J. Math. Soc. Japan
Vol. 51, No. 1, 1999

On the $L_q - L_r$ estimates of the Stokes semigroup in a two dimensional exterior domain

Dedicated to Professor Rentaro Agemi on the occasion of his sixtieth birthday

By Wakako Dan and Yoshihiro Shibata

(Received Nov. 11, 1996)
(Revised Apr. 17, 1997)

Abstract. We proved $L_q - L_r$ type estimates of the Stokes semigroup in a two dimensional exterior domain. Our proof is based on the local energy decay estimate obtained by investigation of the asymptotic behavior of the resolvent of the Stokes operator near the origin.

§1. Introduction.

Let Ω be an unbounded domain in the 2-dimensional Euclidean space $\boldsymbol{R}^2$ having a compact and smooth boundary $\partial\Omega$ contained in the ball $B_{b_0} = \{x \in \boldsymbol{R}^2 \mid |x| \leqq b_0\}$. In $(0, \infty) \times \Omega$, we consider the nonstationary Stokes initial boundary value problem concerning the velocity field $\mathbf{u} = \mathbf{u}(t, x) = {}^t(u_1, u_2)$ and the scalar pressure $\mathfrak{p} = \mathfrak{p}(t, x)$:

$$\text{(NS)} \qquad \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \mathfrak{p} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } (0, \infty) \times \Omega,$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, \infty) \times \partial\Omega, \qquad \mathbf{u}(0, x) = \mathbf{f}(x) \quad \text{in } \Omega,$$

where $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\boldsymbol{R}^2$, $\nabla = (\partial_1, \partial_2)$ with $\partial_j = \partial/\partial x_j$ is the gradient, and $\nabla \cdot \mathbf{u} = \operatorname{div} \mathbf{u} = \partial_1 u_1 + \partial_2 u_2$ is the divergence of $\mathbf{u}$.

For the corresponding nonlinear Navier-Stokes equations in two dimensional exterior domain, we know the uniqueness of the Leray-Hopf weak solutions which was proved by Lions and Prodi [**23**]. Masuda [**27**] proved that if $\mathbf{u}(x)$ is a weak solution with $\int_0^\infty \|\nabla \mathbf{u}(t)\|_{L_2(\Omega)}^2 \, dt < \infty$, $\|\mathbf{u}(t)\|_{L_2(\Omega)}$ tends to zero as $t \to \infty$. The decay rate of a weak solution was investigated by Borchers & Miyakawa [**3**] and Maremonti [**24**]. In 1993, Kozono and Ogawa [**19**] proved a unique existence theorem of global strong solutions with initial data in $L_2(\Omega)$, which satisfy the following decay rate:

$$\text{(D)} \qquad \begin{gathered} \|\mathbf{u}(t)\|_{L_q(\Omega)} = o(t^{-(1/2 - 1/q)}) \quad 2 \leqq q < \infty, \quad \|\mathbf{u}(t)\|_{L_\infty(\Omega)} = o(t^{-1/2}\sqrt{\log t}), \\ \|\nabla \mathbf{u}(t)\|_{L_2(\Omega)} = o(t^{-1/2}) \end{gathered}$$

as $t \to \infty$.

1991 Mathematics Subject Classification. 35Q30.

Key words and phrases. Stokes semigroup, $L_q - L_r$ estimates, boundary value problem in a two dimensional exterior domain, low frequency asymptotic behavior, local energy decay.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 50114088), Ministry of Education, Science and Culture, Japan.

But it is surprising that we do not know any $L_q - L_r$ estimate of the Stokes semigroup in a two dimensional exterior domain like Iwashita [**12**] for the space dimension $n \geqq 3$. Borchers and Varnhorn [**5**, **36**] investigated the behavior of the resolvent of the Stokes operator A in a two dimensional exterior domain by using the classical potential theory, which implied the boundedness of the Stokes semigroup $\{e^{-tA}\}_{t \geqq 0}$ in L_q for any $1 < q < \infty$. But, it dose not seem that the $L_q - L_r$ decay estimates of the Stokes semigroup follow from their results, because we do not know the estimate:

$$\|\nabla e^{-tA}\mathbf{f}\|_{L_q(\Omega)} \leqq \|A^{1/2}e^{-tA}\mathbf{f}\|_{L_q(\Omega)}, \quad t > 0$$

in the two dimensional case, which was proved by Giga and Sohr [**10**] when $n \geqq 3$.

The purpose of this paper is to show the $L_q - L_r$ estimates which is an extension of Iwashita's to two dimensional case. If we apply the $L_q - L_r$ estimates to Kato's argument, we also obtain all of estimates in (D) except L_∞ decay for the corresponding nonlinear Navier-Stokes equations.

To discuss our results more precisely, first we outline at this point our notation used throughout the paper. To denote the special sets, we use the following symbols:

$$D_b = \{x \in \boldsymbol{R}^2 \mid b - 1 \leqq |x| \leqq b\}, \quad S_b = \{x \in \boldsymbol{R}^2 \mid |x| = b\}, \quad \Omega_b = \Omega \cap B_b.$$

Let $W_q^m(D)$ denote the Sobolev space of order m on a domain D in the L_q sense and $\|\cdot\|_{q,m,D}$ its usual norm. For simplicity, we use the following abbreviation:

$$\|\cdot\|_{q,D} = \|\cdot\|_{q,0,D}, \quad \|\cdot\|_{q,m} = \|\cdot\|_{q,m,\Omega}, \quad \|\cdot\|_q = \|\cdot\|_{q,0,\Omega}.$$

Moreover, we put

$$L_{q,b}(D) = \{u \in L_q(D) \mid u(x) = 0 \ \forall x \notin B_b\},$$

$$W_{q,b}^m(D) = \{u \in W_q^m(D) \mid u(x) = 0 \ \forall x \notin B_b\},$$

$$W_{q,loc}^m(\boldsymbol{R}^2) = \{u \in \mathscr{S}' \mid \partial_x^\alpha u \in L_q(B_b) \ ^\forall\alpha, |\alpha| \leq m \text{ and } ^\forall b > 0\},$$

$$W_{q,loc}^m(D) = \{u \mid {}^\exists U \in W_{q,loc}^m(\boldsymbol{R}^2) \text{ such that } u = U \text{ on } D\}, \quad L_{q,loc}(D) = W_{q,loc}^0(D),$$

$$\dot{W}_q^m(D) = \text{the completion of } C_0^\infty(D) \text{ with respect to } \|\cdot\|_{q,m,D},$$

$$\dot{W}_{q,a}^m(D) = \left\{u \in \dot{W}_q^m(D) \,\middle|\, \int_D u(x)\,dx = 0\right\},$$

$$\hat{W}_q^m(D) = \{u \in W_{q,loc}^m(D) \mid \|\partial_x^m u\|_{q,D} < \infty\},$$

$$(\mathbf{u}, \mathbf{v})_D = \int_D \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)}\,dx, \quad (\cdot,\cdot) = (\cdot,\cdot)_\Omega.$$

To denote function spaces of two dimensional column vector-valued functions, we use the bold letters. For example, $\boldsymbol{L}_q(D) = \{\mathbf{u} = {}^t(u_1, u_2) \mid u_j \in L_q(D), j = 1, 2\}$. Likewise for $\boldsymbol{C}_0^\infty(D)$, $\boldsymbol{L}_{q,b}(D)$, $\boldsymbol{W}_{q,loc}^m(D)$, $\boldsymbol{L}_{q,loc}(D)$, $\boldsymbol{W}_q^m(D)$, $\boldsymbol{W}_{q,b}^m(D)$, $\dot{\boldsymbol{W}}_q^m(D)$ and $\hat{\boldsymbol{W}}_q^m(D)$.

Moreover, we put

$$\boldsymbol{J}_q(D) = \text{the completion in } \boldsymbol{L}_q(D) \text{ of the set } \{\mathbf{u} \in \boldsymbol{C}_0^\infty(D) \mid \nabla \cdot \mathbf{u} = 0 \text{ in } D\},$$

$$\boldsymbol{G}_q(D) = \{\nabla p \mid p \in \hat{W}_q^1(D)\}.$$

For exterior domains in $\boldsymbol{R}^3$ Miyakawa [**28**] proved that the Banach space $\boldsymbol{L}_q(D)$ admits the Helmholtz decomposition: $\boldsymbol{L}_q(D) = \boldsymbol{J}_q(D) \oplus \boldsymbol{G}_q(D)$, where $\oplus$ denotes the direct sum. His method carries over to arbitrary space dimensions $n \geq 2$. Let $\boldsymbol{P}_D$ be a continuous projection from $\boldsymbol{L}_q(D)$ onto $\boldsymbol{J}_q(D)$. The Stokes operator $\boldsymbol{A}_D$ is defined by $\boldsymbol{A}_D = -\boldsymbol{P}_D \Delta$ with dense domain $\mathscr{D}_q(\boldsymbol{A}_D) = \boldsymbol{J}_q(D) \cap \dot{\boldsymbol{W}}_q^1(D) \cap \boldsymbol{W}_q^2(D)$. For simplicity, we write: $\boldsymbol{P} = \boldsymbol{P}_\Omega$, $\boldsymbol{A} = \boldsymbol{A}_\Omega$. It is known that $-\boldsymbol{A}$ generates an analytic semigroup $e^{-t\boldsymbol{A}}$ in $\boldsymbol{J}_q(\Omega)$ [**9, 5, 36**], [**4** for $n \geqq 3$]. To denote various constants we use the same letter C, and by $C_{A,B,\cdots}$ we denotes the constant depending on the quantities $A, B, \cdots$. The constants C and $C_{A,B,\cdots}$ may change from line to line. For two Banach spaces X and Y, $\mathscr{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y and $\|\cdot\|_{\mathscr{L}(X,Y)}$ means its operator norm. In particular, we put $\mathscr{L}(X) = \mathscr{L}(X, X)$. $\mathscr{A}(I, X)$ denotes the set of all X-valued analytic functions in I.

Now we state our main results.

Theorem 1.1. (Local energy decay) *Let* $1 < q < \infty$. *For any* $b > b_0$ *and any integer* $m \geqq 0$, *there exists a constant* $C = C_{q,b,m} > 0$ *such that*

$$\|\partial_t^m e^{-t\boldsymbol{A}} \mathbf{f}\|_{q,2,\Omega_b} \leq C t^{-1-m} (\log t)^{-2} \|\mathbf{f}\|_q, \quad t \to \infty \tag{1.1}$$

for any $\mathbf{f} \in \boldsymbol{J}_q(\Omega) \cap \boldsymbol{L}_{q,b}(\Omega) =: \boldsymbol{J}_{q,b}(\Omega)$.

Theorem 1.2. ($L_q - L_r$ estimates) (1) *Let* $1 < q \leqq r < \infty$. *Then the following estimate holds for any* $\mathbf{f} \in \boldsymbol{J}_q(\Omega)$:

$$\|e^{-t\boldsymbol{A}} \mathbf{f}\|_r \leqq C_{q,r} t^{-(1/q-1/r)} \|\mathbf{f}\|_q, \quad t > 0. \tag{1.2}$$

(2) *Let* $1 < q \leqq r \leqq 2$. *Then, for* $\mathbf{f} \in \boldsymbol{J}_q(\Omega)$

$$\|\nabla e^{-t\boldsymbol{A}} \mathbf{f}\|_r \leq C_{q,r} t^{-(1/q-1/r)-1/2} \|\mathbf{f}\|_q, \quad t > 0. \tag{1.3}$$

And let $1 < q \leqq r$ *and* $2 < r < \infty$, *then, for* $\mathbf{f} \in \boldsymbol{J}_q(\Omega)$

$$\|\nabla e^{-t\boldsymbol{A}} \mathbf{f}\|_r \leqq \begin{cases} C_{q,r} t^{-(1/q-1/r)-1/2} \|\mathbf{f}\|_q, & 0 < t < 1, \\ C_{q,r} t^{-1/q} \|\mathbf{f}\|_q, & t \geqq 1. \end{cases} \tag{1.4}$$

Our proof of Theorem 1.2 is based on the local energy decay estimate (1.1) obtained by investigation of the asymptotic behavior of the resolvent of the Stokes operator near the origin. We combine (1.1) with the $L_q - L_r$ estimates in the whole space by cut-off techniques. We are aware of the related work of P. Maremonti and V. A. Solonnikov [**25**]. In their paper, they also obtained $L_q - L_r$ estimates of Stokes semigroup in n-dimensional exterior domain ($n \geqq 2$) by a different method. Their arguments rely on energy estimates, imbedding theorems, $L_q - L_r$ estimates in the whole space and duality arguments.

§2. Preliminaries.

Let us first consider the stationary Stokes equation in $\boldsymbol{R}^2$:

$$(\lambda - \Delta)\mathbf{u} + \nabla \mathfrak{p} = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \boldsymbol{R}^2. \tag{2.1}$$

When $\lambda \in \Sigma = \boldsymbol{C} \backslash \{\lambda \leqq 0\}$, put

$$A_\lambda \mathbf{f} = \mathscr{F}^{-1}\left[\frac{(1 - P(\xi))\hat{\mathbf{f}}(\xi)}{|\xi|^2 + \lambda}\right](x) = E_\lambda * \mathbf{f},$$

$$\Pi \mathbf{f} = \mathscr{F}^{-1}\left[\frac{\xi \cdot \hat{\mathbf{f}}(\xi)}{i|\xi|^2}\right](x) = \mathbf{p} * \mathbf{f}$$

for $\mathbf{f} \in \boldsymbol{L}_q(\boldsymbol{R}^2)$, where $i = \sqrt{-1}$, $P(\xi) = (\xi_j\xi_k/|\xi|^2)_{j,k=1,2}$,

$$\hat{\mathbf{f}}(\xi) = \int_{\boldsymbol{R}^2} e^{-ix\cdot\xi}\mathbf{f}(x)\,dx, \quad \mathscr{F}^{-1}\mathbf{f}(x) = \frac{1}{(2\pi)^2}\int_{\boldsymbol{R}^2} e^{i\xi\cdot x}\mathbf{f}(\xi)\,d\xi$$

and

$$\begin{aligned}
E_\lambda &= E_\lambda(x) = (E_{jk}^\lambda(x))_{j,k=1,2},\\
E_{jk}^\lambda(x) &= (2\pi)^{-1}\{\delta_{jk}K_0(\sqrt{\lambda}|x|) - \lambda^{-1}\partial_j\partial_k(\log|x| + K_0(\sqrt{\lambda}|x|))\}\\
&= (2\pi)^{-1}\left\{\delta_{jk}e_1(\sqrt{\lambda}|x|) + \frac{x_jx_k}{|x|^2}e_2(\sqrt{\lambda}|x|)\right\},\\
\mathbf{p} &= \mathbf{p}(x) = \frac{1}{2\pi}\left(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}\right).
\end{aligned} \tag{2.2}$$

Here, K_n $(n \in \boldsymbol{N} \cup \{0\})$ denotes the modified Bessel function of order n and

$$\begin{aligned}
e_1(\kappa) &= K_0(\kappa) + \kappa^{-1}K_1(\kappa) - \kappa^{-2}\\
&= -\frac{1}{2}\left(\gamma + \frac{1}{2} - \log 2 + \log\kappa\right) + O(\kappa^2)\log\kappa \quad \text{as } \kappa \to 0,\\
&\qquad \text{where } \gamma \text{ is Euler's constant},\\
e_2(\kappa) &= -K_0(\kappa) - 2\kappa^{-1}K_1(\kappa) + 2\kappa^{-2}\\
&= \frac{1}{2} + O(\kappa^2)\log\kappa \quad \text{as } \kappa \to 0.
\end{aligned}$$

These are calculated in [**5**, **36**]. Then, for $1 < q < \infty$ and any integer $m \geqq 0$, by the L_q boundedness of Fourier multiplier (cf. [Theorem 7.9.5 of **11**]), we have

$$A_\lambda \in \mathscr{A}(\Sigma, \mathscr{L}(\boldsymbol{W}_q^{2m}(\boldsymbol{R}^2), \boldsymbol{W}_q^{2m+2}(\boldsymbol{R}^2))), \quad \Pi \in \mathscr{L}(\boldsymbol{W}_q^{2m}(\boldsymbol{R}^2), \hat{W}_q^{2m+1}(\boldsymbol{R}^2)), \tag{2.3}$$

and the pair of $\mathbf{u} = A_\lambda\mathbf{f}$ and $\mathfrak{p} = \Pi\mathbf{f}$ solves (2.1) for $\lambda \in \Sigma$. When $\mathbf{f} \in \boldsymbol{L}_{q,b}(\boldsymbol{R}^2)$, we have

$$A_\lambda\mathbf{f} = O(|x|^{-2}), \quad \Pi\mathbf{f} = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{2.4}$$

For $\lambda = 0$, put

$$A_0\mathbf{f} = E_0 * \mathbf{f} \quad \text{for } \mathbf{f} \in \boldsymbol{W}_q^{2m}(\boldsymbol{R}^2), \tag{2.5}$$

where

$$E_0 = E_0(x) = (E_{jk}^0(x))_{j,k=1,2},$$
$$E_{jk}^0(x) = \frac{1}{4\pi}\left\{-\delta_{jk}\log|x| + \frac{x_j x_k}{|x|^2}\right\}$$

(cf. [IV.2 of **7**]). Then the pair of $\mathbf{u} = A_0\mathbf{f}$ and $\mathfrak{p} = \Pi\mathbf{f}$ solves (2.1) for $\lambda = 0$. We have the following facts for $1 < q < \infty$:

$$\begin{aligned} &A_0 \in \mathscr{L}(\boldsymbol{W}_q^{2m}(\boldsymbol{R}^2), \hat{\boldsymbol{W}}_q^{2m+2}(\boldsymbol{R}^2)), \\ &A_0\mathbf{f} = O(\log|x|) \quad \text{as } |x| \to \infty \quad \text{for } \mathbf{f} \in \boldsymbol{L}_{q,b}(\boldsymbol{R}^2). \end{aligned} \tag{2.6}$$

From (2.2) and (2.5), it follows that

$$E_\lambda(x) = E_0(x) - \frac{1}{4\pi}(c + \log\sqrt{\lambda})I_2 + H_\lambda(x), \tag{2.7}$$

where I_2 is the 2×2 identity matrix, $H_\lambda(x) = O(\lambda|x|^2)\log(\sqrt{\lambda}|x|)$ and $c = \gamma + 1/2 - \log 2$.

