J. Math. Soc. Japan
Vol. 51, No. 3, 1999

Dimension groups for subshifts and simplicity of the associated C^*-algebras

By Kengo Matsumoto

(Received Feb. 12, 1997)
(Revised Oct. 13, 1997)

Abstract. We describe the notion of the dimension group for subshifts in terms of symbolic dynamical system and show that the dimension group is a conjugacy invariant. We will show that the the hereditary subsets invariant under the dimension group automorphism exactly corresponds to the gauge invariant ideals of the associated C^*-algebra $\mathcal{O}_\Lambda$ for the subshift (Λ, σ). As a result, we give the conditions that the C^*-algebra $\mathcal{O}_\Lambda$ becomes simple and purely infinite in terms of symbolic dynamics.

1. Introduction.

The study of the dimension groups for topological Markov shifts was initiated by W. Krieger in [**Kr1**] and [**Kr2**]. He defined the notion of the dimension groups for topological Markov shifts based on ideas in the theory of the C^*-algebras. The dimension group has been playing an important role as a conjugacy invariant in the theory of topological Markov shifts. For a square matrix A with entries in $\{0, 1\}$, we denote by Λ_A the topological Markov shift determined by the matrix A. Let $\mathcal{O}_A$ be the Cuntz-Krieger algebra for the matrix A. Then the (future) dimension group $DG(\Lambda_A)$ for the Markov shift Λ_A appears as the K_0-group of the AF-algebra consisting of the fixed elements under the gauge action of the C^*-algebra $\mathcal{O}_A$ ([**CK**], [**C2**], [**Kr**], [**Kr2**]).

In [**Ma**], the author has introduced and studied the C^*-algebra $\mathcal{O}_\Lambda$ from a general subshift Λ keeping in mind that the class of the topological Markov shifts is a subclass of the class of the subshifts. In his studies, many structural properties of the Cuntz-Krieger algebras have been generalized to the C^*-algebras $\mathcal{O}_\Lambda$ associated with subshifts (cf. [**Ma**], [**Ma2**], [**Ma3**]). In particular, as a generalization for the notion of the dimension group of the topological Markov shifts, he defined the (future) dimension group for subshift Λ as the K_0-group $K_0(\mathcal{F}_\Lambda)$ of the AF-algebra $\mathcal{F}_\Lambda$ consisting of the fixed elements under the gauge action of the C^*-algebra $\mathcal{O}_\Lambda$ as an ordered group ([**Ma2**]).

In this paper, we first describe the dimension group for general subshifts in terms of the symbolic dynamical system. For a (two-sided) subshift Λ over $\Sigma = \{1, 2, \dots, n\}$ with shift transformation σ, we denote by X_Λ the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the one-sided subshift for Λ and is simply written by X_Λ. For a natural number $l \in \mathbf{N}$, let Λ_l be the set of all words appearing in the sequences in the subshift Λ of length less than or equal to l. Put $\Lambda_l(x) = \{\mu \in \Lambda_l \mid \mu x \in X_\Lambda\}$ for $x \in X_\Lambda$. We define equivalence relations in the space X_Λ. Two

---

1991 *Mathematics Subject Classification.* Primary 46L35; Secondary 58F03, 54H20.

Key words and phrases. C^*-algebra, subshift, dimension group.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 09740094), Ministry of Education, Science and Culture, Japan

points $x, y \in X_\Lambda$ are said to be l-past equivalent if $\Lambda_l(x) = \Lambda_l(y)$. We write this equivalence as $x \sim_l y$. Let F_i^l, $i = 1, 2, \dots, m(l)$ be the set of the l-past equivalence classes of X_Λ. We write $I_{l,l+1}(i,j) = 1$ if $F_j^{l+1} \subset F_i^l$ otherwise $I_{l,l+1}(i,j) = 0$. We denote by $A_{l,l+1}(i,j)$ the cardinality of the set $\{h \in \Sigma \mid hx \in F_i^l \text{ for some } x \in F_j^{l+1}\}$. Hence we have two $m(l) \times m(l+1)$ matrices $I_{l,l+1}$ and $A_{l,l+1}$ with entries in $\{0,1\}$ and with entries in non-negative integers respectively. We denote by $\boldsymbol{Z}_\Lambda, \boldsymbol{Z}_\Lambda^+$ the inductive limits

$$\boldsymbol{Z}_\Lambda = \varinjlim\{I_{l,l+1}^t : \boldsymbol{Z}^{m(l)} \to \boldsymbol{Z}^{m(l+1)}\},$$

$$\boldsymbol{Z}_\Lambda^+ = \varinjlim\{I_{l,l+1}^t : \boldsymbol{Z}_+^{m(l)} \to \boldsymbol{Z}_+^{m(l+1)}\}$$

where $\boldsymbol{Z}_+^{m(l)}$ is the vectors with entries in non-negative integers. The sequence $A_{l,l+1}^t$ of matrices induces an ordered homomorphism on $\boldsymbol{Z}_\Lambda$ which we denote by λ_Λ. We will show

THEOREM 1.1 (Theorem 4.11). *The (future) dimension group $(K_0(\mathscr{F}_\Lambda), K_0(\mathscr{F}_\Lambda)_+)$ for subshift Λ is realized to be the following ordered group $(DG(\Lambda), DG(\Lambda)_+)$:*

$$DG(\Lambda) = \varinjlim\{\lambda_\Lambda : \boldsymbol{Z}_\Lambda \to \boldsymbol{Z}_\Lambda\},$$

$$DG(\Lambda)_+ = \varinjlim\{\lambda_\Lambda : \boldsymbol{Z}_\Lambda^+ \to \boldsymbol{Z}_\Lambda^+\}.$$

The homomorphism λ_Λ yields a natural automorphism on $DG(\Lambda)$. We call it the *dimension group automorphism* and write it as δ_Λ. The triple $(DG(\Lambda), DG(\Lambda)_+, \delta_\Lambda)$ is called the *dimension triplet* for (Λ, σ).

We will next introduce some properties for subshifts as symbolic dynamical systems. They are condition (I), irreducibility and aperiodicity in some sense. If a subshift is a topological Markov shift Λ_A determined by a matrix A with entries in $\{0,1\}$, their properties coincide with those of the matrix (condition (I) in the sense of Cuntz-Krieger (cf. [**CK**]), irreducibility and aperiodicity) respectively.

DEFINITION.

(i) A subshift (X_Λ, σ) satisfies the *condition* (I) if for any $l \in \boldsymbol{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$ and $y \sim_l x$.

(ii) A subshift (X_Λ, σ) is *irreducible in past equivalence* if for any $l \in \boldsymbol{N}, y \in X_\Lambda$ and a sequence $(x^k)_{k \in \boldsymbol{N}}$ of X_Λ with $x^k \sim_k x^{k+1}, k \in \boldsymbol{N}$, there exist a number N and a word μ of length N in a sequence of X_Λ such that $y \sim_l \mu x^{l+N}$.

(iii) A subshift (X_Λ, σ) is *aperiodic in past equivalence* if for any $l \in \boldsymbol{N}$, there exists a number N such that for any pair $x, y \in X_\Lambda$ there exists a word μ of length N in a sequence of X_Λ such that $y \sim_l \mu x$.

We know that if a subshift (X_Λ, σ) is aperiodic in past equivalence or irreducible in past equivalence with an aperiodic point, then it satisfies the condition (I).

We will see that the dimension triplet is a conjugacy invariant under the hypothesis of the condition (I):

THEOREM 1.2 (Theorem 5.4). *Suppose that both one-sided subshifts (X_{Λ_1}, σ) and (X_{Λ_2}, σ) satisfy the condition* (I). *If they are conjugate, there exists an isomorphism* *between $(DG(\Lambda_1), DG(\Lambda_1)_+)$ and $(DG(\Lambda_2), DG(\Lambda_2)_+)$ as ordered groups which intertwines δ_{Λ_1} and δ_{Λ_2}.*

It is well-known that the dimension group completely determines the algebraic structure of the associated AF-algebra $\mathscr{F}_\Lambda$ (cf. [**Br**], [**El**]). In particular, hereditary subsets correspond to the ideals of the AF-algebra ([**Br**]). We will show that

Theorem 1.3 (Theorem 6.8). *There exists an inclusion preserving bijective correspondences between the set of all δ_Λ-invariant hereditary subsets of the dimension group $DG(\Lambda)$ and the set of all gauge invariant ideals of the C^*-algebra $\mathcal{O}_\Lambda$.*

We will see that any nonzero ideal of $\mathcal{O}_\Lambda$ contains a gauge invariant nonzero ideal of $\mathcal{O}_\Lambda$ under the hypothesis of the condition (I) for (X_Λ, σ). Hence we see that (X_Λ, σ) is irreducible in past equivalence if and only if there exists no gauge invariant ideal of $\mathcal{O}_\Lambda$. Therefore we will obtain a description of the criterion that the C^*-algebra $\mathcal{O}_\Lambda$ become simple and purely infinite in terms of the subshift as follows:

Theorem 1.4 (Corollary 6.11). *If a subshift X_Λ is irreducible in past equivalence and has an aperiodic point, then the C^*-algebra $\mathcal{O}_\Lambda$ is simple. In particular, if a subshift X_Λ is aperiodic in past equivalence, the C^*-algebra $\mathcal{O}_\Lambda$ is simple and purely infinite.*

The above theorem is a generalization of the result of the Cuntz-Krieger's theorem ([**CK; 2.14** Theorem]).

We remark that the condition for X_Λ to be irreducible in past equivalence written in the paper [Ma3] is weaker than the corresponding condition in this paper. The former is not sufficient for $\mathcal{O}_\Lambda$ to be simple.

2. Notation and Dimension groups.

Throughout this paper, a finite set $\Sigma = \{1, 2, \dots, n\}$ is fixed.

Let $\Sigma^{\boldsymbol{Z}}$, $\Sigma^{\boldsymbol{N}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology respectively. The transformation σ on $\Sigma^{\boldsymbol{Z}}, \Sigma^{\boldsymbol{N}}$ given by $(\sigma(x))_i = x_{i+1}, i \in \boldsymbol{Z}, \boldsymbol{N}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\boldsymbol{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ for simplicity. We denote by X_Λ the set of all right-infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the one-sided subshift for Λ.

A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, \dots, \mu_k)$ is said to occur or appear in $x = (x_i) \in \Sigma^{\boldsymbol{Z}}$ if $x_m = \mu_1, \dots, x_{m+k-1} = \mu_k$ for some $m \in \boldsymbol{Z}$. For a subshift (Λ, σ) and a number $k \in \boldsymbol{N}$, let Λ^k be the set of all words of length k in $\Sigma^{\boldsymbol{Z}}$ occurring in some $x \in \Lambda$. Put $\Lambda_l = \bigcup_{k=0}^{l} \Lambda^k, \Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word.

