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Spherical functions and local densities on hermitian forms

Dedicated to Professor Ichiro Satake on his seventieth birthday

By Yumiko Hironaka

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Abstract. First we give a formula of spherical functions on certain spherical homogeneous spaces. Then, applying it, we complete the theory of the spherical functions on the space X of nondegenerate unramified hermitian forms on a $\mathfrak{p}$-adic number field. More precisely, we give an explicit expression for the spherical functions, prove theorems on the spherical Fourier transforms on the space of Schwartz-Bruhat functions on X, and parametrize of all spherical functions on X. Finally, as an application, we give explicit expressions of local densities of representations of hermitian forms.

§0. Introduction.

The aim of the present paper is to complete the theory of the spherical functions on the space of nondegenerate unramified hermitian forms on a $\mathfrak{p}$-adic number field, which has been studied in a series of papers [**H1-4**], and to apply it to calculate local densities of unramified hermitian forms.

Let k be a nonarchimedian local field of characteristic 0, $\mathcal{O}_k$ the ring of integers in k, and $*$ an involution on k. We assume that k is unramified over the fixed field k_0 by the involution $*$. For a matrix $v = (v_{ij}) \in M_{mn}(k)$, we denote by v^* the matrix $(v_{ji}^*) \in M_{nm}(k)$. For a positive integer n, we denote $G = GL_n(k)$, $K = GL_n(\mathcal{O}_k)$ and $X = \{x \in G \mid x^* = x\}$. G acts on X by $g \cdot x = gxg^*$ $(g \in G, x \in X)$.

Denote by $\mathcal{H}(G,K)$ the Hecke algebra of G with respect to K. Let $\mathcal{C}^\infty(K\backslash X)$ be the space of all K-invariant complex valued functions on X and $\mathcal{S}(K\backslash X)$ be the subspace of $\mathcal{C}^\infty(K\backslash X)$ consisting of all compactly supported functions in $\mathcal{C}^\infty(K\backslash X)$. They are $\mathcal{H}(G,K)$-modules by the action

$$(f * \Psi)(x) = \int_G f(g)\Psi(g^{-1}\cdot x)\,dg, \quad (f \in \mathcal{H}(G,K), \Psi \in \mathcal{C}^\infty(K\backslash X)).$$

A nonzero function Ψ in $\mathcal{C}^\infty(K\backslash X)$ is called *a spherical function on* X if it is an $\mathcal{H}(G,K)$-common eigenfunction. We consider a typical one introduced in [**H1**]:

$$\omega(x;s) = \int_K \prod_{i=1}^n |d_i(k\cdot x)|^{s_i}\,dk \quad (x \in X, s = (s_1,\dots,s_n) \in \boldsymbol{C}^n),$$

where $|\ |$ is the normalized absolute value on k_0. Our main results are the following:

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[1] To give an explicit formula for $\omega(x; s)$ (Theorem 1).

[2] Employing spherical functions as kernel functions, we give an $\mathcal{H}(G,K)$-module isomorphism (spherical Fourier transform)

$$\mathcal{S}(K\backslash X) \xrightarrow{\sim} \boldsymbol{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n},$$

where $z = (z_1, \dots, z_n)$ is a variable related to s by

$$\begin{cases} s_i = -z_i + z_{i+1} - 1 - \dfrac{\pi\sqrt{-1}}{\log q} & (1 \le i \le n-1) \\ s_n = -z_n + \dfrac{n-1}{2} - \dfrac{\pi\sqrt{-1}}{\log q}. \end{cases}$$

Especially, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n (Theorem 2).

[3] To give the Plancherel measure and the inversion formula for the spherical Fourier transform (Theorems 3 and 4).

[4] To parametrize all spherical functions on X (Theorem 5).

[5] As an application, to give explicit expressions for local densities and primitive local densities (Theorems 6 and 7).

Similar results have been obtained in [**HS1**] for alternating forms, which are rather easy, since $\mathcal{S}(K\backslash X)$ is generated by only one element as $\mathcal{H}(G,K)$-module. As for the present hermitian case, the rank of $\mathcal{S}(K\backslash X)$ as $\mathcal{H}(G,K)$-module is 2^n according to the size n, so it is greater than 1, and things become more complicated than the previous case. We cannot use a similar method as before, and we need to decompose spherical functions according to the orbit decomposition of X by the action of a minimal parabolic subgroup of G. Then a method of representation theory based on W. Casselman [**Cas**] is useful to investigate spherical functions on spherical homogeneous spaces. W. Casselman and J. Shalika [**CasS**] carried forward this methods to obtain explicit expressions of Whittaker functions associated to p-adic reductive groups. In a similar method, S. Kato, A. Murase and T. Sugano obtained expressions for spherical functions of certain spherical homogeneous spaces (cf. [**K2**], [**KMS**]). For the spaces they investigated, the dimension of the space of the spherical functions is 1. On the other hand, in this paper we will give an expression of spherical functions of certain spherical homogeneous spaces for which the dimension of the space of the spherical functions is not necessarily 1 (cf. Proposition 1.9).

As an application of spherical functions to local densities, we will give explicit formulae for local densities and primitive local densities along the same lines as in [**HS1**]. Formulae here are more explicit than those in alternating forms given in [**HS1**].

Here we shall explain about local densities and their relation to spherical functions.

Let $m \ge n$. For nondegenerate unramified hermitian matrices x, y of size m, n, respectively, with entries in $\mathcal{O}_k$, we consider the congruence

$$v x v^* \equiv y \pmod{\mathfrak{p}^d}.$$

Let $N_d(y,x)$ (resp. $N_d^{pr}(y,x)$) be the number of solutions (resp. primitive solutions) of the congruence above. The *local density* $\mu(y,x)$ (resp. the *primitive local density*

$\mu^{pr}(y,x)$) of integral representation of y by x is defined to be the limit

$$\mu(y,x) = \lim_{d\to\infty} \frac{N_d(y,x)}{q^{dn(2m-n)}} \quad \left(\text{resp. } \mu^{pr}(y,x) = \lim_{d\to\infty} \frac{N_d^{pr}(y,x)}{q^{dn(2m-n)}}\right).$$

In some sense, the spherical functions on X can be regarded as generating functions of local densities and primitive local densities ([**H1**, §2 Theorem], which is quoted as Lemma 3.1). And we can derive explicit expressions of local densities and primitive local densities (Theorems 6 and 7) from the explicit formulae for spherical functions on X and some properties of Hall-Littlewood symmetric polynomials.

The organization of the present paper is as follows. In Section 1, we explain a method of calculating spherical functions on spherical homogeneous spaces based on Casselmann. From Section 2 we treat the space of unramified hermitian forms. The result of Section 1 is applied in Section 2 to obtain an explicit formula for spherical functions $\omega(x;s)$ on X. We also prove in Section 2 theorems on the spherical Fourier transform on $\mathscr{S}(K\backslash X)$ and parametrization of all spherical functions. An application to local densities is given in Section 3. We need some notations and results concerning Hall-Littlewood polynomials, so we collect them as Appendix in Section 4.

Notation. Throughout this paper, we denote by k a nonarchimedian local field of characteristic 0. So k is a finite extension of the p-adic number field $\boldsymbol{Q}_p$ for some prime p. Denote by $\mathcal{O}$ the ring of integers in k, $\mathfrak{p}$ the maximal ideal in $\mathcal{O}$ and π_k a prime element of k.

As usual, we denote by $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{Z}$ and $\boldsymbol{N}$, respectively, the complex number field, the real number field, the rational number field, the ring of rational integers, and the set of natural numbers.

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§1. A formula for spherical functions on spherical homogeneous spaces.

§1.1. Let

$\boldsymbol{G}$ be an algebraic group defined over k, $G = \boldsymbol{G}(k)$

K be a compact open subgroup of G,

$\boldsymbol{P}$ be a closed subgroup defined over k and $P = \boldsymbol{P}(k)$ for which $G = KP = PK$,

dg (resp. dk) be the left invariant Haar measure on G (resp. K) normalized by $\int_K dg = 1$ (resp. $\int_K dk = 1$),

dp be the left invariant Haar measure on P normalized by $dg = dpdk$ corresponding to $G = PK$,

$\delta(q)$ be the character of P for which $d(pq) = \delta(q)^{-1} dp$ $(p, q \in P)$.

Let $\boldsymbol{X}$ be an affine algebraic variety defined over k which is a $\boldsymbol{G}$-homogeneous space, where we write the action as $g \cdot x$ $(g \in \boldsymbol{G}, x \in \boldsymbol{X})$, and set $X = \boldsymbol{X}(k)$.

Let $d_i(x)$ $(1 \le i \le r)$ be nonzero regular functions on $\boldsymbol{X}$ defined over k which are relative $\boldsymbol{P}$-invariants, namely

(1.1) $d_i(p \cdot x) = \psi_i(p) d_i(x) \quad (p \in \boldsymbol{P})$ for some rational characters ψ_i of $\boldsymbol{P}(1 \le i \le r)$.

The characters ψ_i are automatically k-rational.

Consider the following integral for $x \in X$ and $s = (s_1, \dots, s_r) \in \boldsymbol{C}^r$

$$\omega(x; s) = \int_K \prod_{i=1}^{r} |d_i(k \cdot x)|^{s_i} dk. \tag{1.2}$$

The right hand side is absolutely convergent if $\mathrm{Re}(s_i) \ge 0$, $(1 \le i \le r)$, and has an analytic continuation to a rational function in $|\pi_k|^{s_i}$ $(1 \le i \le r)$ (cf. Remark 1.1 below).

Remark 1.1. The theorems on rationality of p-adic integrals ([**Den1**, Th.3.2, 7.4], [**Den2**, Th.3.1], [**Des**, Part II (2.5.1)]) can be generalized to the following statement, which can be proved in the same manner as in [**Den1**, **2**, **Des**], see also [**Sf**, Lemma 2.1].

*Let $f_1, \dots, f_r$ be regular functions on G defined over k and U be the intersection of G and a semialgebraic subset of $M_n(k)$ in the sense of Denef, embedding $\boldsymbol{G}$ into $M_n(\bar{k})$ (cf. [**Den2**, §2], [**BS**, §2]). For $s = (s_1, \dots, s_r) \in \boldsymbol{C}^r$ with $\mathrm{Re}(s_i) \ge 0$, consider the correspondence*

$$\mathscr{S}(G) \ni \phi \mapsto \int_U \prod_{i=1}^{r} |f_i(g)|^{s_i} \phi(g)\, dg,$$

where $\mathscr{S}(G)$ is the Schwartz-Bruhat space on G. Then it can be analytically continued to a distribution on G meromorphically depending on $s_1, \dots, s_r (\in \boldsymbol{C})$.

It is easy to see that $\omega(x; s)$ is a K-invariant function on X, so it is contained in the space $\mathscr{C}^\infty(K \backslash X) = \{\Phi : X \to \boldsymbol{C} \mid \Phi(k \cdot x) = \Phi(x) \ (k \in K, x \in X)\}$.

Recall the Hecke algebra $\mathscr{H}(G, K) = \left\{ f : G \to \boldsymbol{C} \;\middle|\; \begin{array}{l} f(k_1 g k_2) = f(g) (k_1, k_2 \in K, g \in G) \\ \text{compactly supported} \end{array} \right\}$.

$\mathscr{H}(G, K)$ acts on $\mathscr{C}^\infty(K \backslash X)$ by

$$f * \Phi(x) = \int_G f(g) \Phi(g^{-1} \cdot x)\, dg \quad (f \in \mathscr{H}(G, K), \Phi \in \mathscr{C}^\infty(K \backslash X)). \tag{1.3}$$

Definition. We call a nonzero $\mathscr{H}(G, K)$-common eigenfunction in $\mathscr{C}^\infty(K \backslash X)$ *a spherical function on X.*

Proposition 1.1. *When $\omega(x; s)$ is a nonzero function on X, then $\omega(x; s)$ is a spherical function on X, namely there is a $\boldsymbol{C}$-algebra homomorphism $\lambda : \mathscr{H}(G, K) \to \boldsymbol{C}$ satisfying $(f * \omega(\ ; s))(x) = \lambda(f) \omega(x; s)$ for every $f \in \mathscr{H}(G, K)$. Indeed, $\lambda(f)$ is given by*

$$\lambda(f) = \int_G f(g) \prod_{i=1}^{r} |\psi_i(p(g))|^{-s_i} \delta(p(g))\, dg,$$

where $g = p(g)k \in G = PK$.