Let D be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary ∂D and $\Sigma_0 = \Sigma \cup \{0\}$. We now consider the stationary Stokes equations with parameter $\lambda \in \Sigma_0$ in D:

$$(\lambda - \Delta)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{f} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } D, \tag{2.8}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial D.$$

The existence, uniqueness and regularity of solutions to (2.8) are well known.

Proposition 2.1. *Let $1 < q < \infty$ and let m be an integer $\geqq 0$. Then, for any $\mathbf{f} \in \boldsymbol{W}_q^m(D)$ and $\lambda \in \Sigma_0$, there exists a unique $\mathbf{u} \in \boldsymbol{W}_q^{m+2}(D)$ which together with some $\mathfrak{p} \in W_q^{m+1}(D)$ solves* (2.8); *$\mathfrak{p}$ is unique up to an additive constant. Moreover, the following estimate is valid*:

$$\|\mathbf{u}\|_{q,m+2,D} + \|\nabla\mathfrak{p}\|_{q,m,D} \leqq C_{q,m,D}\|\mathbf{f}\|_{q,m,D}. \tag{2.9}$$

Proof. See Giga [**9**], Ladyzhenskaya [p. 62, Theorem 2 of **21**], Solonnikov [**31**] and Temam [p. 33, Proposition 2.2 of **32**].

The following results in bounded domain D are used later.

Proposition 2.2. *Let $1 < q < \infty$.* (1) *The following relation holds*:

$$\|v\|_{q,D} \leqq C_D\left(\|\nabla v\|_{q,D} + \left|\int_D v(x)\,dx\right|\right), \quad \text{for } v \in W_q^1(D). \tag{2.10}$$

(2) *Let m be an integer $\geqq 0$. Then, for any $u \in W_q^m(D)$, there exists a $v \in W_q^m(\boldsymbol{R}^2)$ such that $u = v$ in D and $\|v\|_{q,m,\boldsymbol{R}^2} \leqq C_{q,m,D}\|u\|_{q,m,D}$, where $C_{q,m,D}$ is a constant independent of u and v.*

Proof. See [II.4 of **7**] for (1) and [II.2 of **7**] for (2).

Proposition 2.3. *Let $1 < q < \infty$ and let m be an integer $\geqq 0$. Then, there exists a linear bounded operator* $\mathbf{B} : \dot{W}^m_{q,a}(D) \to \dot{\boldsymbol{W}}^{m+1}_q(D)$ *such that*

$$\nabla \cdot \mathbf{B}[f] = f \quad in\ D, \quad \|\mathbf{B}[f]\|_{q,m+1,D} \leqq C_{q,m,D}\|f\|_{q,m,D}. \tag{2.11}$$

Proof. See Bogovskii [**1**, **2**] (also Giga and Sohr [Lemma 2.1 of **10**], Iwashita [Proposition 2.5 of **12**] and Galdi [III.3 of **7**]).

Proposition 2.4. *Let $1 < q < \infty$. Let $G = \Omega$ or $\boldsymbol{R}^2$ and let m be an integer $\geqq 1$. Let φ be a function of $C^\infty(\boldsymbol{R}^2)$ such that $\varphi(x) = 1$ for $|x| \leqq b-1$ and $\varphi(x) = 0$ for $|x| \geqq b$, where $b \geqq b_0$. If $\mathbf{u} \in \boldsymbol{W}^m_{q,loc}(G)$, $\nabla \cdot \mathbf{u} = 0$ in G and $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ when $G = \Omega$, then $(\nabla\varphi) \cdot \mathbf{u} \in \dot{W}^m_{q,a}(D_b)$. As a result, $\mathbf{B}[(\nabla\varphi) \cdot \mathbf{u}] \in \dot{\boldsymbol{W}}^{m+1}_q(D_b)$, $\nabla \cdot \mathbf{B}[(\nabla\varphi) \cdot \mathbf{u}] = (\nabla\varphi) \cdot \mathbf{u}$ and*

$$\|\mathbf{B}[(\nabla\varphi) \cdot \mathbf{u}]\|_{q,m+1,\boldsymbol{R}^2} \leqq C_{q,m,\varphi,b}\|\mathbf{u}\|_{q,m,D_b}. \tag{2.12}$$

Proposition 2.5. *Let $1 < q < \infty$. Let $\mathbf{u} \in \hat{\boldsymbol{W}}^2_q(\Omega)$ and $\mathfrak{p} \in \hat{W}^1_q(\Omega)$ satisfy the homogeneous equations*:

$$-\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0} \quad and \quad \nabla \cdot \mathbf{u} = 0 \quad in\ \Omega, \quad \mathbf{u} = \mathbf{0} \quad on\ \partial\Omega. \tag{2.13}$$

Assume that $\mathbf{u}(x)$ and $\mathfrak{p}(x)$ satisfy the following:

$$\mathbf{u}(x) = O(1), \quad \mathfrak{p}(x) = O(|x|^{-1}) \quad as\ |x| \to \infty.$$

Then, $\mathbf{u} = \mathbf{0}$ and $\mathfrak{p} = 0$.

Proof. First of all we shall show that $\mathbf{u} \in \boldsymbol{W}^3_{q,loc}(\Omega)$ and $\mathfrak{p} \in W^2_{q,loc}(\Omega)$. To do this, we use the same cut function φ as in Proposition 2.4. If we put $\mathbf{w} = \varphi\mathbf{u} - \mathbf{B}[(\nabla\varphi) \cdot \mathbf{u}]$ by Proposition 2.3, then $\mathbf{w} \in \boldsymbol{W}^2_q(\Omega)$, $\operatorname{supp}\mathbf{w} \subset \Omega_b$ and $\mathbf{w}$ satisfies the following equations:

$$-\Delta\mathbf{w} + \nabla(\varphi\mathfrak{p}) = \mathbf{g} \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega_b,$$
$$\mathbf{w} = \mathbf{0} \qquad\qquad\qquad \text{on } \partial\Omega_b,$$

where $\mathbf{g} = \nabla\varphi\mathfrak{p} - 2(\nabla\varphi \cdot \nabla)\mathbf{u} + \Delta\mathbf{B}[(\nabla\varphi) \cdot \mathbf{u}]$. Noting that $\mathbf{g} \in \boldsymbol{W}^1_q(\Omega_b)$, we know that $\mathbf{w} \in \boldsymbol{W}^3_q(\Omega_b)$ and $\varphi\mathfrak{p} \in W^2_q(\Omega_b)$ by Proposition 2.1, which means that $\mathbf{u} \in \boldsymbol{W}^3_{q,loc}(\Omega)$ and $\mathfrak{p} \in W^2_{q,loc}(\Omega)$. By Proposition 2.2 (2), $\mathbf{u}$ and $\mathfrak{p}$ have the extensions $\tilde{\mathbf{u}} \in \boldsymbol{W}^3_{q,loc}(\boldsymbol{R}^2)$, $\mathfrak{q} \in W^2_{q,loc}(\boldsymbol{R}^2)$ such that $\mathbf{u} = \tilde{\mathbf{u}}$, $\mathfrak{p} = \mathfrak{q}$ in Ω. Let $\mathcal{O} = \boldsymbol{R}^2 \backslash \bar{\Omega}$. Noting that $\tilde{\mathbf{u}} = \mathbf{0}$ on $\partial\mathcal{O}$, we can apply Proposition 2.3 to find $\mathbf{B}[\nabla \cdot \tilde{\mathbf{u}}] \in \dot{\boldsymbol{W}}^3_q(\bar{\mathcal{O}})$. If we set $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{B}[\nabla \cdot \tilde{\mathbf{u}}]$, then we have $\nabla \cdot \mathbf{v} = 0$ in $\boldsymbol{R}^2$ and $\mathbf{u} = \mathbf{v}$ in Ω.

At this point we prepare the following lemma:

Lemma 2.6. *Assume that $\mathbf{u}$ and $\mathfrak{p} \in \mathcal{S}'$ satisfy $-\Delta\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0}$, $\nabla \cdot \mathbf{u} = 0$ in $\boldsymbol{R}^2$ and $|\mathbf{u}(x)| = O(\log|x|)$, $|\mathfrak{p}(x)| = O(|x|^{-1})$ as $|x| \to \infty$. Then $\mathbf{u} =$ constant and $\mathfrak{p} = 0$.*

Proof. Since $\mathbf{u}$ and $\mathfrak{p}$ satisfy $|\xi|^2\hat{\mathbf{u}} + i\xi\hat{\mathfrak{p}} = \mathbf{0}$ and $i\xi \cdot \hat{\mathbf{u}} = 0$, we have $\operatorname{supp}\hat{\mathbf{u}}$, $\operatorname{supp}\hat{\mathfrak{p}} \subset \{0\}$, which means that $\hat{\mathbf{u}}$ and $\hat{\mathfrak{p}}$ depend on x polynomially. Considering that $|\mathbf{u}(x)| = O(\log|x|)$ and $|\mathfrak{p}(x)| = O(|x|^{-1})$ as $|x| \to \infty$, we have $\mathbf{u} =$ constant and $\mathfrak{p} = 0$. □

We continue the proof of Proposition 2.5. We set $\mathbf{f} = -\Delta\mathbf{v} + \nabla\mathfrak{q}$. Since $\mathbf{f} \in \boldsymbol{W}^1_{q,loc}(\boldsymbol{R}^2)$ and $\operatorname{supp}\mathbf{f} \subset \bar{\mathcal{O}}$, then $\mathbf{f} \in \boldsymbol{L}_2(\boldsymbol{R}^2)$. If we put $\mathbf{z} = A_0\mathbf{f}$ and $\mathfrak{r} = \Pi\mathbf{f}$, then we have $-\Delta(\mathbf{z} - \mathbf{v}) + \nabla(\mathfrak{r} - \mathfrak{q}) = \mathbf{0}$ and $\nabla \cdot (\mathbf{z} - \mathbf{v}) = 0$ in $\boldsymbol{R}^2$. Since $\mathbf{z} = O(\log|x|)$ and $\mathfrak{r} = O(|x|^{-1})$ as $|x| \to \infty$ and $\mathbf{v} = \mathbf{u} = O(1)$ as $|x| \to \infty$, we know $\mathbf{z} - \mathbf{v} = O(\log|x|)$ and $\mathfrak{r} - \mathfrak{q} = O(|x|^{-1})$ as $|x| \to \infty$. By Lemma 2.6, we have $\mathbf{z} = \mathbf{v} + \text{constant} = O(1)$ and $\mathfrak{r} = \mathfrak{q}$. From the fact: $\mathbf{z} = E_0(x)\int_{\boldsymbol{R}^2}\mathbf{f}(y)\,dy + O(|x|^{-1})$, we have $\int_{\boldsymbol{R}^2}\mathbf{f}(y)\,dy = \mathbf{0}$, which means that $\mathbf{z} = O(|x|^{-1})$, $\nabla\mathbf{z} = O(|x|^{-2})$, $\mathfrak{r} = O(|x|^{-2})$ and $\nabla\mathfrak{r} = O(|x|^{-3})$. Therefore we have

$$\mathbf{u} = O(1), \quad \nabla\mathbf{u} = O(|x|^{-2}), \quad \mathfrak{p} = O(|x|^{-2}) \quad \text{and} \quad \nabla\mathfrak{p} = O(|x|^{-3}) \tag{2.14}$$

as $|x| \to \infty$, which implies that

$$\|\nabla\mathbf{u}\|_2 = 0. \tag{2.15}$$

In fact, let us consider the formula:

$$\begin{aligned} 0 &= (-\Delta\mathbf{u} + \nabla\mathfrak{p}, \mathbf{u})_{\Omega_R} \\ &= \left(-\left(\frac{x}{|x|}\cdot\nabla\right)\mathbf{u}, \mathbf{u}\right)_{|x|=R} + \left(\frac{x}{|x|}\mathfrak{p}, \mathbf{u}\right)_{|x|=R} + (\nabla\mathbf{u}, \nabla\mathbf{u})_{\Omega_R}. \end{aligned}$$

By (2.14) the first and the second terms of right hand side tend to 0 as $R \to \infty$, thus we have (2.15), which implies that $\nabla\mathbf{u} = \mathbf{0}$. From the boundary condition it follows $\mathbf{u} = \mathbf{0}$ and $\nabla\mathfrak{p} = \mathbf{0}$. By the assumption, we have $\mathfrak{p} = 0$. □

Proposition 2.7. *Let $1 < q < \infty$ and $G = \boldsymbol{R}^2$ or Ω. Let $\mathbf{u} \in \hat{\boldsymbol{W}}^2_q(G)$ and $\mathfrak{p} \in \hat{W}^1_q(G)$ satisfy the equations*:

$$(\lambda - \Delta)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0} \quad \textit{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \textit{in } \Omega, \quad \mathbf{u} = \mathbf{0} \quad \textit{on } \partial\Omega \quad \textit{if } G = \Omega$$

for $\lambda \in \Sigma$. Assume that $\mathfrak{p} = O(|x|^{-1})$. Then, $\mathbf{u}(x) = \mathbf{0}$ and $\mathfrak{p}(x) = 0$.

Proof. When $G = \boldsymbol{R}^2$, since $\mathbf{u}$ and $\mathfrak{p}$ satisfy $(\lambda + |\xi|^2)\hat{\mathbf{u}} + i\xi\hat{\mathfrak{p}} = \mathbf{0}$ and $i\xi\cdot\hat{\mathbf{u}} = 0$, $\operatorname{supp}\{(\lambda + |\xi|^2)\hat{\mathbf{u}}\} = \operatorname{supp}(i\xi\hat{\mathfrak{p}}) = \emptyset$. In view of $\lambda + |\xi|^2 \neq 0$ for $\lambda \in \Sigma$, $\mathbf{u} = \mathbf{0}$ and $\mathfrak{p} =$ constant. From the assumption $\mathfrak{p} = O(|x|^{-1})$, we have $\mathfrak{p} = 0$.