For a one-sided subshift X_Λ, put

$$\Lambda_l(x) = \{\mu \in \Lambda_l \mid \mu x \in X_\Lambda\} \quad \text{for } x \in X_\Lambda, \quad l \in \boldsymbol{N}.$$

We define equivalence relations in the space X_Λ. For $l \in \boldsymbol{N}$, two points $x, y \in X_\Lambda$ are said to be *l-past equivalent* if $\Lambda_l(x) = \Lambda_l(y)$. We write this equivalence as $x \sim_l y$. We denote by $\Omega_l = X_\Lambda / \sim_l$ the l-past equivalence classes of X_Λ.

Lemma 2.1. *For $x, y \in X_\Lambda$ and $\mu \in \Lambda^k$,*
(i) *if $x \sim_l y$, we have $x \sim_m y$ for $m < l$.*
(ii) *if $x \sim_l y$ and $\mu x \in X_\Lambda$, we have $\mu y \in X_\Lambda$ and $\mu x \sim_{l-k} \mu y$ for $l > k$.*

Hence we have the following sequence of surjections in a natural way:

$$\Omega_1 \hookleftarrow \Omega_2 \hookleftarrow \cdots \hookleftarrow \Omega_l \hookleftarrow \Omega_{l+1} \hookleftarrow \cdots.$$

Set

$$\Omega_\Lambda = \varprojlim \Omega_l$$

the projective limit as topological space of the above sequence of surjections.

Some properties of subshifts are characterized in terms of the sequences of the spaces of l-past equivalence classes as follows:

Lemma 2.2. *For a subshift (Λ, σ), we have*
(i) *(Λ, σ) is a topological Markov subshift if and only if $\Omega_1 = \Omega_l$ for all $l \in \boldsymbol{N}$.*
(ii) *(Λ, σ) is a sofic subshift if and only if $\Omega_l = \Omega_{l+1}$ for some $l \in \boldsymbol{N}$.*

For a fixed $l \in \boldsymbol{N}$, let F_i^l, $i = 1, 2, \ldots, m(l)$ be the set of the l-past equivalence classes of X_Λ. Hence X_Λ is a disjoint union of the set F_i^l, $i = 1, 2, \ldots, m(l)$. For $h \in \Sigma$ and $i = 1, 2, \ldots, m(l)$, $j = 1, 2, \ldots, m(l+1)$, we know $hx \in F_i^l$ for some $x \in F_j^{l+1}$ if and only if $hy \in F_i^l$ for all $y \in F_j^{l+1}$ by Lemma 2.1. We write $I_{l,l+1}(i,j) = 1$ if $F_j^{l+1} \subset F_i^l$ otherwise $I_{l,l+1}(i,j) = 0$. Let $A_{l,l+1}(i,j)$ be the cardinality of the set $\{h \in \Sigma \mid hx \in F_i^l$ for some $x \in F_j^{l+1}\}$. We have two sequences $I_{l,l+1}$ and $A_{l,l+1}$ of $m(l) \times m(l+1)$-matrices with entries in $\{0, 1\}$ and with non-negative entries respectively.

A system of sequence $\{[A_{l,l+1}(i,j)]_{i=1,2,\ldots,m(l)}^{j=1,2,\ldots,m(l+1)}\}_{l \in \boldsymbol{N}}$ of matrices is said to be *aperiodic* if for any $l \in \boldsymbol{N}$, there exists a number $N \in \boldsymbol{N}$ such that the all entries of the product $A_{l,l+1} \cdot A_{l+1,l+2} \cdots A_{l+N-1,l+N}$ of the matrices are strictly positive (cf. [**Br**] [**Ev2**]). We then easily have

Lemma 2.3. *For a subshift Λ, the following three assertions are equivalent*:
(i) *X_Λ is aperiodic in past equivalence.*
(ii) *The system $\{[A_{l,l+1}(i,j)]_{i=1,2,\ldots,m(l)}^{j=1,2,\ldots,m(l+1)}\}_{l \in \boldsymbol{N}}$ of matrices is aperiodic.*
(iii) *For any $l \in \boldsymbol{N}$, there exists a number $N \in \boldsymbol{N}$ such that*

$$F_i^l \cap \sigma^{-N}(x) \neq \emptyset \quad \text{for } i = 1, 2, \ldots, m(l), \quad x \in X_\Lambda.$$

We denote by $\boldsymbol{Z}_\Lambda, \boldsymbol{Z}_\Lambda^+$ the inductive limits:

$$\boldsymbol{Z}_\Lambda = \varinjlim \{I_{l,l+1}^t : \boldsymbol{Z}^{m(l)} \to \boldsymbol{Z}^{m(l+1)}\},$$

$$\boldsymbol{Z}_\Lambda^+ = \varinjlim \{I_{l,l+1}^t : \boldsymbol{Z}_+^{m(l)} \to \boldsymbol{Z}_+^{m(l+1)}\}.$$

The sequence $A_{l,l+1}^t$ of matrices induces an ordered homomorphism on $\boldsymbol{Z}_\Lambda$ which we denote by λ_Λ.

We now define an ordered abelian group $DG(\Lambda)$ with positive cone $DG(\Lambda)_+$ for subshift Λ as follows:

Notation 2.4.

$$DG(\Lambda) = \varinjlim\{\lambda_\Lambda : \boldsymbol{Z}_\Lambda \to \boldsymbol{Z}_\Lambda\},$$

$$DG(\Lambda)_+ = \varinjlim\{\lambda_\Lambda : \boldsymbol{Z}_\Lambda^+ \to \boldsymbol{Z}_\Lambda^+\}.$$

The group $DG(\Lambda)$ can be identified with the space of all sequences $(x_1, x_2, \dots)$ for $x_l \in \boldsymbol{Z}_\Lambda$ such that $\lambda_\Lambda(x_l) = x_{l+1}$ eventually for all l large, and two sequences which eventually agree are identified. Then there is an induced homomorphism δ_Λ acting on $DG(\Lambda)$ by shifting the sequences to the left:

$$\delta_\Lambda(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Since we see

$$\delta_\Lambda^{-1}(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots),$$

δ_Λ yields an isomorphism on $DG(\Lambda)$. We call it the *dimension group automorphism*. The triple $(DG(\Lambda), DG(\Lambda)_+, \delta_\Lambda)$ is called the *dimension triplet* for (Λ, σ).

We will, in Section 4, see that the above group is actually isomorphic to the K_0-group for the canonical AF-algebra $\mathscr{F}_\Lambda$ inside of the C^*-algebra $\mathscr{O}_\Lambda$. Hence the dimension group for subshift Λ will be realized as the ordered group $(DG(\Lambda), DG(\Lambda)_+)$.

Remark. (i) If a subshift Λ is a topological Markov shift Λ_A, the dimension group $DG(\Lambda_A)$ as a subshift coincides with the dimension group for Markov shift defined by W. Krieger in [**Kr**] and [**Kr2**].

(ii) The notion of the dimension group for general subshifts has appeared in [**Ma2**] in terms of K-theory for the C^*-algebras constructed from subshifts. Jungseob Lee independently studied the dimension group for general subshifts in terms of symbolic dynamics ([**Le**]).

3. AF-algebras from subshifts.

We henceforth fix a subshift (Λ, σ). Take natural numbers $l, k \in \boldsymbol{N}$ with $l \geq k$. Put

$$\Lambda_l^k(i) = \{\mu \in \Lambda^k \mid \mu x \in X_\Lambda \text{ for } x \in F_i^l\} \quad \text{for } i = 1, 2, \dots, m(l).$$

The set $\Lambda_l^k(i)$ is independent of $x \in F_i^l$ such that $\mu x \in X_\Lambda$. Set

$$\tilde{X}_{\Lambda_l^k} = \coprod_{i=1}^{m(l)} \Lambda_l^k(i) : \text{ disjoint union.}$$

For $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, and $h \in \Sigma = \{1, 2, \dots, n\}$, define

$$A_l(i, h, j) = \begin{cases} 1 & \text{if } hx \in F_i^l \quad \text{for} \quad x \in F_j^{l+1} \\ 0 & \text{otherwise.} \end{cases}$$

Thus one sees $A_{l,l+1}(i, j) = \sum_{h=1}^n A_l(i, h, j)$.

Let us consider the following imbeddings.

(i) $\iota^k : \nu \in (\Lambda_{l+1}^k(j) \subset)\tilde{X}_{\Lambda_{l+1}^k} \hookrightarrow \nu \in (\Lambda_l^k(j') \subset)\tilde{X}_{\Lambda_l^k}$ if $F_j^{l+1} \subset F_{j'}^l$

(ii) $\eta : \mu \in \tilde{X}_{\Lambda_{l+1}^{k+1}} \hookrightarrow \mu h \in X_{\Lambda_l^k}$ if $\mu \in \Lambda_l^k(i), \mu h \in \Lambda_{l+1}^{k+1}(j)$, and $A_l(i, h, j) = 1$.

We set the projective limits as topological spaces:

$$\tilde{X}_{\Lambda^k_\infty} = \varprojlim(\tilde{X}_{\Lambda^k_l}, \iota^k) \quad \text{for } k \in \boldsymbol{Z}_+, \qquad \tilde{X}_{\Lambda(m)} = \varprojlim(\tilde{X}_{\Lambda^k_{k+m}}, \eta) \quad \text{for } m \in \boldsymbol{Z}_+.$$

Since the diagram

$$\begin{array}{ccc} \tilde{X}_{\Lambda^{k+1}_{l+2}} & \xrightarrow{\iota^{k+1}} & \tilde{X}_{\Lambda^{k+1}_{l+1}} \\ \eta \downarrow & & \downarrow \eta \\ \tilde{X}_{\Lambda^k_{l+1}} & \xrightarrow{\iota^k} & \tilde{X}_{\Lambda^k_l} \end{array}$$

is commutative, we have an isomorphism of topological spaces:

$$\varprojlim(\tilde{X}_{\Lambda^k_\infty}, k) \cong \varprojlim(\tilde{X}_{\Lambda(m)}, m).$$

We denote by $\tilde{X}_\Lambda$ the above projective limit.

Let $m^k_l(i) = |\Lambda^k_l(i)|$: be the cardinality of the set $\Lambda^k_l(i)$. Now we define AF-algebras as in the following way. Set

$$\mathscr{M}^l_k = M_{m^k_l(1)} \oplus \cdots \oplus M_{m^k_l(m(l))}.$$

Thus we have

$$D(\mathscr{M}^l_k) \cong C(\tilde{X}_{\Lambda^k_l})$$

where $D(\mathscr{M}^l_k)$ is the algebra of all diagonal elements of $\mathscr{M}^l_k$.

