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Associated variety, Kostant-Sekiguchi correspondence, and locally free $U(\mathfrak{n})$-action on Harish-Chandra modules

Dedicated to Professor Takeshi Hirai on his sixtieth birthday

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. For each irreducible Harish-Chandra $(\mathfrak{g}, \mathfrak{k})$-module $\mathbf{X}$, we construct a family of nilpotent Lie subalgebras $\mathfrak{n}(\mathcal{O})$ of $\mathfrak{g}$ whose universal enveloping algebras $U(\mathfrak{n}(\mathcal{O}))$ act on $\mathbf{X}$ locally freely. The Lie subalgebras $\mathfrak{n}(\mathcal{O})$ are parametrized by the nilpotent orbits $\mathcal{O}$ in the associated variety of $\mathbf{X}$, and they are obtained by making use of the Cayley tranformation of $\mathfrak{sl}_2$-triples (Kostant-Sekiguchi correspondence). As a consequence, it is shown that an irreducible Harish-Chandra module has the possible maximal Gelfand-Kirillov dimension if and only if it admits locally free $U(\mathfrak{n}_m)$-action for $\mathfrak{n}_m = \mathfrak{n}(\mathcal{O}_{max})$ attached to a principal nilpotent orbit $\mathcal{O}_{max}$ in $\mathfrak{p}$.

Introduction.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the symmetric decomposition of $\mathfrak{g}$ defined by an involutive automorphism θ of $\mathfrak{g}$. By a Harish-Chandra module associated to the pair $(\mathfrak{g}, \mathfrak{k})$, we mean a $U(\mathfrak{g})$-module $\mathbf{X}$ of finite length on which the subalgebra $U(\mathfrak{k})$ acts locally finitely. Here $U(\mathfrak{q})$ denotes the universal enveloping algebra of a complex Lie algebra $\mathfrak{q}$.

The main purpose of this paper is to give for each irreducible Harish-Chandra module $\mathbf{X}$ a family of nilpotent Lie subalgebras $\mathfrak{n}(\mathcal{O})$ of $\mathfrak{g}$ whose enveloping algebras $U(\mathfrak{n}(\mathcal{O}))$ act on $\mathbf{X}$ locally freely. The Lie subalgebras $\mathfrak{n}(\mathcal{O})$ are parametrized by the nilpotent K_C^{ad}-orbits $\mathcal{O}$ contained in the associated variety $\mathcal{V}(\mathbf{X}) \subset \mathfrak{p}$ of $\mathbf{X}$, where K_C^{ad} denotes the analytic subgroup of the adjoint group $G_C^{ad} = \mathrm{Int}(\mathfrak{g})$ of $\mathfrak{g}$ corresponding to the Lie subalgebra $\mathfrak{k}$. We construct $\mathfrak{n}(\mathcal{O})$ from a K_C^{ad}-orbit $\mathcal{O}$ through the Cayley transformation of normal $\mathfrak{sl}_2$-triples that gives the Kostant-Sekiguchi correspondence of nilpotent orbits ([**10**]).

The Harish-Chandra modules are essentially related to infinite-dimensional representations of a real semisimple Lie group as follows. Let $\mathfrak{g}_0$ be any real form of $\mathfrak{g}$, and let G be a connected linear Lie group with Lie algebra $\mathfrak{g}_0$. We choose an involution θ of $\mathfrak{g}$ so that the real form $\mathfrak{g}_0$ is θ-stable and that $\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0$ coincides with the Lie algebra of a maximal compact subgroup K of G ([**4**, Ch. III, §4]). By fundamental

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results of Harish-Chandra ([**3**], see also [**13**, Ch. 3]), any admissible Hilbert representation $(\pi, \mathbf{H})$ of G of finite length yields a Harish-Chandra module $\mathbf{X}$ by passing to the K-finite part of $\mathbf{H}$ through differentiation. The irreducibility is preserved by the assignment $\mathbf{H} \to \mathbf{X}$. Accordingly, we may say that the present work reveals some new algebraic aspects of representations of the group G.

We now explain the results of this article in more detail.