When $G = \Omega$, let the pair of $(\mathbf{v}, \mathfrak{q})$ be an extension of $(\mathbf{u}, \mathfrak{p})$ to $\boldsymbol{R}^2$ such that $\mathbf{v} \in \boldsymbol{W}^3_{q,loc}(\boldsymbol{R}^2)$, $\mathfrak{q} \in W^2_{q,loc}(\boldsymbol{R}^2)$ and $\nabla\cdot\mathbf{v} = 0$ in $\boldsymbol{R}^2$ (cf. proof of Proposition 2.5). We set $\mathbf{f} = (\lambda - \Delta)\mathbf{v} + \nabla\mathfrak{q}$, then $\operatorname{supp}\mathbf{f} \subset \bar{\mathcal{O}}$ and $\mathbf{f} \in \boldsymbol{L}_2(\boldsymbol{R}^2)$. If we put $\mathbf{z} = A_\lambda\mathbf{f}$ and $\mathfrak{r} = \Pi\mathbf{f}$, in view of the result for $G = \boldsymbol{R}^2$, we have $\mathbf{u} = \mathbf{v} = \mathbf{z} = O(|x|^{-2})$ and $\mathfrak{p} = \mathfrak{q} = \mathfrak{r} = O(|x|^{-1})$ as $|x| \to \infty$ by (2.4). Therefore from the same argument as Proposition 2.5 we have $\mathbf{u} = \mathbf{0}$, $\mathfrak{p} = 0$. □

Proposition 2.8. *Let $1 < q < \infty$ and let $\boldsymbol{A}$ be the Stokes operator in $\boldsymbol{J}_q(\Omega)$ and m be any integer $\geqq 0$.*

(1) *Assume that $\mathbf{u} \in \mathcal{D}_q(\boldsymbol{A})$ and $\boldsymbol{A}\mathbf{u} \in \boldsymbol{W}^m_q(\Omega)$. Then $\mathbf{u} \in \boldsymbol{W}^{m+2}_q(\Omega)$ and for some constant $C_{q,m} > 0$,*

$$\|\mathbf{u}\|_{q,m+2} \leqq C_{q,m}(\|\boldsymbol{A}\mathbf{u}\|_{q,m} + \|\mathbf{u}\|_q).$$

(2) *If* $\mathbf{u} \in \mathscr{D}_q(\boldsymbol{A}^m)$, *then*

$$\|\mathbf{u}\|_{q,2m} \leqq C_{q,m}(\|\boldsymbol{A}^m\mathbf{u}\|_q + \|\mathbf{u}\|_q),$$

$$\|\boldsymbol{A}^m\mathbf{u}\|_q \leqq C_{q,m}\|\mathbf{u}\|_{q,2m}.$$

Proof. See [Proposition 2.7, 2.8 of **12**].

§3. Asymptotic behavior of the resolvent around the origin

Let us consider the stationary problem for the Stokes equation with parameter $\lambda \in \Sigma$ in Ω:

$$\text{(S)} \qquad (\lambda - \Delta)\mathbf{u} + \nabla \mathfrak{p} = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega.$$

In terms of the Stokes operator $\boldsymbol{A}$, (S) is written in the form:

$$\text{(S}'\text{)} \qquad (\lambda + \boldsymbol{A})\mathbf{u} = \mathbf{f}.$$

Giga [**9**] and Borchers and Varnhorn [**5**, **36**] proved that Σ belongs to the resolvent set $\rho(\boldsymbol{A})$ of $\boldsymbol{A}$ and

$$\|(\lambda + \boldsymbol{A})^{-1}\|_{\mathscr{L}(\boldsymbol{J}_q(\Omega))} \leqq C_{q,\tau}|\lambda|^{-1}, \tag{3.1}$$

when $|\arg \lambda| \leqq \tau$ for any $0 < \tau < \pi$.

Let $b > b_0 + 4$ and $1 < q < \infty$. Contracting the domain of $(\lambda + \boldsymbol{A})^{-1}$ from $\boldsymbol{J}_q(\Omega)$ to $\boldsymbol{J}_{q,b}(\Omega)$, we shall investigate the asymptotic behavior of $(\lambda + \boldsymbol{A})^{-1}$ as $|\lambda| \to 0$. Put $\Sigma_{\tau,\varepsilon} = \{\lambda \in \Sigma \mid |\arg \lambda| \leqq \tau, |\lambda| \leqq \varepsilon\}$.

Proposition 3.1. *Let* $1 < q < \infty$ *and* m *be any integer* $\geqq 0$. *There exist operator valued functions* R_λ *and* P_λ *possessing the following properties*:

(1)
$$R_\lambda \in \mathscr{A}(\Sigma, \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), \boldsymbol{W}^{2m+2}_q(\Omega_b))),$$
$$P_\lambda \in \mathscr{A}(\Sigma, \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), W^{2m+1}_q(\Omega_b))),$$

(2) *the pair of* $\mathbf{u} = R_\lambda\mathbf{f}$ *and* $\mathfrak{p} = P_\lambda\mathbf{f}$ *is a solution to* (S) *and*

$$R_\lambda\mathbf{f} \in \boldsymbol{W}^{2m+2}_q(\Omega), \quad P_\lambda\mathbf{f} \in \hat{W}^{2m+1}_q(\Omega), \quad P_\lambda\mathbf{f} = O(|x|^{-1}) \quad \text{as } |x| \to \infty \tag{3.2}$$

for $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$, $\lambda \in \Sigma$, *and we have*

$$R_\lambda = (\lambda + \boldsymbol{A})^{-1} \quad \text{on } \boldsymbol{J}_{q,b}(\Omega) \quad \text{for } \lambda \in \Sigma, \tag{3.3}$$

(3) *for any* $0 < \tau < \pi$, *there exists an* $\varepsilon = \varepsilon(\tau)$ *such that for* $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$ *and* $\lambda \in \Sigma_{\tau,\varepsilon}$,

$$\begin{pmatrix} R_\lambda \\ P_\lambda \end{pmatrix}\mathbf{f} = \lambda^s \begin{pmatrix} M(\log\lambda)/L(\log\lambda) \\ \tilde{M}(\log\lambda)/\tilde{L}(\log\lambda) \end{pmatrix}\mathbf{f} + O(\lambda^{s+1}(\log\lambda)^\beta), \tag{3.4}$$

where s *is an integer* (*not necessarily positive*); L *and* $\tilde{L}$ *are polynomials with constant* *coefficients and M (resp. $\tilde{M}$) is a polynomial, not identically zero, whose coefficients belong to $\mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), \boldsymbol{W}^{2m+2}_{q}(\Omega_b))$ (resp. $\mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), W^{2m+1}_{q}(\Omega_b))$); β is an integer. The order symbol O is used in the sense that*

$$\|R_\lambda \mathbf{f} - \lambda^s (M(\log\lambda)/L(\log\lambda))\mathbf{f}\|_{q,2m+2,\Omega_b} \leqq C_{q,m,b}|\lambda^{s+1}(\log\lambda)^\beta|\,\|\mathbf{f}\|_{q,2m},$$

$$\|P_\lambda \mathbf{f} - \lambda^s (\tilde{M}(\log\lambda)/\tilde{L}(\log\lambda))\mathbf{f}\|_{q,2m+1,\Omega_b} \leqq C_{q,m,b}|\lambda^{s+1}(\log\lambda)^\beta|\,\|\mathbf{f}\|_{q,2m}.$$

Proof. At first, we introduce some symbols. Let φ be a function of $C^\infty(\boldsymbol{R}^2)$ such that $\varphi(x) = 0$ for $|x| \geqq b-1$ and $\varphi(x) = 1$ for $|x| \leqq b-2$. For $\mathbf{f} \in \boldsymbol{L}_q(\Omega)$ let us denote the restriction of $\mathbf{f}$ on Ω_b by $\pi_b\mathbf{f}$ and define the extension $\imath\mathbf{f}$ of $\mathbf{f}$ to whole $\boldsymbol{R}^2$ by the relation: $\imath\mathbf{f}(x) = \mathbf{f}(x)$ for $x \in \Omega$ and $\imath\mathbf{f}(x) = \mathbf{0}$ for $x \in \boldsymbol{R}^2\backslash\Omega$. Let $L_{b\lambda}$ and $\mathfrak{p}_{b\lambda}$ be the operators defined by the relations: $L_{b\lambda}\mathbf{g} = \mathbf{w}$ and $\mathfrak{p}_{b\lambda}\mathbf{g} = \mathfrak{q}$ where the pair of $\mathbf{w}$ and $\mathfrak{q}$ is the solution of the following Stokes equation in Ω_b:

$$(\lambda - \Delta)\mathbf{w} + \nabla\mathfrak{q} = \mathbf{g} \quad \text{and} \quad \nabla\cdot\mathbf{w} = 0 \quad \text{in } \Omega_b, \quad \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega_b, \tag{3.5}$$

where $\partial\Omega_b = S_b \cup \partial\Omega$ and $\lambda \in \Sigma_0$. $\mathfrak{p}_{b\lambda}\mathbf{g}$ is not decided uniquely at this moment, that is we have freedom to choose any additive constant, which will be chosen in (3.8) below. By Proposition 2.1 we know that

$$\|L_{b\lambda}\mathbf{g}\|_{q,2m+2,\Omega_b} + \|\nabla\mathfrak{p}_{b\lambda}\mathbf{g}\|_{q,2m,\Omega_b} \leqq C_{q,m,b,\lambda}\|\mathbf{g}\|_{q,2m,\Omega_b}. \tag{3.6}$$

Let us construct R_λ and P_λ from a compact perturbation of the following operators:

$$\begin{aligned} \Phi_\lambda\mathbf{f} &= (1-\varphi)(A_\lambda\imath\mathbf{f}) + \varphi L_{b\lambda}\pi_b\mathbf{f} + \mathbf{B}[(\nabla\varphi)\cdot A_\lambda\imath\mathbf{f}] - \mathbf{B}[(\nabla\varphi)\cdot L_{b\lambda}\pi_b\mathbf{f}], \\ \Psi_\lambda\mathbf{f} &= (1-\varphi)(\Pi\imath\mathbf{f}) + \varphi\mathfrak{p}_{b\lambda}\pi_b\mathbf{f}, \end{aligned} \tag{3.7}$$

for $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$, where we have used Proposition 2.4. Now, $\mathfrak{p}_{b\lambda}$ is chosen so that

$$\int_{\Omega b} (\mathfrak{p}_{b\lambda}\pi_b\mathbf{f} - \Pi\imath\mathbf{f})(x)\,dx = \mathbf{0}. \tag{3.8}$$

We know that there exists an $a > 0$ such that $L_{b\lambda}$ and $\mathfrak{p}_{b\lambda}$ are analytic with respect to $\lambda \in \boldsymbol{C}\backslash(-\infty, -a]$ (cf. [Proposition 2.6 of **18**]). From the construction, we have

$$(\lambda - \Delta)\Phi_\lambda\mathbf{f} + \nabla\Psi_\lambda\mathbf{f} = (1 + F_\lambda)\mathbf{f} \quad \text{in } \Omega, \tag{3.9}$$

$$\nabla\cdot\Phi_\lambda\mathbf{f} = 0 \quad \text{in } \Omega, \quad \Phi_\lambda\mathbf{f} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{3.10}$$

where

$$\begin{aligned} F_\lambda\mathbf{f} = {} & 2(\nabla\varphi\cdot\nabla)A_\lambda\imath\mathbf{f} + \Delta\varphi A_\lambda\imath\mathbf{f} - 2(\nabla\varphi\cdot\nabla)L_{b\lambda}\pi_b\mathbf{f} - \Delta\varphi L_{b\lambda}\pi_b\mathbf{f} \\ & + (\lambda-\Delta)\mathbf{B}[\nabla\varphi\cdot A_\lambda\imath\mathbf{f}] - (\lambda-\Delta)\mathbf{B}[\nabla\varphi\cdot L_{b\lambda}\pi_b\mathbf{f}] - \nabla\varphi\Pi\imath\mathbf{f} + \nabla\varphi\mathfrak{p}_{b\lambda}\pi_b\mathbf{f}. \end{aligned}$$

Contracting the domain of A_λ and Π, and considering those ranges in wider spaces, we have

$$A_\lambda\imath \in \mathscr{A}(\Sigma, \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), \boldsymbol{W}^{2m+2}_{q}(\Omega_b))) \quad \text{and} \quad \Pi\imath \in \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), W^{2m+1}_{q}(\Omega_b)).$$

At each point $\lambda \in \Sigma$, F_λ is a compact operator from $\boldsymbol{W}^{2m}_{q,b}(\Omega)$ into itself and F_λ is analytic in $\lambda \in \Sigma$. We know that $(1 + F_\lambda)^{-1}$ is analytic in Σ. In fact, in view of

Fredholm alternative theorem [VI §4 of **35**], it is sufficient that $1+F_\lambda$ is injective for $\lambda \in \Sigma$. Let $\mathbf{f}$ be an element of $\boldsymbol{W}^{2m}_{q,b}(\Omega)$ such that $(1+F_\lambda)\mathbf{f} = \mathbf{0}$. Since $\Phi_\lambda \mathbf{f}$ and $\Psi_\lambda \mathbf{f}$ satisfy the condition of Proposition 2.7, we see that $\Phi_\lambda \mathbf{f} = \mathbf{0}$ and $\Psi_\lambda \mathbf{f} = 0$. Therefore, employing the same argument as in the proof of Lemma 3.5 in Iwashita [**12**], we can show that $\mathbf{f} = \mathbf{0}$. Thus $(1+F_\lambda)^{-1} \in \mathscr{A}(\Sigma, \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega)))$. Put

$$R_\lambda = \Phi_\lambda (1+F_\lambda)^{-1} \quad \text{and} \quad P_\lambda = \Psi_\lambda (1+F_\lambda)^{-1}, \tag{3.11}$$

then the pair of $\mathbf{u} = R_\lambda \mathbf{f}$ and $\mathfrak{p} = P_\lambda \mathbf{f}$ solves (S) as $\lambda \in \Sigma$. By Proposition 2.7, when $\mathbf{f} \in \boldsymbol{J}_{q,b}(\Omega)$, $R_\lambda \mathbf{f} = (\lambda + \boldsymbol{A})^{-1}\mathbf{f}$ for $\lambda \in \Sigma$.

