\tilde{S}_\lambda;[R]). \end{array}$$

Because $\mathscr{C}_{0,R}$ and $\mathscr{P}_R$ are different mappings, the above diagram for $\lambda = 0$ is different from (1).

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