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The reducibility of linear almost periodic systems with sufficiently small coefficient matrices

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Abstract. In this paper, we shall obtain a reducible theorem for a linear almost periodic system with an almost zero coefficient matrix. This reducible theorem states that the system can be transforms into two systems with size smaller than the original system. Of course, the transformation is linear and almost periodic.

§ 1. Introduction.

Let us consider a linear almost periodic system

$$\dot{x} = \varepsilon^N A(t, \varepsilon)x, \quad {}^{\boldsymbol{\cdot}} = d/dt, \quad x \in \boldsymbol{C}^n, \quad t \in \boldsymbol{R}. \tag{1}$$

Here N is a positive integer, ε is a complex parameter sufficiently close to 0 and $A(t, \varepsilon)$ is a matrix function continuous in

$$-\infty < t < \infty, \quad |\varepsilon| \leqq \rho,$$

almost periodic in t uniformly for $|\varepsilon| \leqq \rho$ and holomorphic in $|\varepsilon| < \rho$. From the holomorphic property of $A(t, \varepsilon)$ we get the analytical expression

$$A(t, \varepsilon) = \sum_{k=0}^{\infty} A_k(t)\varepsilon^k.$$

Now we denote by Λ the mean of $A_0(t)$. If $\tilde{A}_0(t) = A_0(t) - \Lambda$, then $\tilde{A}_0(t)$ is an almost periodic matrix function whose mean is zero. Suppose that Λ has a Jordan's normal form $\mathrm{diag}(\Lambda_1, \Lambda_2)$ where
(i) Λ_1 is a $z \times z$ matrix whose diagonal entries are arranged as $\lambda_1, \dots, \lambda_z$ and whose $(i, i+1)$th entries are denoted by ℓ_i $(i = 1, \dots, z-1)$,
(ii) Λ_2 is a $(n-z) \times (n-z)$ matrix whose diagonal entries are arranged as $\lambda_{z+1}, \dots, \lambda_n$ and whose $(i, i+1)$th entries are denoted by ℓ_{z+i} $(i = 1, \dots, n-z-1)$,
(iii) if $i = 1, \dots, z$, $j = z+1, \dots, n-1$, then

$$\lambda_i \neq \lambda_j.$$

Needless to say,

$$\ell_i = 0 \quad \text{or} \quad 1 \ \ (i = 1, \dots, z-1, z+1, \dots, n-1).$$

Moreover we adopt a convention

$$\ell_z = 0.$$

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Recall that a system

$$\dot{x} = A(t)x \tag{2}$$

is called reducible with a projection P, if (2) is kinematically similar to a system

$$\dot{x} = B(t)x \tag{3}$$

where $B(t)P = PB(t)$. Here (2) is said to be kinematically similar to (3), if there exists a continuously differentiable invertible matrix function $S(t)$ bounded as well as its inverse $S^{-1}(t)$ such that the transformation $x = S(t)y$ transforms (2) to (3).

Let $P(z)$ be a diagonal matrix. The first z diagonal entries of this are 1 and the others are 0. The purpose of this paper is to show that (1) is reducible with $P(z)$ under suitable suppositions. For this goal we shall use the idea stated in the proof of Proposition 1 of [**1**, p. 42]. As in there we define

$$\{M\}_1 = P(z)MP(z) + (I - P(z))M(I - P(z))$$

$$\{M\}_2 = P(z)M(I - P(z)) + (I - P(z))MP(z)$$

for any matrix M. Moreover we put

$$B(t,\varepsilon) = \sum_{k=0}^{\infty} B_k(t)\varepsilon^k = A(t,\varepsilon) - \Lambda.$$

Therefore we get

$$B_0(t) = \tilde{A}_0(t), \quad B_k(t) = A_k(t).$$

From the part of [**1**] mentioned above, it follows that if we get a solution $H = H(t,\varepsilon)$ of

$$\dot{H} = \varepsilon^N(\Lambda H - H\Lambda + \{(I - H)B(t,\varepsilon)(I + H)\}_2) \tag{4}$$

$$\{H\}_1 = 0, \tag{5}$$

then the transformation $x = (I + H(t,\varepsilon))y$ transforms (1) to

$$\dot{y} = \varepsilon^N(\Lambda + \{B(t,\varepsilon)(I + H(t,\varepsilon))\}_1)y. \tag{6}$$

Therefore if $H(t,\varepsilon)$ is bounded, then (1) is reducible with $P(z)$.