Proof. To simplify the notation we put

$$|d(y)|^s = \prod_{i=1}^{r} |d_i(y)|^{s_i} \quad \text{and} \quad |\psi(p)|^s = \prod_{i=1}^{r} |\psi_i(p)|^{s_i}.$$

Then, we get for every $f \in \mathscr{H}(G,K)$

$$\begin{aligned}(f * \omega(\ ;s))(x) &= \int_G f(g) \int_K |d(kg^{-1} \cdot x)|^s dk dg \\ &= \int_K \int_G f(gk) |d(g^{-1} \cdot x)|^s dg dk \\ &= \int_G f(g) |d(g^{-1} \cdot x)|^s dg \\ &= \int_K dk \int_P f(kp) |d(p^{-1}k^{-1} \cdot x)|^s d_r p \quad (d_r p = \delta(p)\, dp) \\ &= \int_K dk \int_P f(p) |\psi(p)|^{-s} |d(k^{-1} \cdot x)|^s d_r p \\ &= \int_P f(p) |\psi(p)|^{-s} \delta(p)\, dp \cdot \omega(x;s).\end{aligned}$$

Put

$$\lambda(f) = \int_P f(p) |\psi(p)|^{-s} \delta(p)\, dp = \int_G f(g) |\psi(p(g))|^{-s} \delta(p(g))\, dg,$$

where $g = p(g)k \in G = PK$.

For $f_1, f_2 \in \mathscr{H}(G,K)$, we see

$$\begin{aligned}\lambda(f * g)\omega(x;s) &= ((f_1 * f_2) * \omega(\ ;s))(x) \\ &= (f_1 * \lambda(f_2)\omega(\ ;s))(x) \\ &= \lambda(f_1)\lambda(f_2)\omega(x;s),\end{aligned}$$

hence λ is a $\boldsymbol{C}$-algebra homomorphism. □

§1.2. We note here some results from representation theory (cf. [**Cas**] or [**Car**]). Assume that $\boldsymbol{G}$ is connected and reductive, K is a special good maximal bounded subgroup of G, and $\boldsymbol{P}$ is a minimal parabolic subgroup of $\boldsymbol{G}$ defined over k.

Further let

$\boldsymbol{A}$ be a maximal split torus of $\boldsymbol{G}$ in $\boldsymbol{P}$ defined over k, $A = \boldsymbol{A}(k)$,

$\boldsymbol{W}$ be the Weyl group $(\cong N_G(A)/Z_G(A))$,

Σ be the roots of $\boldsymbol{G}$ with respect to $\boldsymbol{A}$,

Σ^+ be the set of positive roots with respect to $\boldsymbol{P}$,

B be the Iwahori subgroup in K which is compatible with P.

Let χ be an unramified and regular character of $M = Z_G(A)$, i.e. $\chi|_{M \cap K} \equiv 1$, and $\sigma\chi = \chi$ implies $\sigma = 1$ for every $\sigma \in W$. Here, taking a representative $x_\sigma \in N_G(A)$ for $\sigma \in W$ through the isomorphism $W \cong N_G(A)/M$, we set $(\sigma\chi)(m) = \chi(x_\sigma^{-1} m x_\sigma)$.

Since $M \cong P/R_u(P)$, the character χ determines a character of P, for which we use the same symbol. We denote by $I(\chi)$ the principal series representation of G induced by χ;

$$(1.4)\qquad I(\chi) = \mathrm{Ind}_P^G \chi = \left\{ \phi : G \to \boldsymbol{C} \;\middle|\; \begin{array}{l} \text{locally constant} \\ \phi(pg) = \chi\delta^{1/2}(p)\phi(g) \quad (p \in P, g \in G) \end{array} \right\},$$

with left G-module structure

$$(1.5)\qquad g \cdot \phi(x) = \phi(xg), \quad (\phi \in I(\chi),\ g, x \in G).$$

Let $\mathscr{S}(G)$ be the Schwartz-Bruhat space on G, namely the space of locally constant compactly supported functions on G, and $\mathscr{C}^\infty(G)$ be the space of locally constant functions on G. Both spaces are right G-modules by $f^g(x) = f(gx)$ $(f \in \mathscr{C}^\infty(G), g, x \in G)$ and left G-module by the same action as in (1.5). Denote by $\mathscr{D}(G)$ the space of distributions on G, namely $\mathscr{D}(G) = \mathrm{Hom}_{\boldsymbol{C}}(\mathscr{S}(G), \boldsymbol{C})$. We write $\langle T, \phi \rangle = T(\phi)$ for $T \in \mathscr{D}(G)$ and $\phi \in \mathscr{S}(G)$. We regard a function $f \in \mathscr{C}^\infty(G)$ as an element of $\mathscr{D}(G)$ by

$$\langle f, \phi \rangle = \int_G f(g)\phi(g)\, dg \quad (\phi \in \mathscr{S}(G)).$$

The right G-action on $\mathscr{C}^\infty(G)$ can be extended to $\mathscr{D}(G)$ by

$$(1.6)\qquad \langle T^g, \phi \rangle = \langle T, \phi^{g^{-1}} \rangle \quad (T \in \mathscr{D}(G),\ \phi \in \mathscr{S}(G),\ g \in G).$$

There is a surjection $\mathscr{P}_\chi : \mathscr{S}(G) \to I(\chi)$ as left G-modules given by

$$(1.7)\qquad \mathscr{P}_\chi(\phi)(x) = \int_P \chi^{-1}\delta^{1/2}(p)\phi(px)\, dp.$$

We set $\phi_{K,\chi} = \mathscr{P}_\chi(ch_K)$, where ch_K is the characteristic function of K.

Let $I(\chi^{-1})^* = \mathrm{Hom}_{\boldsymbol{C}}(I(\chi^{-1}), \boldsymbol{C})$. Then $\mathscr{P}_{\chi^{-1}} : \mathscr{S}(G) \to I(\chi^{-1})$ induces an injective G-morphism

$$\mathscr{P}_{\chi^{-1}}{}^* : I(\chi^{-1})^* \longrightarrow \mathscr{D}(G).$$

Lemma 1.2. *Put*

$$\mathscr{D}(G)_\chi = \{T \in \mathscr{D}(G) \mid T^p = \chi\delta^{1/2}(p)T \quad (p \in P)\}.$$

Then $\mathscr{P}_{\chi^{-1}}{}^*$ *gives a* G*-module isomorphism*

$$I(\chi^{-1})^* \xrightarrow{\sim} \mathscr{D}(G)_\chi.$$

Proof. First we see that the image of $\mathscr{P}_{\chi^{-1}}{}^*$ is contained in $\mathscr{D}(G)_\chi$. Take any $\psi \in I(\chi^{-1})^*$ and put $T = \mathscr{P}_{\chi^{-1}}{}^*(\psi)$. Then for $\phi \in \mathscr{S}(G)$ and $p \in P$, we have

$$\langle T^p, \phi \rangle = \langle T, \phi^{p^{-1}} \rangle = \psi(\mathscr{P}_{\chi^{-1}}(\phi^{p^{-1}})).$$

Since

$$\begin{aligned}\mathscr{P}_{\chi^{-1}}(\phi^{p^{-1}})(g) &= \int_P \chi\delta^{1/2}(q)\phi(p^{-1}qg)\,dq \\ &= \chi\delta^{1/2}(p)\int_P \chi\delta^{1/2}(q)\phi(qg)\,dq \\ &= \chi\delta^{1/2}(p)\mathscr{P}_{\chi^{-1}}(\phi)(g),\end{aligned}$$

we obtain

$$\langle T^p, \phi\rangle = \chi\delta^{1/2}(p)\langle T, \phi\rangle.$$

Now we have to prove that every element of $\mathscr{D}(G)_\chi$ comes from $I(\chi^{-1})^*$. To see this, it is enough to show that every $T \in \mathscr{D}(G)_\chi$ kills Ker $\mathscr{P}_{\chi^{-1}}$. Take any $\phi \in \operatorname{Ker}\mathscr{P}_{\chi^{-1}}$ and take an open compact subgroup K_0 of G for which

$$\phi(kgk') = \phi(g) \quad (k, k' \in K_0,\ g \in G).$$

Now we see that

$$0 \equiv \int_P \chi\delta^{1/2}(p)\phi(pg)\,dp = \operatorname{vol}(P_0)\sum_{p \in P_0\backslash P} \chi\delta^{1/2}(p)\phi(pg), \tag{1.8}$$

where $P_0 = K_0 \cap P$. Since $G = PK$, there are only a finite number of representatives $\{g_1, \dots, g_n\}$ of $P\backslash G/K_0$. Denote by ϕ_i the function defined by

$$\phi_i(g) = \begin{cases} \phi(g) & \text{if } g \in Pg_iK_0 \\ 0 & \text{otherwise.}\end{cases}$$

Then, it is obvious that ϕ_i's are in $\operatorname{Ker}\mathscr{P}_{\chi^{-1}}$ and $\phi = \phi_1 + \cdots + \phi_n$. Therefore we may assume that the support of ϕ is contained in some double coset Pg_0K_0. Since the support of ϕ is compact, $P \cap \operatorname{supp}(\phi)g_0^{-1}$ consists of only a finite number of P_0-cosets, say $\{P_0p_1, \dots, P_0p_r\}$. Then by (1.8), we have

$$\sum_{i=1}^{r} \chi\delta^{1/2}(p_i)\phi(p_ig_0) = 0. \tag{1.9}$$

We will show that

$$\sum_{i=1}^{r} \chi\delta^{1/2}(p_i)\phi^{p_i} \equiv 0. \tag{1.10}$$

If $g \notin Pg_0K_0$, clearly we have $\sum_{i=1}^{r}\chi\delta^{1/2}(p_i)\phi(p_ig) = 0$. Let $g \in Pg_0K_0$ and write $g = pg_0k$ $(p \in P, k \in K_0)$. Then $\phi(p_ig) = \phi(p_ipg_0k) = \phi(p_ipg_0)$. Hence, if $\phi(p_ig) \neq 0$, then p_ip belongs to some P_0p_j $(1 \le j \le r)$, and there exists a permutation $\sigma \in S_r$ and $p_i' \in P_0$ such that

$$p_ip = p_i'p_{\sigma(i)} \quad (1 \le i \le r).$$

Since χ and δ are trivial on P_0, we obtain by (1.9) that

$$\begin{aligned}\sum_{i=1}^{r}\chi\delta^{1/2}(p_i)\phi(p_ig) &= \sum_{i=1}^{r}\chi\delta^{1/2}(p_i'p_{\sigma(i)}p^{-1})\phi(p_i'p_{\sigma(i)}g_0)\\ &= \chi\delta^{1/2}(p^{-1})\sum_{i=1}^{r}\chi\delta^{1/2}(p_{\sigma(i)})\phi(p_{\sigma(i)}g_0)\\ &= 0.\end{aligned}$$

Thus we have proved (1.10). Applying $T\in\mathscr{D}(G)_\chi$ to the both side of (1.10), we get

$$0=\sum_{i=1}^{r}\chi\delta^{1/2}(p_i)\langle T,\phi^{p_i}\rangle=\sum_{i=1}^{r}\chi\delta^{1/2}(p_i)\langle T^{p_i^{-1}},\phi\rangle=\sum_{i=1}^{r}\langle T,\phi\rangle=r\langle T,\phi\rangle.$$

This implies that $\langle T,\phi\rangle=0$ for any $\phi\in\operatorname{Ker}\mathscr{P}_{\chi^{-1}}$, which completes a proof. □

By Lemma 1.2, we see that the representation space of the smooth dual of $I(\chi^{-1})$ is

$$\mathscr{C}^\infty(G)\cap I(\chi^{-1})^*\cong\mathscr{C}^\infty(G)\cap\mathscr{D}(G)_\chi=I(\chi).$$

Hence we can calculate the pairing on $I(\chi)\times I(\chi^{-1})$ in the following way. Let $f_1\in I(\chi)$, $f_2\in I(\chi^{-1})$ and take $\phi\in\mathscr{S}(G)$ for which $\mathscr{P}_{\chi^{-1}}(\phi)=f_2$. Then

$$\begin{aligned}\langle f_1,f_2\rangle &= \langle(\mathscr{P}_{\chi^{-1}}{}^*)^{-1}f_1,f_2\rangle_{I(\chi^{-1})^*\times I(\chi^{-1})}=\langle f_1,\phi\rangle_{\mathscr{D}(G)\times\mathscr{S}(G)}\\ &= \int_G f_1(g)\phi(g)\,dg=\int_K dk\int_P f_1(pk)\phi(pk)\,dp\\ &= \int_K f_1(k)\,dk\int_P\chi\delta^{1/2}(p)\phi(pk)\,dp=\int_K f_1(k)\mathscr{P}_{\chi^{-1}}(\phi)(k)\,dk=\int_K f_1(k)f_2(k)\,dk.\end{aligned}$$

Hence, we get the next corollary.

Corollary 1.3. *The representation $I(\chi)$ is isomorphic to the contragredient of $I(\chi^{-1})$ and the pairing on $I(\chi)\times I(\chi^{-1})$ is given by*

$$\langle f_1,f_2\rangle=\int_K f_1(k)f_2(k)\,dk\quad(f_1\in I(\chi),\ f_2\in I(\chi^{-1})).$$

For $\sigma\in W$, there is a unique G-morphism (intertwining operator) $T_\sigma: I(\chi)\to I(\sigma\chi)$ which satisfies

$$T_\sigma(\phi_{K,\chi})=c_\sigma(\chi)\phi_{K,\sigma\chi},\tag{1.11}$$

where

$$\begin{aligned}c_\sigma(\chi)&=\prod_{\alpha\in\Sigma^+,\sigma\alpha<0}c_\alpha(\chi),\\ c_\alpha(\chi)&=\frac{(1-q_{\alpha/2}^{-1/2}q_\alpha^{-1}\chi(a_\alpha))(1+q_{\alpha/2}^{-1/2}\chi(a_\alpha))}{1-\chi(a_\alpha)^2}.\end{aligned}\tag{1.12}$$

For the definition of $a_\alpha \in A$ and the numbers q_α, $q_{\alpha/2}$ ($\alpha \in \Sigma$), see [**Cas**, (9), Remark 1.1 and (12)]. If G is split, each $q_\alpha = \#(\mathcal{O}/\mathfrak{p})(= q_k, \text{say})$ and each $q_{\alpha/2} = 1$, and so

$$c_\alpha(\chi) = \frac{1 - q_k^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}. \tag{1.13}$$

The following properties on T_σ are known.