(**I**) For a nonzero nilpotent K_C^{ad}-orbit $\mathcal{O}$ in $\mathfrak{p}$, take a normal $\mathfrak{sl}_2$-triple $(X, H, Y) \subset \mathfrak{g}$ with $X \in \mathcal{O}$ (see 1.6), and define its Cayley transform (X', H', Y') as in (1.2). Making use of the l-eigenspaces $\mathfrak{g}(l)$ $(l = 1, 2, \dots)$ of $\mathfrak{g}$ with respect to $\mathrm{ad}(H')$, we can construct a nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O}) = (\mathfrak{g}_1^1(1) \oplus \mathfrak{g}_3^3(1)) \oplus (\bigoplus_{l \geq 2} \mathfrak{g}(l))$ of $\mathfrak{g}$ with $\mathfrak{g}_1^1(1) \oplus \mathfrak{g}_3^3(1) \subset \mathfrak{g}(1)$ (see 1.4 and 1.6 for the precise definition of subspaces $\mathfrak{g}_\eta^\kappa(l)$ of $\mathfrak{g}(l)$) such that:

(i) $\dim \mathfrak{n}(\mathcal{O}) = \dim \mathcal{O}$,

(ii) the Killing form B of $\mathfrak{g}$ is nondegenerate on $\mathrm{ad}(X)\mathfrak{k} \times \mathfrak{n}(\mathcal{O})$.

(See Theorem 1.2 and Lemma 3.1.) Up to K_C^{ad}-conjugacy, the Lie subalgebra $\mathfrak{n}(\mathcal{O})$ is independent of the choice of an $\mathfrak{sl}_2$-triple (X, H, Y). In addition, the ideal $\bigoplus_{l \geq 2} \mathfrak{g}(l)$ of $\mathfrak{n}(\mathcal{O})$ becomes stable under the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$, if we construct $\mathfrak{n}(\mathcal{O})$ from a strictly normal $\mathfrak{sl}_2$-triple (Proposition 3.1). We can describe concretely the Lie subalgebras $\mathfrak{n}(\mathcal{O})$ associated to the holomorphic nilpotent orbits $\mathcal{O}$ in $\mathfrak{p}$ (Theorem 3.6), when $\mathfrak{g}_0$ is a noncompact real simple Lie algebra of hermitian type. As we indicate below, the above two properties (i) and (ii) are crucial to establish the local freeness of the $U(\mathfrak{n}(\mathcal{O}))$-action on Harish-Chandra modules.

(**II**) Now let $\mathbf{X}$ be an irreducible Harish-Chandra module. Through the natural increasing filtlation $U_k(\mathfrak{g})$ $(k = 0, 1, \dots)$ of $U(\mathfrak{g})$, we attach to each nonzero vector $v \in \mathbf{X}$ a graded module $\mathbf{M} = \mathrm{gr}(\mathbf{X}; v) := \bigoplus_{k=0}^{\infty} U_k(\mathfrak{g})v / U_{k-1}(\mathfrak{g})v$ over the symmetric algebra $S(\mathfrak{g}) \simeq \bigoplus_{k=0}^{\infty} U_k(\mathfrak{g}) / U_{k-1}(\mathfrak{g})$ of $\mathfrak{g}$, where $U_{-1}(\mathfrak{g}) := \{0\}$. The associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$ is then defined to be the set of the common zeros of elements in the annihilator $\mathrm{Ann}_{S(\mathfrak{g})}(\mathbf{M})$ of $\mathbf{M}$. Here, $\mathscr{V}(\mathbf{X})$ is independent of the choice of a vector v, and we identify $S(\mathfrak{g})$ with the ring of polynomial functions on $\mathfrak{g}$ through the Killing form B.

As is shown by Vogan [**12**], the variety $\mathscr{V}(\mathbf{X})$ associated to $\mathbf{X}$ is a union of finitely many nilpotent K_C^{ad}-orbits in $\mathfrak{p}$ (cf. Lemma 2.2). If $\mathcal{O}$ is a K_C^{ad}-orbit contained in $\mathscr{V}(\mathbf{X})$, the above properties (i) and (ii) imply that the natural projection $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}(\mathcal{O})^\perp$ induces a linear isomorphism from the tangent space $\mathrm{ad}(X)\mathfrak{k}$ of $\mathcal{O}$ at X onto $\mathfrak{g}/\mathfrak{n}(\mathcal{O})^\perp$, where $\mathfrak{n}(\mathcal{O})^\perp$ is the orthogonal of $\mathfrak{n}(\mathcal{O})$ in $\mathfrak{g}$ with respect to B. This allows us to deduce that $\mathbf{M} = \mathrm{gr}(\mathbf{X}; v)$ is a torsion free $S(\mathfrak{n}(\mathcal{O}))$-module for every nonzero $v \in \mathbf{X}$. As a consequence, we establish the main result of this article as follows.