In the previous paper [**2**] we obtained a formal solution

$$H(t,\varepsilon) = \sum_{k=0}^{\infty} H_k(t)\varepsilon^k$$

of (4), (5) in case of $N = 1$ under some suppositions. To tell the truth, the consideration of the reducibility of (1) here arises from the expectation that such a formal solution converges. Actually under the suppositions given in [**2**] we shall show the existence of a solution $H(t,\varepsilon)$ of (4), (5) continuous in $-\infty < t < \infty$, $|\varepsilon| \leqq \mu$, almost periodic in t uniformly for $|\varepsilon| \leqq \mu$ and holomorphic in $|\varepsilon| < \mu$ for some positive constant μ. Therefore we shall find that the formal solution converges, since this was uniquely determined.

§2. Preliminaries.

Concerning $A(t, \varepsilon)$ we assume the same properties as in [**2**]. First we define the following two function spaces:

Definition 1. Let $\boldsymbol{M}_1$ be a set consisting of 0 and real numbers whose absolute values are greater than some positive constant. Suppose $\boldsymbol{M}_1$ is closed under the addition. Then we put

$$\mathscr{F}_1 = \{f : f \in AP(\boldsymbol{C}), \mathrm{Exp}(f) \subset \boldsymbol{M}_1\}$$

where $AP(E)$ denotes the totality of E-valued almost periodic functions and $\mathrm{Exp}(f)$ denotes the exponents of the almost periodic function f.

Recall that $\mathscr{F}_1$ is a set of periodic functions or of almost periodic functions whose exponents consist of 0 and numbers with the definite sign.

Definition 2. Let $\omega \in \boldsymbol{R}^K$ have components linearly independent with respect to integers. Suppose the nonresonance condition

$$|(m, \omega)| \geqq c_0 |m|^{-\sigma} \; (c_0, \sigma : \text{positive constants})$$

where $m = (m_1, \dots, m_K) \in \boldsymbol{Z}^K$, (,) denotes the inner product of vectors and

$$|m| = |m_1| + \cdots + |m_K|.$$

Then we define

$$\boldsymbol{M}_2 = \{(m, \omega) : m \in \boldsymbol{Z}^K\},$$

$$\mathscr{F}_2 = \{f : f \in AP(\boldsymbol{C}) \cap \mathcal{O}(\boldsymbol{R}), \mathrm{Mod}(f) \subset \boldsymbol{M}_2\}$$

where $\mathcal{O}(\boldsymbol{R})$ denotes the totality of real analytic functions and $\mathrm{Mod}(f)$ denotes the smallest module of real numbers containing $\mathrm{Exp}(f)$.

If $f \in \mathscr{F}_2$, then f is quasiperiodic. Hence there exists a function $\tilde{f}(\theta)$ of $\theta = (\theta_1, \dots, \theta_K)$ with the period 2π in every $\theta_i (i = 1, \dots, K)$ such that

$$f(t) = \tilde{f}(\omega t).$$

Definition 3. Let us call $\tilde{f}$ the extension of f.

Consider the case when all entries of $A(t, \varepsilon)$ belong to $\mathscr{F}_1$ or $\mathscr{F}_2$. Moreover we must define operators $\mathscr{M}$ and $\mathscr{L}$.

Definition 4. For any almost periodic function $h(t)$, we denote by $\mathscr{M}h$ the mean value of $h(t)$. Furthermore $\mathscr{L}h$ denotes the almost periodic solution of

$$\dot{x} = h(t) - \mathscr{M}h$$

whose mean value is zero, if this exists. Furthermore if $H(t) = [h_{ij}(t)]$ is an almost periodic matrix function, then we define

$$\mathscr{L}H(t) = [\mathscr{L}h_{ij}(t)], \quad \mathscr{M}H = [\mathscr{M}h_{ij}].$$

DEFINITION 5. Let $\tilde{\mathscr{F}}_2(v)$ be a set of functions such that if $\tilde{f} \in \tilde{\mathscr{F}}_2(v)$, then $\tilde{f} = \tilde{f}(\theta)(\theta = (\theta_1, \dots, \theta_K))$ is continuous in $|\mathrm{Im}\,\theta_i| \leqq v$, holomorphic in its interior and is the extension of some function $f \in \mathscr{F}_2$.

The following is obtained by Lemma 2.3 of [**2**].