PROPOSITION 1.4 ([**Cas**, 3.5]).
(i) *The operator* T_σ *is an isomorphism if and only if* $c_\sigma(\chi)c_{\sigma^{-1}}(\sigma\chi) \neq 0$.
(ii) $I(\chi)$ *is irreducible if and only if* $c_{w_l}(\chi)c_{w_l}(w_l\chi) \neq 0$, *where* w_l *is the longest element of* W.

By $T_{\sigma^{-1}} : I(\sigma\chi^{-1}) \to I(\chi^{-1})$ we get the adjoint G-morphism $T_{\sigma^{-1}}{}^* : \mathcal{D}(G)_\chi \to \mathcal{D}(G)_{\sigma\chi}$, which induces a G-morphism from $I(\chi)$ to $I(\sigma\chi)$ by Lemma 1.2 and Corollary 1.3.

LEMMA 1.5. *Let* $\sigma \in W$. *If* $c_\sigma(\chi) \neq 0$, *then*

$$T_{\sigma^{-1}}{}^* = \frac{c_{\sigma^{-1}}(\sigma\chi^{-1})}{c_\sigma(\chi)} T_\sigma.$$

PROOF. Since we have assumed that χ is regular, $T_{\sigma^{-1}}{}^*$ is a constant multiple of T_σ (cf. [**Car**, Cor.3.3]). We get by (1.11) and Corollary 1.3,

$$\langle T_\sigma(\phi_{K,\chi}), \phi_{K,\sigma\chi^{-1}} \rangle = c_\sigma(\chi) \int_K \phi_{K,\sigma\chi}(k)\phi_{K,\sigma\chi^{-1}}(k)\, dk = c_\sigma(\chi).$$

On the other hand, we get also

$$\langle T_{\sigma^{-1}}{}^*(\phi_{K,\chi}), \phi_{K,\sigma\chi^{-1}} \rangle = \langle \phi_{K,\chi}, T_{\sigma^{-1}}(\phi_{K,\sigma\chi^{-1}}) \rangle = c_{\sigma^{-1}}(\sigma\chi^{-1}).$$

The result follows from this. □

As an immediate consequence of this lemma, we have the next proposition.

PROPOSITION 1.6. *Let* $\sigma \in W$. *If* $c_\sigma(\chi)c_{\sigma^{-1}}(\sigma\chi^{-1}) \neq 0$, *then*

$$\tilde{T}_\sigma = \frac{c_\sigma(\chi)}{c_{\sigma^{-1}}(\sigma\chi^{-1})} T_{\sigma^{-1}}{}^* : \mathcal{D}(G)_\chi \to \mathcal{D}(G)_{\sigma\chi}$$

is an extension of $T_\sigma : I(\chi) \to I(\sigma\chi)$.

For a subgroup U of G, we denote $I(\chi)^U$ the fixed subspace by the action of U.
There exists a basis $\{f_{\sigma,\chi} \mid \sigma \in W\}$ of $I(\chi)^B$ which satisfies

$$T_\sigma(f_{\tau,\chi})(1) = \delta_{\sigma,\tau}, \tag{1.14}$$

where $\delta_{\sigma,\tau}$ is the Kronecker delta ([**Cas**, p. 402]). For a compact open subgroup U of G, let $\mathcal{P}_U$ be the operator on $\mathcal{D}(G)$ defined by

$$\langle \mathcal{P}_U(T), \phi \rangle = \int_U \langle u \cdot T, \phi \rangle\, du = \int_U \langle T, u^{-1} \cdot \phi \rangle\, du,$$

where du is the Haar measure on U normalized by $\int_U du = 1$. Then it is known ([**Cas**,

p. 403–4]) that

$$\mathscr{P}_K(f_{\sigma,\chi})(1) = \frac{\gamma(\sigma\chi)}{Q \cdot c_\sigma(\chi)}, \tag{1.15}$$

where

$$\begin{aligned} \gamma(\chi) &= c_{w_l}(w_l\chi) = \prod_{\alpha \in \Sigma^+} \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} \chi(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2} \chi(a_\alpha)^{-1})}{1 - \chi(a_\alpha)^{-2}}, \\ Q &= \sum_{\sigma \in W} [B\sigma B : B]^{-1}. \end{aligned} \tag{1.16}$$

The following property will be used in the next subsection.

Proposition 1.7. *Let U be a compact open subgroup of G and $\sigma \in W$ and assume that $c_\sigma(\chi) c_{\sigma^{-1}}(\sigma\chi) \neq 0$. Then $\mathscr{P}_U \circ \tilde{T}_\sigma = T_\sigma \circ \mathscr{P}_U$.*

Proof. We will show the commutativity of the following diagram

$$\begin{array}{ccccccc} I(\chi^{-1})^* & \cong & \mathscr{D}(G)_\chi & \xrightarrow{\tilde{T}_\sigma} & \mathscr{D}(G)_{\sigma\chi} & \cong & I(\sigma\chi^{-1})^* \\ & & \downarrow{\scriptstyle \mathscr{P}_U} & & \downarrow{\scriptstyle \mathscr{P}_U} & & \\ & & (\mathscr{D}(D)_\chi)^U & & (\mathscr{D}(G)_{\sigma\chi})^U & & \\ & & \| & & \| & & \\ & & I(\chi)^U & \xrightarrow{T_\sigma} & I(\sigma\chi)^U. & & \end{array}$$

For simplicity, we set $c = c_\sigma(\chi)/c_{\sigma^{-1}}(\sigma\chi^{-1})$. For every $S \in \mathscr{D}(G)_\chi$ and $\phi \in \mathscr{S}(G)$, we get

$$\begin{aligned} \langle \mathscr{P}_U \circ \tilde{T}_\sigma(S), \phi \rangle &= \int_U \tilde{T}_\sigma(S)(u^{-1} \cdot \phi)\, du = c \int_U S(T_{\sigma^{-1}}(u^{-1} \cdot \phi))\, du \\ &= c \int_U S(u^{-1} \cdot T_{\sigma^{-1}}(\phi))\, du = c\mathscr{P}_U(S)(T_{\sigma^{-1}}(\phi)) \\ &= \tilde{T}_\sigma \circ \mathscr{P}_U(S)(\phi) = T_\sigma \circ \mathscr{P}_U(S)(\phi). \end{aligned}$$

This completes a proof. □

§1.3. In general, the group $\mathfrak{X}(\boldsymbol{P})$ of k-rational characters of $\boldsymbol{P}$ corresponding to relative invariants is a free abelian group of finite rank. When $\{\psi_i \mid 1 \leq i \leq r\}$ forms a basis of $\mathfrak{X}(\boldsymbol{P})$, the corresponding set $\{d_i(x) \mid 1 \leq i \leq r\}$ is called a set of basic relative $\boldsymbol{P}$-invariants defined over k.

We assume that

(A1) $\boldsymbol{X}$ has an open $\boldsymbol{P}$-orbit $\boldsymbol{X}'$ and finite P-orbit decomposition $\boldsymbol{X}'(k) = \bigsqcup_u X_u$, $\{d_i(x) \mid 1 \leq i \leq r\}$ forms a set of basic relative $\boldsymbol{P}$-invariants defined over k, and k-rank$(\boldsymbol{P}) = \text{rank}(\mathfrak{X}(\boldsymbol{P}))$.

Hereafter we fix a G-orbit Ω in $\boldsymbol{X}(k)$ and denote by $\mathscr{U}$ the set of indices u for which $X_u \subset \Omega$.

For $x \in \Omega$, $g \in G$ and $s \in \boldsymbol{C}^r$ with $\mathrm{Re}(s_i) \geq 0$, put

$$
(1.17)\qquad
\begin{aligned}
d_u^s(g; x) &= ch_{X_u}(g \cdot x) \prod_{i=1}^{r} |d_i(g \cdot x)|^{s_i}, \\
\widetilde{d_u^s}(g; x) &= \mathscr{P}_B(d_u^s(\ ; x))(g) = \int_B d_u^s(gb; x)\, db.
\end{aligned}
$$

For general $s \in \boldsymbol{C}^r$, distributions $d_u^s(g; x)$ and $\widetilde{d_u^s}(g; x)$ on G are defined by analytic continuation (cf. Remark 1.1). We set also

$$
(1.18)\qquad \omega_u^s(x) = \int_K d_u^s(k; x)\, dk = \int_K \widetilde{d_u^s}(k; x)\, dk = \mathscr{P}_K(\widetilde{d_u^s}(\ ; x))(1),
$$

then we have

$$
(1.19)\qquad \omega(x; s) = \sum_u \omega_u^s(x).
$$

We note here that $\{\omega_u^s(x) \mid u \in \mathscr{U}\}$ are linearly independent functions for generic s by the injectivity of Poisson integral (cf. [**K1**]).

We define a character $\chi = \chi_s$ of P by

$$
\chi(p) = \prod_{i=1}^{r} |\psi_i(p)|^{s_i} \delta^{-1/2}(p) \qquad (\text{cf. } (1.1)).
$$

Then it satisfies

$$
d_u^s(pg; x) = \chi\delta^{1/2}(p) d_u^s(g; x)
$$

and we see that $d_u^s(g; x) \in \mathscr{D}(G)_\chi$ and $\widetilde{d_u^s}(g; x) \in I(\chi)^B$. Since χ is uniquely determined by s, we write $d_u^\chi(g; x) = d_u^s(g; x)$, $\widetilde{d_u^\chi}(g; x) = \widetilde{d_u^s}(g; x)$ and $\omega_u^\chi(x) = \omega_u^s(x)$.

Setting $\boldsymbol{H} = \boldsymbol{G}_x$ for $x \in \boldsymbol{X}(k)$, we have $\boldsymbol{X} \cong \boldsymbol{G}/\boldsymbol{H}$, where we consider the compatible action of $\boldsymbol{P} \times \boldsymbol{H}$ on $\boldsymbol{G}$ with the original action of $\boldsymbol{P}$ on $\boldsymbol{X}$. Then by the assumption (A1), there exists an open $(\boldsymbol{P} \times \boldsymbol{H})$-orbit $\boldsymbol{Y}$. For each $g \in \boldsymbol{G}$, denote $\boldsymbol{P}_{(g)}$ the image of the stabilizer $(\boldsymbol{P} \times \boldsymbol{H})_g$ by the projection $\boldsymbol{P} \times \boldsymbol{H} \to \boldsymbol{P}$.

We assume that

(A2)
(i) $\boldsymbol{G} \backslash \boldsymbol{Y}$ decomposes into a finite *number of* $\boldsymbol{P}$-orbits,
(ii) for any $g \in \boldsymbol{G} \backslash \boldsymbol{Y}$, there exists $\psi \in \mathfrak{X}(\boldsymbol{P})$ whose restriction to the identity component of $\boldsymbol{P}_{(g)}$ is nontrivial.

We note here that the condition (A2) is satisfied for every $x \in \boldsymbol{X}(k)$, if it is satisfied for some $x_0 \in \boldsymbol{X}(k)$.

Lemma 1.8. *For each $x \in \Omega$ and generic s, the set $\{d_u^{\sigma\chi}(g; x) \mid u \in \mathscr{U}\}$ forms a basis for $(\mathscr{D}(G)_{\sigma\chi})^{G_x}$ for any $\sigma \in W$.*

Proof. Let S be a $(\boldsymbol{P}\times\boldsymbol{H})(k)$-orbit not contained in $\boldsymbol{Y}$ and let $T\in(\mathscr{D}(D)_{\sigma\chi})^{G_x}$ whose support is contained in S. Then it is known ([**Sf**, Lemma 2.3]) that, under the assumption (A2), s must satisfy a finite number of linear relations of type $\sum_{i=1}^{r} m_i s_i - \alpha \in (2\pi\sqrt{-1}/\log|\pi_k|)\boldsymbol{Z}$ with $m_i\in\boldsymbol{Z}$, $\alpha\in\boldsymbol{C}$. Hence, if we avoid such s's, the support of nonzero $T\in(\mathscr{D}(G)_{\sigma\chi})^{G_x}$ is contained in $\boldsymbol{Y}(k)$, and those distributions are spanned by the linearly independent set $\{d_u^\chi(g;x)\,|\,u\in\mathscr{U}\}$. Since W is a finite group, we obtain the result. □

We assume that s is generic for which Lemma 1.8 holds, s is neither a pole nor a zero of $\omega_u^s(x)(u\in\mathscr{U})$, $\chi=\chi_s$ is regular, and $c_\sigma(\chi)c_{\sigma^{-1}}(\sigma\chi^{-1})\neq 0$ for any $\sigma\in W$.