The preceding two inclusions of the spaces $\{\tilde{X}_{\Lambda^l_k}\}_{k \le l}$ yield the inclusions of the algebras $\{\mathscr{M}^l_k\}_{k \le l}$ as follows:

(i) $\iota^{k^*} : \mathscr{M}^l_k \hookrightarrow \mathscr{M}^{l+1}_k$ induced from $\iota^k : \tilde{X}_{\Lambda^k_{l+1}} \hookrightarrow \tilde{X}_{\Lambda^k_l}$.

(ii) $\eta^* : \mathscr{M}^l_k \hookrightarrow \mathscr{M}^{l+1}_{k+1}$ induced from $\eta : \tilde{X}_{\Lambda^{k+1}_{l+1}} \hookrightarrow \tilde{X}_{\Lambda^k_l}$.

Take the inductive limits of the sequences of the finite dimensional algebras as follows:

$$\mathscr{M}^\infty_k = \varinjlim(\mathscr{M}^l_k, \iota^{k^*}) \quad \text{for } k \in \boldsymbol{Z}_+, \qquad \mathscr{M}_\Lambda(m) = \varinjlim(\mathscr{M}^{k+m}_k, \eta^*) \quad \text{for } m \in \boldsymbol{Z}_+.$$

As in a previous discussion, we have an isomorphism : $\varinjlim \mathscr{M}^\infty_k \cong \varinjlim \mathscr{M}_\Lambda(m)$. We denote this C^*-algebra by $\mathscr{M}^\infty_\Lambda$. One thus has

PROPOSITION 3.1.

$$(K_0(\mathscr{M}^\infty_\Lambda), K_0(\mathscr{M}^\infty_\Lambda)_+) \cong (DG(\Lambda), DG(\Lambda)_+).$$

PROOF. The ordered group $(K_0(\mathscr{M}^\infty_\Lambda), K_0(\mathscr{M}^\infty_\Lambda)_+)$ is isomorphic to the inductive limit $\varinjlim(K_0(\mathscr{M}^\infty_k), K_0(\mathscr{M}^\infty_k)_+)$ by the induced map η^*_*. The natural imbedding $j^l_k : \mathscr{M}^l_k \hookrightarrow \mathscr{M}^\infty_k$ makes the following diagram commutative

$$\begin{array}{ccc} K_0(\mathscr{M}^l_k) & \xrightarrow{A_{l,l+1}} & K_0(\mathscr{M}^{l+1}_{k+1}) \\ j^l_{k*} \downarrow & & \downarrow j^{l+1}_{k+1*} \\ K_0(\mathscr{M}^\infty_k) & \xrightarrow{\eta^*} & K_0(\mathscr{M}^\infty_{k+1}). \end{array}$$

As the ordered group $K_0(\mathscr{M}_k^\infty)$ is isomorphic to the ordered group $\boldsymbol{Z}_\Lambda$, we have $\varinjlim(K_0(\mathscr{M}_k^\infty), K_0(\mathscr{M}_k^\infty)_+)$ is isomorphic to $(DG(\Lambda), DG(\Lambda)_+)$.

4. The C^*-algebras associated with subshifts.

We will review the construction of the C^*-algebra $\mathscr{O}_\Lambda$ associated with subshift (Λ, σ).

Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of the n-dimensional Hilbert space $\boldsymbol{C}^n$. Set

$F_\Lambda^0 = \boldsymbol{C}e_0$ (e_0: vacuum vector)

$F_\Lambda^k =$ the Hilbert space spanned by the vectors $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$,

$F_\Lambda = \bigoplus_{k=0}^\infty F_\Lambda^k$ (Hilbert space direct sum)

We denote by $T_\nu, (\nu \in \Lambda^*)$ the creation operator on F_Λ of $e_\nu, \nu \in \Lambda^*$ ($\nu \neq \emptyset$) defined by

$$T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu, & (\nu\mu \in \Lambda^*) \\ 0 & \text{else} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. Let $\boldsymbol{P}_0$ be the projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^n T_i T_i^* + \boldsymbol{P}_0 = 1$. We then easily see that for $\mu, \nu \in \Lambda^*$, the operator $T_\mu \boldsymbol{P}_0 T_\nu^*$ is the partial isometry from the vector e_ν to e_μ. Hence, the C^*-algebra generated by elements of the form $T_\mu \boldsymbol{P}_0 T_\nu^*, \mu, \nu \in \Lambda^*$ is the C^*-algebra $\mathscr{K}(F_\Lambda)$ of all compact operators on F_Λ. Let $\mathscr{T}_\Lambda$ be the C^*-algebra on F_Λ generated by the elements $T_\nu, \nu \in \Lambda^*$.

Definition ([**Ma**]). *The C^*-algebra $\mathscr{O}_\Lambda$ associated with subshift (Λ, σ) is defined as the quotient C^*-algebra $\mathscr{T}_\Lambda / \mathscr{K}(F_\Lambda)$ of $\mathscr{T}_\Lambda$ by $\mathscr{K}(F_\Lambda)$.*

We denote by S_i, S_μ the quotient image of the operator $T_i, i \in \Sigma, T_\mu, \mu \in \Lambda^*$. Hence $\mathscr{O}_\Lambda$ is generated by n partial isometries $S_1, \dots, S_n$ with relation $\sum_{i=1}^n S_i S_i^* = 1$.

If (Λ, σ) is a topological Markov shift, the C^*-algebra $\mathscr{O}_\Lambda$ is nothing but the Cuntz-Krieger algebra associated with the topological Markov shift (cf. [**CK**], [**EFW**], [**Ev**]).

Put $a_\mu = S_\mu^* S_\mu, \mu \in \Lambda^*$. Since $T_\nu T_\nu^*$ commutes with $T_\mu^* T_\mu, \mu, \nu \in \Lambda^*$, the following identities hold

$$(*) \qquad a_\mu S_\nu = S_\nu a_{\mu\nu}, \quad \mu, \nu \in \Lambda^*.$$

For $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$, we have $S_\mu^* S_\nu \neq 0$ if and only if $\mu = \nu$.

We will use the following notation. Let k, l be natural numbers with $k \leq l$.

$A_l =$ The C^*-subalgebra of $\mathscr{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda_l$.

$A_\Lambda =$ The C^*-subalgebra of $\mathscr{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda^*$.

$\mathscr{F}_k^l =$ The C^*-subalgebra of $\mathscr{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in A_l$.

$\mathscr{F}_k^\infty =$ The C^*-subalgebra of $\mathscr{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in A_\Lambda$.

$$\mathscr{F}_A = \text{The } C^*\text{-subalgebra of } \mathcal{O}_A \text{ generated by } S_\mu a S_\nu^*,$$
$$\mu, \nu \in \Lambda^*, \quad |\mu| = |\nu|, \quad a \in A_\Lambda.$$

The projections $\{T_\mu^* T_\mu; \mu \in \Lambda^*\}$ are mutually commutative so that the C^*-algebras $A_l, l \in \boldsymbol{N}$ are commutative. Thus we easily see the following lemma (cf. [**Ma**; Section 3]).

Lemma 4.1.
(i) *A_l is finite dimensional and commutative.*
(ii) *A_l is naturally embedded into A_{l+1} so that $A_\Lambda = \varinjlim A_l$ is a commutative AF-algebra.*
(iii) *Each element of $\mathscr{F}_k^l$ is a finite linear combination of elements of the form $S_\mu a S_\nu^*, \mu, \nu \in \Lambda^k, a \in A_l$. Hence $\mathscr{F}_k^l$ is finite dimensional.*
(iv) *There are two embeddings in $\{\mathscr{F}_k^l\}_{k \le l}$:*
(iv-a) *$\iota_l : \mathscr{F}_k^l \subset \mathscr{F}_k^{l+1}$ through the embedding $A_l \subset A_{l+1}$ and*
(iv-b) *$\eta_k : \mathscr{F}_k^l \subset \mathscr{F}_{k+1}^{l+1}$ through the identity*

$$S_\mu a S_\nu^* = \sum_{j=1}^n S_{\mu j} S_j^* a S_j S_{\nu j}^*, \quad \mu, \nu \in \Lambda^k, \ a \in A_l.$$

(v) *Both $\mathscr{F}_k^\infty = \lim_{l \to \infty} \mathscr{F}_k^l$ and $\mathscr{F}_\Lambda = \lim_{k \to \infty} \mathscr{F}_k^\infty$ are AF-algebras.*

In the preceding Hilbert space F_Λ, the transformation $e_\mu \to z^k e_\mu, \mu \in \Lambda^k, z \in \boldsymbol{T} = \{z \in \boldsymbol{C}; |z| = 1\}$ on each base e_μ yields a unitary representation which leaves $\mathscr{K}(F_\Lambda)$ invariant. Thus it gives rise to an action α of $\boldsymbol{T}$ on the C^*-algebra $\mathcal{O}_\Lambda$. It is called the gauge action and satisfies $\alpha_z(S_i) = z S_i, \ i = 1, 2, \dots, n$.

Each element X of the $*$-subalgebra of $\mathcal{O}_\Lambda$ algebraically generated by S_μ, S_ν^*, $\mu, \nu \in \Lambda^*$ is written as a finite sum

$$X = \sum_{|\nu| \ge 1} X_{-\nu} S_\nu^* + X_0 + \sum_{|\mu| \ge 1} S_\mu X_\mu \quad \text{for some } X_{-\nu}, X_0, \ X_\mu \in \mathscr{F}_\Lambda$$

because of the relation $(*)$. The map $E(X) = \int_{z \in \boldsymbol{T}} \alpha_z(X) dz, X \in \mathcal{O}_\Lambda$ defines a projection of norm one onto the fixed point algebra $\mathcal{O}_\Lambda^\alpha$ under α. We then have (cf. [**Ma**; Proposition 3.11])

Lemma 4.2. *$\mathscr{F}_\Lambda = \mathcal{O}_\Lambda^\alpha$.*

We denote by D_Λ the commutative C^*-algebra of all diagonal elements of $\mathscr{F}_\Lambda$ that is the C^*-algebra generated by elements of the form $S_\mu a_\nu S_\mu^*, \mu, \nu \in \Lambda^*$. Let $\mathfrak{D}_\Lambda$ be the C^*-subalgebra of D_Λ generated by $S_\mu S_\mu^*, \mu \in \Lambda^*$, that is isomorphic to the C^*-algebra $C(X_\Lambda)$ of all complex valued continuous functions on the space X_Λ. Put $\phi_\Lambda(X) = \sum_{j=1}^n S_j X S_j^*, \ X \in \mathfrak{D}_\Lambda$ that corresponds to the shift σ on X_Λ.

Consider the following condition called (I_Λ) in [**Ma**].