LEMMA 1. (a) *$\mathscr{L}$ is a bounded linear operator of $\mathscr{F}_1$ into $\mathscr{F}_1$.*
(b) *$\mathscr{L}$ is a bounded linear operator of $\tilde{\mathscr{F}}_2(v)$ into $\tilde{\mathscr{F}}_2(v/2)$ where $\mathscr{L}\tilde{f}$ ($\tilde{f} \in \tilde{\mathscr{F}}_2(v)$) is defined to be equal to the extension of $\mathscr{L}f$.*

Furthermore, from Lemma 2.4 (c) of [**2**], we get

LEMMA 2. *Let $f(t, \varepsilon) = \sum_{k=0}^{\infty} f_k(t)\varepsilon^k$ be a function continuous in $-\infty < t < \infty$, $|\varepsilon| \leqq \rho$, almost periodic in t uniformly for $|\varepsilon| \leqq \rho$, holomorphic in $|\varepsilon| < \rho$ and $f_k(t)$ belong to $\mathscr{F}_1$ or $\mathscr{F}_2$. Then*

$$\mathscr{L}\left(\sum_{k=0}^{\infty} f_k(t)\varepsilon^k\right) = \sum_{k=0}^{\infty} \mathscr{L}f_k(t)\varepsilon^k$$

which converges uniformly for all $t \in \boldsymbol{R}$.

The assumption that entries of $A(t, \varepsilon)$ belong to $\mathscr{F}_1$ or $\mathscr{F}_2$ is given for ensuring the boundedness of $\mathscr{L}$. The discussions can be carried out more easily in the case when entries of $A(t, \varepsilon)$ belong to $\mathscr{F}_1$ than in the case when these belong to $\mathscr{F}_2$. So the discussions of the former case will be omitted.

SUPPOSITION A. *All the entries of $A(t, \varepsilon)$ belong to $\mathscr{F}_2$.*

Under this supposition we must assume the more.

SUPPOSITION B. *There exists the extension $\tilde{A}(\theta, \varepsilon)$ of $A(t, \varepsilon)$ which is continuous in the set*

$$|\mathrm{Im}\,\theta_i| \leqq v, \quad |\varepsilon| \leqq \rho$$

and is holomorphic in $|\mathrm{Im}\,\theta_i| < v$ where v is a constant independent of ε.

Since $\tilde{A}(\theta, \varepsilon)$ has the period 2π in θ_i and hence is bounded, this is also holomorphic in $|\varepsilon| < \rho$.

§ 3. The main discussions.

If (4) has an almost periodic solution, then we get

$$H = \varepsilon^N \mathscr{L}(\Lambda H - H\Lambda + \{(I - H)B(t, \varepsilon)(I + H)\}_2) + C \tag{7}$$

$$\mathscr{M}(\Lambda H - H\Lambda + \{(I - H)B(t, \varepsilon)(I + H)\}_2) = \mathbf{0}, \tag{8}$$

where C is a constant matrix. Here if

$$H = G + C$$

where G is an almost periodic matrix function with the mean zero and $C = \mathscr{M}H$, then from (8) we obtain

$$\Lambda C - C\Lambda + \mathscr{M}(\{(I-H)B(t,\varepsilon)(I+H)\}_2) = \mathbf{0}. \tag{9}$$

Conversely if (7), (9) have an almost periodic solution, then we have (4). Namely (4) is equivalent to (7), (9).

To solve (9), we require

Lemma 3. *If M is a matrix with $\{M\}_1 = \mathbf{0}$, then there exists uniquely a solution X of*

$$\Lambda X - X\Lambda = M \tag{10}$$

such that

$$\{X\}_1 = \mathbf{0}.$$

Proof. Suppose that X, M are partitioned as

$$X = [X_{ij}]_{i,j=1,2}, \quad M = [M_{ij}]_{i,j=1,2}$$

where if $n_1 = z$, $n_2 = n - z$, then X_{ij}, M_{ij} are $n_i \times n_j$ matrices and from $\{M\}_1 = \mathbf{0}$ we get

$$M_{11} = M_{22} = \mathbf{0}.$$

In this case, from (10) we obtain

$$\Lambda_1 X_{11} - X_{11}\Lambda_1 = \mathbf{0} \tag{11}$$

$$\Lambda_1 X_{12} - X_{12}\Lambda_2 = M_{12} \tag{12}$$

$$\Lambda_2 X_{21} - X_{21}\Lambda_1 = M_{21} \tag{13}$$

$$\Lambda_2 X_{22} - X_{22}\Lambda_2 = \mathbf{0}. \tag{14}$$

$X_{11} = X_{22} = \mathbf{0}$ satisfies (11) and (14). If we put

$$X_{12} = [x_{i\,z+j}], \quad M_{12} = [m_{i\,z+j}]$$

$$(i = 1, \dots, z,\ j = 1, \dots, n-z),$$

then we get from (12)

$$(\lambda_i - \lambda_{z+j})x_{i\,z+j} + \ell_i x_{i+1\,z+j} - x_{i\,z+j-1}\ell_{z+j-1} = m_{i\,z+j}.$$