For $u\in\mathscr{U}$, write

$$\widetilde{d_u^\chi}(g;x)=\sum_{\sigma\in W} a_{u,\sigma}(x;\chi)f_{\sigma,\chi}, \tag{1.20}$$

where $\{f_{\sigma,\chi}\,|\,\sigma\in W\}$ is the basis of $I(\chi)^B$ (cf. (1.14)). Then we obtain

$$\begin{aligned} a_{u,\sigma}(x;\chi) &= T_\sigma(\widetilde{d_u^\chi}(\ ;x))(1) \\ &= T_\sigma\circ\mathscr{P}_B(d_u^\chi(\ ;x))(1) \\ &= \mathscr{P}_B\circ\tilde{T}_\sigma(d_u^\chi(\ ;x))(1) \quad \text{(by Proposition 1.7).}\end{aligned} \tag{1.21}$$

By Lemma 1.8, we see that there exists an invertible matrix $A_\sigma(\chi)$ satisfying

$$\big(\widetilde{T}_\sigma(d_u^\chi(\ ;x))\big)_{u\in\mathscr{U}} = A_\sigma(\chi)\big(d_u^{\sigma\chi}(\ ;x)\big)_{u\in\mathscr{U}}. \tag{1.22}$$

Here $A_\sigma(\chi)$ depends only on the G-orbit Ω containing x, since $\widetilde{T}_\sigma$ is G-morphism.

Now we obtain

$$\begin{aligned}\big(\omega_u^\chi(x)\big)_{u\in\mathscr{U}} &= \sum_{\sigma\in W}\mathscr{P}_K(f_{\sigma,\chi})(1)\big(a_{u,\sigma}(x;s)\big)_u \quad \text{(by (1.18) and (1.20))}\\ &= \sum_{\sigma\in W}\frac{\gamma(\sigma\chi)}{Q\cdot c_\sigma(\chi)}\big(\mathscr{P}_B\circ\tilde{T}_\sigma(d_u^\chi(\ ;x))(1)\big)_u \quad \text{(by (1.15) and (1.21))}\\ &= \frac{1}{Q}\sum_{\sigma\in W}\frac{\gamma(\sigma\chi)}{c_\sigma(\chi)}\cdot A_\sigma(\chi)\big(\mathscr{P}_B(d_u^{\sigma\chi}(\ ;x))(1)\big)_u \quad \text{(by (1.22)).}\end{aligned}$$

Set $B_\sigma(\chi)=c_\sigma(\chi)^{-1}A_\sigma(\chi)$. Then

$$\big(\omega_u^\chi(x)\big)_u = \frac{1}{Q}\sum_{\sigma\in W}\gamma(\sigma\chi)\cdot B_\sigma(\chi)\big(\mathscr{P}_B(d_u^{\sigma\chi}(\ ;x))(1)\big)_u,$$

and invertible matrices $\{B_\sigma(\chi)\,|\,\sigma\in W\}$ satisfy cocycle relations

$$B_{\sigma\tau}(\chi)=B_\tau(\chi)B_\sigma(\tau\chi)\quad(\sigma,\tau\in W). \tag{1.23}$$

Hence we obtain, for each $\tau\in W$

$$\begin{aligned}\big(\omega_u^{\tau\chi}(x)\big)_{u\in\mathscr{U}} &= \frac{1}{Q}\sum_{\sigma\in W}\gamma(\sigma\tau\chi)B_\sigma(\tau\chi)\big(\mathscr{P}_B(d_u^{\sigma\tau\chi}(\ ;x))(1)\big)_u\\ &= B_\tau(\chi)^{-1}\big(\omega_u^\chi(x)\big)_u \quad \text{(by (1.23)).}\end{aligned}$$

The matrix $B_\tau(\chi)$ is determined by the above relation, since $\{\omega_u^{\tau\chi}(x) \mid u \in \mathscr{U}\}$ are linearly independent for each $\tau \in W$.

We shall summarize the above argument as a proposition.

Proposition 1.9. *For $x \in \Omega$, generic s and $\chi = \chi_s$, we have*

$$\left(\omega_u^\chi(x)\right)_{u \in \mathscr{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma\chi) B_\sigma(\chi) \left(\mathscr{P}_B(d_u^{\sigma\chi}(\ ;x))(1)\right)_{u \in \mathscr{U}},$$

where

$$Q = \sum_{\sigma \in W} [B\sigma B : B]^{-1},$$

$$\gamma(\chi) = \prod_{\alpha \in \Sigma^+} \frac{(1 - q_{\alpha/2}^{-1/2} q_\alpha^{-1} \chi(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2} \chi(a_\alpha)^{-1})}{1 - \chi(a_\alpha)^{-2}} \left(= \prod_{\alpha \in \Sigma^+} \frac{1 - q_k^{-1} \chi(a_\alpha)^{-1}}{1 - \chi(a_\alpha)^{-1}} \ \textit{if } G \textit{ is split} \right),$$

and $B_\sigma(\chi)$ is the invertible matrix determined by

$$\left(\omega_u^\chi(x)\right)_{u \in \mathscr{U}} = B_\sigma(\chi) \left(\omega_u^{\sigma\chi}(x)\right)_{u \in \mathscr{U}}.$$

Remark 1.2. We can apply Proposition 1.9 to the spaces of hermitian forms and symmetric forms.

§2. Spherical functions on the space of hermitian forms.

§2.1. Hereafter we fix an involution $*$ on k and assume that k is unramified over the fixed field k_0 by $*$. We take a prime element π of k in k_0 and let $q^2 = \#(\mathcal{O}/\pi\mathcal{O})$. Denote by $|\ |$ the absolute value on k_0 normalized by $|\pi| = q^{-1}$.

For a matrix $A = (a_{ij})$ in $M_{mn}(k)$, we denote by A^* the matrix (a_{ji}^*) in $M_{nm}(k)$.

The group $G = GL_n(k)$ can be regarded as the k_0-rational points of the algebraic group $\boldsymbol{G} = R_{k/k_0}(GL_n)$ defined over k_0, where R_{k/k_0} is the restriction functor of the base field (cf. §2.2 below). Let $\boldsymbol{P}$ be the parabolic subgroup of $\boldsymbol{G}$ defined over k_0 for which $P = \boldsymbol{P}(k_0) = \{g \in G \mid g_{ij} = 0 \text{ unless } i \geq j\}$. Then

$$\delta(p) = \prod_{i=1}^n |p_i|^{-2(n-2i+1)} \quad \text{for } p = \begin{pmatrix} p_1 & & \mathbf{0} \\ & \ddots & \\ * & & p_n \end{pmatrix} \in P.$$

Further set

$$K = GL_n(\mathcal{O}) \quad \text{and} \quad B = \{(b_{ij}) \in K \mid b_{ii} \in \mathcal{O}^\times, \text{ for } \forall i, \ b_{ij} \in \mathfrak{p} \text{ if } i < j\}.$$

Let ι be the involution on G given by $\iota(x) = (x^*)^{-1}$. Then ι can be extended to an involution on $\boldsymbol{G}$ defined over k_0, which we denote by the same symbol ι. Now set

$$\boldsymbol{X} = \boldsymbol{X}_n = \{x \in \boldsymbol{G} \mid \iota(x) = x^{-1}\},$$

then we see that

$$X = \boldsymbol{X}(k_0) = \{x \in G \mid x^* = x\}.$$

The group $\boldsymbol{G}$ acts on $\boldsymbol{X}$ by $g \cdot x = gx\iota(g)^{-1}$ $(g \in \boldsymbol{G}, x \in \boldsymbol{X})$.

For $g \in G$, let $d_i(g)$ be the determinant of the upper left i by i block of g. We shall consider the following integral:

$$\omega(x;s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i} dk \quad (x \in X,\ s = (s_1, \dots, s_n) \in \boldsymbol{C}^n). \tag{2.1}$$

This has been introduced in [**H1**], where $\omega(x;s) = \zeta(x;s/2)$ in the notation of [**H1**]. The right hand side of (2.1) is absolutely convergent if $\mathrm{Re}(s_1), \dots, \mathrm{Re}(s_{n-1})$ are non negative, and has an analytic continuation to a rational function in $q^{s_1}, \dots, q^{s_n}$.

We introduce a new variable $z \in \boldsymbol{C}^n$ which is related to s by

$$\begin{cases} s_i = -z_i + z_{i+1} - 1 - \dfrac{\pi\sqrt{-1}}{\log q} & (1 \le i \le n-1) \\ s_n = -z_n + \dfrac{n-1}{2} - \dfrac{\pi\sqrt{-1}}{\log q}, \end{cases} \tag{2.2}$$

and we write $\omega(x;z) = \omega(x;s)$.

We note that the results in [**H1-3**] are still valid for $p = 2$, since we have assumed that k/k_0 is unramified. We have to take notice that $\omega(x;z) = \zeta(x;z/2) = \zeta(x;s/2)$ in the previous notation. Especially we have known the following:

[1] $\omega(x;s)$ is an $\mathscr{H}(G,K)$-common eigenfunction ([**H1**]), more precisely

$$(f * \omega(\ ;z))(x) = \tilde{f}(z)\omega(x;z), \quad f \in \mathscr{H}(G,K),$$

where

$$\begin{aligned} \mathscr{H}(G,K) &\xrightarrow{\sim} \boldsymbol{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^{S_n} \\ f &\longmapsto \tilde{f}(z) = \int_G f(g)\psi_z(g)\, dg. \end{aligned} \tag{2.3}$$

Here the right hand side is the ring of symmetric Laurent polynomials in $q^{2z_1}, \dots, q^{2z_n}$ with complex coefficients and

$$\psi_z(g) = \prod_{i=1}^n |p_i|^{2z_{n-i+1}-(n-2i+1)}, \quad g = k\begin{pmatrix} p_1 & & * \\ & \ddots & \\ \mathbf{0} & & p_n \end{pmatrix} \text{ with } k \in K.$$

[2] As for functional equations and the location of poles of $\omega(x;z)$, we have ([**H3**, §2])

$$\prod_{1 \le i < j \le n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i - 1}} \cdot \omega(x;z) \in \boldsymbol{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}. \tag{2.4}$$

By the property [1], $\omega(\boldsymbol{x};\boldsymbol{z})$ is a spherical function on X. We will give an explicit formula for $\omega(\boldsymbol{x};\boldsymbol{z})$ in Theorem 1. Since $\omega(\boldsymbol{x};\boldsymbol{z})$ is K-invariant, it suffices to consider the values at representatives of K-orbits in X. A complete set of representatives of K-orbits in X is given as follows (cf. [**Ja**]):

$$\left\{ \pi^{\lambda} = \begin{pmatrix} \pi^{\lambda_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \pi^{\lambda_n} \end{pmatrix} \;\middle|\; \lambda \in \Lambda_n \right\},$$

where

$$\Lambda_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \boldsymbol{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}.$$

§2.2. We will apply the results in §1 to this $\boldsymbol{X}$.

Denote by $\bar{k}$ the algebraic closure of k. Let $k = k_0(u)$ with $u^2 \in k_0$, and consider the map

$$\rho : k \longrightarrow M_2(k_0)$$
$$a + bu \longmapsto \begin{pmatrix} a & bu^2 \\ b & a \end{pmatrix}.$$

Then we can identify k with the image of ρ, and a realization of $\boldsymbol{G}$ and $\boldsymbol{X}$ are given by

$$\boldsymbol{G} = \left\{ (g_{ij})_{1 \leq i,j \leq n} \in GL_{2n}(\bar{k}) \;\middle|\; g_{ij} = \begin{pmatrix} a_{ij} & b_{ij}u^2 \\ b_{ij} & a_{ij} \end{pmatrix} \in M_2(\bar{k}) \quad (\forall i, j) \right\}$$

$$\boldsymbol{X} = \left\{ (x_{ij})_{1 \leq i,j \leq n} \in \boldsymbol{G} \;\middle|\; x_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_{ji} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\forall i, j) \right\}.$$

For $g \in \boldsymbol{G}$, let $\widehat{d_i}(g)$ be the determinant of the upper left $2i$ by $2i$ block of g.

First, we show that the assumption (A1) is satisfied. The set $\boldsymbol{X}' = \{x \in \boldsymbol{X} \mid \prod_{i=1}^n \widehat{d_i}(x) \neq 0\}$ is a Zariski open $\boldsymbol{P}$-orbit,

$$X' = \boldsymbol{X}'(k_0) = \left\{ x \in X \;\middle|\; \prod_{i=1}^n d_i(x) \neq 0 \right\},$$

and the P-orbit decomposition of X' is

$$X' = \bigsqcup_{u \in \{0,1\}^n} X_u \quad \text{with } X_u = \{x \in X' \mid v_p(d_i(x)) \equiv u_1 + \cdots + u_i \pmod 2\},$$

where $v_\pi(\;)$ is the additive value on k_0. There are two G-orbits Ω_0, Ω_1 in X with $\Omega_j = \{x \in X \mid v_\pi(\det x) \equiv j \pmod 2\}$, and the corresponding set $\mathscr{U}_j$ of indices is given by $\mathscr{U}_j = \{u \in \{0,1\}^n \mid \sum_{i=1}^n u_i \equiv j \pmod 2\}$.