Thus we know the analyticity of R_λ in Σ, but our purpose is to investigate the asymptotic behavior of at $\lambda = 0$. If we recall (2.7), then we have the following formula:

$$A_\lambda \imath \mathbf{f} = A_0 \imath \mathbf{f} - \frac{1}{4\pi}(c + \log\sqrt{\lambda}) T\mathbf{f} + B_\lambda \mathbf{f}, \tag{3.12}$$

where $T\mathbf{f} = \int_{\boldsymbol{R}^2} \imath \mathbf{f}\, dx$ and $B_\lambda \mathbf{f} = H_\lambda * \imath \mathbf{f} \in \boldsymbol{W}^{2m+2}_q(\Omega_b)$ for $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$, $\lambda \in \Sigma$. In investigating the asymptotic behavior of R_λ at $\lambda = 0$, difficulties arise from logarithmic singularity. But this singularity appears only in the coefficients of finite dimensional operators. To make the above point clear, let us consider the auxiliary operator:

$$\begin{aligned} \Phi_0 \mathbf{f} &= (1-\varphi) A_0 \imath \mathbf{f} + \varphi L_{b0} \pi_b \mathbf{f} + \mathbf{B}[(\nabla\varphi) \cdot A_0 \imath \mathbf{f}] - \mathbf{B}[(\nabla\varphi) \cdot L_{b0}\pi_b \mathbf{f}], \\ \Psi_0 \mathbf{f} &= (1-\varphi)(\Pi \imath \mathbf{f}) + \varphi \mathfrak{p}_{b0} \pi_b \mathbf{f}, \end{aligned} \tag{3.13}$$

for $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$. Then,

$$-\Delta \Phi_0 \mathbf{f} + \nabla \Psi_0 \mathbf{f} = (1 + S_0)\mathbf{f} \quad \text{and} \quad \nabla \cdot \Phi_0 \mathbf{f} = 0, \tag{3.14}$$

where

$$\begin{aligned} S_0 \mathbf{f} &= 2(\nabla\varphi \cdot \nabla)(A_0 \imath \mathbf{f}) + (\Delta\varphi) A_0 \imath \mathbf{f} - 2(\nabla\varphi \cdot \nabla)(L_{b0}\pi_b \mathbf{f}) - (\Delta\varphi) L_{b0} \pi_b \mathbf{f} \\ &\quad - \Delta \mathbf{B}[(\nabla\varphi) \cdot A_0 \imath \mathbf{f}] + \Delta \mathbf{B}[(\nabla\varphi) \cdot L_{b0}\pi_b \mathbf{f}] - (\nabla\varphi) \Pi \imath \mathbf{f} + (\nabla\varphi)\mathfrak{p}_{b0}\pi_b \mathbf{f}. \end{aligned}$$

We see that S_0 is a compact operator from $\boldsymbol{W}^{2m}_{q,b}(\Omega)$ into itself. Taking (3.12) into account, we have the following formula:

$$(1+F_\lambda)\mathbf{f} = (1+S_\lambda)\mathbf{f} - \frac{1}{4\pi}\Delta\varphi(c + \log\sqrt{\lambda}) T\mathbf{f} + \frac{1}{4\pi}(c + \log\sqrt{\lambda}) \Delta \mathbf{B}[\nabla\varphi \cdot T\mathbf{f}], \tag{3.15}$$

where

$$\begin{aligned} S_\lambda \mathbf{f} &= S_0 \mathbf{f} + (\nabla\varphi \cdot \nabla)(B_\lambda \mathbf{f}) + \Delta B_\lambda \mathbf{f} - 2(\nabla\varphi \cdot \nabla)(L_{b\lambda} - L_{b0})\pi_b \mathbf{f} - (\Delta\varphi)(L_{b\lambda} - L_{b0})\pi_b \mathbf{f} \\ &\quad + \lambda \mathbf{B}[(\nabla\varphi) \cdot A_\lambda \imath \mathbf{f}] - \Delta \mathbf{B}[(\nabla\varphi) \cdot B_\lambda \mathbf{f}] - \lambda \mathbf{B}[(\nabla\varphi) \cdot L_{b\lambda}\pi_b \mathbf{f}] \\ &\quad + \Delta \mathbf{B}[\nabla\varphi \cdot (L_{b\lambda} - L_{b0})\pi_b \mathbf{f}] + \nabla\varphi(\mathfrak{p}_{\mathbf{b}\lambda} - \mathfrak{p}_{\mathbf{b0}})\pi_{\mathbf{b}} \mathbf{f}. \end{aligned}$$

S_λ is continuous at $\lambda = 0$, i.e.

$$\|S_\lambda - S_0\|_{\mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega))} \to 0 \quad \text{as } |\lambda| \to 0. \tag{3.16}$$

In order to investigate the behavior of $(1+F_\lambda)^{-1}$, modifying $1+S_\lambda$ in terms of some finite dimensional operators, we will construct inverse of the modified operator. To do this, we would like to start with the following lemma.

LEMMA 3.2. *$1+S_0$ is one to one on the domain $X=\{\mathbf{f}\in \boldsymbol{W}^{2m}_{p,b}(\Omega) \mid T\mathbf{f}=\mathbf{0}\}$.*

PROOF. Assume that $\mathbf{f}\in X$ satisfies $(1+S_0)\mathbf{f}=\mathbf{0}$. Since $\int_{\boldsymbol{R}^2} \iota\mathbf{f}\,dx=0$, we have

$$\Phi_0\mathbf{f}=(1-\varphi)\int_{\boldsymbol{R}^2}(E_0(x-y)-E_0(x))\iota\mathbf{f}(y)\,dy+\varphi L_{b0}\pi_b\mathbf{f}$$
$$+\mathbf{B}[(\nabla\varphi)\cdot A_0\iota\mathbf{f}]-\mathbf{B}[(\nabla\varphi)\cdot L_{b0}\pi_b\mathbf{f}].$$

Thus, $\Phi_0\mathbf{f}=O(|x|^{-1})$. On the other hand, from (3.14) it follows that

$$-\Delta\Phi_0\mathbf{f}+\nabla\Psi_0\mathbf{f}=\mathbf{0}\quad\text{and}\quad\nabla\cdot\Phi_0\mathbf{f}=0\quad\text{in }\Omega,\qquad\Phi_0\mathbf{f}=\mathbf{0}\quad\text{on }\partial\Omega.$$

Since $\Phi_0\mathbf{f}$ and $\Psi_0\mathbf{f}$ satisfy the condition of Proposition 2.5, we have $\Phi_0\mathbf{f}=\mathbf{0}$ and $\Psi_0\mathbf{f}=0$, which means $\mathbf{f}=\mathbf{0}$. □

LEMMA 3.3. $\dim\mathrm{Ker}(1+S_0)\leqq 2$.

PROOF. Suppose that $\dim\mathrm{Ker}(1+S_0)\geqq 3$. Pick up non-zero two dimensional vectors of functions $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3\in\mathrm{Ker}(1+S_0)$. Since $T\mathbf{k}_j$ $j=1,2,3$ are two dimensional numerical vectors, there exist constants α_1, α_2 and α_3 such that $(\alpha_1,\alpha_2,\alpha_3)\neq(0,0,0)$ and $\mathbf{0}=\sum_{j=1}^3\alpha_jT\mathbf{k}_j=T(\sum_{j=1}^3\alpha_j\mathbf{k}_j)$, which together with Lemma 3.2 implies that $\sum_{j=1}^3\alpha_j\mathbf{k}_j=\mathbf{0}$. This completes the proof of the lemma. □

When $\dim\mathrm{Ker}(1+S_0)\neq 0$, in view of Lemma 3.2 we can find a $\mathbf{k}={}^t(k_1,k_2)\in\mathrm{Ker}(1+S_0)$ such that $T\mathbf{k}\neq 0$, so that without loss of generality we may assume that $Tk_1=1$. Since the dimension of the kernel of a Fredholm operator coincides with that of its cokernel, we can choose $\mathbf{m}_1$ and $\mathbf{m}_2\notin\mathrm{Im}\,(1+S_0)$ so that

$$\boldsymbol{W}^{2m}_{q,b}(\Omega)=\mathrm{Im}\ (1+S_0)\oplus\boldsymbol{C}\mathbf{m}_1\oplus\boldsymbol{C}\mathbf{m}_2, \tag{3.17}$$

where $\mathbf{m}_2=\mathbf{0}$ if $\dim\mathrm{Ker}(1+S_0)=1$ and $\mathbf{m}_1=\mathbf{m}_2=\mathbf{0}$ if $\dim\mathrm{Ker}(1+S_0)=0$. Let us define the operator:

$$G_0\mathbf{f}=(1+S_0)\mathbf{f}+(Tf_1)\mathbf{m}_1+(Tf_2)\mathbf{m}_2$$

for $\mathbf{f}={}^t(f_1,f_2)\in\boldsymbol{W}^{2m}_{q,b}(\Omega)$.

LEMMA 3.4. *G_0 is bijective Fredholm operator, so that inverse G_0^{-1} is continuous, too.*

PROOF. From the construction, obviously G_0 is a Fredholm operator. In order to prove bijectivity, it is sufficient to prove injectivity of G_0. When $\dim\mathrm{Ker}(1+S_0)=0$, it is trivial. Next we consider the case that $\dim\mathrm{Ker}(1+S_0)=2$. If $G_0\mathbf{f}=\mathbf{0}$, then $(1+S_0)\mathbf{f}=-Tf_1\mathbf{m}_1-Tf_2\mathbf{m}_2$. In view of (3.17), $T\mathbf{f}=\mathbf{0}$ and $(1+S_0)\mathbf{f}=\mathbf{0}$, so that we have $\mathbf{f}=\mathbf{0}$ by Lemma 3.2. Finally we consider the case that $\dim\mathrm{Ker}(1+S_0)=1$. If $G_0\mathbf{f}=\mathbf{0}$, then $(1+S_0)\mathbf{f}=-Tf_1\mathbf{m}_1$. From (3.17) it follows that $Tf_1=0$ and $(1+S_0)\mathbf{f}=\mathbf{0}$. Since $\mathbf{f}\in\mathrm{Ker}\,(1+S_0)$, there extists α such that $\mathbf{f}=\alpha\mathbf{k}$. Then $0=Tf_1=\alpha Tk_1=\alpha$, which implies that $\mathbf{f}=\mathbf{0}$. □

Set

$$G_\lambda \mathbf{f} = (I + S_\lambda)\mathbf{f} + (Tf_1)\mathbf{m}_1 + (Tf_2)\mathbf{m}_2.$$

LEMMA 3.5. *For any* $0 < \tau < \pi$, *there exists an* $\varepsilon = \varepsilon(\tau) > 0$ *such that*

$$G_\lambda^{-1} = G_0^{-1} \sum_{j=0}^{\infty} [(S_\lambda - S_0)G_0^{-1}]^j \quad \lambda \in \Sigma_{\tau,\varepsilon}. \tag{3.18}$$

PROOF. For $\lambda \neq 0$, G_λ can be represented in the form

$$\begin{aligned} G_\lambda &= G_\lambda - G_0 + G_0 = G_0 + (S_\lambda - S_0) \\ &= \{I + (S_\lambda - S_0)G_0^{-1}\}G_0. \end{aligned}$$

For any $0 < \tau < \pi$, by (3.16) there exists an $\varepsilon = \varepsilon(\tau) > 0$ such that

$$\|S_\lambda - S_0\|_{\mathscr{L}(W_{q,b}^{2m}(\Omega))} \|G_0^{-1}\|_{\mathscr{L}(W_{q,b}^{2m}(\Omega))} \leqq 1/2$$

for $\lambda \in \Sigma_{\tau,\varepsilon}$, which completes a proof. □

Using G_λ, we shall investigate the behavior of $(I + F_\lambda)^{-1}$. In terms of G_λ we have

$$(1 + F_\lambda)\mathbf{f} = G_\lambda \mathbf{f} + N_\lambda(T\mathbf{f}), \tag{3.19}$$

where

$$N_\lambda \mathbf{d} = -d_1\mathbf{m}_1 - d_2\mathbf{m}_2 - \frac{1}{4\pi}\Delta\varphi(c + \log\sqrt{\lambda})\,\mathbf{d} + \frac{1}{4\pi}(c + \log\sqrt{\lambda})\Delta\mathbf{B}[\nabla\varphi \cdot \mathbf{d}], \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Thus we consider the equation:

$$G_\lambda \mathbf{f} + N_\lambda(T\mathbf{f}) = \mathbf{g} \quad \text{for } \mathbf{g} \in W_{q,b}^{2m}(\Omega).$$

By Lemma 3.5 we have

$$\mathbf{f} + G_\lambda^{-1} N_\lambda(T\mathbf{f}) = G_\lambda^{-1}\mathbf{g}. \tag{3.20}$$

Let $\rho \in C_0^\infty(\Omega_b)$ be a function such that $T\rho = 1$. Let us decompose $\mathbf{f}$ as follows:

$$\mathbf{f} = \mathbf{f}_a + (T\mathbf{f})\rho, \quad \mathbf{f}_a = \mathbf{f} - (T\mathbf{f})\rho,$$

where $T\mathbf{f}_a = \mathbf{0}$. In the same way, we write

$$\begin{aligned} G_\lambda^{-1} N_\lambda(T\mathbf{f}) &= (G_\lambda^{-1} N_\lambda(T\mathbf{f}))_a + (TG_\lambda^{-1} N_\lambda(T\mathbf{f}))\rho, \\ G_\lambda^{-1}\mathbf{g} &= (G_\lambda^{-1}\mathbf{g})_a + (TG_\lambda^{-1}\mathbf{g})\rho, \end{aligned}$$

where $T(G_\lambda^{-1} N_\lambda(T\mathbf{f}))_a = \mathbf{0}$ and $T(G_\lambda^{-1}\mathbf{g})_a = \mathbf{0}$. Thus from (3.20) we have

$$\mathbf{f}_a + (G_\lambda^{-1} N_\lambda(T\mathbf{f}))_a + ((T\mathbf{f}) + TG_\lambda^{-1} N_\lambda(T\mathbf{f}))\rho = (G_\lambda^{-1}\mathbf{g})_a + (TG_\lambda^{-1}\mathbf{g})\rho.$$

Applying T, we have

$$L_\lambda(T\mathbf{f}) = TG_\lambda^{-1}\mathbf{g},$$

where $L_\lambda = I + TG_\lambda^{-1}N_\lambda$ is a linear operator from $\boldsymbol{C}^2$ to $\boldsymbol{C}^2$. From (3.18) and (3.19) it follows that the elements of $\tilde{L}_\lambda = \lambda L_\lambda$ can be represented as numerical series, absolutely and uniformly convergent in $\Sigma_{\tau,\varepsilon}$, of the form

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\alpha_{jk}(\log\lambda)^k\right)\lambda^j.$$

In particular,

$$\det\tilde{L}_\lambda = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}d_{jk}(\log\lambda)^k\right)\lambda^j = \sum_{j=0}^{\infty}\lambda^j D_j(\log\lambda),$$

where $D_j(t) = \sum_{k=0}^{j} d_{jk}t^k$ is a polynomial of degree j. If $D_j(t) \equiv 0$ for all j, that is, $\det\tilde{L}_\lambda \equiv 0 \equiv \det L_\lambda$, then there exists a $\mathbf{d} \neq \mathbf{0}$ such that $L_\lambda\mathbf{d} = \mathbf{0}$. Put

$$\mathbf{z} = -G_\lambda^{-1}N_\lambda\mathbf{d}.$$

Then $T\mathbf{z} = \mathbf{d}$. By (3.19), $(1 + F_\lambda)\mathbf{z} = \mathbf{0}$, which implies that $\mathbf{z} = \mathbf{0}$, that is $\mathbf{d} = \mathbf{0}$. This leads to a contradiction. Hence, there is an $a < \infty$ such that $D_a(t) \not\equiv 0$ and $D_j(t) \equiv 0$ for $j < a$. Then

$$\det\tilde{L}_\lambda = \lambda^a D_a(\log\lambda)\left[1 + \sum_{s=1}^{\infty}\frac{\lambda^s D_{a+s}(\log\lambda)}{D_a(\log\lambda)}\right].$$

Since in this formula the sum over s tends to zero when $|\lambda| \to 0$, for suffciently small $\varepsilon = \varepsilon(\tau) > 0$ we have

$$(\det\tilde{L}_\lambda)^{-1} = \frac{\lambda^{-a}}{D_a(\log\lambda)}\sum_{r=0}^{\infty}\left[-\sum_{s=1}^{\infty}\frac{\lambda^s R_{s(a+1)}(\log\lambda)}{(D_a(\log\lambda))^s}\right]^r \quad \text{for } \lambda \in \Sigma_{\tau,\varepsilon},$$

where $R_{s(a+1)} = D_{a+s}(D_a)^{s-1}$ is a polynomial of degree not greater than $s(a+1)$. Since all the series that take part in these formulae converge absolutely and uniformly when $\lambda \in \Sigma_{\tau,\varepsilon}$, if we collect together the terms in the same powers of $\lambda(D_a(\log\lambda))^{-1}$, we have

$$(\det\tilde{L}_\lambda)^{-1} = \frac{\lambda^{-a}}{D_a(\log\lambda)}\sum_{s=0}^{\infty}\left\{P_{s(a+1)}(\log\lambda)\left[\frac{\lambda}{D_a(\log\lambda)}\right]^s\right\},$$

where P_j is a polynomial of degree not greater than j. Thus we know the behavior of $T\mathbf{f}$ as $|\lambda| \to 0$ by the formula $T\mathbf{f} = \tilde{L}_\lambda^{-1}\lambda TG_\lambda^{-1}\mathbf{g}$. On the other hand, we have

$$\mathbf{f}_a = -(G_\lambda^{-1}N_\lambda(T\mathbf{f}))_a + (G_\lambda^{-1}\mathbf{g})_a.$$

If we substitute the $T\mathbf{f}$ into the above formula, we know the behavior of $\mathbf{f}_a$. Thus we obtain the behavior of $\mathbf{f}$, i.e. behavior of $(I + F_\lambda)^{-1}$. Therefore, the assertions of Proposition 3.1 follow immediately from (3.11). □

Proposition 3.1 says that the operators (R_λ, P_λ) can be expanded by the series of polynomials of $\log\lambda$ and λ. Next task is to determine s, M and L of (3.4), exactly. The strategy follows Kleinman and Vainberg [**17**]. Let q, m, τ, and ε be the same as in Proposition 3.1.