(I_Λ): For any $l, k \in \boldsymbol{N}$ with $l \ge k$, there exists a projection q_k^l in $\mathfrak{D}_\Lambda$ such that
(i) $q_k^l a \ne 0$ for any nonzero $a \in A_l$,
(ii) $q_k^l \phi_\Lambda^m(q_k^l) = 0, \ 1 \le m \le k$.

Lemma 4.3 ([**Ma**; Theorem 4.9 and 5.2]). *Let $\mathscr{A}$ be a unital C^*-algebra. Suppose that there is a unital *-homomorphism π from A_Λ* to *$\mathscr{A}$ and there are n partial isometries*

$s_1, \ldots, s_n \in \mathscr{A}$ *satisfying the following relations*

(a) $$\sum_{j=1}^{n} s_j s_j^* = 1, \quad s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu}, \quad \mu, \nu \in \Lambda^*,$$

(b) $$s_\mu^* s_\mu = \pi(S_\mu^* S_\mu), \quad \mu \in \Lambda^*$$

where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}, \mu = (\mu_1, \ldots, \mu_k)$. *Then there exists a unital *-homomorphism* $\tilde{\pi}$ *from* $\mathcal{O}_\Lambda$ *to* $\mathscr{A}$ *such that* $\tilde{\pi}(S_i) = s_i, i = 1, \ldots, n$ *and its restriction to* A_Λ *coincides with* π. *In addition, if the* C^**-algebra* $\mathcal{O}_\Lambda$ *satisfy the condition* (I_Λ), *this extended homomorphism* $\tilde{\pi}$ *becomes injective whenever* π *is injective.*

We say the operator λ_Λ on A_Λ (defined by $\lambda_\Lambda(X) = \sum_{j=1}^n S_j^* X S_j, X \in A_\Lambda$) to be *irreducible* if there exists no non-trivial ideal of A_Λ invariant under λ_Λ. It is also said to be *aperiodic* if for any number l, there exists $N \in \boldsymbol{N}$ such that $\lambda_\Lambda^N(p) \geq 1$ for any minimal projection $p \in A_l$.

Lemma 4.4 ([**Ma**; Theorem 6.3 and Theorem 7.5]). *If the* C^**-algebra* $\mathcal{O}_\Lambda$ *satisfies the condition* (I_Λ) *and* λ_Λ *is irreducible on* A_Λ, *then* $\mathcal{O}_\Lambda$ *is simple. In addition, if* λ_Λ *is aperiodic,* $\mathcal{O}_\Lambda$ *is purely infinite.*

We notice that the following:

Lemma 4.5 (cf. [**Ma**; Proposition 5.8] and [**CK**; 2.17 Proposition]). *Let* (Λ_1, σ_1) *and* (Λ_2, σ_2) *be subshifts such that both the associated* C^**-algebras* $\mathcal{O}_{\Lambda_1}$ *and* $\mathcal{O}_{\Lambda_2}$ *satisfy the condition* (I_Λ). *If the associated one-sided subshifts* $(X_{\Lambda_1}, \sigma_1)$ *and* $(X_{\Lambda_2}, \sigma_2)$ *are topologically conjugate, then there exists an isomorphism* Φ *from* $\mathcal{O}_{\Lambda_1}$ *onto* $\mathcal{O}_{\Lambda_2}$ *such that* $\Phi \circ \alpha_z^1 = \alpha_z^2 \circ \Phi$, $z \in \boldsymbol{T}$ *where* α^i *is the gauge action on* $\mathcal{O}_{\Lambda_i}, i = 1, 2$ *respectively. Furthermore* Φ *maps* $\mathfrak{D}_{\Lambda_1}$ *and* D_{Λ_1} *onto* $\mathfrak{D}_{\Lambda_2}$ *and* D_{Λ_2} *respectively and satisfies* $\Phi \circ \lambda_{\Lambda_1} = \lambda_{\Lambda_2} \circ \Phi$ *on* D_{Λ_1}.

We will here give a proof of Lemma 4.5 for the sake of completeness. The proof is based on the proof of [**CK**; 2.17 Proposition]

Proof. For a subshift (X_Λ, σ), a finite partition $Z(1), Z(2), \ldots, Z(m)$ of X_Λ is called a generator for σ if the characteristic functions of the sets $\sigma^{-k}(Z(i))$, $i = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots,$ generate the C^*-algebra $C(X_\Lambda)(= \mathfrak{D}_\Lambda)$. Put

$$U_\mu = \{(x_1, x_2, \ldots) \in X_\Lambda \mid x_1 = \mu_1, x_2 = \mu_2, \ldots, x_k = \mu_k\}$$

the cylinder set for $\mu = \mu_1 \cdots \mu_k \in \Lambda^k$. We denote by χ_{U_μ} the characteristic function of U_μ. It is easy to see that the cylinder sets $Z(i) = U_{\{i\}}$, $i = 1, 2, \ldots, n$ form a generator for σ in X_Λ. We indeed have

$$S_\mu S_\mu^* = \chi_{Z(\mu_1)} \cdot \chi_{\sigma^{-1}(Z(\mu_2))} \cdots \chi_{\sigma^{-(k-1)}(Z(\mu_k))}$$

for $\mu = \mu_1 \mu_2 \cdots \mu_k$. We may assume that the σ_1 and σ_2 act on the same space X and that $\sigma_1 = \sigma_2 = \sigma$. Let $Z_1(i)$, $i \in \Sigma_1$ and $Z_2(j), j \in \Sigma_2$ be generators for σ_1 and σ_2 respectively. Put $W_{i,j} = Z_1(i) \cap Z_2(j)$ for $(i, j) \in \Sigma_1 \times \Sigma_2 = \Sigma^w$. Then the non-empty sets among the sets $W_{i,j}, (i, j) \in \Sigma^w$ form a generator for σ. We define a subshift

(Λ_w, σ) over Σ^w for which forbidden blocks are

$$\bigcup_{k=1}^{\infty}\{(w_1, w_2, \dots, w_k) \in \Sigma^w \times \Sigma^w \times \cdots \times \Sigma^w \mid W_{w_1} \cap \sigma^{-1}(W_{w_2}) \cap \cdots \cap \sigma^{-(k-1)}(W_{w_k}) = \emptyset\}.$$

The subshift (X_{Λ_w}, σ) may be identified with the original ones $(X_{\Lambda_i}, \sigma_i)$, $i = 1, 2$. Let $V_{i,j}, (i, j) \in \Sigma^w$ be the generating partial isometries of the C^*-algebra $\mathcal{O}_{\Lambda_w}$ associated with the subshift Λ_w. Put

$$T_{1_i} = \sum_{j \in \Sigma_2} V_{i,j}, \quad T_{2_j} = \sum_{i \in \Sigma_1} V_{i,j} \in \mathcal{O}_{\Lambda_w}.$$

We first construct an isomorphism from $\mathcal{O}_{\Lambda_1}$ onto $\mathcal{O}_{\Lambda_w}$. Since we know that $V_{i,j}V_{i,k}^* = 0$ for $j \neq k$, the operators $T_{1_i}, i \in \Sigma_1$ are partial isometries satisfying the relations

$$T_{1_i}T_{1_i}^* = \sum_{j \in \Sigma_2} V_{i,j}V_{i,j}^*.$$

For $\mu = (\mu_1, \dots, \mu_k) \in \Lambda_1^k$, we have

$$\begin{aligned} T_{1_\mu}^* T_{1_\mu} &= \sum_{(\nu_1, \dots, \nu_k) \in \Lambda_2^k} V_{\mu_k, \nu_k}^* \cdots V_{\mu_1, \nu_1}^* V_{\mu_1, \nu_1} \cdots V_{\mu_k, \nu_k} \\ &= \sum_{(\nu_1, \dots, \nu_k) \in \Lambda_2^k} \chi_{\sigma^k(W_{(\mu_1, \dots, \mu_k)(\nu_1, \dots, \nu_k)})} \\ &= \chi_{\sigma^k(Z_1(\mu_1, \dots, \mu_k))} \end{aligned}$$

so that the correspondences $S_\mu^* S_\mu \leftrightarrow T_{1_\mu}^* T_{1_\mu}$, $\mu \in \Lambda_1^*$ gives rise to an isomorphism between the C^*-algebra A_{Λ_1} and the C^*-algebra $C^*(T_{1_\mu}^* T_{1_\mu}; \mu \in \Lambda_1^*)$ generated by $T_{1_\mu}^* T_{1_\mu}$, $\mu \in \Lambda_1^*$. Hence by Lemma 4.3, the map $\Phi_1 : S_i \in \mathcal{O}_{\Lambda_1} \to T_{1_i} \in \mathcal{O}_{\Lambda_w}$ yields an isomorphism from $\mathcal{O}_{\Lambda_1}$ to the C^*-algebra $C^*(T_{1_i}; i \in \Sigma_1)$ generated by $T_{1_i}, i \in \Sigma_1$. The identities $V_{i,j} = V_{i,j}V_{i,j}^* T_{1_i}$ hold. Since we know $V_{i,j}V_{i,j}^*$ is contained in the C^*-algebra $C^*(T_{1_\mu}T_{1_\mu}^*; \mu \in \Lambda_1^*)$, which is regarded as the C^*-algebra $C(X_{\Lambda_1})(= C(X))$, hence $\mathcal{O}_{\Lambda_w}$ is generated by $T_{1_i}, i \in \Sigma_1$. Thus the map Φ_1 yields an isomorphism from $\mathcal{O}_{\Lambda_1}$ onto $\mathcal{O}_{\Lambda_w}$. Similarly we have an isomorphism Φ_2 from $\mathcal{O}_{\Lambda_2}$ onto $\mathcal{O}_{\Lambda_w}$. Put $\Phi = \Phi_2^{-1} \circ \Phi_1$ an isomorphism from $\mathcal{O}_{\Lambda_1}$ onto $\mathcal{O}_{\Lambda_2}$. As we see

$$T_{1_\mu}T_{1_\mu}^* = \chi_{Z_1(\mu)} \quad \text{and} \quad T_{2_\nu}T_{2_\nu}^* = \chi_{Z_2(\nu)},$$

we know that the restriction of Φ to $\mathfrak{D}_{\Lambda_1}$ coincides the isomorphism from $\mathfrak{D}_{\Lambda_1}(= C(X_{\Lambda_1}))$ onto $\mathfrak{D}_{\Lambda_2}(= C(X_{\Lambda_2}))$ induced from the conjugacy from X_{Λ_2} to X_{Λ_1}. We can show that $D_{\Lambda_i} = \mathfrak{D}_{\Lambda_i}' \cap \mathcal{F}_\Lambda, i = 1, 2$ as in [**Ma5**; Proposition 3.3] (cf. [**CK**; 2.18 Remark]), the isomorphism Φ maps D_{Λ_1} onto D_{Λ_2}. Since the identity $\Phi \circ \lambda_{\Lambda_1} = \lambda_{\Lambda_2} \circ \Phi$ holds on $\mathfrak{D}_{\Lambda_1}$, it does on D_{Λ_1}.