The corresponding character χ of P to $d^s(g; x)$ is given by

$$\chi(p) = \prod_{i=1}^n |p_i|^{-2z_i} \quad p = \begin{pmatrix} p_1 & & \mathbf{0} \\ & \ddots & \\ * & & p_n \end{pmatrix},$$

and the Weyl group $W = S_n$ acts on χ by permuting $z_1, \ldots, z_n$. We see that

$$c_\sigma(z) = c_\sigma(\chi) = \prod_{\substack{i>j \\ \sigma(i)<\sigma(j)}} \frac{q^{2z_j} - q^{2z_i - 2}}{q^{2z_j} - q^{2z_i}} \quad (\sigma \in S_n).$$

We see that $\boldsymbol{X} \cong \boldsymbol{G}/\boldsymbol{H}$ with $\boldsymbol{H} = \{h \in \boldsymbol{G} \mid \iota(h) = h\}$. We consider the following action on $\boldsymbol{G}$ of $\boldsymbol{P} \times \boldsymbol{H}$

$$(p, h) \cdot g = pgh^{-1} \quad (p \in \boldsymbol{P},\ h \in \boldsymbol{H},\ g \in \boldsymbol{G}),$$

then $\boldsymbol{Y} = \boldsymbol{Y}_n = \{g \in \boldsymbol{G} \mid g\iota(g)^{-1} \in \boldsymbol{X}'\}$ is an open $(\boldsymbol{P} \times \boldsymbol{H})$-orbit.

Now we show that the assumption (A2) is satisfied by induction on n. The case $n = 1$ is clear, so let $n > 1$. Take any $g \in \boldsymbol{G}_n \backslash \boldsymbol{Y}_n$. If $\widehat{d_1}(g\iota(g)^{-1}) \neq 0$, then $\boldsymbol{PgH}$ contains an element g_1 of the form

$$g_1 = \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & g_0 \end{array}\right), \quad g_0 \in \boldsymbol{G}_{n-1} \backslash \boldsymbol{Y}_{n-1}.$$

It is easy to see that

$$(\boldsymbol{P} \times \boldsymbol{H})_{g_1} \supset \left\{ \left(\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & x \end{array}\right), \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & y \end{array}\right) \right) \;\middle|\; x g_0 y^{-1} = g_0 \right\},$$

hence it reduces to the induction hypothesis.

Next, let $\widehat{d_1}(gg^*) = 0$, then $\boldsymbol{PgH}$ contains an element g_1 satisfying for some l with $2 \leq l \leq n$

$$(i,j)\text{-block of } g_1 g_1^* = \begin{cases} 1_2 & \text{if } (i,j) = (1,l), (l,1) \\ 0_2 & \text{if } i \text{ or } j = 1, l \text{ but } (i,j) \neq (1,l), (l,1). \end{cases}$$

Then it is easy to see that

$$\boldsymbol{P}_{(g_1)} \supset \left\{ \begin{pmatrix} \hat{a} & & & \mathbf{0} \\ & I_{2(l-2)} & & \\ & & \hat{a}^{-1} & \\ \mathbf{0} & & & I_{2(n-l)} \end{pmatrix} \;\middle|\; \hat{a} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in GL_2(\bar{k}) \right\}.$$

Hence we stand the position to apply Proposition 1.9 to X.

Lemma 2.1. *For $x \in \Omega_j$ $(j = 0, 1)$, generic s and $\chi = \chi_s$, we have*

$$\left(\omega_u^s(x)\right)_u = \prod_{i=1}^n \frac{1 - q^{-2}}{1 - q^{-2i}} \sum_{\sigma \in S_n} \gamma(\sigma z) B_\sigma(z) \left(\mathscr{P}_B(d_u^{\sigma s}(\ ; x))(1)\right)_u,$$

where u runs over $\mathscr{U}_j$ and

$$\gamma(z) = \prod_{1 \leq j < i \leq n} \frac{q^{2z_i} - q^{2z_j - 2}}{q^{2z_i} - q^{2z_j}},$$

$B_\sigma(z) \in GL_{2^{n-1}}(\boldsymbol{C}(q^{z_1}, \ldots, q^{z_n}))$ *is determined by* $\left(\omega_u^z(x)\right)_u = B_\sigma(z)\left(\omega_u^{\sigma z}(x)\right)_u$.

LEMMA 2.2. *Set* $J = \begin{pmatrix} \mathbf{0} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \mathbf{0} \end{pmatrix} \in K$. *Then for each* $\lambda \in \Lambda_n$, *we get the following identity*:

$$\mathscr{P}_B(d_u^s(\ ; J \cdot \pi^\lambda))(1) = ch_{X_u}(J \cdot \pi^\lambda)(-1)^{\sum_{i=1}^n i\lambda_i} q^{-\sum_{i=1}^n (n-2i+1)\lambda_i/2} q^{\langle j(z), \lambda\rangle},$$

where

$$\langle z, \lambda\rangle = \sum_{i=1}^n z_i\lambda_i \quad \textit{and} \quad j(z) = (z_n, \dots, z_1).$$

PROOF. It is easy to see that $B = (B \cap P)N = (K \cap P)N$ with $N = \{(b_{ij}) \in B \mid b_{ii} = 1 \text{ for } \forall i,\ \ b_{ij} = 0 \text{ if } i > j\}$. By the definition of $\mathscr{P}_B$, we see that, with suitably normalized measures db on $B \cap P$ and dn on N,

$$\begin{aligned} \mathscr{P}_B(d_u^s(\ ; J \cdot \pi^\lambda))(1) &= \int_N \int_{B \cap P} ch_{X_u}(bn \cdot J \cdot \pi^\lambda) \prod_{i=1}^n |d_i(bn \cdot J \cdot \pi^\lambda)|^{s_i} db dn \\ &= \int_N ch_{X_u}(n \cdot J \cdot \pi^\lambda) \prod_{i=1}^n |d_i(n \cdot J \cdot \pi^\lambda)|^{s_i} dn \\ &= ch_{X_u}(J \cdot \pi^\lambda) \prod_{i=1}^n |d_i(J \cdot \pi^\lambda)|^{s_i}, \end{aligned}$$

and we obtain the result. □

§2.3. Now we state our main results on spherical functions on X. As for the combinatorial notations and Hall-Littlewood polynomials, we refer to Appendix.

THEOREM 1. *For each* $\lambda \in \Lambda_n$, *we have*

$$\omega(\pi^\lambda; z) = (-1)^{\sum_{i=1}^n i\lambda_i} q^{-\sum_{i=1}^n (n-2i+1)\lambda_i/2} \prod_{i=1}^n \frac{1 - q^{-2}}{1 - q^{-2i}} \prod_{1 \le i < j \le n} \frac{q^{z_j} - q^{z_i - 1}}{q^{z_j} + q^{z_i}}$$

$$\times \sum_{\sigma \in S_n} \sigma\left(q^{\langle z, \lambda\rangle} \prod_{1 \le i < j \le n} \frac{q^{z_i} + q^{z_j - 1}}{q^{z_i} - q^{z_j}}\right),$$

where $\sigma \in S_n$ *acts on* $z = (z_1, \dots, z_n)$ *by* $\sigma(z) = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$.

PROOF. Let $\chi = (\chi_1, \dots, \chi_n) \in \{(k_0^\times / N(k^\times))^\wedge\}^n$, namely χ_i is the trivial character or the character χ^* on $k_0^\times$ determined by $\chi^*(x) = (-1)^{v_\pi(x)}$. As a matter of convenience, we put $\chi_i(0) = 0$.

Define

$$\begin{aligned} L(x; \chi; s) &= L(x; \chi; z) \\ &= \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i} \chi_i(d_i(k \cdot x))\, dk. \end{aligned}$$

Then, clearly we have, for $x \in \Omega_j$

$$L(x;\chi;s) = \sum_{u \in \{0,1\}^n} \chi(u)\omega_u^s(x) = \sum_{u \in \mathscr{U}_j} \chi(u)\omega_u^s(x), \quad \text{with } \chi(u) = \prod_{i=1}^n \chi_i(\pi^{(u_1+\cdots+u_i)}).$$

Determine $e_{\chi_i} \in \{0,1\}$ by $\chi_i(\pi) = (-1)^{e_{\chi_i}}$, and set $s_i^{(\chi)} = s_i + (e_{\chi_i}\pi\sqrt{-1})/\log q$. We denote $z^{(\chi)}$ the corresponding variable to $s^{(\chi)}$ under the relation (2.2), so we have $z_i^{(\chi)} = z_i + \sum_{j=i}^n (e_{\chi_j}\pi\sqrt{-1})/\log q$.
Then we see that

$$L(x;\chi;z) = \omega(x;s^{(\chi)}) = \omega(x;z^{(\chi)}),$$

and hence we obtain a functional equation of $L(x;\chi;z)$ by using (2.4), which is expressed in the following way. By the relation of variables s and z, we see that for $\sigma \in S_n$ and χ, there exists a character ψ for which we get $\omega(x;\sigma(z^{(\chi)})) = L(x;\psi;\sigma(z))$. We denote this ψ by $\sigma(\chi)$. Then we obtain the following identity for each $\sigma = S_n$

$$L(x;\sigma(\chi);\sigma(z)) = f_\chi(\sigma;z)L(x;\chi;z),$$

where

$$f_\chi(\sigma;z) = \prod_{i<j} \frac{\{(-1)^{\sum_{r \geq \sigma(j)} e_{\chi_r}}\}q^{z_{\sigma(j)}} - \{(-1)^{\sum_{r \geq \sigma(i)} e_{\chi_r}}\}q^{z_{\sigma(i)}-1}}{\{(-1)^{\sum_{r \geq j} e_{\chi_r}}\}q^{z_j} - \{(-1)^{\sum_{r \geq i} e_{\chi_r}}\}q^{z_i-1}}.$$

Now setting

$$F(\sigma,z) = Diag(f_\chi(\sigma;z))_\chi, \quad A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u},$$

which are matrices of size 2^n, we obtain

$$A\big(\omega_u^z(x)\big)_u = F(\sigma;z)^{-1}\sigma A\big(\omega_u^{\sigma(z)}(x)\big)_u.$$

By Lemma 2.1 and Lemma 2.2, we get, for $\lambda \in \Lambda_n$

$$\begin{aligned}
\omega(\pi^\lambda;z) &= \sum_u \mathbf{1}(u)\omega_u^z(\pi^\lambda) \\
&= (-1)^{\sum_{i=1}^n i\lambda_i} q^{-\sum_{i=1}^n (n-2i+1)\lambda_i/2} \prod_{i=1}^n \frac{1-q^{-2}}{1-q^{-2i}} \\
&\quad \times \sum_{\sigma \in S_n} \sigma\left(\prod_{i<j} \frac{q^{2z_j} - q^{2z_i-2}}{q^{2z_j} - q^{2z_i}}\right) q^{\langle \sigma j(z), \lambda \rangle} f_{\mathbf{1}}(\sigma;z)^{-1} \\
&= (-1)^{\sum_{i=1}^n i\lambda_i} q^{-\sum_{i=1}^n (n-2i+1)\lambda_i/2} \prod_{i=1}^n \frac{1-q^{-2}}{1-q^{-2i}} \prod_{i<j} \frac{q^{z_j} - q^{z_i-1}}{q^{z_j} + q^{z_i}} \\
&\quad \times \sum_\sigma q^{\langle \sigma(z), \lambda \rangle} \prod_{i<j} \sigma\left(\frac{q^{z_i} + q^{z_j-1}}{q^{z_i} - q^{z_j}}\right).
\end{aligned}$$

This completes a proof. □

Remark 2.1. Employing the notation of Hall-Littlewood polynomial (cf. Appendix), we can rewrite the formula in Theorem 1 as follows:

$$\omega(\pi^\lambda; z) = (-1)^{n(\lambda)+|\lambda|} q^{n(\lambda)-((n-1)/2)|\lambda|} (1-q^{-1})^n \frac{w_\lambda^{(n)}(-q^{-1})}{w_n(q^{-2})} \tag{2.5}$$

$$\times \prod_{1 \le i < j \le n} \frac{q^{z_j} - q^{z_i - 1}}{q^{z_j} + q^{z_i}} \times P_\lambda(q^{z_1}, \dots, q^{z_n}; -q^{-1}).$$

Now we consider the following subspace of $\mathscr{C}^\infty(K \backslash X)$:

$$\mathscr{S}(K\backslash X) = \left\{ \varphi : X \to \boldsymbol{C} \;\middle|\; \begin{array}{ll} \varphi(k \cdot x) = \varphi(x) & (x \in X, k \in K) \\ \varphi \text{ is compactly supported} & \end{array} \right\}.$$

Set $\Psi_z(x) = \omega(x; z)/\omega(1_n; z)$, then by (2.5) we get for each $\lambda \in \Lambda_n$

$$\Psi_z(\pi^\lambda) = (-1)^{n(\lambda)+|\lambda|} q^{n(\lambda)-((n-1)/2)|\lambda|} \frac{w_\lambda^{(n)}(-q^{-1})}{w_n(-q^{-1})} P_\lambda(q^{z_1}, \dots, q^{z_n}; -q^{-1}). \tag{2.6}$$

We define the spherical Fourier transform on $\mathscr{S}(K\backslash X)$ as follows:

$$\wedge : \mathscr{S}(K\backslash X) \longrightarrow \boldsymbol{C}(q^{z_1}, \dots, q^{z_n})$$

$$\varphi \longmapsto \hat{\varphi}(z) = \int_X \varphi(x) \Psi_z(x^{-1})\, dx,$$

where dx is the G-invariant measure on X normalized by $\int_{K \cdot 1_n} dx = 1$.