THEOREM. (Theorem 3.2) *Let* $\mathbf{X}$ *be an irreducible Harish-Chandra module. The enveloping algebra* $U(\mathfrak{n}(\mathcal{O}))$ *of nilpotent Lie subalgebra* $\mathfrak{n}(\mathcal{O})$ *acts on* $\mathbf{X}$ *locally freely for every nilpotent* K_C^{ad}*-orbit* $\mathcal{O} \subset \mathfrak{p}$ *contained in the associated variety* $\mathscr{V}(\mathbf{X})$ *of* $\mathbf{X}$.

We remark that, by the Hilbert-Serre theorem, $\mathbf{X}$ is a torsion free $U(\mathfrak{n})$-module for a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ only if $\dim \mathfrak{n} \leq \dim \mathscr{V}(\mathbf{X})$.

Bearing this remark in mind, we derive two interesting conclusions of the above theorem. First, we find that the nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O}_{max})$ associated to a maximal K_C^{ad}-orbit $\mathcal{O}_{max}$ in $\mathscr{V}(\mathbf{X})$ realizes a maximal Lie subalgebra of $\mathfrak{g}$ among those having locally free action on $\mathbf{X}$ (Theorem 3.3). Second, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}_m$ be a complexified Iwasawa decomposition of $\mathfrak{g}_0$. Then it can be shown that an irreducible Harish-Chandra module $\mathbf{X}$ is large i.e., $\dim \mathscr{V}(\mathbf{X}) = \dim \mathfrak{n}_m$, if and only if $\mathbf{X}$ is a torsion free $U(\mathfrak{n}_m)$-module (Theorem 3.4).

(**III**) The organization of this paper is as follows.

In section 1, we first study certain fine structure on finite-dimensional $SL(2,\boldsymbol{C})$-modules equipped with involutive linear transformations (see Proposition 1.2 and Theorem 1.1). The properties (i) and (ii) stated in (I) for the nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O})$ of $\mathfrak{g}$ are shown by applying Theorem 1.1 to the adjoint representation of $\mathfrak{s} := \boldsymbol{C}X + \boldsymbol{C}H + \boldsymbol{C}Y \simeq \mathfrak{sl}(2,\boldsymbol{C})$ on $\mathfrak{g}$.

Section 2 is devoted to giving a simple criterion for $\mathbf{X}$ to be a torsion free $U(\mathfrak{n})$-module. More precisely we shall consider a much more general situation, where $\mathfrak{g}$ is an arbitrary complex Lie algebra, $\mathfrak{k}$ and $\mathfrak{n}$ are any two Lie subargebras of $\mathfrak{g}$, and $\mathbf{X}$ is a locally $U(\mathfrak{k})$-finite, irreducible $U(\mathfrak{g})$-module. Our criterion (Theorem 2.1) is given by means of the Lie subalgebras $\mathfrak{k}$, $\mathfrak{n}$ and the associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$.

In section 3, the main result of this paper, Theorem 3.2, is established by using Theorems 1.2 and 2.1. Then we deduce two important consequences (Theorems 3.3 and 3.4) of Theorem 3.2. In addition, the Lie subalgebras $\mathfrak{n}(\mathcal{O})$ associated to the holomorphic nilpotent K_C^{ad}-orbits $\mathcal{O}$ are described explicitly in 3.3.

1. $SL(2,\boldsymbol{C})$-modules with involution $\tilde{\sigma}$.

In this section, we begin with investigating in 1.1–1.5 certain fine structure on finite-dimensional $SL(2,\boldsymbol{C})$-modules V equipped with an involutive linear transformation $\tilde{\sigma} \in GL(V)$, compatible with a nontrivial involution σ of $SL(2,\boldsymbol{C})$. The results are summarized as Proposition 1.2 and Theorem 1.1.