We will next connect the discussions of Section 2 and Section 3 and the preceding discussion on C^*-algebras. The following lemma is key in our studies.

Lemma 4.6. *Regard a minimal projection of A_l as a characteristic function on the subshift X_Λ. Then the support of the function is one of the set $\{F_i^l\}_{i=1,2,\dots,m(l)}$ of all l-past* *equivalence classes of X_Λ. Hence the set of all minimal projection $\{E_i^l\}_i$ exactly corresponds to the set $\{F_i^l\}_i$ of all l-past equivalence classes Ω_l.*

Corollary 4.7. *The C^*-algebra A_l is isomorphic to the C^*-algebra $C(\Omega_l)$ of all complex valued continuous functions on Ω_l. Hence A_Λ is isomorphic to $C(\Omega_\Lambda)$.*

For a word $\mu \in \Lambda^k$, it belongs to $\Lambda_l^k(i)$ if and only if the inequality $S_\mu^* S_\mu \geq E_i^l$ holds, that is also equivalent to the condition $S_\mu E_i^l S_\mu^* \neq 0$. Thus we have

Lemma 4.8.

(i) $\mathscr{F}_k^l \cong M_{m_l^k(1)} \oplus \cdots \oplus M_{m_l^k(m(l))} (= \mathscr{M}_k^l)$.

(ii) $D(\mathscr{F}_k^l) \cong C(\tilde{X}_{\Lambda_l^k})$.

We easily have

Lemma 4.9. $\lambda_\Lambda(E_i^l) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,j) E_j^{l+1}$.

Proof. The assertion follows from the identity: $S_h^* E_i^l S_h = \sum_{j=1}^{m(l+1)} A_l(i,h,j) E_j^{l+1}$.

Since the identity $S_\mu E_i^l S_\mu^* = \sum_{j=1}^{m(l+1)} \sum_{h=1}^{n} A_l(i,h,j) S_{\mu h} E_j^{l+1} S_{\mu h}^*$ holds, we see

Lemma 4.10. *We have $\mathscr{F}_k^\infty \cong \mathscr{M}_k^\infty$ for $k \in \boldsymbol{N}$ and hence $\mathscr{F}_\Lambda \cong \mathscr{M}_\Lambda^\infty$.*

Therefore by Proposition 3.1 and Lemma 4.9, we obtain

Theorem 4.11.

$$(DG(\Lambda), DG(\Lambda)_+, \delta_\Lambda) \cong (K_0(\mathscr{F}_\Lambda), K_0(\mathscr{F}_\Lambda)_+, \lambda_{\Lambda *}).$$

The following proposition is deduced from Lemma 4.6 and Lemma 4.9.

Proposition 4.12.
(i) *(X_Λ, σ) is irreducible in past equivalence if and only if λ_Λ is irreducible on A_Λ.*
(ii) *(X_Λ, σ) is aperiodic in past equivalence if and only if λ_Λ is aperiodic on A_Λ.*

Proof. (i) It is easy to see that λ_Λ is irreducible on A_Λ if and only if for any $l \in \boldsymbol{N}$, $i = 1, 2, \ldots, m(l)$ and $\omega \in \Omega_\Lambda$, there exists an number $N \in \boldsymbol{N}$ such that

$$\lambda_\Lambda^N(E_i^l)(\omega) \neq 0 \tag{4.1}$$

where $\lambda_\Lambda^N(E_i^l)$ is regarded as a function on Ω_Λ. We denote the element ω by the sequence $(\omega_1, \omega_2, \ldots) \in \Omega_\Lambda (= \varprojlim \Omega_l)$ where $\omega_l \in \Omega_l$. We may identify Ω_l with the set $\{1, 2, \ldots, m(l)\}$. Then the condition (4.1) is equivalent to the condition: there exists a word $\mu \in \Lambda^N$ such that

$$\mu x \in F_i^l \quad \text{for all } x \in F_{\omega_{l+N}}^{l+N}.$$

Hence we have that λ_Λ is irreducible if and only if for any $l \in \boldsymbol{N}, y \in X_\Lambda$ and $(\omega_1, \omega_2, \ldots) \in \Omega_\Lambda$, there exist $N \in \boldsymbol{N}$ and $\mu \in \Lambda^N$ such that $y \sim_l \mu x$ for all $x \in F_{\omega_{l+N}}^{l+N}$. Let $\overline{X}_\Lambda$ be the set of all sequences $(x^k)_{k \in \boldsymbol{N}}$ of X_Λ such that $x^k \sim_k x^{k+1}$ for all $k \in \boldsymbol{N}$. Hence each $\bar{x} = (x^k)_{k \in \boldsymbol{N}} \in \overline{X}_\Lambda$ gives rise to an element $\omega(\bar{x}) = (\omega_k(\bar{x}))_{k \in \boldsymbol{N}}$ of

Ω_Λ that satisfy $x^k \in F^k_{\omega_k(\bar{x})}$. As the map from $\overline{X}_\Lambda$ to Ω_Λ is surjective, we easily see that the irreducibility for subshift is equivalent to the irreducibility for λ_Λ.

(ii) The assertion is direct from Lemma 4.9.

REMARK. Y. Watatani informed the author that the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshifts can be regarded as C^*-algebras constructed from Hilbert C^*-modules considered in [**K**], [**Pi**], [**KPW**], [**KT**]. In [**KT**], Katayama-Takehana have recently defined and studied an aperiodicity for Hilbert C^*-modules that is a generalization of our aperiodicity for subshifts.

5. Condition (I) for subshifts and conjugacy invariance for $DG(\Lambda)$.

In this section, we introduce the condition (I) for subshifts, which is a generalization of the condition (I) for topological Markov shifts in the sense of Cuntz-Krieger (cf. [**CK**]). The condition (I) for subshift (Λ, σ) is an equivalent condition to the condition (I_Λ) for the associated C^*-algebra $\mathcal{O}_\Lambda$. Hence we will show that, under the condition (I), the dimension triple $(DG(\Lambda), DG(\Lambda)_+, \delta_\Lambda)$ (and in particular the dimension group $DG(\Lambda)$) is a conjugacy invariant. We may always assume that all the letters $\Sigma = \{1, 2, \dots, n\}$ are admissible in the subshift Λ and $n \geq 2$. Hence the space X_Λ may not be a single point.

LEMMA 5.1. *The following six conditions are equivalent*:

(i) *$\tilde{X}_\Lambda$ does not have an isolated point.*

(ii) *$\tilde{X}_\Lambda(m)$ does not have an isolated point for some $m \in \boldsymbol{N}$.*

(iii) *$\tilde{X}_\Lambda(m)$ does not have an isolated point for all $m \in \boldsymbol{N}$.*

(iv) *For any $l, m \in \boldsymbol{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y_N \neq x_N$ for some $N > m$, $y_j = x_j$ for $j = 1, 2, \dots, m$ and $y \sim_l x$.*

(v) *For any $l \in \boldsymbol{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$ and $y \sim_l x$.*

(vi) *For any pair $l, k \in \boldsymbol{N}$ with $l \geq k$, there exists $y_i \in F^l_i$ for $i = 1, 2, \dots, m(l)$ such that $\sigma^m(y_i) \neq y_j$ for all $i, j = 1, 2, \dots, m(l)$ and $m = 1, 2, \dots, k$.*

PROOF. (iv) $\Rightarrow$ (v): trivial.

(v) $\Rightarrow$ (iv): For any $x = (x_j)_{j \in \boldsymbol{N}} \in X_\Lambda$ and $l, m \in \boldsymbol{N}$, put $\mu = (x_1, x_2 \cdots, x_m) \in \Lambda^m$, $x' = (x_{m+1}, x_{m+2}, \cdots) \in X_\Lambda$. By the condition (v), there exists $w \in X_\Lambda$ such that $x' \neq w$ and $x' \sim_{l+m} w$. By putting $y = \mu w \in X_\Lambda$, One sees that

$$x_j = y_j \quad \text{for all } 1 \leq j \leq m, \quad x_N = y_N \quad \text{for some } N > m, \quad \text{and } x \sim_l y.$$

(iv) $\Rightarrow$ (vi) We first show that for a fixed $l \geq k$ and $i = 1, 2, \dots, m(l)$, there exists $y \in F^l_i$ satisfying $\sigma^n(y) \neq y$ for $1 \leq n \leq k$. Take an element $x \in F^l_i$ for some $i = 1, 2, \dots, m(l)$. If $\sigma(x) = x$, we may find $y \in X_\Lambda$ such that $\sigma(y) \neq y$ by the condition (iv). Hence assume that $\sigma(x) \neq x$. Now we suppose that $\sigma^n(x) \neq x$ for all $1 \leq n \leq K$. Take $k_n \in \boldsymbol{N}$ such that $x_{k_n} \neq x_{n+k_n}$. Put $M = \mathrm{Max}\{n + k_n; n = 1, 2, \dots, K\} > K + 1$. By the condition (iv), there exists $y \in F^l_i$ such that

$$x_j = y_j \text{ for all } 1 \leq j \leq M \quad \text{and} \quad x_N \neq y_N \text{ for some } N > M.$$

Hence we have $\sigma^n(y) \neq y$ for all $1 \leq n \leq K$. If both the condition $\sigma^{K+1}(x) = x$ and $\sigma^{K+1}(y) = y$ hold, it contradicts to the condition $x_N \neq y_N$ for some $N > M$. Hence

$x(\text{or } y) \in F_i^l$ satisfies $\sigma^n(x) \neq x(\text{or } \sigma^n(y) \neq y)$ for all $1 \leq n \leq K+1$. Thus the induction is completed. By a similar argument to this, we can prove the condition (vi).

(vi) $\Rightarrow$ (v) Suppose that there exist $y \in X_\Lambda, l \in \boldsymbol{N}$ such that $y \sim_l z \in X_\Lambda$ implies $y = z$. Let $F_{i(l)}^l$ be the equivalence class belonging y. As $F_{i(l)}^l = \{y\}$, one sees $F_{i(m)}^m = \{y\}$ for any $m \geq l$. By the condition (vi), we have $\sigma(y) \neq y$ so that $\sigma(y) \in F_j^{l+1}$ for some $j \neq i(l+1)$. Then for any $z \in F_j^{l+1}$, we have $y \sim_l y_1 z$ where $y = (y_1, y_2, \ldots)$, $z = (z_1, z_2, \ldots)$ and $y_1 z = (y_1, z_1, z_2, \ldots)$. Thus we obtain $z = \sigma(y)$ and see $F_j^{l+1} = \{\sigma(y)\}$. Since $F_{i(l+1)}^{l+1} = \{y\}$ and $j \neq i(l+1)$, This contradicts to the condition (vi). Other implications are easily proved.