Theorem 2. *The spherical Fourier transform* $\wedge$ *gives an* $\mathscr{H}(G,K)$*-module isomorphism*

$$\mathscr{S}(K\backslash X) \xrightarrow{\sim} \boldsymbol{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n},$$

where the right hand side is regarded as $\mathscr{H}(G,K)$*-module through the* **C***-algebra isomorphism* (2.3). *Especially,* $\mathscr{S}(K\backslash X)$ *is a free* $\mathscr{H}(G,K)$*-module of rank* 2^n.

Proof. We have shown in [**H1**] that the spherical Fourier transform $\wedge$: $\mathscr{S}(K\backslash X) \to \boldsymbol{C}(q^{z_1}, \dots, q^{z_n})$ is an $\mathscr{H}(G,K)$-module monomorphism. By Theorem 1, we see that the image coincides with $\boldsymbol{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}$. □

Remark 2.2. Theorem 2 is the affirmative answer of our conjecture proposed in [**H3**, §2].

Since the G-invariant measure dx is normalized by $\int_{K \cdot 1_n} dx = 1$, we see that

$$v(K \cdot \pi^\lambda) = \int_{K \cdot \pi^\lambda} dx = q^{n|\lambda|} \frac{\mu(1_n, 1_n)}{\mu(\pi^\lambda, \pi^\lambda)},$$

and so we obtain, by [**H1**, (2.3)]

$$v(K \cdot \pi^\lambda) = q^{-2n(\lambda)+(n-1)|\lambda|} \frac{w_n(-q^{-1})}{w_\lambda^{(n)}(-q^{-1})}. \tag{2.7}$$

Employing the explicit expression of spherical functions in Theorem 1 and (2.7), we can prove the following theorem on Plancherel formula by the same argument as in [**M1**, C5 (5.1)].

THEOREM 3. (*Plancherel formula*)
Let $\mathfrak{a}^* = \left\{\sqrt{-1}\left(\boldsymbol{R}\Big/\frac{2\pi}{\log q}\boldsymbol{Z}\right)\right\}^n$, *and define the measure* $d\mu(z)$ *on* $\mathfrak{a}^*$ *by*

$$d\mu(z) = \frac{1}{n!}\cdot\frac{w_n(-q^{-1})}{(1+q^{-1})^n}\cdot\frac{dz}{|c(z)|^2},$$

where dz *is the Haar measure on* $\mathfrak{a}^*$ *normalized by* $\int_{\mathfrak{a}^*} dz = 1$, *and*

$$c(z) = \prod_{1\le i<j\le n}\frac{q^{z_i}+q^{z_j-1}}{q^{z_i}-q^{z_j}}.$$

Then for any $\varphi,\ \psi \in \mathscr{S}(K\backslash X)$, *we have*

$$\int_X \varphi(x)\overline{\psi(x)}\ dx = \int_{\mathfrak{a}^*}\hat{\varphi}(z)\overline{\hat{\psi}(z)}\,d\mu(z).$$

THEOREM 4. (*inversion formula*)
For each $\varphi \in \mathscr{S}(K\backslash X)$, *we have*

$$\varphi(x) = (-1)^{(n+1)v_\pi(\det x)}\int_{\mathfrak{a}^*}\varphi(z)\Psi_z(x)\,d\mu(z) \quad (x\in X).$$

PROOF. Let $\varphi \in \mathscr{S}(K\backslash X)$. Take any $x\in X$ and denote by ψ the characteristic function of $K\cdot x$. Then we see that

$$\varphi(x) = \frac{1}{v(K\cdot x)}\int_{\mathfrak{a}^*}\hat{\varphi}(z)\overline{\hat{\psi}(z)}\,d\mu(z) = \int_{\mathfrak{a}^*}\hat{\varphi}(z)\overline{\Psi_z(x^{-1})}\,d\mu(z).$$

By (2.6), we have

$$\overline{\Psi_z(x^{-1})} = (-1)^{(n+1)v_\pi(\det x)}\Psi_z(x),$$

and the result follows from this. □

THEOREM 5. *Eigenvalues for spherical functions are parametrized by* $z\in \boldsymbol{C}^n/S_n$ *through* $\mathscr{H}(G,K)\to \boldsymbol{C}, f\mapsto \tilde{f}(z)$. *The set* $\{\Psi_{z+\varepsilon}(x)\,|\,\varepsilon\in\{0,\pi\sqrt{-1}/\log q\}^n\}$ *forms a basis of the space of spherical functions on* X *corresponding to* $z\in \boldsymbol{C}^n/S_n$.

PROOF. Put $\mathscr{E} = \{0,\pi\sqrt{-1}/\log q\}^n$. Then by (2.5), we see that $\{\Psi_{z+\varepsilon}(x)\,|\,\varepsilon\in\mathscr{E}\}$ are linearly independent over $\boldsymbol{C}$, and since $\tilde{f}(z+\varepsilon) = \tilde{f}(z)$, we get

$$(f*\Psi_{z+\varepsilon})(x) = \tilde{f}(z)\Psi_{z+\varepsilon}(x) \quad f\in\mathscr{H}(G,K).$$

Now we obtain the result, since $\mathscr{S}(K\backslash X)$ is a free $\mathscr{H}(G,K)$-module of rank 2^n. □

§3. Local densities.

Hereafter let $m,\ n\in \boldsymbol{N}$ with $m>n$. We set

$$X_m = \{x\in GL_m(k)\,|\,x^* = x\},\quad X_m(\mathcal{O}) = X_m\cap M_m(\mathcal{O}),\quad K_m = CL_m(\mathcal{O}),$$

we also set for X_n, $X_n(\mathcal{O})$, K_n in a similar way. For $x \in X_m(\mathcal{O})$ and $y \in X_n(\mathcal{O})$, and we define local density $\mu(y,x)$ and primitive local density $\mu^{pr}(y,x)$ by

$$\mu(y,x) = \lim_{d\to\infty} \frac{N_d(y,x)}{q^{dn(2m-n)}},$$

(3.1)

$$\mu^{pr}(y,x) = \lim_{d\to\infty} \frac{N_d^{pr}(y,x)}{q^{dn(2m-n)}},$$

where

$$N_d(y,x) = \#\{v \in M_{n,m}(\mathcal{O}/\mathfrak{p}^d) \mid vxv^* \equiv y \pmod{\mathfrak{p}^d}\},$$

$$N_d^{pr}(y,x) = \#\{v = (1_n\, 0)\tilde{v} \in M_{n,m}(\mathcal{O}/\mathfrak{p}^d) \mid \tilde{v} \in GL_m(\mathcal{O}/\mathfrak{p}^d),\ vxv^* \equiv y \pmod{\mathfrak{p}^d}\}.$$

It is easy to see by definition that $\mu(y',x') = \mu(y,x)$ and $\mu^{pr}(y',x') = \mu^{pr}(y,x)$ if $x' \in K_m \cdot x$ and $y' \in K_n \cdot y$. So it suffices to consider densities only for the representatives of K_m-orbits and K_n-orbits. A complete set of representatives of K_m-orbits in $X_m(\mathcal{O})$ is given by the set $\{\pi^\xi \mid \xi \in \Lambda_m^+\}$, where $\Lambda_m^+ = \{\xi \in \Lambda_m \mid \xi_m \geq 0\}$.

To describe our results, we need some notations concerning Hall-Littlewood polynomials which are listed in Appendix. First we recall the following fact ([**H1**, §2 Theorem]), which gives a close relation between spherical functions and local densities.

Lemma 3.1. *For every $\xi \in \Lambda_m^+$, we have*

$$\omega(\pi^\xi; s_1, \ldots, s_n, 0, \ldots, 0)$$

$$= c_{m,n} \sum_{\lambda \in \Lambda_n^+} \frac{\mu^{pr}(\pi^\lambda, \pi^\xi)}{w_n(\lambda)} \cdot q^{-2n(\lambda)-|\lambda|} \omega(\pi^\lambda; s_1, \ldots, s_n)$$

$$= c_{m,n} \prod_{i=1}^{n} (1 - q^{-2s_i - \cdots - 2s_n - 2m + 2i - 2}) \sum_{\lambda \in \Lambda_n^+} \frac{\mu(\pi^\lambda, \pi^\xi)}{w_n(\lambda)} \cdot q^{-2n(\lambda)-|\lambda|} \omega(\pi^\lambda, s_1, \ldots, s_n),$$

where

$$c_{m,n} = \frac{w_n(q^{-2}) w_{m-n}(q^{-2})}{w_m(q^{-2})},$$

$$w_n(\lambda) = w_\lambda^{(n)}(-q^{-1}) \quad \textit{for } \lambda \in \Lambda_n^+.$$

Theorem 6. *For every $\xi \in \Lambda_m^+$ and $\lambda \in \Lambda_n^+$, we have*

$$\mu^{pr}(\pi^\lambda, \pi^\xi) = (-1)^{n(\xi)+n(\lambda)} q^{n(\xi)+n(\lambda)} \times \frac{w_m(\xi) b_\lambda(-q^{-1})}{w_{m-n}(-q^{-1})}$$

$$\times \sum_\mu \left\{ \frac{(-1)^{(m-n)|\mu|} q^{-(m-n-1)|\mu|}}{b_\mu(-q^{-1})} \right.$$

$$\times P_{\xi/\mu}(1, -q^{-1}, \ldots, (-q^{-1})^{m-n-1}; -q^{-1})$$

$$\left. \times P_{\lambda/\mu}(1, -q^{-1}, \ldots, (-q^{-1})^{m-n-1}; -q^{-1}) \right\},$$

where μ *runs* Λ_n^+ *satisfying* $0 \le {}^t\xi_i - {}^t\mu_i \le m-n$ *and* $0 \le {}^t\lambda_i - {}^t\mu_i \le m-n$ *for every* $i \ge 1$.

Proof. Let $(z_1, \dots, z_n)$ be the variable corresponding to $(s_1, \dots, s_n)$ by (2.2), and $(z_1^{(m)}, \dots, z_n^{(m)})$ be the one corresponding to $(s_1, \dots, s_n, 0, \dots, 0)$. Then we have

$$z_i^{(m)} = \begin{cases} z_i + \dfrac{n-m}{2} + (n-m)\dfrac{\pi\sqrt{-1}}{\log q} & \text{if } 1 \le i \le n \\ -\dfrac{m-2i+1}{2} - (m-i+1)\dfrac{\pi\sqrt{-1}}{\log q} & \text{if } n < i \le m, \end{cases}$$

and so

$$q^{z_i^{(m)}} = \begin{cases} (-1)^{n-m} q^{(n-m)/2} q^{z_i} & \text{if } 1 \le i \le n \\ (-1)^{n-m} q^{(n-m)/2} (-1)^{j-1} q^{(n+2j-1)/2} & \text{if } j = i-n \ge 1. \end{cases} \tag{3.2}$$

By Lemma 3.1 and (2.6), we get the following identity.

$$\begin{aligned} &\omega(\pi^{\xi}; s_1, \dots, s_m, 0, \dots, 0)/\omega(1_m; s_1, \dots, s_m) \\ &\quad = \frac{c_{m,n}}{w_n(-q^{-1})} \sum_{\lambda \in \Lambda_n^+} \mu^{pr}(\pi^{\lambda}, \pi^{\xi}) (-1)^{n(\lambda)+|\lambda|} q^{-n(\lambda)-((n+1)/2)|\lambda|} P_{\lambda}(z), \end{aligned} \tag{3.3}$$

where we use the abbreviation $P_{\lambda}(z)$ for $P_{\lambda}(q^{z_1}, \dots, q^{z_n}; -q^{-1})$. We set $[A]$ the left hand side of (3.3). Then, by (2.5), we obtain

$$\begin{aligned} [A] = {} & (-1)^{n(\xi)+|\xi|} q^{n(\xi)-((m-1)/2)|\xi|} (1-q^{-1})^{m-n} \frac{w_m(\xi) w_n(q^{-2})}{w_m(q^{-2}) w_n(-q^{-1})} \\ & \times [B] \times [P_{\xi}] \times \prod_{1 \le i < j \le n} \frac{1 + q^{z_i - z_j}}{1 - q^{z_i - z_j - 1}}, \end{aligned} \tag{3.4}$$

where we set

$$[B] = \prod_{1 \le i < j \le m} \frac{1 - q^{z_i^{(m)} - z_j^{(m)} - 1}}{1 + q^{z_i^{(m)} - z_j^{(m)}}}, \quad [P_{\xi}] = P_{\xi}(q^{z_1^{(m)}}, \dots, q^{z_m^{(m)}}; -q^{-1}).$$