Thus $x_{i\,z+j}$ are determined uniquely in the order

$$\begin{aligned}(i,j) = {} & (z,1), (z,2), \dots, (z, n-z),\\ & (z-1,1), (z-1,2), \dots, (z-1, n-z),\\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ & (1,1), (1,2), \dots, (1, n-z).\end{aligned}$$

Similarly X_{21} is determined uniquely by (13). Hence the proof is completed.

Definition 6. Let us denote by $\mathscr{S}M$ the unique solution of (10).

Notice that $\mathscr{S}$ is a bounded linear operator.

Returning to (9), we get

$$C = -\mathscr{S}\mathscr{M}(\{(I-H)B(t,\varepsilon)(I+H)\}_2). \tag{15}$$

From (7) and (15), we obtain

$$H = \varepsilon^N \mathscr{L}(\Lambda H - H\Lambda + \{(I-H)B(t,\varepsilon)(I+H)\}_2)$$
$$- \mathscr{S}\mathscr{M}(\{(I-H)B(t,\varepsilon)(I+H)\}_2).$$

Here we denote by $\mathscr{T}H$ the right side of this. Furthermore we put

$$H_0 = \mathbf{0}, \quad H_k = \mathscr{T}H_{k-1} \quad (k = 1, 2, \dots). \tag{16}$$

The norm $\| \ \|_v$ we shall use is defined as

$$\|H(\theta,\varepsilon)\|_v = \sup_{|\mathrm{Im}\,\theta_i| \leqq v} |H(\theta,\varepsilon)|,$$

where $| \ |$ denotes a matrix norm with $|GH| \leqq |G||H|$ for matrices G, H and $H(\theta,\varepsilon)$ is a matrix function defined in a region of $\boldsymbol{C}^2$. Moreover we define

$$\tilde{B}(\theta,\varepsilon) = \tilde{A}(\theta,\varepsilon) - \Lambda.$$

Since $\tilde{B}(\theta,\varepsilon)$ has the period 2π in θ_i and is continuous in $|\mathrm{Im}\,\theta_i| \leqq v$ $(i = 1,\dots,K)$, $|\varepsilon| \leqq \rho$, there exists a constant Ω such that

$$\|\tilde{B}(\theta,\varepsilon)\|_v \leqq \Omega$$

over $|\varepsilon| \leqq \rho$.

Applying Lemma 1 (b) to (16), it follows from the inductional argument that H_k has the extension holomorphic in $|\mathrm{Im}\,\theta_i| \leqq v/2^k$ which is equal to the extension of $\mathscr{T}H_{k-1}$. The extension of H_k will be also denoted by H_k. Moreover from the inductional argument,

$$H_k = O(\varepsilon),$$

because if $H_{k-1} = O(\varepsilon)$, then

$$\mathscr{S}\mathscr{M}(\{(I-H_{k-1})B(t,\varepsilon)(I+H_{k-1})\}_2) \tag{17}$$
$$= \mathscr{S}\mathscr{M}(\{B_0(t) + O(\varepsilon)\}_2) = \mathscr{S}\mathscr{M}(O(\varepsilon)) = O(\varepsilon).$$

Now suppose that

$$\|H_{k-1}\|_{v/2^{k-1}} \leqq 1.$$

Then for $|\varepsilon| < 1$ we get

$$\|H_k\|_{v/2^k} = \|\mathscr{T}H_{k-1}\|_{v/2^k}$$
$$\leqq |\varepsilon||\mathscr{L}|(2|\Lambda| + 4c\Omega) + 4|\mathscr{S}||\mathscr{M}|c\Omega,$$

where $|\ |$ denotes a norm of operators together with a matrix norm and c is a constant satisfying

$$|\{M\}_2| \leqq c|M|.$$

However, from (17) we obtain

$$\|\mathscr{S}\mathscr{M}(\{(I - H_{k-1})B(t,\varepsilon)(I + H_{k-1})\}_2\|_{v/2^{k-1}}$$
$$\leqq 4|\mathscr{S}||\mathscr{M}|c\Omega \frac{|\varepsilon|}{\rho - |\varepsilon|}.$$