Definition. *A subshift* (X_Λ, σ) *satisfies the condition* (I) *if it satisfies one of the six equivalent conditions of the preceding lemma.*

A topological Markov shift (X_A, σ) satisfies the Cuntz-Krieger's condition (I) (cf. [**CK**]) if and only if it satisfies the condition (I) in our sense.

Proposition 5.2.

(i) *If* (X_Λ, σ) *is aperiodic in past equivalence, it satisfies the condition* (I).

(ii) *If* (X_Λ, σ) *is irreducible in past equivalence and has an aperiodic point, it satisfies the condition* (I).

Proof. (i) For $l, m \in \boldsymbol{N}$, put $l' = l + m$. As (X_Λ, σ) is aperiodic in past equivalence, for the above l', there exists $N \in \boldsymbol{N}$ satisfying the condition of the aperiodicity. For an element $x = (x_j)_{j \in \boldsymbol{N}} \in X_\Lambda$, put $\gamma = (x_1, x_2, \ldots, x_m) \in \Lambda^m$ and

$$x' = (x_{m+1}, x_{m+2}, \ldots,), \quad x'' = (x_{m+N+1}, x_{m+N+2}, \ldots,) \in X_\Lambda.$$

Take a point $w \in X_\Lambda$ such that $x'' \neq w$. By assumption, there exists a word $\mu \in \Lambda^N$ such that $x' \sim_{l'} \mu w$ so that one sees $\gamma x' \sim_l \gamma\mu w$. Put $y = \gamma\mu w$, which satisfies the condition:

$$x \sim_l y, \quad x_j = y_j \quad \text{for all } 1 \leq j \leq m, \qquad x_K \neq y_K \quad \text{for some } K > m.$$

(ii) For $l, m \in \boldsymbol{N}$, put $l' = l + m$. For an element $x = (x_j)_{j \in \boldsymbol{N}} \in X_\Lambda$, put

$$\gamma = (x_1, x_2, \ldots, x_m) \in \Lambda^m, \quad x' = (x_{m+1}, x_{m+2}, \ldots,).$$

Case 1: x is aperiodic.

Since (X_Λ, σ) is irreducible in past equivalence, for the $l' \in \boldsymbol{N}$ and $x, x' \in X_\Lambda$, we can find a word $\mu \in \Lambda^k$ for some k such that $x' \sim_{l'} \mu x$. Put $y = \gamma\mu x \in X_\Lambda$ so that one has $x \sim_l y$ and $x_j = y_j, 1 \leq j \leq m$. As x is aperiodic, we see that $x \neq y$. Thus the condition (I) is satisfied.

Case 2: x is not aperiodic.

By the assumption, there exists an aperiodic point $w \in X_\Lambda$. Since (X_Λ, σ) is irreducible in past equivalence, for the $l' \in \boldsymbol{N}$ and $w, x' \in X_\Lambda$, we can find a word $\nu \in \Lambda^k$ for some k such that $x' \sim_{l'} \nu w$. Put $y = \gamma\nu w \in X_\Lambda$ so that one has $x \sim_l y$ and $x_j = y_j$, $1 \leq j \leq m$. As w is aperiodic, we see that $x \neq y$. Thus the condition (I) is satisfied.

Lemma 5.3. *A subshift* (X_Λ, σ) *satisfies the condition* (I) *if and only if the* C^*-*algebra* $\mathcal{O}_\Lambda$ *satisfies the condition* (I_Λ).

Proof. Suppose that (X_Λ,σ) satisfies the condition (I). Fix $l \geq k$. We can find $y_i^l \in F_i^l$ with $\sigma^n(y_i^l) \neq y_j^l$ for all $i,j = 1,2,\dots,m(l)$ and $n = 1,2,\dots,k$. Put $Y = \{y_i^l \mid i = 1,2,\dots,m(l)\} \subset X_\Lambda$. Since $\sigma^{-n}(Y) \cap Y = \emptyset$, $1 \leq n \leq k$ and X_Λ is Hausdorff, we may find a clopen set $V \subset X_\Lambda$ containing Y such that $V \cap \sigma^{-j}(V) = \emptyset, 1 \leq j \leq k$. We denote by q_k the characteristic function of V. It satisfies the following conditions:

(i) $q_k E_i^l \neq 0$ for all $i = 1,2,\dots,m(l)$,

(ii) $q_k \phi_\Lambda^m(q_k) = 0$, $1 \leq m \leq k$.

Thus $\mathcal{O}_\Lambda$ satisfies the condition (I_Λ).

Assume that $\mathcal{O}_\Lambda$ satisfies the condition (I_Λ). Fix $l \geq k$. There exists a projection $q_k \in \mathfrak{D}_\Lambda (= C(X_\Lambda))$ satisfying the above conditions (i) and (ii). Take elements $y_i^l \in F_i^l$ contained in the support of q_k, that satisfy the condition (vi) of Lemma 5.1.

Now we reach the following theorem:

Theorem 5.4. *Suppose that both one-sided subshifts (X_{Λ_1},σ) and (X_{Λ_2},σ) satisfy the condition* (I). *If they are conjugate, the dimension triples $(DG(\Lambda_1), DG(\Lambda_1)_+, \delta_{\Lambda_1})$ and $(DG(\Lambda_2), DG(\Lambda_2)_+, \delta_{\Lambda_2})$ are isomorphic.*

Proof. The assertion is deduced from Lemma 4.5, Theorem 4.11 and Lemma 5.3.

6. Gauge invariant ideals of $\mathcal{O}_\Lambda$.

Throughout this section, we mean a closed two-sided ideal of a C^*-algebra by an ideal for simplicity. The term "gauge invariant" means (globally) invariant under gauge action. In this section, we will see that the δ_Λ-invariant hereditary subsets of $DG(\Lambda)$ corresponds to the gauge invariant ideals of $\mathcal{O}_\Lambda$.

The following proposition is basic in our discussions.

Proposition 6.1. *If I is a nonzero gauge invariant ideal of $\mathcal{O}_\Lambda$, we have $I \cap A_\Lambda \neq 0$ and $I \cap D_\Lambda \neq 0$ where D_Λ is the algebra of all diagonal elements of $\mathcal{F}_\Lambda$.*

Proof. Let π_I be the canonical quotient map from $\mathcal{O}_\Lambda$ to the quotient $\mathcal{O}_\Lambda/I$. Since I is gauge invariant, the gauge action α naturally yields an action on $\mathcal{O}_\Lambda/I$ which we denote by $\bar{\alpha}$. Hence the map E_I defined by $E_I(X) = \int_{\boldsymbol{T}} \bar{\alpha}_t(X)dt, X \in \mathcal{O}_\Lambda/I$ gives rise to a faithful projection of norm one from $\mathcal{O}_\Lambda/I$ onto $\mathcal{F}_\Lambda/I$. Now we suppose that $I \cap A_\Lambda = \{0\}$. If $\mathcal{F}_\Lambda \cap I \neq \{0\}$, there exists an element $S_\mu E_i^l S_\nu^* (\neq 0)$ in $\mathcal{F}_\Lambda \cap I$ for some $\mu, \nu \in \Lambda^k$ and $i = 1,2,\dots,m(l)$. The identity $E_i^l = S_\mu^* S_\mu E_i^l S_\nu^* S_\nu$ holds so that E_i^l belongs to I, a contradiction. Hence we have $\mathcal{F}_\Lambda \cap I = \{0\}$. This means that the restriction of π_I to $\mathcal{F}_\Lambda$ is injective. Take an element $X \in \mathcal{O}_\Lambda$ with $\pi_I(X) = 0$. Since one has $\pi_I \circ E = E_I \circ \pi_I$, one sees that $\pi_I(E(X^*X)) = 0$ so that $E(X^*X) = 0$. As E is faithful, we obtain $X = 0$. Thus we conclude that π_I is injective and hence the ideal I is trivial. This contradicts the hypothesis. Therefore we have $I \cap A_\Lambda \neq \{0\}$. We also conclude that $I \cap \mathcal{F}_\Lambda \neq \{0\}$ so that $I \cap D_\Lambda \neq \{0\}$ because $\mathcal{F}_\Lambda$ is an AF-algebra.

Recall that $[I_{l,l+1}(i,j)]_{i=1,\dots,m(l)}^{j=1,\dots,m(l+1)}$ is the $m(l) \times m(l+1)$-matrix with entries in $\{0,1\}$ such that $I_{l,l+1}(i,j) = 1$ if and only if $F_j^{l+1} \subset F_i^l$. Thus we know that

$$E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1} \quad \text{for } i = 1,\dots,m(l).$$

Set

$$\Gamma_A = \{(i,l) \mid i = 1,\dots,m(l), l \in \boldsymbol{Z}_+\}.$$

We define two kinds of partial orders $\succ (\prec)$ and $\geq (\leq)$ in Γ_A as follows:

(i) $(i,l) \geq (j,l+1)$ if $I_{l,l+1}(i,j) = 1$.

(ii) $(i,l) \succ (j,l+1)$ if $A_{l,l+1}(i,j) \neq 0$.

For (i,l) and $(j,l+m)$, we define $(i,l) \geq (j,l+m)$ if there exist $(i_1,l+1),(i_2,l+2),\dots \in \Gamma_A$ such that

$$(i,l) \geq (i_1,l+1) \geq \cdots \geq (j,l+m).$$

Similarly $(i,l) \succ (j,l+m)$ is defined.

A subset $H \subset \Gamma_A$ is said to be hereditary in $\geq$ (resp. $\succ$) if $(i,l) \geq$ (resp. $\succ$) $(j,k) \in \Gamma_A$ and $(i,l) \in H$ implies $(j,k) \in H$. If H is hereditary in both the orders $\geq$ and $\succ$, it is said to be hereditary in Γ_A.

We will show that there exists a bijective correspondence between the set of all hereditary subsets of Γ_A and the set of all gauge-invariant ideals of $\mathcal{O}_A$.

Lemma 6.2. *For a gauge invariant ideal I of $\mathcal{O}_A$, put*

$$H_I = \{(i,l) \in \Gamma_A \mid E_i^l \in A_A \cap I\}.$$

Then H_I is hereditary in Γ_A.

Proof. We may assume that $I \neq \{0\}$ so that $I \cap A_A \neq \{0\}$ by the previous proposition. As A_A is an AF-algebra, we can find $E_i^l \in I \cap A_A$ for some $i = 1,\dots,m(l)$. Suppose that $(j,l+1) \prec (i,l)$ and $(i,l) \in H_I$. As $\lambda_A(E_i^l) \geq E_j^{l+1}$, it follows that $E_j^{l+1} \lambda_A(E_i^l) = c_j^{l+1} E_j^{l+1}$ for some $c_j^{l+1} (\neq 0)$ a scalar. Hence E_j^{l+1} belongs to I because $\lambda_A(E_i^l)$ belongs to I. Hence H_I is hereditary in $\succ$. It is clear that H_I is hereditary in $\geq$. Thus H_I is hereditary in Γ_A.