We then apply the results to Lie algebra case in 1.6, where $V = \mathfrak{g}$ is a complex semisimple Lie algebra with an involution $\tilde{\sigma} = \theta$, and $SL(2,\boldsymbol{C})$ acts on $\mathfrak{g}$ through the adjoint representation of a θ-stable, simple Lie subalgebra $\mathfrak{s} \simeq \mathfrak{sl}(2,\boldsymbol{C})$ of $\mathfrak{g}$. This gives us a new kind of decomposition of $\mathfrak{g}$ (Theorem 1.2(3)), which is, in a sense, comparable with the (complexified) generalized Iwasawa decompositions of $\mathfrak{g}$. The nilpotent Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ appearing in this decomposition will play an essential role in §3 for studying locally free $U(\mathfrak{n})$-action on Harish-Chandra modules.

1.1. $\mathfrak{sl}_2$-triples and Cayley transformation.

Let $\mathfrak{s} = \boldsymbol{C}X + \boldsymbol{C}H + \boldsymbol{C}Y \simeq \mathfrak{sl}(2,\boldsymbol{C})$ be a three-dimensional, complex simple Lie algebra with commutation relation:

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H. \tag{1.1}$$

We denote by $S \simeq SL(2,\boldsymbol{C})$ the simply connected Lie group with Lie algebra $\mathfrak{s}$. Setting

$$X' = \frac{i}{2}(H - X + Y), \quad H' = X + Y, \quad Y' = -\frac{i}{2}(H + X - Y), \tag{1.2}$$

one gets another $\mathfrak{sl}_2$-triple (X', H', Y') in $\mathfrak{s}$ which satisfies the same relation (1.1). If we identify $\mathfrak{s}$ with $\mathfrak{sl}(2,\boldsymbol{C})$ by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$X' = \frac{i}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y' = \frac{i}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

is a basis of the real form $\mathfrak{su}(1,1)$ of $\mathfrak{s}$, and the Cayley transformation:

$$(1.3) \qquad \tilde{c} : \mathfrak{s} \ni Z \mapsto \mathrm{Ad}(c)Z = cZc^{-1} \in \mathfrak{s} \quad \text{with } c = \frac{1}{1+i}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in SL(2, \boldsymbol{C})$$

sends the $\mathfrak{sl}_2$-triple (X, H, Y) to (X', H', Y'). Note that the center of S contains a unique nontrivial element $\varepsilon = \exp(\pi i H')$ corresponding to the matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$.

Now let σ be the involutive automorphism of $\mathfrak{s}$ defined by

$$(1.4) \qquad \sigma X = -X, \quad \sigma H = H, \quad \sigma Y = -Y.$$

It then follows that $\sigma X' = -Y'$, $\sigma Y' = -X'$ and $\sigma H' = -H'$. Extend σ to an automorphism of S through the exponential map, which we denote again by σ. Let

$$(1.5) \qquad w := \exp\frac{\pi}{2}(X' - Y') = \exp X' \cdot \exp(-Y') \cdot \exp X'$$

be the element of S corresponding to the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ which represents the nontrivial element of the Weyl group of $\mathfrak{s}$ with respect to the Cartan subalgebra $\boldsymbol{C}H'$.

Direct computation in $S \simeq SL(2, \boldsymbol{C})$ immediately gives the following lemma.

LEMMA 1.1. *One has the equalities*:
(1) $\sigma(w) = w$, $w^2 = \varepsilon$,
(2) $\sigma(s) = wsw^{-1}$ $(s \in S)$, *and* σ *equals* $\mathrm{Ad}(w)$ *on* $\mathfrak{s}$,
(3) $\mathrm{Ad}(\exp(-iY'))X = iX'/2$.

1.2. Irreducible *S*-modules.

For each nonnegative integer d, let (τ_d, V_d) be an irreducible S-module of dimension $d+1$. The Lie algebra $\mathfrak{s}$ acts on V_d through differentiation. Take a nonzero highest weight vector $v_d^{(d)} \in V_d$ such that

$$(1.6) \qquad \tau_d(H')v_d^{(d)} = dv_d^{(d)}, \quad \tau_d(X')v_d^{(d)} = 0,$$

and set

$$(1.7) \qquad v_{d-2j}^{(d)} = \frac{1}{j!}\tau_d(Y')^j v_d^{(d)} \quad (j = 0, 1, \ldots, d).$$