Here if we suppose

$$|\varepsilon||\mathscr{L}|(2|\Lambda| + 4c\Omega) \leqq \frac{1}{2},\ \ 4|\mathscr{S}||\mathscr{M}|c\Omega \frac{|\varepsilon|}{\rho - |\varepsilon|} \leqq \frac{1}{2},$$

then

(18) $$|\varepsilon| \leqq \min\left(\frac{1}{4|\mathscr{L}|(|\Lambda| + 2c\Omega)}, \frac{\rho}{8|\mathscr{S}||\mathscr{M}|c\Omega + 1}, 1\right)$$

since $|\varepsilon| < 1$. If ε satisfies this, then

(19) $$\|H_k\|_{v/2^k} \leqq 1.$$

Namely, from the induction, (19) is valid for $k = 1, 2, \ldots$.

Furthermore we get

$$\begin{aligned}\|H_k - H_{k-1}\|_{v/2^k} \leqq |\varepsilon|^N |\mathscr{L}| \| &\Lambda H_{k-1} - H_{k-1}\Lambda \\ &+ \{(I - H_{k-1})B(t,\varepsilon)(I + H_{k-1})\}_2 \\ &- \Lambda H_{k-2} + H_{k-2}\Lambda - \{(I - H_{k-2})B(t,\varepsilon)(I + H_{k-2})\}_2\|_{v/2^{k-1}} \\ &+ |\mathscr{S}||\mathscr{M}| \|\{(I - H_{k-1})B(t,\varepsilon)(I + H_{k-1})\}_2 \\ &- \{(I - H_{k-2})B(t,\varepsilon)(I + H_{k-2})\}_2\|_{v/2^{k-1}}.\end{aligned}$$

On the other hand

(20) $$\begin{aligned}&\|\{(I - H_{k-1})B(t,\varepsilon)(I + H_{k-1})\}_2 \\ &\quad - \{(I - H_{k-2})B(t,\varepsilon)(I + H_{k-2})\}_2\|_{v/2^{k-1}}\end{aligned}$$

is not greater than

$$4c\Omega\|H_{k-1} - H_{k-2}\|_{v/2^{k-1}}.$$

Therefore (20) is not greater than

$$4c\Omega\|H_{k-1} - H_{k-2}\|_{v/2^{k-1}} \frac{|\varepsilon|}{\rho - |\varepsilon|},$$

since (20) is equal to $O(\varepsilon)$. Consequently we have

$$\|H_k - H_{k-1}\|_{\nu/2^k} \leqq \left\{|\varepsilon||\mathscr{L}|(2|\Lambda| + 4c\Omega) + 4|\mathscr{S}||\mathscr{M}|c\Omega\frac{|\varepsilon|}{\rho - |\varepsilon|}\right\}\|H_{k-1} - H_{k-2}\|_{\nu/2^{k-1}}.$$

If we suppose

$$|\varepsilon||\mathscr{L}|(2|\Lambda| + 4c\Omega) \leqq \frac{1}{4}, \quad 4|\mathscr{S}||\mathscr{M}|c\Omega\frac{|\varepsilon|}{\rho - |\varepsilon|} \leqq \frac{1}{4},$$

then

$$|\varepsilon| \leqq \min\left(\frac{1}{8|\mathscr{L}|(|\Lambda| + 2c\Omega)}, \frac{\rho}{16|\mathscr{S}||\mathscr{M}|c\Omega + 1}\right). \tag{21}$$

Therefore if $|\varepsilon| \leqq \mu$ where

$$\mu = \min\left(\frac{1}{8|\mathscr{L}|(|\Lambda| + 2c\Omega)}, \frac{\rho}{16|\mathscr{S}||\mathscr{M}|c\Omega + 1}, 1\right),$$

then (18) and (21) are satisfied. In this case,

$$\|H_k - H_{k-1}\|_{\nu/2^k} \leqq \frac{1}{2}\|H_{k-1} - H_{k-2}\|_{\nu/2^{k-1}}.$$

Hence we have

$$\begin{aligned}\|H_k - H_{k-1}\|_{\nu/2^k} &\leqq \left(\frac{1}{2}\right)^{k-1}\|H_1 - H_0\|_{\nu/2} \\ &\leqq \left(\frac{1}{2}\right)^{k-1}.\end{aligned} \tag{22}$$