Proposition 3.6. *Let R_λ be the same as in Proposition* 3.1. *Then we have*

$$\begin{pmatrix} R_\lambda \\ P_\lambda \end{pmatrix}\mathbf{f} = \begin{pmatrix} V_0 \\ Q_0 \end{pmatrix}\mathbf{f} + (\log\lambda)^{-1}\begin{pmatrix} V_1 \\ Q_1 \end{pmatrix}\mathbf{f} + O(\log\lambda)^{-2} \quad as\ \lambda \in \Sigma_{\tau,\varepsilon}, \tag{3.21}$$

where $V_j \in \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), \boldsymbol{W}^{2m+2}_q(\Omega_b))$ *and* $Q_j \in \mathscr{L}(\boldsymbol{W}^{2m}_{q,b}(\Omega), W^{2m+1}_q(\Omega_b))$ $(j = 0, 1)$ *are independent of* λ.

To prove this proposition, we use the cut-off function $\eta \in C^\infty(\boldsymbol{R}^2)$ such that $\eta(x) = 0$ for $|x| < b-2$ and $\eta(x) = 1$ for $|x| > b-1$.

Put $\mathbf{u} = R_\lambda\mathbf{f}$, $\mathfrak{p} = P_\lambda\mathbf{f}$ and $\mathbf{z} = \eta\mathbf{u} - \mathbf{B}[\nabla\eta\cdot\mathbf{u}]$ for $\mathbf{f} \in \boldsymbol{W}^{2m}_{q,b}(\Omega)$ and $\lambda \in \Sigma_{\tau,\varepsilon}$. Then,

$$(\lambda - \Delta)\mathbf{z} + \nabla(\eta\mathfrak{p}) = \eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) - \lambda\mathbf{B}[\nabla\eta\cdot\mathbf{u}] \quad \text{and} \quad \nabla\cdot\mathbf{z} = 0 \quad \text{in } \boldsymbol{R}^2,$$

where

$$\mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) = -2(\nabla\eta\cdot\nabla)\mathbf{u} - \Delta\eta\mathbf{u} + \nabla\eta\mathfrak{p} + \Delta\mathbf{B}[\nabla\eta\cdot\mathbf{u}].$$

Obviously, $\operatorname{supp}\mathbf{g} \subset D_{b-1}$.

Lemma 3.7. *Let* $\mathbf{u}, \mathfrak{p}$ *and* $\mathbf{z}$ *be as above. Then, the following formula is valid*:

$$\mathbf{z} = A_\lambda(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) - \lambda\mathbf{B}[\nabla\eta\cdot\mathbf{u}]) \quad and \tag{3.22}$$

$$\eta\mathfrak{p} = \Pi(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) - \lambda\mathbf{B}[\nabla\eta\cdot\mathbf{u}]) \quad in\ \boldsymbol{R}^2,$$

for $\lambda \in \Sigma_{\tau,\varepsilon}$.

Proof. Put $\mathbf{v} = A_\lambda(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) - \lambda\mathbf{B}[\nabla\eta\cdot\mathbf{u}])$ and $\mathfrak{q} = \Pi(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{u},\mathfrak{p})) - \lambda\mathbf{B}[\nabla\eta\cdot\mathbf{u}])$. By (2.3), (2.4) and (3.2), $\mathbf{z} - \mathbf{v}$ and $\eta\mathfrak{p} - \mathfrak{q}$ satisfy the condition of Proposition 2.7, thus we have (3.22). □

Now we start to prove Proposition 3.6.

Proof of Proposition 3.6. To determine s of (3.4), we employ the contradiction argument. We may assume that $\mathbf{f} \not\equiv \mathbf{0}$ and we put $\mathbf{w}_{(\lambda)} = (M(\log\lambda)/L(\log\lambda))\mathbf{f}$, $\mathfrak{r}_{(\lambda)} = (\tilde{M}(\log\lambda)/\tilde{L}(\log\lambda))\mathbf{f}$ in (3.4) and ${}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}) \not\equiv {}^t(\mathbf{0}, 0)$. At first we shall prove $s \leqq 0$. If $s > 0$, then by (3.4) $\mathbf{u}$ and $\mathfrak{p}$ tend to 0 in Ω_b as $|\lambda| \to 0$, thus we have $\mathbf{0} = \mathbf{f}$ in Ω_b by (S). From $\operatorname{supp}\mathbf{f} \subset \Omega_b$ it follows $\mathbf{f} \equiv \mathbf{0}$, which contradicts the assumption.

Let us suppose that $s < 0$. By substituting (3.4) into (S) and equating the terms which contain the multiplier λ^s in both sides of (S), we have

$$-\Delta\mathbf{w}_{(\lambda)} + \nabla\mathfrak{r}_{(\lambda)} = \mathbf{0} \quad \text{and} \quad \nabla\cdot\mathbf{w}_{(\lambda)} = 0 \quad \text{in } \Omega_b, \quad \mathbf{w}_{(\lambda)} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.23}$$

To investigate the behavior of solution as $|x|$ is large, we use the following formula, which is obtained by substituting (3.4) into (3.22):

$$\begin{aligned}
&\eta(\lambda^s\mathbf{w}_{(\lambda)} + O(\lambda^{s+1}(\log\lambda)^\beta) - \mathbf{B}[\nabla\eta\cdot(\lambda^s\mathbf{w}_{(\lambda)} + O(\lambda^{s+1}(\log\lambda)^\beta))] \\
&\quad = \left\{A_0 - \frac{1}{4\pi}(c + \log\sqrt{\lambda})T + B_\lambda\right\}(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})\lambda^s) + O(\lambda^{s+1}(\log\lambda)^{\beta'})), \\
&\eta(\lambda^s\mathfrak{r}_{(\lambda)} + O(\lambda^{s+1}(\log\lambda)^\beta)) = \Pi(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})\lambda^s) + O(\lambda^{s+1}(\log\lambda)^{\beta'})) \quad \text{in } \Omega_b,
\end{aligned} \tag{3.24}$$

where β' is an integer. Equating the terms which contain the multiplier λ^s in both sides of (3.24), we obtain

$$\begin{aligned} &\eta \mathbf{w}_{(\lambda)} = \mathbf{B}[\nabla\eta \cdot \mathbf{w}_{(\lambda)}] + \left\{A_0 - \frac{1}{4\pi}(c+\log\sqrt{\lambda})T\right\}\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})), \\ &\eta \mathfrak{r}_{(\lambda)} = \Pi\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})) \quad \text{in } \Omega_b. \end{aligned} \tag{3.25}$$

Since the right hand sides of (3.25) depend only on values of $(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})$ in Ω_b, (3.25) allows us to continue them to the whole domain Ω. Thus we obtain $(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})$ which satisfies (3.23) and

$$\begin{aligned} &\eta \mathbf{w}_{(\lambda)} = \mathbf{B}[\nabla\eta \cdot \mathbf{w}_{(\lambda)}] + \left\{A_0 - \frac{1}{4\pi}(c+\log\sqrt{\lambda})T\right\}\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})), \\ &\eta \mathfrak{r}_{(\lambda)} = \Pi\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})) \quad \text{in } \Omega. \end{aligned} \tag{3.26}$$

Since $\mathbf{B}[\nabla\eta \cdot \mathbf{w}_{(\lambda)}] = \mathbf{0}$ for $|x| > b-1$, when $|x| > b-1$, we have

$$\begin{aligned} -\Delta \mathbf{w}_{(\lambda)} + \nabla \mathfrak{r}_{(\lambda)} &= -\Delta(\eta \mathbf{w}_{(\lambda)}) + \nabla(\eta \mathfrak{r}_{(\lambda)}) \\ &= -\Delta(A_0 \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) + \nabla(\Pi\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) \\ &= \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})) = \mathbf{0}, \\ \nabla \cdot \mathbf{w}_{(\lambda)} &= \nabla \cdot (\eta \mathbf{w}_{(\lambda)}) = \nabla \cdot (A_0 \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) = 0, \end{aligned}$$

which together with (3.23) implies

$$-\Delta \mathbf{w}_{(\lambda)} + \nabla \mathfrak{r}_{(\lambda)} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{w}_{(\lambda)} = 0 \quad \text{in } \Omega, \quad \mathbf{w}_{(\lambda)} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.27}$$

Moreover by (3.26)

$$\begin{aligned} &\mathbf{w}_{(\lambda)} - \left\{E_0(x) - \frac{1}{4\pi}(c+\log\sqrt{\lambda})\right\}T\mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})) \to \mathbf{0}, \\ &\mathfrak{r}_{(\lambda)} = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \end{aligned} \tag{3.28}$$

By the definition of ${}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})$, there exist an integer ν, ${}^t(\mathbf{w}_0, \mathfrak{r}_0)$ and ${}^t(\mathbf{w}_1, \mathfrak{r}_1)$ such that ${}^t(\mathbf{w}_0, \mathfrak{r}_0) \not\equiv (\mathbf{0}, 0)$ and

$$\begin{pmatrix} \mathbf{w}_{(\lambda)} \\ \mathfrak{r}_{(\lambda)} \end{pmatrix} = (\log\lambda)^{\nu}\begin{pmatrix} \mathbf{w}_0 \\ \mathfrak{r}_0 \end{pmatrix} + (\log\lambda)^{\nu-1}\begin{pmatrix} \mathbf{w}_1 \\ \mathfrak{r}_1 \end{pmatrix} + O((\log\lambda)^{\nu-2}) \quad \text{in } \Omega_b \quad \text{as } |\lambda| \to 0. \tag{3.29}$$

We multiply both sides of (3.27) by $(\log\lambda)^{-\nu}$ and take the limit as $|\lambda| \to 0$, we have

$$-\Delta \mathbf{w}_0 + \nabla \mathfrak{r}_0 = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{w}_0 = 0 \quad \text{in } \Omega_b, \quad \mathbf{w}_0 = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.30}$$

Substituting (3.29) into (3.26) and equating the terms of $(\log\lambda)^{\nu+1}$ and $(\log\lambda)^{\nu}$ in both sides, we have

$$\mathbf{0} = -\frac{1}{8\pi}T\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)), \tag{3.31}$$

$$\begin{aligned} &\eta \mathbf{w}_0 = \mathbf{B}[\nabla\eta \cdot \mathbf{w}_0] + \left(A_0 - \frac{c}{4\pi}T\right)\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)) - \frac{1}{8\pi}T\mathbf{g}({}^t(\mathbf{w}_1, \mathfrak{r}_1)), \\ &\eta \mathfrak{r}_0 = \Pi\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)) \quad \text{in } \Omega_b. \end{aligned} \tag{3.32}$$

If we continue $\mathbf{w}_0$ and $\mathfrak{r}_0$ to the whole domain Ω by (3.32) as in the same way of (3.26), we have $-\Delta\mathbf{w}_0 + \nabla\mathfrak{r}_0 = \mathbf{0}$ and $\nabla\cdot\mathbf{w}_0 = 0$ as $|x| > b-1$, which combined with (3.30) implies

$$-\Delta\mathbf{w}_0 + \nabla\mathfrak{r}_0 = \mathbf{0} \quad \text{and} \quad \nabla\cdot\mathbf{w}_0 = 0 \quad \text{in } \Omega, \quad \mathbf{w}_0 = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.33}$$

By (3.31) and (3.32) for $|x| > b-1$,

$$\begin{aligned} \mathbf{w}_0(x) &= \int_{\mathbf{R}^2} (E_0(x-y) - E_0(x))\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0))(y)\,dy - \frac{1}{8\pi}T\mathbf{g}({}^t(\mathbf{w}_1, \mathfrak{r}_1)) = O(1), \\ \mathfrak{r}_0(x) &= \Pi\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)) = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \end{aligned} \tag{3.34}$$

Thus from Proposition 2.5 it follows that $(\mathbf{w}_0, \mathfrak{r}_0) = (\mathbf{0}, 0)$. This contradiction proves that $s = 0$. Now we have

$$\begin{pmatrix} \mathbf{u} \\ \mathfrak{p} \end{pmatrix} = \begin{pmatrix} \mathbf{w}_{(\lambda)} \\ \mathfrak{r}_{(\lambda)} \end{pmatrix} + O(\lambda(\log\lambda)^{\beta}) \quad \text{in } \Omega_b.$$

Let us determine v of (3.29). Employing the same argument as in (3.23)–(3.28), we can continue $\mathbf{w}_{(\lambda)}$ and $\mathfrak{r}_{(\lambda)}$ to Ω as follows:

$$\begin{aligned} \eta\mathbf{w}_{(\lambda)} &= \mathbf{B}[\nabla\eta\cdot\mathbf{w}_{(\lambda)}] + \left\{A_0 - \frac{1}{4\pi}(c + \log\sqrt{\lambda})T\right\}(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))), \\ \eta\mathfrak{r}_{(\lambda)} &= \Pi(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) \quad \text{in } \Omega, \end{aligned} \tag{3.35}$$

and we have

$$-\Delta\mathbf{w}_{(\lambda)} + \nabla\mathfrak{r}_{(\lambda)} = \mathbf{f} \quad \text{and} \quad \nabla\cdot\mathbf{w}_{(\lambda)} = 0 \quad \text{in } \Omega, \quad \mathbf{w}_{(\lambda)} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{3.36}$$

$$\begin{aligned} \mathbf{w}_{(\lambda)} - \left\{E_0(x) - \frac{1}{4\pi}(c + \log\sqrt{\lambda})\right\}T(\eta\mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) \to \mathbf{0}, \\ \mathfrak{r}_{(\lambda)} = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \end{aligned} \tag{3.37}$$

If $v < 0$, taking a limit as $|\lambda| \to 0$ leads a contradiction $\mathbf{0} = \mathbf{f}$, which implies $v \geqq 0$. Suppose that $v > 0$. If we multiply both sides of (3.36) by $(\log\lambda)^{-v}$ and take the limit as $|\lambda| \to 0$, we have (3.30). Substituting (3.29) into (3.35) and equating the terms of $(\log\lambda)^{v+1}$ and $(\log\lambda)^{v}$ in both sides, we obtain (3.31) and

$$\begin{aligned} \eta\mathbf{w}_0 &= \mathbf{B}[\nabla\eta\cdot\mathbf{w}_0] + \left(A_0 - \frac{c}{4\pi}T\right)\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)) - \frac{1}{8\pi}T(\eta\mathbf{f}^v + \mathbf{g}({}^t(\mathbf{w}_1, \mathfrak{r}_1))), \\ \eta\mathfrak{r}_0 &= \Pi\mathbf{g}({}^t(\mathbf{w}_0, \mathfrak{r}_0)) \quad \text{in } \Omega_b, \end{aligned} \tag{3.38}$$

where

$$\mathbf{f}^v = \begin{cases} \mathbf{f} & v = 1, \\ \mathbf{0} & v \geqq 2. \end{cases}$$

If we continue $\mathbf{w}_0$ and $\mathfrak{r}_0$ to the whole domain Ω by (3.38), we have (3.33). Employing the same argument as (3.34), by Proposition 2.5 we have $(\mathbf{w}_0, \mathfrak{r}_0) = (\mathbf{0}, 0)$. This contradiction implies $v = 0$. Thus we have (3.21) and complete the proof of Proposition 3.6. □