To simplify notations, we set for $\xi, \mu \in \Lambda_m^+$ with $\xi \supset \mu$ and r $(1 \le r \le m)$

$$P_{\xi}(t; m) = P_{\xi}(1, t, \dots, t^{m-1}; t),$$

$$P_{\xi/\mu}(t; r) = P_{\xi/\mu}(1, t, \dots, t^{r-1}; t) \quad (\text{cf. (A.5)}).$$

By (3.2) and (A.11), we obtain

$$\begin{aligned} [P_{\xi}] &= (-1)^{(n-m)|\xi|} q^{((n-m)/2)|\xi|} \\ &\quad \times P_{\xi}(q^{z_1}, \dots, q^{z_n}, q^{(n+1)/2}, -q^{(n+3)/2}, \dots, (-1)^{m-n-1} q^{m-((n+1)/2)}; -q^{-1}) \\ &= (-1)^{|\xi|} q^{((m-1)/2)|\xi|} \sum_{\mu \in \Lambda_n^+} (-1)^{(m-n-1)|\mu|} q^{-(m-(n+1)/2)|\mu|} P_{\xi/\mu}(-q^{-1}; m-n) P_{\mu}(z). \end{aligned}$$

We also get, by (3.2)

$$[B] = \prod_{1\le i<j\le n} \frac{1-q^{z_i-z_j-1}}{1+q^{z_i-z_j}} \times [B'] \times \prod_{i=1}^{m-n-1}\prod_{j=i+1}^{m-n} \frac{1-(-1)^{i-j}q^{i-j-1}}{1+(-1)^{i-j}q^{i-j}}$$

$$= \prod_{1\le i<j\le n} \frac{1-q^{z_i-z_j-1}}{1+q^{z_i-z_j}} \times [B'] \times \frac{w_{m-n}(q^{-2})}{w_{m-n}(-q^{-1})(1-q^{-1})^{m-n}},$$

where we set

$$[B'] = \prod_{\substack{1\le i\le n\\ n<j\le m}} \frac{1-q^{z_i^{(m)}-z_j^{(m)}-1}}{1+q^{z_i^{(m)}-z_j^{(m)}}}.$$

And we obtain by using (3.2) and (A.9)

$$[B'] = \prod_{\substack{1\le i\le n\\ 1\le j\le m-n}} \frac{1-(-q^{-1})(-q^{-(n+1)/2})q^{z_i}(-q^{-1})^{j-1}}{1-(-q^{-(n+1)/2})q^{z_i}(-q^{-1})^{j-1}}$$

$$= \sum_{\nu\in\Lambda_n^+\cap\Lambda_{m-n}^+} (-q^{-(n+1)/2})^{|\nu|} b_\nu(-q^{-1})P_\nu(-q^{-1};m-n)P_\nu(z).$$

By these calculation together with (A.4) and (A.10), we rewrite (3.4) in the following way.

$$[A] = (-1)^{n(\xi)}q^{n(\xi)}c_{m,n}\cdot\frac{w_m(\xi)}{w_n(-q^{-1})w_{m-n}(-q^{-1})}\sum_{\lambda\in\Lambda_n^+}\Big\{(-1)^{|\lambda|}q^{-((n+1)/2)|\lambda|}P_\lambda(z)$$

$$\times\sum_{\mu,\nu} f_{\mu\nu}^\lambda(-q^{-1})(-1)^{(m-n)|\mu|}q^{(-m+n+1)|\mu|}P_{\xi/\mu}(-q^{-1};m-n)b_\nu(-q^{-1})P_\nu(-q^{-1};m-n)\Big\}$$

$$= (-1)^{n(\xi)}q^{n(\xi)}c_{m,n}\cdot\frac{w_m(\xi)}{w_n(-q^{-1})w_{m-n}(-q^{-1})}\sum_{\lambda\in\Lambda_n^+}\Big\{(-1)^{|\lambda|}q^{-((n+1)/2)|\lambda|}b_\lambda(-q^{-1})P_\lambda(z)$$

$$\times\sum_{\mu}\frac{(-1)^{(m-n)|\mu|}q^{(-m+n+1)|\mu|}}{b_\mu(-q^{-1})}P_{\xi/\mu}(-q^{-1};m-n)P_{\lambda/\mu}(-q^{-1};m-n)\Big\},$$

where μ runs Λ_n^+ satisfying $0\le {}^t\xi_i-{}^t\mu_i\le m-n$ and $0\le {}^t\lambda_i-{}^t\mu_i\le m-n$ for every $i\ge 1$. Comparing the coefficient of $P_\lambda(z)$ above with the one in the right hand side of (3.3), we obtain the result. □

Theorem 7. *For every $\xi\in\Lambda_m^+$ and $\lambda\in\Lambda_n^+$, we have*

$$\mu(\pi^\lambda,\pi^\xi) = (-1)^{n(\xi)+n(\lambda)+(m-n)|\lambda|}q^{n(\xi)+n(\lambda)-(m-n-1)|\lambda|}\times\frac{w_m(\xi)}{w_{m-n}(-q^{-1})}$$

$$\times\sum_\nu\Big\{(-1)^{n(\nu)+(m-n)|\nu|}q^{-n(\nu)+(m-n-1)|\nu|}\Big(\sum_\kappa(-1)^{(m-n-1)|\kappa|}q^{(m-n)|\kappa|}N_\kappa^\nu(-q^{-1})\Big)$$

$$\times\Big(\sum_\mu f_{\mu\nu}^\lambda(-q^{-1})P_{\xi/\mu}(1,-q^{-1},\dots,(-q^{-1})^{m-n-1};-q^{-1})\Big)\Big\},$$

where μ, ν *and* κ *run* Λ_n^+ *satisfying* $\mu \subset \lambda$, $\kappa \subset \nu \subset \lambda$, $|\lambda| = |\mu| + |\nu|$ *and* $0 \leq {}^t\xi_i - {}^t\mu_i \leq m - n$ *for every* $i \geq 1$.

Proof. By Lemma 3.1 and (2.6), we get the following identity:

$$\prod_{i=1}^{n}(1 - q^{-2s_i-\cdots-2s_n-2m+2i-2})^{-1}\frac{\omega(\pi^{\xi}; s_1, \ldots, s_n, 0, \ldots, 0)}{\omega(1_n; s_1, \ldots, s_n)} \tag{3.5}$$

$$= \frac{c_{m,n}}{w_n(-q^{-1})}\sum_{\lambda \in \Lambda_n^+}(-1)^{n(\lambda)+|\lambda|}q^{-n(\lambda)-((n+1)/2)|\lambda|}\mu(\pi^{\lambda}, \pi^{\xi})P_{\lambda}(z).$$

We denote by $[A']$ the left hand side of (3.5). Recalling the calculation in the proof of Theorem 6, we have

$$[A'] = (-1)^{n(\xi)+|\xi|}q^{n(\xi)-((m-1)/2)|\xi|}c_{m,n}\cdot\frac{w_m(\xi)}{w_n(-q^{-1})w_{m-n}(-q^{-1})} \tag{3.6}$$

$$\times [B'] \times \prod_{i=1}^{n}(1 - q^{-2s_i-\cdots-2s_n-2m+2i-2})^{-1} \times [P_{\xi}].$$

Now we modify $[B']$ as follows.

$$[B'] = \prod_{1\leq i\leq n}\prod_{1\leq j\leq m-n}\frac{1 + (-1)^j q^{z_i-((n+2j+1)/2)}}{1 + (-1)^{j-1}q^{z_i-((n+2j-1)/2)}} = \prod_{1\leq i\leq n}\frac{1 + (-1)^{m-n}q^{z_i+((n-2m-1)/2)}}{1 + q^{z_i-((n+1)/2)}}.$$

Then we obtain, by using (A.4), (A.6) and (A.8)

$$[B'] \times \prod_{i=1}^{n}(1 - q^{-2s_i-\cdots-2s_n-2m+2i-2})^{-1}$$

$$= \prod_{1\leq i\leq n}\frac{1}{(1 + q^{z_i-((n+1)/2)})(1 - (-1)^{m-n}q^{z_i+((n-2m-1)/2)})}$$

$$= \sum_{\lambda,\kappa\in\Lambda_n^+}(-q^{-1})^{n(\lambda)+n(\kappa)}(-q^{-(n+1)/2})^{|\lambda|}((-1)^{m-n}q^{(n-2m-1)/2})^{|\kappa|}P_{\lambda}(z)P_{\kappa}(z)$$

$$= \sum_{\nu\in\Lambda_n^+}(-1)^{|\nu|}q^{-((n+1)/2)|\nu|}P_{\nu}(z)$$

$$\times \sum_{\kappa}(-1)^{n(\kappa)+(m-n-1)|\kappa|}q^{-n(\kappa)-(m-n)|\kappa|}\sum_{\lambda}(-q^{-1})^{n(\lambda)}f_{\lambda\kappa}^{\nu}(-q^{-1})$$

$$= \sum_{\nu}(-1)^{n(\nu)+|\nu|}q^{-n(\nu)-((n+1)/2)|\nu|}P_{\nu}(z)\sum_{\kappa}(-1)^{(m-n-1)|\kappa|}q^{(-m+n)|\kappa|}N_{\kappa}^{\nu}(-q^{-1}).$$

Then, employing the calculation of $[P_{\xi}]$ in the proof of Theorem 6 together with (A.4), we obtain

$$[A'] = (-1)^{n(\xi)+(n-m+1)|\xi|} q^{n(\xi)+(n-m+1)|\xi|} c_{m,n} \cdot \frac{w_m(\xi)}{w_n(-q^{-1}) w_{m-n}(-q^{-1})}$$
$$\times \sum_{\lambda \in \Lambda_n^+} \left\{ (-1)^{(m-n-1)|\lambda|} q^{(-m+(n+1)/2)|\lambda|} P_\lambda(z) \sum_{\nu} \left\{ (-1)^{n(\nu)+(m-n)|\nu|} q^{-n(\nu)+(m-n-1)|\nu|} \right.\right.$$
$$\left.\left. \times \left(\sum_{\mu} f_{\mu\nu}^{\lambda}(-q^{-1}) P_{\xi/\mu}(-q^{-1}; m-n) \right) \left(\sum_{\kappa} (-1)^{(m-n-1)|\kappa|} q^{(-m+n)|\kappa|} N_\kappa^\nu(-q^{-1}) \right) \right\} \right\},$$

where μ, ν and κ run Λ_n^+ satisfying $\mu \subset \lambda$, $\kappa \subset \nu \subset \lambda$, $|\lambda| = |\mu| + |\nu|$ and $0 \le {}^t\xi_i - {}^t\mu_i \le m-n$ for every $i \ge 1$. Comparing the coefficient of $P_\lambda(z)$ above with the one in the right hand side of (3.5), we obtain the result. □

The next corollary follows easily from Theorems 6 and 7.

Corollary 3.2. *Let $\xi \in \Lambda_m^+$ and $\lambda \in \Lambda_n^+$.*
(i) $\mu^{pr}(\pi^\lambda, \pi^\xi) = 0$ *unless* $-m+n \le {}^t\xi_i - {}^t\lambda_i \le m-n$ *for every* $i \ge 1$.
(ii) $\mu(\pi^\lambda, \pi^\xi) = 0$ *unless* ${}^t\xi_i - {}^t\lambda_i \le m-n$ *for every* $i \ge 1$.

Remark 3.1. A classical result of Johnson [Jo] asserts that the condition in (ii) is necessary and sufficient to be $\mu(\pi^\lambda, \pi^\xi) \ne 0$. He proved it by using lattices.

Corollary 3.3. *Let $\lambda \in \Lambda_n^+$.*

(i)
$$\mu^{pr}(\pi^\lambda, 1_m) = \frac{w_m(-q^{-1})}{w_{l(\lambda)}(-q^{-1})},$$

where $l(\lambda) = \#\{i \mid \lambda_i = 0,\, i \le m-n\}$.

(ii)
$$\mu(\pi^\lambda, 1_m) = \frac{w_m(-q^{-1})}{w_{m-n}(-q^{-1})} \times \sum_{\substack{\kappa \in \Lambda_n^+ \\ \kappa \subset \lambda}} (-1)^{(m-n-1)|\kappa|} q^{(n-m)|\kappa|} N_\kappa^\lambda(-q^{-1}).$$

Remark 3.2. The formulae in Theorems 6 and 7 are more explicit than those for alternating forms given in [**HS1**]. Along the same line, we can obtain similar results for alternating forms, which we shall note down in the following.