Conversely we have

Lemma 6.3. *For a hereditary subset H in Γ_A put*

$$I_H = \overline{\mathrm{span}}\{S_\mu E_i^l S_\nu^* \mid (i,l) \in H\}.$$

Then I_H is a gauge invariant ideal of $\mathcal{O}_A$ generated by $E_i^l, (i,l) \in H$.

Proof. Since it is clear that I_H is gauge invariant, it suffices to show that I_H is an ideal of $\mathcal{O}_A$. As in the discussions of Section 4, the C^*-algebra $\mathcal{O}_A$ is spanned by linear combinations of elements of the form $S_\xi a_\eta S_\zeta^*, \xi,\eta,\zeta \in \Lambda^*$. It is enough to show that $S_\xi a_\eta S_\zeta^* \cdot S_\mu E_i^l S_\nu^*$ belongs to I_H for $\mu,\nu \in \Lambda^*, (i,l) \in H$. Since H is hereditary in Γ_A, by the identities:

$$S_\mu E_i^l S_\mu^* = \sum_{j=1}^{m(l+1)} \sum_{h=1}^{n} A_l(i,h,j) S_{\mu h} E_j^{l+1} S_{\mu h}^*,$$

$$S_\mu E_i^l S_\mu^* = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) S_\mu E_j^{l+1} S_\mu^*,$$

we assume that $|\zeta| < |\mu|$ and $|\eta| + |\mu|, |\zeta| + |\mu| < l$. For $S_\xi a_\eta S_\zeta^* \cdot S_\mu E_h^{l+1} S_\nu^* \neq 0$, we have

$S_\zeta^* S_\mu \neq 0$ so that $S_\zeta^* S_\mu = a_\zeta S_{\mu'} = S_{\mu'} a_\mu$ for some $\mu' \in \Lambda^{|\mu|-|\zeta|}$ with $\mu = \zeta\mu'$. It follows that $S_\xi a_\eta S_\zeta^* \cdot S_\mu E_h^{l+1} S_\nu^* = S_{\xi\mu'} a_{\eta\mu'} a_\mu E_i^l S_\nu^* = S_{\xi\mu'} E_i^l S_\nu^*$. Thus $S_\xi a_\eta S_\zeta^* \cdot S_\mu E_h^{l+1} S_\nu^*$ belongs to the ideal I_H.

Lemma 6.4. *For a gauge invariant ideal I of $\mathcal{O}_\Lambda$, we have $I_{H_I} = I$.*

Proof. The inclusion relation $I_{H_I} \subset I$ is clear. We will prove the other inclusion relation. Let π be the canonical quotient map $\mathcal{O}_\Lambda / I_{H_I} \to \mathcal{O}_\Lambda / I$. It is easy to see that $A_\Lambda \cap I_{H_I} = A_\Lambda \cap I$. Hence the restriction of π to the AF-algebra $\mathcal{F}_\Lambda / (\mathcal{F}_\Lambda \cap I_{H_I}) \to \mathcal{F}_\Lambda / (\mathcal{F}_\Lambda \cap I)$ is an isomorphism by a similar argument of the proof of Proposition 6.1. Now both the ideals I and I_{H_I} are gauge invariant so that the gauge action α_t on $\mathcal{O}_\Lambda$ induces actions on the both quotients $\mathcal{O}_\Lambda / I_{H_I}$ and $\mathcal{O}_\Lambda / I$. We write these actions as $\alpha_t^{H_I}$ and α_t^I respectively. We can define faithful expectations E_{H_I} and E_I from $\mathcal{O}_\Lambda / I_{H_I}$ onto $\mathcal{F}_\Lambda / \mathcal{F}_\Lambda \cap I_{H_I}$ and $\mathcal{O}_\Lambda / I$ onto $\mathcal{F}_\Lambda / \mathcal{F}_\Lambda \cap I$ by averaging the actions $\alpha_t^{H_I}$ and α_t^I respectively. As one sees that π intertwines E_{H_I} and E_I, one conclude that π is injective and hence $I_{H_I} = I$.

Corollary 6.5. *There exists a bijective correspondence between the set of all gauge invariant ideals of $\mathcal{O}_\Lambda$ and the set of all hereditary subsets of Γ_Λ through the map $I \to H_I$ and $H \to I_H$.*

We notice that any ideal I of $\mathcal{O}_\Lambda$ is invariant under both ϕ_Λ and λ_Λ.

We will next describe gauge invariant ideals of $\mathcal{O}_\Lambda$ in terms of the dimension group for the subshift.

Lemma 6.6. *For a nonzero ideal I of $\mathcal{F}_\Lambda$ invariant under both ϕ_Λ and λ_Λ, we have*
(i) *$I \cap A_\Lambda$ is a nonzero λ_Λ-invariant ideal of A_Λ.*
(ii)

$$I = \overline{\mathrm{span}}\{S_\mu E_i^l S_\nu^* \mid E_i^l \in I \cap A_\Lambda,\ |\mu| = |\nu|\}.$$

Proof. (i) Put

$$\mathcal{P}_I = \{E_i^l \in A_\Lambda \mid S_\mu E_i^l S_\nu^* \in I \text{ for any } \mu, \nu \text{ with } |\mu| = |\nu|, S_\mu E_i^l S_\nu^* \neq 0\}.$$

We can find a subalgebra $\mathcal{F}_k^l$ for some $k \le l$ such that $I \cap \mathcal{F}_k^l \neq 0$ so that there exists $\xi, \eta \in \Lambda^k$ such that $S_\xi E_i^l S_\eta^* \in I \cap \mathcal{F}_k^l$. Then for any $\mu, \nu \in \Lambda^k$, one sees

$$S_\mu E_i^l S_\nu^* = S_\mu E_i^l S_\xi^* \cdot S_\xi E_i^l S_\eta^* \cdot S_\eta E_i^l S_\nu^*.$$

Thus $S_\mu E_i^l S_\eta^* \in I \cap \mathcal{F}_k^l$. Hence one has $\mathcal{P}_I \neq \emptyset$. We then have $\mathcal{P}_I = \{E_i^l \in I \cap A_\Lambda\}$. We indeed see that, for $E_i^l \in \mathcal{P}_I$, the element $S_\mu E_i^l S_\nu^*$ belongs to I for some $|\mu| = |\nu| = k$. As one has the identity $E_i^l = \lambda_\Lambda^k (S_\mu E_i^l S_\nu^*)$, E_i^l belongs to I and $I \cap A_\Lambda$ because I is λ_Λ-invariant. Conversely, for $E_i^l \in I \cap A_\Lambda$, one has $\sum_{\mu \in \Lambda^k} S_\mu E_i^l S_\mu^* \in I$ because I is ϕ_Λ-invariant. Hence by the identity

$$S_\mu E_i^l S_\nu^* = S_\mu E_i^l S_\nu^* \cdot \sum_{\xi \in \Lambda^k} S_\xi E_i^l S_\xi^* \cdot S_\nu E_i^l S_\nu^*,$$

we obtain $E_i^l \in \mathcal{P}_I$. Thus $\mathcal{P}_I = \{E_i^l \in I \cap A_\Lambda\}$ so that $I \cap A_\Lambda \neq \{0\}$.

(ii) Put

$$I' = \overline{\mathrm{span}}\{S_\mu E_i^l S_\nu^* \mid E_i^l \in I \cap A_\Lambda, |\mu| = |\nu|\}.$$

The inclusion $I' \subset I$ is easily seen from the relation: $\mathscr{P}_I = \{E_i^l \in I \cap A_\Lambda\}$. Conversely, as $\mathscr{F}_\Lambda$ is an AF-algebra, we may assume that any element of $\mathscr{F}_\Lambda \in I$ is of the form: $S_\mu E_i^l S_\nu^*$. Hence E_i^l belongs to $\mathscr{P}_I$.

We conversely have

Lemma 6.7. *For a nonzero ideal J of A_Λ invariant under λ_Λ, Put*

$$I_J = \overline{\mathrm{span}}\{S_\mu E_i^l S_\nu^* \mid E_i^l \in J, |\mu| = |\nu|\}.$$

Then we have
(i) *I_J is an ideal of $\mathscr{F}_\Lambda$ invariant under both ϕ_Λ and λ_Λ.*
(ii) *$I_J \cap A_\Lambda = J$.*

Proof. (i) The invariance of I_J under ϕ_Λ is clear. For $\mu = \mu_1\mu', \nu = \nu_1\nu' \in \Lambda^k$ with $\mu_1, \nu_1 \in \Sigma$ and $\mu', \nu' \in \Lambda^{k-1}$, it follows that

$$\lambda_\Lambda(S_\mu E_i^l S_\nu^*) = S_{\mu'} E_i^l S_{\nu'}^*.$$

This implies that I_J is invariant under λ_Λ.

(ii) By the previous lemma, we know that $\mathscr{P}_{I_J} = \{E_i^l \in I_J \cap A_\Lambda\}$. As we easily see that

$$\mathscr{P}_{I_J} = \{E_i^l \in A_\Lambda \mid 0 \neq S_\mu E_i^l S_\nu^* \in I_J, \quad \text{for all } \mu, \nu \in \Lambda^* \text{ with } |\mu| = |\nu|\},$$

we obtain that $I_J \cap A_\Lambda = J$.

We consequently have the following theorem.

Theorem 6.8. *There exist inclusion relation preserving bijective correspondences between the following five sets*:
(i) *gauge invariant ideals of $\mathcal{O}_\Lambda$*
(ii) *ideals of $\mathscr{F}_\Lambda$ invariant under both ϕ_Λ and λ_Λ*
(iii) *order ideals of $DG(\Lambda)$ invariant under δ_Λ*
(iv) *λ_Λ-invariant ideals of A_Λ*
(v) *hereditary subsets in Γ_Λ.*

Proof. The correspondence between (ii) and (iii) follows from a general theory of K-theory of AF-algebras (cf. [**Ef**]). All other correspondences follow from the previous discussions.

We will finally mention a relationship between simplicity for $\mathcal{O}_\Lambda$ and the dimension group $DG(\Lambda)$.