By $\mathbf{u}_0$ (2-dimensional column vector) and $\mathfrak{q}_0$ (scalar) we denote the solution of the problem:

$$-\Delta \mathbf{u}_0 + \nabla \mathfrak{q}_0 = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u}_0 = 0 \quad \text{in } \Omega, \quad \mathbf{u_0} = \mathbf{0} \quad \text{on } \partial\Omega,$$
$$\mathbf{u}_0 = O(1), \quad \mathfrak{q}_0 = O(|x|^{-1}) \quad \text{as } |x| \to \infty, \tag{3.39}$$

where $\mathbf{f} \in \boldsymbol{L}_{q,b}(\Omega)$. By $U_1 = (\mathbf{u}_1^1\ \mathbf{u}_1^2)$ (2×2 matrix) and $\mathbf{q}_1$ (2-dimensional row vector) we denote the solution of the problem:

$$-\Delta U_1 + \nabla \mathbf{q}_1 = (\mathbf{0}\ \mathbf{0}) \quad \text{and} \quad \nabla \cdot \mathbf{u}_1^i = 0\ (i = 1, 2) \quad \text{in } \Omega, \quad U_1 = (\mathbf{0}\ \mathbf{0}) \text{ on } \partial\Omega,$$
$$U_1 - E_0 = O(1), \quad \mathbf{q}_1 = O(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{3.40}$$

The uniqueness of (3.39) follows from Proposition 2.5 and the existence will be proved below. The unique solvability of (3.40) follows from that of (3.39) (see [**17**]). Since we can show that the solution $\mathbf{u}_0$ of (3.39) converges to some constant vector later on, we define the constant vector and matrix as follows:

$$\mathbf{b} = \lim_{|x| \to \infty} \mathbf{u}_0 \quad \text{and} \quad L = \lim_{|x| \to \infty} (U_1 - E_0). \tag{3.41}$$

Corollary 3.8.

$$R_\lambda \mathbf{f} = \mathbf{u}_0 + U_1 \left(-\frac{1}{4\pi} (c + \log\sqrt{\lambda}) I_2 - L \right)^{-1} \mathbf{b} + O(\lambda (\log \lambda)^\beta), \tag{3.42}$$

for $\mathbf{f} \in \boldsymbol{L}_{q,b}(\Omega)$ *and* $\lambda \in \Sigma_{\tau,\varepsilon}$, *where* $\mathbf{u}_0$, U_1, $\mathbf{b}$ *and* L *are defined in* (3.39)–(3.41), β *is an integer and the order symbol* O *is used in the sense that*

$$\left\| R_\lambda \mathbf{f} - \mathbf{u}_0 - U_1 \left(-\frac{1}{4\pi} (c + \log\sqrt{\lambda}) I_2 - L \right)^{-1} \mathbf{b} \right\|_{q,2,\Omega_b} \leqq C_{q,b} |\lambda (\log \lambda)^\beta| \, \|\mathbf{f}\|_q.$$

Proof. Since $\nu = 0$ in (3.29) by (3.21), employing the same argument as in the proof of Proposition 3.6, we have

$$-\Delta \mathbf{w}_0 + \nabla \mathfrak{r}_0 = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{w}_0 = 0 \quad \text{in } \Omega, \quad \mathbf{w}_0 = \mathbf{0} \quad \text{on } \partial\Omega,$$

$$\mathbf{w}_0 \to -\frac{1}{8\pi} T \mathbf{g}({}^t(\mathbf{w}_1, \mathfrak{r}_1)) \quad \text{and} \quad \mathfrak{r}_0 = O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

Thus putting $(\mathbf{u}_0, \mathfrak{q}_0) = (\mathbf{w}_0, \mathfrak{r}_0)$, we have the existence of the solution of (3.39) and $\mathbf{w}_0$ tends to a constant as $|x| \to \infty$. Hence as noted previously, the solution of (3.40): $(U_1, \mathbf{q}_1)$ also exists and the limits of (3.41) are constant. If we recall that ${}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})$ satisfies (3.36) and (3.37), then

$$\mathbf{w}_{(\lambda)} - U_1 T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)})))$$
$$\to \left\{ -\frac{1}{4\pi} (c + \log\sqrt{\lambda}) I_2 - L \right\} T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))),$$

$$\mathfrak{r}_{(\lambda)} - \mathbf{q}_1 T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) = O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

From Proposition 2.5 it follows that

$$\mathbf{w}_{(\lambda)} - U_1 T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) = \mathbf{u}_0 \quad \text{and} \quad \mathfrak{r}_{(\lambda)} - \mathbf{q}_1 T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) = \mathfrak{q}_0.$$

Since $(-1/4\pi)(c + \log\sqrt{\lambda})I_2 - L$ is invertible as $|\lambda| \to 0$,

$$T(\eta \mathbf{f} + \mathbf{g}({}^t(\mathbf{w}_{(\lambda)}, \mathfrak{r}_{(\lambda)}))) = \left(-\frac{1}{4\pi}(c + \log\sqrt{\lambda})I_2 - L\right)^{-1}\mathbf{b}.$$

Thus we have $\mathbf{u} = \mathbf{w}_{(\lambda)} + O(\lambda(\log\lambda)^{\beta}) = \mathbf{u}_0 + U_1((-1/4\pi)(c + \log\sqrt{\lambda})I_2 - L)^{-1}\mathbf{b} + O(\lambda(\log\lambda)^{\beta})$, which implies (3.42). □

§4. Proof of Theorem 1.1.

In this section, we shall obtain the order of local energy decay of $e^{-tA}\mathbf{f}$. To this end, we use the result of Proposition 3.6. Let $\tau > 3\pi/4$ and $\varepsilon = \varepsilon(\tau)$ be fixed in Proposition 3.1.

PROOF OF THEOREM 1.1. Let the curve $\Gamma \subset \boldsymbol{C}$ consist of three curves $\Gamma_1^{\pm}$ and Γ_0, where

$$\begin{aligned}
\Gamma_1^{\pm} &= \{\lambda \in \boldsymbol{C} \mid \arg\lambda = \pm 3\pi/4,\ |\lambda| \geqq \varepsilon\}, \\
\Gamma_0 &= \Gamma_2^{+} \cup \Gamma_3 \cup \Gamma_2^{-}, \\
\Gamma_2^{\pm} &= \{\lambda \in \boldsymbol{C} \mid \arg\lambda = \pm 3\pi/4,\ 2/t \leqq |\lambda| \leqq \varepsilon\}, \\
\Gamma_3 &= \{\lambda \in \boldsymbol{C} \mid |\lambda| = 2/t,\ -3\pi/4 \leqq \arg\lambda \leqq 3\pi/4\}
\end{aligned}$$

and $0 < 2/t < \varepsilon$. Then, by (3.1), the semigroup e^{-tA} admits the representation

$$e^{-tA} = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda + A)^{-1}\, d\lambda, \quad t > 0 \tag{4.1}$$

(cf. [**15**]). By (3.3) we shall estimate

$$J_1^{\pm}(t)\mathbf{f} = \frac{1}{2\pi i}\int_{\Gamma_1^{\pm}} e^{\lambda t}(\lambda + A)^{-1}\mathbf{f}\, d\lambda, \quad J_0(t)\mathbf{f} = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t} R_\lambda \mathbf{f}\, d\lambda.$$

Since by (3.1) and Proposition 2.8

$$\|(\lambda + A)^{-1}\mathbf{f}\|_{q,2} \leqq C_{q,\varepsilon}\|\mathbf{f}\|_q \quad \text{as } \lambda \in \Gamma_1^{\pm},$$

we have

$$\|\partial_t^m J_1^{\pm}(t)\mathbf{f}\|_{q,2} \leqq C_{q,m,\varepsilon} e^{-(\varepsilon/2\sqrt{2})t}\|\mathbf{f}\|_q.$$

In view of (3.21) we have

$$\begin{aligned}
\partial_t^m J_0(t)\mathbf{f} &= \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\lambda^m (V_0\mathbf{f} + (\log\lambda)^{-1}V_1\mathbf{f})\, d\lambda + \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\lambda^m M_\lambda \mathbf{f}\, d\lambda \\
&= K_0^1(t)\mathbf{f} + K_0^2(t)\mathbf{f},
\end{aligned}$$

where

$$\|M_\lambda \mathbf{f}\|_{q,2,\Omega_b} \leqq C_{q,m,b} |\log \lambda|^{-2} \|\mathbf{f}\|_q.$$

On the term $K_0^1(t)\mathbf{f}$, in view of Cauchy's integral theorem we can replace Γ_0 by $\tilde{\Gamma}_0 = \tilde{\Gamma}_1^+ \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_1^-$:

$$\tilde{\Gamma}_1^\pm = \{\lambda = -\varepsilon/\sqrt{2} \pm i\ell \mid 0 \leqq \ell \leqq \varepsilon/\sqrt{2}\},$$

$\tilde{\Gamma}_2 =$ a smooth loop joining the points $\lambda = (\varepsilon/\sqrt{2})e^{i\pi}$ and $\lambda = (\varepsilon/\sqrt{2})e^{-i\pi}$ and going around the cut in Σ and connecting $\tilde{\Gamma}_1^+$ and $\tilde{\Gamma}_1^-$.

Then we have

$$\left\| \int_{\tilde{\Gamma}_1^+ \cup \tilde{\Gamma}_1^-} e^{\lambda t} \lambda^m (V_0 \mathbf{f} + (\log \lambda)^{-1} V_1 \mathbf{f})\, d\lambda \right\|_{q,2,\Omega_b} \leqq C_{q,m,b,\varepsilon} e^{-(\varepsilon/\sqrt{2})t} \|\mathbf{f}\|_q.$$

Since $\int_{\tilde{\Gamma}_2} e^{\lambda t} \lambda^m \, d\lambda = 0$, if we apply Lemma 7 of [p. 369, **35**] to $\int_{\tilde{\Gamma}_2} e^{\lambda t} \lambda^m (\log \lambda)^{-1} \, d\lambda$, we obtain

$$\|K_0^1(t)\mathbf{f}\|_{q,2,\Omega_b} \leqq C_{q,m,b,\varepsilon} t^{-m-1} (\log t)^{-2} \|\mathbf{f}\|_q \quad \text{as } t \to \infty.$$

On the term $K_0^2(t)\mathbf{f}$, employing the same argument as in the proof of Lemma 8 of [p. 370, **35**], we have

$$\|K_0^2(t)\mathbf{f}\|_{q,2,\Omega_b} \leqq C_{q,m,b} t^{-m-1} (\log t)^{-2} \|\mathbf{f}\|_q, \quad \text{as } t \to \infty,$$

which completes the proof of Theorem 1.1. □

Corollary 4.1. *Let $1 < q < \infty$, $b > b_0$ and m be a positive integer. Assume that $\mathbf{f} \in \mathscr{D}_q(\boldsymbol{A}^m) \cap \boldsymbol{J}_{q,b}(\Omega)$. Then,*

$$\|e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2m,\Omega_b} \leqq C_{q,m,b} (1 + t(\log t)^2)^{-1} \|\mathbf{f}\|_{q,2m} \quad \textit{for } t \geqq 0, \tag{4.2}$$

$$\|\partial_t e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2(m-1),\Omega_b} \leqq C_{q,m,b} (1 + t^2(\log t)^2)^{-1} \|\mathbf{f}\|_{q,2m} \quad \textit{for } t \geqq 0. \tag{4.3}$$

Proof. When t is bounded, by Proposition 2.8

$$\begin{aligned}
\|e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2m,\Omega_b} &\leqq C\|e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2m} \\
&\leqq C(\|\boldsymbol{A}^m e^{-t\boldsymbol{A}}\mathbf{f}\|_q + \|e^{-t\boldsymbol{A}}\mathbf{f}\|_q) \\
&\leqq C(\|\boldsymbol{A}^m \mathbf{f}\|_q + \|\mathbf{f}\|_q) \leqq C\|\mathbf{f}\|_{q,2m}, \\
\|\partial_t e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2(m-1)} &\leqq C\|\boldsymbol{A} e^{-t\boldsymbol{A}}\mathbf{f}\|_{q,2(m-1)} \leqq C\|\mathbf{f}\|_{q,2m}.
\end{aligned}$$

When $\lambda \in \Gamma_1^\pm$, by Proposition 2.8 and (3.1) we have

$$\|(\lambda + \boldsymbol{A})^{-1}\mathbf{f}\|_{q,2m+2} \leqq C_{q,m,\varepsilon,\tau} \|\mathbf{f}\|_{q,2m} \tag{4.4}$$

for $\mathbf{f} \in \mathscr{D}_q(\boldsymbol{A}^m)$. Therefore, by (4.4) and (3.21), if we employ the same argument as in the proof of Theorem 1.1, we can prove (4.2) and (4.3) for $t \to \infty$. □

§5. Proof of Theorem 1.2

We start with $L_q - L_r$ estimate in the whole space case. Put

$$E(t)\mathbf{a} = \frac{1}{4\pi t}\int_{\boldsymbol{R}^2} e^{-|x-y|^2/4t}\mathbf{a}(y)\,dy. \tag{5.1}$$

When $\mathbf{a} \in \boldsymbol{J}_q(\boldsymbol{R}^2)$, $\mathbf{v}(t) = E(t)\mathbf{a}$ solves the nonstationary Stokes equation in $\boldsymbol{R}^2$:

$$\begin{aligned} &\partial_t \mathbf{v}(t) - \Delta \mathbf{v}(t) = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{v}(t) = 0 \quad \text{in } (0,\infty)\times \boldsymbol{R}^2,\\ &\mathbf{v}(0) = \mathbf{a} \quad \text{in } \boldsymbol{R}^2. \end{aligned} \tag{5.2}$$

By Young's inequality and Sobolev's imbedding theorem we have the following estimates.

Lemma 5.1. *Let* $1 \leqq q \leqq r \leqq \infty$. *Then,*

$$\|\partial_t^j \partial_x^\alpha \mathbf{v}(t)\|_{r,\boldsymbol{R}^2} \leqq C_{q,r,j,\alpha} t^{-(1/q-1/r)-j-|\alpha|/2}\|\mathbf{a}\|_{q,\boldsymbol{R}^2} \quad t \geqq 1, \tag{5.3}$$

$$\|\partial_t^j \partial_x^\alpha \mathbf{v}(t)\|_{r,\boldsymbol{R}^2} \leqq C_{q,r,j,\alpha}(1+t)^{-(1/q-1/r)-j-|\alpha|/2}\|\mathbf{a}\|_{q,[2(1/q-1/r)]+1+|\alpha|+2j,\boldsymbol{R}^2} \quad t \geqq 0, \tag{5.4}$$

where $[\cdot]$ *is the Gauss symbol.*

Now we shall prove Theorem 1.2. Set $\mathbf{b} = e^{-A}\mathbf{f}$ for $\mathbf{f} \in \boldsymbol{J}_q(\Omega)$. Then, $\mathbf{b} \in \mathscr{D}_q(\boldsymbol{A}^N)$ for any integer $N \geqq 0$, and in view of Proposition 2.8 for any integer $N \geqq 0$,

$$\|\mathbf{b}\|_{q,2N} \leqq C_{q,N}\|\mathbf{f}\|_q. \tag{5.5}$$

Put $\mathbf{u}(t) = e^{-tA}\mathbf{b} = e^{-(t+1)A}\mathbf{f}$. Then $\mathbf{u}(t)$ is smooth in t and x and satisfies the following equations with some $\mathfrak{p}(t)$:

$$\partial_t \mathbf{u}(t) - \Delta \mathbf{u}(t) + \nabla \mathfrak{p}(t) = \mathbf{0} \quad \text{and} \quad \nabla \cdot \mathbf{u}(t) = 0 \quad \text{in } (0,\infty)\times\Omega,$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } (0,\infty)\times\partial\Omega, \qquad \mathbf{u}(0) = \mathbf{b} \quad \text{in } \Omega.$$

Obviously, the asymptotic behavior of $e^{-tA}\mathbf{f}$ for large $t > 0$ follows from that of $\mathbf{u}(t)$, so that we shall start with the following step.