Take $t \in \boldsymbol{R}$. Then from (22) we get

$$\lim_{k\to\infty} H_k(t,\varepsilon) = \lim_{k\to\infty}\left(H_0(t,\varepsilon) + \sum_{r=1}^{k}(H_r(t,\varepsilon) - H_{r-1}(t,\varepsilon))\right) = H(t,\varepsilon),$$

where the convergence of the limit is uniform. Therefore $H(t,\varepsilon)$ is a function continuous in $-\infty < t < \infty$, $|\varepsilon| \leqq \mu$, almost periodic in t uniformly for $|\varepsilon| \leqq \mu$ and holomorphic in $|\varepsilon| < \mu$ such that

$$H = \mathscr{T}H.$$

Consequently $H(t,\varepsilon)$ is a solution of (7), (9) and hence of (4). Moreover since $H_k = O(\varepsilon)$, we obtain

$$H(t,\varepsilon) = O(\varepsilon).$$

From (16) we get

$$\begin{aligned}\{H_k\}_1 &= \{\varepsilon^N\mathscr{L}(\Lambda H_{k-1} - H_{k-1}\Lambda)\}_1 \\ &= \varepsilon^N\mathscr{L}(\{\Lambda H_{k-1} - H_{k-1}\Lambda\}_1).\end{aligned}$$

Consequently if $\{H_{k-1}\}_1 = \mathbf{0}$, then

$$\{H_k\}_1 = \mathbf{0}.$$

Namely we get

$$\{H\}_1 = \mathbf{0}.$$

Since we have just shown that $H(t,\varepsilon)$ is a solution of (4), (5), we now conclude that $x = (I + H(t,\varepsilon))y$ transforms (1) to (6).

Theorem. (1) *is reducible with* $P(z)$.

Since $H(t,\varepsilon)(= O(\varepsilon))$ is holomorphic in $|\varepsilon| < \mu$, we write

$$H(t,\varepsilon) = \sum_{k=0}^{\infty} H_k(t)\varepsilon^k \quad (H_0(t) = \mathbf{0}). \tag{23}$$

In the same manner as in [**2**], we shall obtain recurrence formulas for determining $H_k(t)$. Substituting (23) into (4), we get

$$\dot{H}_k(t) = 0 \quad (k = 1, 2, \dots, N-1) \tag{24}$$

$$\begin{aligned} \dot{H}_k(t) = {} & \Lambda H_{k-N}(t) - H_{k-N}(t)\Lambda + \{B_{k-N}(t) \\ & - \sum_{q=0}^{k-N-1} H_{k-N-q}(t)B_q(t) + \sum_{q=0}^{k-N-1} B_q(t)H_{k-N-q}(t) \\ & - \sum_{p_1+p_2+p_3=k-N} H_{p_1}(t)B_{p_2}(t)H_{p_3}(t)\}_2 \quad (k = N, N+1, \dots), \end{aligned} \tag{25}$$

where the sum $\sum_{q=0}^{-1}$ is supposed to be equal to zero.

It follows from (24) and (25) that $H_k(t)$ are determined to have the form

$$H_k(t) = G_k(t) + C_k, \quad \{H_k(t)\}_1 = \mathbf{0}$$

where $G_k(t)$ are almost periodic matrix functions whose means are zero and C_k are constant matrices such that the mean of the right side of (25) vanishes if the index k is changed for $k + N$. Consequently we get

$$\begin{aligned} G_k(t) &= \mathbf{0} \quad (k = 0, 1, \dots, N-1), \\ G_k(t) &= \mathscr{L}(\Lambda G_{k-N}(t) - G_{k-N}(t)\Lambda + F_{k-N}(t)) \quad (k = N, N+1, \dots) \\ C_k &= -\mathscr{S}\mathscr{M}F_k(t) \quad (k = 0, 1, \dots). \end{aligned}$$

Here

$$\begin{aligned} F_k(t) = \{ & B_k(t) - G_k(t)B_0(t) - \sum_{q=1}^{k-1}(G_{k-q}(t) + C_{k-q})B_q(t) \\ & + B_0(t)G_k(t) + \sum_{q=1}^{k-1} B_q(t)(G_{k-q}(t) + C_{k-q}) \\ & - \sum_{p_1+p_2+p_3=k} (G_{p_1}(t) + C_{p_1})B_{p_2}(t)(G_{p_3}(t) + C_{p_3})\}_2 \end{aligned}$$

where $p_1 \neq 0$, $p_3 \neq 0$. It is noteworthy that these can be obtained directly from $H = \mathscr{T}H$.

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