Lemma 6.9. *Assume that (X_Λ, σ) satisfies the condition* (I). *Then any nonzero ideal of $\mathcal{O}_\Lambda$ contains a nonzero gauge invariant ideal of $\mathcal{O}_\Lambda$.*

Proof. Let J be a nonzero ideal of $\mathcal{O}_\Lambda$. As $\mathcal{O}_\Lambda$ satisfies the condition (I_Λ), we have $J \cap A_\Lambda \neq \{0\}$. Hence we can find a projection E_i^l in $J \cap A_\Lambda$. Set

$$I_J = \overline{\mathrm{span}}\{S_\mu E_i^l S_\nu^* \mid E_i^l \in J \cap A_\Lambda, \mu, \nu \in \Lambda^*\}.$$

It is clear that I_J is a nonzero gauge invariant ideal of $\mathcal{O}_\Lambda$ contained in J.

Thus we conclude

Proposition 6.10. *Suppose that* X_Λ *satisfies the condition* (I). *Then the following five conditions are equivalent*:

(i) $\mathcal{O}_\Lambda$ *is simple*.
(ii) *There exists no gauge invariant ideal of* $\mathcal{O}_\Lambda$.
(iii) *There exists no proper ideal of* $\mathcal{F}_\Lambda$ *invariant under* ϕ_Λ *and* λ_Λ.
(iv) *There exists no* δ_Λ*-invariant order ideal of* $DG(\Lambda)$.
(v) *There exists no* λ_Λ*-invariant ideal* A_Λ.

Hence, by Proposition 5.2, we have the following corollary, which is a generalization of the Cuntz-Krieger's theorem [**CK**; 2.14 Theorem].

Corollary 6.11.
(i) *If a subshift* (X_Λ, σ) *is irreducible in past equivalence and* X_Λ *has an aperiodic point, the* C^**-algebra* $\mathcal{O}_\Lambda$ *is simple*.
(ii) *If in particular a subshift* (X_Λ, σ) *is aperiodic in past equivalence, the* C^**-algebra* $\mathcal{O}_\Lambda$ *is simple and purely infinite and the AF-algebra* $\mathcal{F}_\Lambda$ *is simple*.

Remark. (i) The above corollary is also deduced directly from Lemma 4.4 and Proposition 4.12 and Proposition 5.2.
(ii) Very recently, C. Anantharaman-Delaroche presented a criterion for simplicity and purely infiniteness of C^*-algebras constructed from groupoids of subshifts ([**An**]). The C^*-algebras of the groupoids are definitely isomorphic to our C^*-algebras. The criterion are similar to ours.
(iii) Ideal structure of the Cuntz-Krieger algebras discussed in [**C2**] and recently in [**aHR**]. A related topics is also seen in [**H**]. In [**KPW**], Kajiwara-Pinzari-Watatani study ideal structure and simplicity condition of C^*-algebras constructed from Hilbert C^*-modules.

7. Examples.

Example 7.1 (Full shifts).
Let (Λ_n, σ) be the full n-shift over $\Sigma = \{1, 2, \dots, n\}$. It is aperiodic hence satisfies the condition (I). In fact, any two points in X_{Λ_n} are l-past equivalent so that the equivalence class Ω_l is a singleton for each $l \in \boldsymbol{N}$. The matrix $A_{l,l+1}$ is n-times multiplication on $\boldsymbol{Z}$. Thus the dimension group is

$$\boldsymbol{Z} \xrightarrow{n} \boldsymbol{Z} \xrightarrow{n} \cdots \xrightarrow{n} \boldsymbol{Z}[1/n] = \left\{ \frac{m}{n^k} \,\middle|\, m, k \in \boldsymbol{Z} \right\}.$$

and namely $DG(\Lambda_n) = \boldsymbol{Z}[1/n]$ in $\boldsymbol{R}$. The corresponding simple purely infinite C^*-algebra $\mathcal{O}_{\Lambda_n}$ is the Cuntz-algebra $\mathcal{O}_n$ of order n ([**C**]). The AF-algebra $\mathcal{F}_{\Lambda_n}$ is the UHF-algebra of type n^∞.

Example 7.2 (Topological Markov shifts).
Let (Λ_A, σ) be the topological Markov shift defined by an $n \times n$ aperiodic matrix $A = [A(i,j)]_{i,j=1,2,\dots,n}$ with entries in $\{0, 1\}$. It is aperiodic in our sense and hence satisfies the condition (I). Then we may easily see that its dimension group is iso-

morphic to the inductive limit:

$$\mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^n \xrightarrow{A} \cdots \xrightarrow{A}$$

and namely $DG(\Lambda_n) = \varinjlim(\mathbf{Z}^n, A)$. The corresponding simple purely infinite C^*-algebra $\mathcal{O}_{\Lambda_A}$ is the Cuntz-Krieger algebra $\mathcal{O}_A$ ([**CK**]).

Example 7.3 (β-shifts).

For an arbitrary real number $\beta > 1$, let (Λ_β, σ) be the β-shift over $\Sigma = \{0, 1, \dots, n-1\}$ where n is the natural number satisfying $n - 1 \le \beta < n$ (cf. [**Pa**], [**Re**]). We know that it is aperiodic in past equivalence and hence satisfies the condition (I). Suppose that the β-shift is sofic. As in [**KMW**], the corresponding AF-algebra $\mathcal{F}_\beta$ has a unique tracial state. Put

$$\mathbf{Z}(l) = \mathbf{Z} + \beta\mathbf{Z} + \beta^2\mathbf{Z} + \cdots + \beta^l\mathbf{Z}$$

for each $l \in \mathbf{N}$. By a discussion in [**KMW**], we see that the matrix $A_{l,l+1}$ is identified with β-times multiplication on $\mathbf{Z}(l)$. Thus the dimension group is isomorphic to the inductive limit

$$\mathbf{Z}(l) \xrightarrow{\beta} \mathbf{Z}(l+1) \xrightarrow{\beta} \cdots \xrightarrow{\beta} \mathbf{Z}[1/\beta] = \left\{ \frac{m_0 + m_1\beta + \cdots + m_l\beta^l}{\beta^k} \,\middle|\, m_i, k, l \in \mathbf{Z} \right\}.$$

and namely $DG(\Lambda_\beta) = \mathbf{Z}[1/\beta]$ in $\mathbf{R}$. The dimension group automorphism δ_β is the multiplication by β on $\mathbf{Z}[1/\beta]$. The corresponding C^*-algebra, denoted by $\mathcal{O}_\beta$, is simple and purely infinite. They are classified in [**KMW**] by the sequences appearing in the β-expansions of 1.

Note Added in Proof. Since the submission of this paper, the author has received the following paper, in which an invariance of the dimension groups for subshifts under topological conjugacy is proved by a method of symbolic dynamical systems. J. Lee, *Equivalence of subshifts*, J. Korean Math. Soc. **33** (1996), 685–692.

It has been proved in [**Ma4**] that the stabilized C^*-algebra $\mathcal{O}_\Lambda \otimes \mathcal{K}$ of $\mathcal{O}_\Lambda$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space, with gauge action is invariant under topological conjugacy as two-side subshift. Hence it is a direct consequence from this fact that the dimension triple is topological conjugacy invariant as two-sided subshifts.

References

[An] C. Anantharaman-Delaroche, Purely infinite C^*-algebras arising from dynamical systems, preprint, 1996.

[aHR] A. an Huef and I. Raeburn, The ideal structure of Cuntz-Krieger algebras, preprint. 1994.

[Br] O. Bratteli, Inductive limits of finite-dimensional C^*-algebras, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[C] J. Cuntz, Simple C^*-algebras generated by isometries, Comm. Math. Phys. **57** (1977), 173–185.

[C2] J. Cuntz, A class of C^*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C^*-algebras, Inventions Math. **63** (1980), 25–40.

[CK] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, Inventions Math. **56** (1980), 251–268.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, Ergodic theory on compact spaces, Springer-Verlag, Berlin, Heidelberg and New York, 1976.

[Ef] E. G. Effros, Dimensions and C^*-algebras, AMS-CBMS REg. Conf., Providence, 1981.

[El] G. A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, J. Algebra **38** (1976), 29–44.

[EFW] M. Enomoto, M. Fujii and Y. Watatani, Tensor algebras on the sub-Fock space associated with $\mathcal{O}_A$, Math. Japon **26** (1981), 171–177.

[Ev] D. E. Evans, Gauge actions on $\mathcal{O}_A$, J. Operator Theory **7** (1982), 79–100.

[Ev2] D. E. Evans, The C^*-algebras of topological Markov chains, Tokyo Metropolitan University Lecture Note, 1982.

[H] D. Huang, The classification of two-component Cuntz-Krieger algebras, Proc. Amer. Soc. **124** (1996), 505–512.

[KPW] T. Kajiwara, C. Pinzari and Y. Watatani, Ideal structure and simplicity of the C^*-algebras generated by Hilbert modules, J. Funct. Anal. **159** (1998), 295–322.

[K] Y. Katayama, A generalized Cuntz-algebra $\mathcal{O}_N^M$, RIMS Kokyuroku **858** (1994), 87–90.

[KMW] Y. Katayama, K. Matsumoto and Y. Watatani, Simple C^*-algebras arising from β-expansion of real numbers, Ergod. Th. & Dynam. Sys. **18** (1998), 937–962.

[KT] Y. Katayama and H. Takehana, On automorphisms of generalized Cuntz algebras, preprint, 1997.

[Kr] W. Krieger, On dimension functions and topological Markov chains, Invent. Math. **56** (1980), 239–250.

[Kr2] W. Krieger, On dimension for a class of homeomorphism groups, Math. Ann. **252** (1980), 87–95.

[Le] J. Lee, Talk in a symposium on "ergodic theory and algorithm", Tsuda College, Japan, Dec. 1995.

[Ma] K. Matsumoto, On C^*-algebras associated with subshifts, Internat. J. Math. **8** (1997), 357–374.

[Ma2] K. Matsumoto, *K*-theory for C^*-algebras associated with subshifts, to appear in Math. Scand. **82** (1998), 237–255.

[Ma3] K. Matsumoto, Relations among generators of the C^*-algebras associated with subshifts, to appear in Internat. J. Math.

[Ma4] K. Matsumoto, Stabilized C^*-algebras constructed from symbolic dynamical systems, to appear in Ergod. Th. & Dynam. Sys..

[Ma5] K. Matsumoto, On automorphisms of C^*-algebras associated with subshifts, to appear in J. Operator Theory.

[Pa] W. Parry, On the β-expansion of real numbers, Acta Math. Acad. Sci. Hung. **11** (1960), 401–416.

[Pi] M. V. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\boldsymbol{Z}$, Free Probability Theory, Fields Institute Communication (1996).

[Re] A. Renyi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung. **8** (1957), 477–493.

[Tu] S.Tuncel, A dimension, dimension modules, and Markov chains, Proc. London Math. Soc. **46** (1983), 100–116.

Kengo MATSUMOTO
Department of Mathematics
Joetsu University of Education
Joetsu 943-8512 Japan
E-mail: matsu@juen.ac.jp