1st step. For any integer $m \geqq 0$, we have the relations:

$$\|\mathbf{u}(t)\|_{q,2m,\Omega_b} + \|\partial_t \mathbf{u}(t)\|_{q,2m,\Omega_b} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q \tag{5.6}$$

for any $t \geqq 0$. In fact, let N be a larger integer $\geqq ([2/q]+2m+6)/2$. Since by Proposition 2.8 $\mathbf{b} \in \mathscr{D}_q(\boldsymbol{A}^N) \subset \boldsymbol{J}_q(\Omega) \cap \dot{\boldsymbol{W}}_q^1(\Omega) \cap \boldsymbol{W}_q^{2N}(\Omega)$, by Propositions 2.2(2) and 2.3 there exists a $\mathbf{c} \in \boldsymbol{W}_q^{2N}(\boldsymbol{R}^2)$ such that $\mathbf{b} = \mathbf{c}$ in Ω, $\nabla \cdot \mathbf{c} = 0$ in $\boldsymbol{R}^2$ and

$$\|\mathbf{c}\|_{q,2N,\boldsymbol{R}^2} \leqq C_{q,N}\|\mathbf{b}\|_{q,2N} \leqq C_{q,N}\|\mathbf{f}\|_q \tag{5.7}$$

(cf. (5.5)). Put $\mathbf{v}(t) = E(t)\mathbf{c}$, where $E(t)$ is the operator defined by (5.1). By Lemma 5.1 and (5.7)

$$\|\partial_t^j \mathbf{v}(t)\|_{\infty,2m+1,\boldsymbol{R}^2} \leqq C_{q,m}(1+t)^{-1/q-j}\|\mathbf{f}\|_q, \quad t \geqq 0, \quad j = 0,1,2, \tag{5.8}$$

because $2N \geqq [2/q] + 2m + 6$. Let φ be a function of $C^\infty(\boldsymbol{R}^2)$ such that $\varphi(x) = 1$ for $|x| \leqq b$ and $\varphi(x) = 0$ for $|x| \geqq b+1$, where b is a fixed number $\geqq b_0$. In view of Proposition 2.4, putting

$$\mathbf{w}(t) = \mathbf{u}(t) - (1-\varphi)\mathbf{v}(t) - \mathbf{B}[(\nabla\varphi)\cdot\mathbf{v}(t)],$$

we see that $\nabla\cdot\mathbf{w}(t) = 0$ in Ω and $\mathbf{w}(t) = \mathbf{0}$ on $\partial\Omega$ for any $t \geqq 0$, and moreover by Proposition 2.4 and (5.8) we have

$$(5.9)\qquad \|\partial_t^j \mathbf{B}[(\nabla\varphi)\cdot\mathbf{v}(t)]\|_{q,2m+2,\boldsymbol{R}^2} \leqq C_{q,m,b}(1+t)^{-1/q-j}\|\mathbf{f}\|_q, \quad t \geqq 0, \quad j = 0,1,2.$$

Since $\operatorname{supp}\mathbf{B}[(\nabla\varphi)\cdot\mathbf{v}(t)] \subset D_{b+1}$ and since $1-\varphi(x) = 0$ for $|x| \leqq b$, $\mathbf{w} = \mathbf{u}$ in Ω_b, so that if we prove that

$$(5.10)\qquad \|\mathbf{w}(t)\|_{q,2m,\Omega_b} + \|\partial_t\mathbf{w}(t)\|_{q,2m,\Omega_b} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q \quad t \geqq 0,$$

then we have (5.6). To get (5.10) we set

$$\mathbf{d} = \varphi\mathbf{b} - \mathbf{B}[(\nabla\varphi)\cdot\mathbf{b}],$$
$$\mathbf{g}(t) = -\{2(\nabla\varphi\cdot\nabla)\mathbf{v}(t) + \Delta\varphi\mathbf{v}(t)\} - (\partial_t - \Delta)\mathbf{B}[(\nabla\varphi)\cdot\mathbf{v}(t)],$$

and then

$$\partial_t\mathbf{w}(t) - \Delta\mathbf{w}(t) + \nabla\mathfrak{p}(t) = \mathbf{g}(t) \quad \text{and} \quad \Delta\cdot\mathbf{w}(t) = 0 \quad \text{in } (0,\infty)\times\Omega,$$
$$\mathbf{w}(t) = \mathbf{0} \quad \text{on } \partial\Omega, \qquad \mathbf{w}(0) = \mathbf{d} \quad \text{in } \Omega.$$

To represent $\mathbf{w}(t)$ by Duhamel's principle and to estimate the resulting formula by using Corollary 4.1, we need the following facts:

$$(5.11)\qquad \mathbf{d} \in D_q(\boldsymbol{A}^N) \cap \boldsymbol{J}_{q,b+1}(\Omega),$$

$$(5.12)\qquad \partial_t^j\mathbf{g}(t) \in \mathscr{D}_q(\boldsymbol{A}^m) \cap \boldsymbol{J}_{q,b+1}(\Omega), \quad t \geqq 0, \quad j = 0,1,$$

$$(5.13)\qquad \|\mathbf{d}\|_{q,2N} \leqq C_{q,N}\|\mathbf{f}\|_q,$$

$$(5.14)\qquad \|\partial_t^j\mathbf{g}(t)\|_{q,2m} \leqq C_{q,m,b}(1+t)^{-1/q-j}\|\mathbf{f}\|_q, \quad t \geqq 0, \quad j = 0,1.$$

Since $\mathbf{b} \in \mathscr{D}_q(\boldsymbol{A}^N)$ $(N \geqq 1)$, $\mathbf{b} \in \boldsymbol{W}_q^{2N}(\Omega) \cap \hat{\boldsymbol{W}}_q^1(\Omega) \cap \boldsymbol{J}_q(\Omega)$, and hence by Proposition 2.4 $\nabla\cdot\mathbf{d} = 0$ in Ω and $\mathbf{d} = \mathbf{b}$ in Ω_{b-1}, and by (5.5), (5.13) holds. Moreover, (5.11) follows from the following lemma.

Lemma 5.2. *Let $1 < q < \infty$. Let U be a neighborhood of $\bar{\mathcal{O}}$ $(\mathcal{O} = \boldsymbol{R}^2\backslash\bar{\Omega})$ in $\boldsymbol{R}^2$ and N an integer $\geqq 1$. If $\mathbf{a} \in \boldsymbol{W}_q^{2N}(\Omega)$ satisfies the condition $\nabla\cdot\mathbf{a} = 0$ in Ω and $\mathbf{a} = \mathbf{0}$ in $\Omega\cap U$, then $\mathbf{a} \in \mathscr{D}_q(\boldsymbol{A}^N)$. As a result, if $\mathbf{a} \in \boldsymbol{W}_q^{2N}(\Omega) \cap \boldsymbol{J}_q(\Omega)$ coincides with some $\mathbf{b} \in \mathscr{D}_q(\boldsymbol{A}^N)$ in $\Omega\cap U$, then $\mathbf{a} \in \mathscr{D}_q(\boldsymbol{A}^N)$.*

Postponing a proof of Lemma 5.2, we shall show (5.12) and (5.14). By (5.8) and (5.9) we have (5.14) as well as $\partial_t^j\mathbf{g}(t) \in \boldsymbol{W}_q^{2m}(\Omega)$ for any $t > 0$ and $j = 0,1$. Moreover, we see easily that $\nabla\cdot\partial_t^j\mathbf{g}(t) = 0$ in Ω and $\operatorname{supp}\partial_t^j\mathbf{g}(t) \subset D_{b+1}$ for any $t > 0$ and $j = 0,1$. Hence by Lemma 5.2 we have (5.12) too.

Proof of Lemma 5.2. For $\mathbf{f} \in \boldsymbol{L}_q(\Omega)$, $\boldsymbol{P}\mathbf{f}$ is defined by $\boldsymbol{P}\mathbf{f} = \mathbf{f} - \nabla \mathfrak{q}$, where $\mathfrak{q}$ is a solution of the boundary value problem:

$$\Delta \mathfrak{q} = \nabla \cdot \mathbf{f} \quad \text{in } \Omega \quad \text{and} \quad (\mathbf{n} \cdot \nabla)\mathfrak{q} = \mathbf{n} \cdot \mathbf{f} \quad \text{on } \partial\Omega, \tag{5.15}$$

where $\mathbf{n}$ is a unit exterior normal of $\partial\Omega$ and the trace to $\partial\Omega$ is justified for functions belonging to the space $\{\mathbf{u} \in \boldsymbol{L}_q(\Omega) \mid \nabla \cdot \mathbf{u} \in L_q(\Omega)\}$ by the same argument of Proposition 1.2 of [**28**]. If $\mathbf{a} \in \boldsymbol{W}_q^{2N}(\Omega)$ satisfies the condition: $\nabla \cdot \mathbf{a} = 0$ in Ω and $\mathbf{a} = \mathbf{0}$ in $\Omega \cap U$, then $\nabla \cdot \{(-\Delta)^M \mathbf{a}\} = \mathbf{0}$ in Ω and $\mathbf{n} \cdot \{(-\Delta)^M \mathbf{a}\} = \mathbf{0}$ on $\partial\Omega$ for any $M = 0, 1, \dots, N-1$, and hence by (5.15) $\boldsymbol{P}(-\Delta)^M \mathbf{a} = (-\Delta)^M \mathbf{a}$. Therefore, by induction on M we see that $\boldsymbol{A}^M \mathbf{a} = (-\Delta)^M \mathbf{a}$ for $M = 0, 1, \dots, N-1$, which implies immediately that $\boldsymbol{A}^M \mathbf{a} \in \mathscr{D}_q(\boldsymbol{A})$ for $M = 0, 1, \dots, N-1$, that is $\mathbf{a} \in \mathscr{D}_q(\boldsymbol{A}^N)$. This completes the proof of the first part of the lemma. Putting $\mathbf{w} = \mathbf{a} - \mathbf{b}$ and applying the first part to $\mathbf{w}$, we also have the second part, which completes the proof of the lemma. □

In view of (5.11) and (5.12), by Duhamel's principle $\mathbf{w}(t)$ is described as the form:

$$\mathbf{w}(t) = e^{-t\boldsymbol{A}}\mathbf{d} + \int_0^t e^{-(t-s)\boldsymbol{A}}\mathbf{g}(s)\,ds.$$

By Corollary 4.1, (5.13) and (5.14), we have

$$\begin{aligned}\|\mathbf{w}(t)\|_{q,2m,\Omega_b} &\leqq C_{q,m,b}(1 + t(\log t)^2)^{-1}\|\mathbf{f}\|_q \\ &\quad + C_{q,m,b}\int_0^t (1 + (t-s)(\log(t-s))^2)^{-1}(1+s)^{-1/q}\,ds\|\mathbf{f}\|_q.\end{aligned}$$

We split the above integral into two parts:

$$\begin{aligned}&\int_0^{t/2} (1 + (t-s)(\log(t-s))^2)^{-1}(1+s)^{-1/q}\,ds \\ &\qquad \leqq \left(1 + \frac{t}{2}\left(\log\left(\frac{t}{2}\right)\right)^2\right)^{-1}\int_0^{t/2}(1+s)^{-1/q}\,ds \leqq C(1+t)^{-1/q} \\ &\int_{t/2}^t (1 + (t-s)(\log(t-s))^2)^{-1}(1+s)^{-1/q}\,ds \\ &\qquad \leqq \left(1+\frac{t}{2}\right)^{-1/q}\int_{t/2}^t (1 + (t-s)(\log(t-s))^2)^{-1}\,ds \leqq C(1+t)^{-1/q},\end{aligned}$$

thus we have

$$\|\mathbf{w}(t)\|_{q,2m,\Omega_b} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q, \quad t \geqq 0.$$

Since

$$\partial_t \int_0^t e^{-(t-s)\boldsymbol{A}}\mathbf{g}(s)\,ds = e^{-t\boldsymbol{A}}\mathbf{g}(0) + \int_0^t e^{-(t-s)\boldsymbol{A}}\partial_s \mathbf{g}(s)\,ds,$$

by Corollary 4.1, (5.13) and (5.14) we have also

$$\|\partial_t \mathbf{w}(t)\|_{q,2m,\Omega_b} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q, \quad t \geqq 0,$$

which completes the proof of (5.10). Therefore we have (5.6).

In view of (5.6), to complete the estimate of $\|\mathbf{u}(t)\|_{q,m}$ for large $t > 0$, it remains to estimate $\|\mathbf{u}(t)\|_{q,m,\{|x|\geqq b\}}$. To this end, we start with the following lemma.

Lemma 5.3. *Let* $\mathfrak{p}(t)$ *be a certain pressure associated with* $\mathbf{u}(t)$. *Then,*

$$\|\mathfrak{p}(t)\|_{q,2m,\Omega_b} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q. \tag{5.16}$$

Proof. From (5.6) it follows that

$$\begin{aligned}\|\nabla \mathfrak{p}(t)\|_{q,2m-1,\Omega_b} &\leqq \|\partial_t \mathbf{u}(t)\|_{q,2m-1,\Omega_b} + \|\Delta \mathbf{u}(t)\|_{q,2m-1,\Omega_b} \\ &\leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q, \quad t \geqq 0.\end{aligned}$$

We can also take $\mathfrak{p}(t,x) - |\Omega_b|^{-1}\int_{\Omega_b}\mathfrak{p}(t,x)\,dx$, $|\Omega_b|$ being the volume of Ω_b as a pressure instead of $\mathfrak{p}(t,x)$, so that by applying Proposition 2.2(1), we are led to (5.16). □

2nd step. Choose $\psi \in C^\infty(\boldsymbol{R}^2)$ so that $\psi(x) = 1$ for $|x| \leqq b-1$ and $\psi(x) = 0$ for $|x| \geqq b$. Put

$$\begin{aligned}\mathbf{z}(t) &= (1-\psi)\mathbf{u}(t) + \mathbf{B}[(\nabla\psi)\cdot\mathbf{u}(t)], \\ \mathbf{e} &= (1-\psi)\mathbf{b} + \mathbf{B}[(\nabla\psi)\cdot\mathbf{b}], \\ \mathbf{h}(t) &= 2(\nabla\psi\cdot\nabla)\mathbf{u}(t) + \Delta\psi\mathbf{u}(t) + (\partial_t - \Delta)\mathbf{B}[(\nabla\psi)\cdot\mathbf{u}(t)] - (\nabla\psi)\mathfrak{p}(t),\end{aligned}$$

and then

$$\partial_t\mathbf{z}(t) - \Delta\mathbf{z}(t) + \nabla((1-\psi)\mathfrak{p}(t)) = \mathbf{h}(t) \quad \text{and} \quad \nabla\cdot\mathbf{z}(t) = 0 \quad \text{in } (0,\infty)\times\boldsymbol{R}^2,$$
$$\mathbf{z}(0) = \mathbf{e} \quad \text{in } \boldsymbol{R}^2.$$

Moreover, by (5.6), (5.7), (5.16) and Proposition 2.4

$$\|\mathbf{h}(t)\|_{q,2m-1,\boldsymbol{R}^2} \leqq C_{q,m,b}(1+t)^{-1/q}\|\mathbf{f}\|_q, \quad m \geqq 1, \tag{5.17}$$

$$\|\mathbf{e}\|_{q,2m,\boldsymbol{R}^2} \leqq C_{q,m,b}\|\mathbf{f}\|_q, \quad m \geqq 0. \tag{5.18}$$

Since $\nabla\cdot\mathbf{e} = 0$, $\mathbf{z}(t)$ is given by the formula:

$$\mathbf{z}(t) = E(t)\mathbf{e} + \mathbf{z}_1(t), \quad \mathbf{z}_1(t) = \int_0^t E(t-s)\boldsymbol{P}_{\boldsymbol{R}^2}\mathbf{h}(s)\,ds. \tag{5.19}$$

Note that $\mathbf{z}(t) = \mathbf{u}(t)$ when $|x| \geqq b$, so that we shall estimate $\mathbf{z}(t)$. At first, we have by (5.4) and (5.18)

$$\|E(t)\mathbf{e}\|_{r,\boldsymbol{R}^2} \leqq C_{q,r}(1+t)^{-(1/q-1/r)}\|\mathbf{f}\|_q. \tag{5.20}$$

Let us estimate $\mathbf{z}_1(t)$. Since supp $\mathbf{h}(t) \subset D_b$ for all $t \geqq 0$, by (5.4), Hölder's inequality and (5.17), we have

$$\begin{aligned}\|\mathbf{z}_1(t)\|_{r,\boldsymbol{R}^2} &\leqq C_r\int_0^t (1+t-s)^{-(1-1/r)}\|\mathbf{h}(s)\|_{1,[2(1-1/r)]+1,\boldsymbol{R}^2}\,ds\\ &\leqq C_{r,q}\int_0^t (1+t-s)^{-(1-1/r)}\|\mathbf{h}(s)\|_{q,[2(1-1/r)]+1,\boldsymbol{R}^2}\,ds\\ &\leqq C_{r,q}\int_0^t (1+t-s)^{-(1-1/r)}(1+s)^{-1/q}\,ds\,\|\mathbf{f}\|_q.\end{aligned}$$

We split the above integral into two parts:

$$\begin{aligned}&\int_0^{t/2}(1+t-s)^{-1+1/r}(1+s)^{-1/q}\,ds\\ &\qquad\leqq\left(1+\frac{t}{2}\right)^{-1+1/r}\int_0^{t/2}(1+s)^{-1/q}\,ds\leqq C(1+t)^{-(1/q-1/r)},\\ &\int_{t/2}^{t}(1+t-s)^{-1+1/r}(1+s)^{-1/q}\,ds\\ &\qquad\leqq\left(1+\frac{t}{2}\right)^{-1/q}\int_{t/2}^{t}(1+t-s)^{-1+1/r}\,ds\leqq C(1+t)^{-(1/q-1/r).}\end{aligned}$$

Thus we have

$$\|\mathbf{z}_1(t)\|_r\leqq C_{q,r}(1+t)^{-(1/q-1/r)}\|\mathbf{f}\|_q,\quad 1<q\leqq r<\infty,\quad t\geqq 0. \tag{5.21}$$

Since $\mathbf{z}(t)=\mathbf{u}(t)$ for $|x|\geqq b$ and $e^{-tA}\mathbf{f}=\mathbf{u}(t-1)$ for $t\geqq 1$, by (5.6), (5.19), (5.20) and (5.21) we have (1.2) for $t\geqq 1$.