Let k be a nonarchimedian local field of characteristic 0 with prime element π and $q = \#(\mathcal{O}_k/\mathfrak{p}_k)$. For each $n \in \boldsymbol{N}$, let X_n be the set of nondegenerate alternating matrices of size $2n$ with entries in k. For $\lambda \in \Lambda_n$, we set

$$\pi^\lambda = \begin{pmatrix} 0 & \pi^{\lambda_1} \\ -\pi^{\lambda_1} & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & \pi^{\lambda_n} \\ -\pi^{\lambda_n} & 0 \end{pmatrix} \in X_n.$$

Let $\xi \in \Lambda_m^+$, $\lambda \in \Lambda_n^+$ with $m > n$. Then

$$\mu^{pr}(\pi^\lambda, \pi^\xi) = q^{2n(\xi)+2n(\lambda)} \times \frac{w_\xi^{(m)}(q^{-2}) b_\lambda(q^{-2})}{w_{m-n}(q^{-2})} \sum_{\mu} \left\{ \frac{q^{-(2m-2n-1)|\mu|}}{b_\mu(q^{-2})} \right.$$
$$\left. \times P_{\xi/\mu}(1, q^{-2}, \ldots, (q^{-2})^{m-n-1}; q^{-2}) P_{\lambda/\mu}(1, q^{-2}, \ldots, (q^{-2})^{m-n-1}; q^{-2}) \right\},$$

where μ runs Λ_n^+ satisfying $0 \le {}^t\xi_i - {}^t\mu_i \le m-n$ and $0 \le {}^t\lambda_i - {}^t\mu_i \le m-n$ for every $i \ge 1$.

$$\mu(\pi^\lambda, \pi^\xi) = q^{2n(\xi)+2n(\lambda)-(2m-2n-1)|\lambda|} \times \frac{w_\xi^{(m)}(q^{-2})}{w_{m-n}(q^{-2})}$$
$$\times \sum_\nu \left\{ q^{-2n(\nu)+(2m-2n-1)|\nu|} \left(\sum_\kappa q^{(2m-2n-1)|\kappa|} N_\kappa^\nu(q^{-2}) \right) \right.$$
$$\left. \times \left(\sum_\mu f_{\mu\nu}^\lambda(q^{-2}) P_{\xi/\mu}(1, q^{-2}, \dots, (q^{-2})^{m-n-1}; q^{-2}) \right) \right\},$$

where μ, ν and κ run Λ_n^+ satisfying $\mu \subset \lambda$, $\kappa \subset \nu \subset \lambda$, $|\lambda| = |\mu| + |\nu|$ and $0 \le {}^t\xi_i - {}^t\mu_i \le m-n$ for every $i \ge 1$.

The formula for local density $\mu(\pi^\lambda, \pi^\xi)$ given in [**HS2**] is more explicit, which does not contain terms such as $f_{\mu\nu}^\lambda$ or $P_{\xi/\mu}$, but not every term of summand is positive. On the other hand, the above expressions are useful, since every term is positive. For example, a necessary and sufficient condition to be $\mu^{pr}(\pi^\lambda, \pi^\xi) \ne 0$ or $\mu(\pi^\lambda, \pi^\xi) \ne 0$ (Theorem 9 in [**HS1**]) is easily seen. Finally, it is easy to obtain

$$\mu^{pr}(\pi^\lambda, J_m) = \frac{w_m(q^{-2})}{w_{l(\lambda)}(q^{-2})},$$

$$\mu(\pi^\lambda, J_m) = \frac{w_m(q^{-2})}{w_{m-n}(q^{-2})} \sum_{\substack{\kappa \in \Lambda_n^+ \\ \kappa \subset \lambda}} q^{-(2m-2n+1)|\kappa|} N_\kappa^\nu(q^{-2}),$$

where $l(\lambda) = \#\{i \mid \lambda_i = 0,\ i \le m-n\}$ and

$$J_m = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in X_m.$$

§4. Appendix.

We collect some notation and known properties of Hall-Littlewood polynomials used in §2 and §3.

For $n \in \boldsymbol{N}$, we put

$$\Lambda_n = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \boldsymbol{Z}^n \mid \lambda_1 \ge \cdots \ge \lambda_n\},$$
$$\Lambda_n^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n \mid \lambda_n \ge 0\}.$$

For $\lambda \in \Lambda_n$, set

(A.1) $$|\lambda| = \sum_{i=1}^n \lambda_i, \quad n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i.$$

For $m \in \boldsymbol{N}$, set $w_m(t) = \prod_{i=1}^m (1-t^i)$ and set $w_0(t) = 1$, and for $\lambda \in \Lambda_n$, put

(A.2)
$$w_\lambda^{(n)}(t) = \prod_{i=-\infty}^{+\infty} w_{m_i(\lambda)}(t),$$
$$b_\lambda(t) = \prod_{i\geq 1} w_{m_i(\lambda)}(t),$$

where $m_i(\lambda) = \#\{j \mid 1 \leq j \leq n,\ \lambda_j = i\}$. In §3, we set for $\lambda \in \Lambda_n^+$,

(A.2′)
$$w_n(\lambda) = w_\lambda^{(n)}(-q^{-1}).$$

For m, n with $m < n$, through the map

$$\Lambda_m^+ \ni \lambda \mapsto (\lambda_1, \dots, \lambda_m, 0, \dots, 0) \in \Lambda_n^+,$$

we regard Λ_m^+ as a subset of Λ_n^+. Then $w_\lambda^{(n)}(t)$ depends on n, though $|\lambda|$, $n(\lambda)$ and $b_\lambda(t)$ are not. Put $\Lambda^+ = \bigcup_{n\in \boldsymbol{N}} \Lambda_n^+$ and let $\lambda, \mu \in \Lambda^+$. We define

$$\lambda \supset \mu \overset{\text{def}}{\iff} \lambda_i \geq \mu_i \quad (\forall i \geq 1),$$
$$\lambda \cap \mu \in \Lambda^+ \quad \text{with } (\lambda \cap \mu)_i = \min\{\lambda_i, \mu_i\} \quad (\forall i \geq 1),$$
$${}^t\lambda \in \Lambda^+ \quad \text{with } {}^t\lambda_i = \#\{j \mid \lambda_j \geq i\} \quad (\forall_i \geq 1).$$

For $\lambda \in \Lambda_n$, *Hall-Littlewood polynomial* $P_\lambda(x;t)$ it defined by

(A.3)
$$P_\lambda(x;t) = P_\lambda(x_1, \dots, x_n; t)$$
$$= \frac{(1-t)^n}{w_\lambda^{(n)}(t)} \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{1\leq i<j\leq n} \frac{x_i - tx_j}{x_i - x_j}\right).$$

Here the symmetric group S_n acts on the set of indeterminates $x_1, \dots, x_n$ by permutations. If λ is in Λ_n^+, $P_\lambda(x;t)$ is actually a polynomial in $x_1, \dots, x_n$ and t, and the set $\{P_\lambda(x;t) \mid \lambda \in \Lambda_n^+\}$ forms a $\boldsymbol{Z}[t]$-basis of the ring $\boldsymbol{Z}[t][x_1, \dots, x_n]^{S_n}$ of symmetric polynomials in $x_1, \dots, x_n$ with coefficients in $\boldsymbol{Z}[t]$. In general, $P_\lambda(x;t)$ is a Laurent polynomial in $x_1, \dots, x_n$, and the set $\{P_\lambda(x;t) \mid \lambda \in \Lambda_n\}$ forms a $\boldsymbol{Z}[t]$-basis of the ring $\boldsymbol{Z}[t][x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$ of symmetric Laurent polynomials in $x_1, \dots, x_n$ with coefficients in $\boldsymbol{Z}[t]$.

Denote by $f_{\mu\nu}^\lambda(t)$ the structure constants with respect to the basis $\{P_\lambda(x;t) \mid \lambda \in \Lambda_n^+\}$, namely

(A.4)
$$P_u(x;t)P_v(x;t) = \sum_{\lambda\in\Lambda_n^+} f_{\mu\nu}^\lambda(t) P_\lambda(x;t).$$

Calculation of these structure functions is carried out in [**M2**, III-§3], especially we note here that

$$f_{\mu\nu}^\lambda(t) = 0 \text{ unless } |\lambda| = |\mu| + |\nu|, \quad \lambda \supset \mu \quad \text{and} \quad \lambda \supset \nu.$$

For λ, $\mu \in \Lambda^+$, $\lambda - \mu$ is called a *horizontal strip* if and only if $0 \leq {}^t\lambda_i - {}^t\mu_i \leq 1$ for every $i \geq 1$; and then we set

$$\psi_{\lambda/\mu}(t) = \prod_{j \in J_{\lambda/\mu}} (1 - t^{m_j(\lambda)}), \quad J_{\lambda/\mu} = \{j \in \boldsymbol{N} \mid \lambda_j = \mu_j,\ \lambda_{j+1} > \mu_{j+1}\}.$$

A sequence $T : \mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$ is called a *tableau of shape* $\lambda - \mu$ *of length* r if and only if $\lambda^{(i)} - \lambda^{(i-1)}$ is a horizontal strip for all i $(1 \leq i \leq r)$; and then we set

$$\psi_T(t) = \prod_{i=1}^{r} \psi_{\lambda^{(i)}/\lambda^{(i+1)}}(t), \quad y^T = \prod_{i=1}^{r} y_i^{|\lambda^{(i)}|-|\lambda^{(i+1)}|}.$$

For $\lambda,\ \mu \in \Lambda^+$ with $\lambda \supset \mu$, we define a polynomial $P_{\lambda/\mu}(y; t)$ in $y_1, \dots, y_r$ and t by

(A.5) $$P_{\lambda/\mu}(y; t) = P_{\lambda/\mu}(y_1, \dots, y_r; t) = \sum_T \psi_T(t) y^T,$$

where T runs over all tableaux T of shape $\lambda - \mu$ of length r. We note here that

$$P_{\lambda/\mu}(y; t) \neq 0 \quad \text{if and only if } 0 \leq {}^t\lambda_i - {}^t\mu_i \leq r \quad \text{for } \forall i \geq 1.$$

For $\lambda, \mu \in \Lambda_n^+$, we set

(A.6) $$N_\mu^\lambda(t) = t^{n(\mu)-n(\lambda)} \sum_{\nu \in \Lambda_n^+} t^{n(\nu)} f_{\mu\nu}^\lambda(t),$$

then it is known ([**HS2**, Lemma 5]) that

$$N_\mu^\lambda(t) = t^{\sum_{i \geq 1}({}^t\mu_i - {}^t\lambda_i){}^t\mu_i} \cdot \prod_{i \geq 1} \begin{bmatrix} {}^t\lambda_i - {}^t\mu_{i+1} \\ {}^t\mu_i - {}^t\mu_{i+1} \end{bmatrix}(t),$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix}(t) = \frac{w_m(t)}{w_r(t) w_{m-r}(t)} \quad (m \geq r).$$

We list several identities concerning theses polynomials which are used in §3. For $\lambda \in \Lambda_n^+$,

(A.7) $$P_\lambda(1, t, \dots, t^{n-1}; t) = \frac{t^{n(\lambda)} w_n(t)}{w_\lambda^{(n)}(t)} \quad ([\mathbf{M2}, \text{III-§ 2 Ex.1}]).$$

(A.8) $$\prod_{1 \leq i \leq n} \frac{1}{1 - x_i} = \sum_{\lambda \in \Lambda_n^+} t^{n(\lambda)} P_\lambda(x_1, \dots, x_n; t) \quad ([\mathbf{M2}, \text{III-§ 4 Ex.1}]).$$

(A.9) $$\prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r}} \frac{1 - t x_i y_j}{1 - x_i y_j} = \sum_{\lambda \in \Lambda_m^+ \cap \Lambda_r^+} b_\lambda(t) P_\lambda(x_1, \dots, x_m; t) P_\lambda(y_1, \dots, y_r; t) \quad ([\mathbf{M2}, \text{III-(4.4)}]).$$

For $\lambda,\ \mu \in \Lambda_n^+$,

(A.10) $$\frac{b_\lambda(t)}{b_\mu(t)} P_{\lambda/\mu}(x; t) = \sum_{\nu \in \Lambda_n^+} f_{\mu\nu}^\lambda(t) b_\nu(t) P_\nu(x; t) \quad ([\mathbf{M2}, \text{III-(5.2), (5.4)}]).$$

For $\lambda \in \Lambda_{m+r}^{+}$,

$$P_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_r; t) = \sum_{\substack{\mu \in \Lambda_m^{+} \\ 0 \le {}^t\lambda_i - {}^t\mu_i \le r}} P_{\lambda/\mu}(y_1, \ldots, y_r; t) P_\mu(x_1, \ldots, x_m; t) \quad ([\mathbf{M2}, \text{III-}(5.5')]). \tag{A.11}$$

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Yumiko Hironaka
Department of Mathematics
Faculty of Science
Shinshu University
Matsumoto, 390-8621 Japan
E-mail: hironaka@math.shinshu-u.ac.jp

Current address
Department of Mathematics
School of Education
Waseda University
Nishi-Waseda Tokyo, 169-8050 Japan
E-mail: hironaka@edu.waseda.ac.jp