3rd step. Let us prove (1.2) for $t<1$. Let $N=[2(1/q-1/r)]$. If N is even, then by Proposition 2.8 we have

$$\|e^{-tA}\mathbf{f}\|_{q,N}\leqq C_{q,r}(\|A^{N/2}e^{-tA}\mathbf{f}\|_q+\|e^{-tA}\mathbf{f}\|_q)\leqq C_{q,r}t^{-N/2}\|\mathbf{f}\|_q.$$

Similarly, $\|e^{-tA}\mathbf{f}\|_{q,N+2}\leqq C_{q,r}t^{-(N+2)/2}\|\mathbf{f}\|_q$. Therefore, we have by Sobolev's imbedding theorem and an interpolation method

$$\begin{aligned}\|e^{-tA}\mathbf{f}\|_r&\leqq C_{q,r}\|e^{-tA}\mathbf{f}\|_{q,2(1/q-1/r)}\leqq C_{q,r}(t^{-N/2-1})^{1-\theta}(t^{-N/2})^{\theta}\|\mathbf{f}\|_q\\ &=C_{q,r}t^{-(1/q-1/r)}\|\mathbf{f}\|_q,\end{aligned}\tag{5.22}$$

where $\theta=\{N+2-2(1/q-1/r)\}/2$. If N is odd, replace N by $N-1$ and employ the same argument as (5.22). Thus we have (1.2).

Next, we shall prove (1.3) and (1.4). Since we have (1.3) and (1.4) for small t with the same method as (5.22), it is sufficient to prove (1.3) and (1.4) for large t. Let us estimate $\mathbf{u}(t)$ for $|x|\geqq b$. Let $\mathbf{z}(t)$ be the same function as in the proof of Theorem 1.2. Then,

$$\nabla\mathbf{z}(t)=\nabla E(t)\mathbf{e}+\nabla\mathbf{z}_1(t),\quad \nabla\mathbf{z}_1(t)=\int_0^t\nabla E(t-s)\boldsymbol{P}_{\boldsymbol{R}^2}\mathbf{h}(s)\,ds.$$

Then we claim

$$(5.23)\qquad \|\nabla \mathbf{z}(t)\|_{r,\boldsymbol{R}^2} \leqq \begin{cases} C_{q,r}(1+t)^{-(1/q-1/r)-1/2}\|\mathbf{f}\|_q & \text{if } 1<r<2, \\ C_{q,r}(1+t)^{-1/q}\|\mathbf{f}\|_q & \text{if } 2<r. \end{cases}$$

In fact, by (5.4) and (5.18) we have

$$\|\nabla E(t)\mathbf{e}\|_{r,\boldsymbol{R}^2} \leqq C_{q,r}(1+t)^{-(1/q-1/r)-1/2}\|\mathbf{f}\|_q.$$

So we shall estimate $\nabla \mathbf{z}_1(t)$. By (5.4), Hölder's inequality and (5.17), we have

$$\begin{aligned} \|\nabla \mathbf{z}_1(t)\|_{r,\boldsymbol{R}^2} &\leqq C_{q,r}\int_0^t (1+t-s)^{-(1-1/r)-1/2}\|\mathbf{h}(s)\|_{1,[2(1-1/r)]+2,\boldsymbol{R}^2}\,ds \\ &\leqq C_{q,r}\int_0^t (1+t-s)^{-(1-1/r)-1/2}\|\mathbf{h}(s)\|_{q,[2(1-1/r)]+2,\boldsymbol{R}^2}\,ds \\ &\leqq C_{q,r}\int_0^t (1+t-s)^{-(3/2-1/r)}(1+s)^{-1/q}\,ds\|\mathbf{f}\|_q. \end{aligned}$$

We split the above integral into two parts. The first part is

$$\begin{aligned} \int_0^{t/2}(1+t-s)^{-(3/2-1/r)}(1+s)^{-1/q}\,ds &\leqq \left(1+\frac{t}{2}\right)^{-(3/2-1/r)}\int_0^{t/2}(1+s)^{-1/q}\,ds \\ &\leqq C(1+t)^{-(1/q-1/r)-1/2}. \end{aligned}$$

On the other part, if $1<r<2$, then we have

$$\begin{aligned} \int_{t/2}^{t}(1+t-s)^{-(3/2-1/r)}(1+s)^{-1/q}\,ds &\leqq \left(1+\frac{t}{2}\right)^{-1/q}\int_{t/2}^{t}(1+t-s)^{-(3/2-1/r)}\,ds \\ &\leqq C(1+t)^{-(1/q-1/r)-1/2}. \end{aligned}$$

If $2<r<\infty$, since we have

$$\int_{t/2}^{t}(1+t-s)^{-(3/2-1/r)}\,ds \leqq C,$$

then

$$\int_{t/2}^{t}(1+t-s)^{-(3/2-1/r)}(1+s)^{-1/q}\,ds \leqq C(1+t)^{-1/q}.$$

Summing up the above results, we obtain (5.23), which implies that

$$(5.24)\qquad \|\nabla \mathbf{u}(t)\|_{r,\{|x|\geqq b\}} \leqq \begin{cases} C_{q,r}(1+t)^{-(1/q-1/r)-1/2}\|\mathbf{f}\|_q, & \text{if } 1<r<2, \\ C_{q,r}(1+t)^{-1/q}\|\mathbf{f}\|_q, & \text{if } 2<r<\infty, \end{cases}$$

for $t\geqq 1$. By (5.24) and (5.6) we have (1.3) and (1.4) for $r\neq 2$.

In the case that $r=2$, we use weighted L_2-method. By the energy method,

$$(5.25)\qquad \frac{1}{2}\|\mathbf{u}(t)\|_2^2 + \int_0^t \|\nabla \mathbf{u}(s)\|_2^2\,ds = \frac{1}{2}\|\mathbf{f}\|_2^2.$$

On the other hand,

$$\frac{d}{dt}(t\|\nabla \mathbf{u}(t)\|_2^2) = \|\nabla \mathbf{u}(t)\|_2^2 + 2t(\nabla \mathbf{u}(t), \nabla \partial_t \mathbf{u}(t))$$
$$= \|\nabla \mathbf{u}(t)\|_2^2 - 2t(\Delta \mathbf{u}(t), \partial_t \mathbf{u}(t)).$$

Applying the equation (NS) to the right-hand side, ws have

$$\frac{d}{dt}(t\|\nabla \mathbf{u}(t)\|_2^2) = \|\nabla \mathbf{u}(t)\|_2^2 - 2t(\nabla \mathfrak{p}(t), \partial_t \mathbf{u}(t)) - 2t\|\partial_t \mathbf{u}(t)\|_2^2 \tag{5.26}$$
$$\leqq \|\nabla \mathbf{u}(t)\|_2^2 + 2t(\mathfrak{p}(t), \nabla \cdot \partial_t \mathbf{u}(t)) = \|\nabla \mathbf{u}(t)\|_2^2.$$

(5.25) and (5.26) imply that

$$\|\nabla \mathbf{u}(t)\|_2 \leqq Ct^{-1/2}\|\mathbf{f}\|_2 \quad \text{for } t > 0.$$

For $1 < q < r = 2$, by (1.2) and the above we have

$$\|\nabla \mathbf{u}(t)\|_2 = \|\nabla e^{-(t/2)A}(e^{-(t/2)A}\mathbf{f})\|_2$$
$$\leqq Ct^{-1/2}\|e^{-(t/2)A}\mathbf{f}\|_2$$
$$\leqq Ct^{-(1/q-1/2)-1/2}\|\mathbf{f}\|_q \quad \text{for } t > 0,$$

which completes the proof.

References

[1] Bogovskii, M. E., Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Sov. Math. Dokl. **20** (1979), 1094–1098.

[2] ———, Solution for some vector analysis problems connected with operators div and grad, Theory of cubature formulas and application of functional analysis to problems of mathematical physics, Trudy Sem. S. L. Sobolev No. 1, Novosibirsk: Acad. Nauk SSSR, Sibirsk. Otdel., Inst. Mat., 1980, 5–40.

[3] Borchers, W. & Miyakawa, T., L^2-decay for Navier-Stokes Flows in unbounded domains, with application to exterior stationary flows, Arch. Rational Mech. Anal. **118** (1992), 273–295.

[4] Borchers, W. & Sohr, H., On the semigroup of Stokes operator for exterior domain in L^q-spaces, Math. Z. **196** (1987), 415–425.

[5] Borchers, W. & Varnhorn, W., On the boundedness of the Stokes semigroup in two-dimensional exterior domains, Math. Z. **213** (1993), 275–299.

[6] Fujiwara, D. & Morimoto, H., An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sci. Univ. Tokyo, Sec., 1 **24** (1977), 685–700.

[7] Galdi, G. P., An Introduction to the Mathematical Theory of the Navier-Stokes Equations Volume 1, linearized Steady Problems, Springer Tracts in Natural Philosophy Volume 38, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1994.

[8] Galdi, G. P., Heywood, J. G. & Shibata, Y., On the global existence and Convergence to steady state of Navier-Stokes flow past an obstacle that is started from rest, preprint.

[9] Giga, Y., Analyticity of the semigroup generated by the Stokes operator in L_r spaces, Math. Z. **178** (1981), 297–329.

[10] Giga, Y. & Sohr, H., On the Stokes operator in exterior domains, J. Fac. Sci. Univ. Tokyo, Sec., IA. Math. **36** (1989), 103–130.

[11] Hörmander, L., The analysis of linear partial differential operators I, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[12] Iwashita, H., $L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces, Math. Ann. **285** (1989), 265–288.

[13] Iwashita, H. & Shibata, Y., On the analyticity of spectral functions for some exterior boundary value problems, Glasnik Mat. **23 (43)** (1988), 291–313.

[14] Kato, T., Strong L^p-solutions of the Navier-Stokes equation in R^m with applications to weak solutions, Math. Z. **187** (1984), 471–480.

[15] ———, Perturbation theory for linear operators, Springer, Berlin-Heidelberg-New York-Tokyo, 1976.

[16] Kato, T. & Fujita, H., On the nonstationary Navier-Stokes system, Rend. Sem. Math. Univ. Padova **32** (1962), 243–260.

[17] Kleinman, R. & Vainberg, B., Full low-frequency asymptotic expansion for second-order elliptic equations in two dimensions, Math. Meth. in the Appl. Sci. **17** (1994), 989–1004.

[18] Kobayashi, T. & Shibata, Y., On the Oseen equation in the three dimensional exterior domains, to appear, Math. Ann..

[19] Kozono, H. & Ogawa, T., Two-dimensional Navier-Stokes flow in unbounded domains, Math. Ann. **297** (1993), 1–31.

[20] ———, Decay properties of strong solutions for the Navier-Stokes in two-dimensional unbounded domains, Arch. Rational Mech. Anal. **122** (1993), 1–17.

[21] Ladyzhenskaya, O. A., The mathematical theory of viscous incompressible flow, Gordon and Breach, New York, 1969.

[22] Lions, J. L. & Magenes, E., Non-homogeneous boundary value problems and applications I, Grundlehren math. Wiss., vol. 181, Springer-Verlag, Berlin et al., 1972.

[23] Lions, J. L. & Prodi, G., Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2, C. R. Acad. Sci. Paris **248** (1959), 3519–3521.

[24] Maremonti, P., Some results on the asymptotic behaviour of Hopf weak solutions to the Navier-Stokes equations in unbounded domains, Math. Z. **210** (1992), 1–22.

[25] Maremonti, P. & V. A. Solonnikov, On nonstationary Stokes problem in exterior domain, preprint (1996).

[26] Masuda, K., On the stability of incompressible viscous fluid motions past objects, J. Math. Soc. Japan **27** (1975), 294–327.

[27] ———, Weak solutions of Navier-Stokes equations, Tôhoku Math. Journ. **36** (1984), 623–646.

[28] Miyakawa, T., On nonstationary solutions of the Navier-Stokes equations in an exterior domain, Hiroshima Math. J. **12** (1982), 115–140.

[29] Shibata, Y., On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain, Tsukuba J. Math. **7** (1983), 1–68.

[30] ———, An exterior initial boundary value problem for Navier-Stokes Equations, to appear, Quart. Appl. Math..

[31] Solonnikov, V. A., General boundary value problems for Douglis-Nirenberg elliptic systems which are elliptic in the sense of Douglis-Nirenberg I, AMS Transl. (2)**56**, 193–232 (1966), Izv. Acad. Nauk SSSR Ser. Mat. **28** (1964), 665–706; II, Russian, Proc. Steklov Inst. Math. **92** (1966), 233–297.

[32] Temam, R., Navier-Stokes equations, 3rd (revised) ed., North-Holland, Amsterdam, 1984.

[33] Vainberg, B., On the analytic properties of the resolvent for a certain class of operator pencils, Mat. Sb. (N.S.) **77** (1968); Math. USSR Sb. **6** (1968), 241–273; English translation.

[34] ———, On the short wave asymptotic behavior of solutions of stationary problems and asymptotic behavior as $t \to +\infty$ of solutions of non-stationary problems, Ushpekhi Mat. Nauk. **30** (1975), 3–55; Russian Math. surveys **30** (1975), 1–58.

[35] ———, Asymptotic Methods in Equations of Mathematical Physics, in Russian, Moscow Univ. Press, 1982; Gordon and Breach Publishers, New York, London, Paris, Montreux, Tokyo, 1989; English translation.

[36] Varnhorn W., The Stokes Equations, Mathematical Research volume 76, Akademie Verlag, Berlin, 1994.

Wakako Dan
Institute of Mathematics,
University of Tsukuba, Tsukuba-shi,
Ibaraki 305-8571, Japan.

Yoshihiro Shibata
Department of Mathematics,
School of Science and Engineering,
Waseda University, Okubo 3-4-1,
Shinjyuku-ku, Tokyo 169-8555, Japan.