Then the vectors $v_{d-2j}^{(d)}$ $(0 \le j \le d)$ form a basis of V_d. The action of X', H', Y' on V_d is described respectively as

$$(1.8) \qquad \begin{cases} \tau_d(X')v_{d-2j}^{(d)} = (d+1-j)v_{d-2(j-1)}^{(d)}, \\ \tau_d(H')v_{d-2j}^{(d)} = (d-2j)v_{d-2j}^{(d)}, \\ \tau_d(Y')v_{d-2j}^{(d)} = (j+1)v_{d-2(j+1)}^{(d)}, \end{cases}$$

where $v^{(d)}_{-d-2} = v^{(d)}_{d+2} = 0$. We note that the element $w \in S$ in (1.5) acts on V_d as

$$\tau_d(w)v^{(d)}_{d-2j} = (-1)^{d-j}v^{(d)}_{-(d-2j)} \quad (j = 0, 1, \ldots, d). \tag{1.9}$$

1.3. Extension $\tilde{\sigma}$ and S-homomorphism J.

Let (τ, V) be any finite-dimensional S-module (and so $\mathfrak{s}$-module). A map $\tilde{\sigma}: V \to V$ is called an *extension* of σ to V if it is an involutive linear isomorphism on V satisfying

$$\tilde{\sigma}\tau(Z)\tilde{\sigma}^{-1} = \tau(\sigma Z) \quad (Z \in \mathfrak{s}). \tag{1.10}$$

The totality of such extensions will be denoted by E_V. If $V = V_d$ is irreducible, $\tilde{\sigma} := i\tau(w)$ for $d \in 2\boldsymbol{Z} + 1$; $\tilde{\sigma} := \tau(w)$ for $d \in 2\boldsymbol{Z}$, gives an extension of σ to V, by Lemma 1.1(2).

In 1.6 we shall consider an extension $\tilde{\sigma}$ arising from an involutive automorphism of a semisimple Lie algebra $\mathfrak{g} = V$, where $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$ acting on V through the adjoint representation.

Let F_V denote the set of all S-homomorphisms J on V such that $J^2 = \tau(\varepsilon)$, where $w^2 = \varepsilon$ is, as in Lemma 1.1, the nontrivial central element of S. Then,

Proposition 1.1. *The assignment $\tilde{\sigma} \mapsto J := \tilde{\sigma}\tau(w)$ gives a bijective correspondence from E_V onto F_V.*

Proof. Let $\tilde{\sigma}$ be in E_V. Then Lemma 1.1 together with (1.10) yields that

$$J\tau(Z) = \tilde{\sigma}\tau(\mathrm{Ad}(w)Z)\tau(w) = \tilde{\sigma}\tau(\sigma Z)\tau(w) = \tau(Z)J$$

for every $Z \in \mathfrak{s}$, and that

$$J^2 = (\tilde{\sigma}\tau(w)\tilde{\sigma})\tau(w) = \tau(\sigma(w))\tau(w) = \tau(w)^2 = \tau(\varepsilon).$$

We thus find that $J \in F_V$, for the group S is connected.

Conversely, if J is in F_V, then $\tilde{\sigma} := J\tau(w)^{-1}$ belongs to E_V. In fact, it follows from Lemma 1.1 that

$$\tilde{\sigma}^2 = J^2\tau(w)^{-2} = \tau(\varepsilon)\tau(\varepsilon)^{-1} = id_V,$$

and that

$$\tilde{\sigma}\tau(Z)\tilde{\sigma}^{-1} = \tau(w)^{-1}\tau(Z)\tau(w) = \tau(\sigma Z)$$

for $Z \in \mathfrak{s}$, where id_V denotes the identity operator on V. These two equalities show that $\tilde{\sigma}$ is an extension of σ to V. $\square$

It should be noticed that

$$J\tilde{\sigma} = \tilde{\sigma}J = \tau(w), \tag{1.11}$$

since $\tilde{\sigma}$ is involutive and it commutes with $\tau(w)$.

We fix once and for all an extension $\tilde{\sigma}$ of σ to V, and the corresponding S-homomorphism $J = \tilde{\sigma}\tau(w)$.

1.4. The subspace U.

For an S-module (τ, V) with $\tilde{\sigma} \in E_V$ and the corresponding $J \in F_V$ in 1.3, let

$$V = V(\tilde{\sigma}, +1) \oplus V(\tilde{\sigma}, -1) \quad \text{with } V(\tilde{\sigma}, \pm 1) := \{v \in V \mid \tilde{\sigma} v = \pm v\} \tag{1.12}$$

be the eigenspace decomposition of V with respect to $\tilde{\sigma}$. The semisimple element $H' \in \mathfrak{s}$ gives a weight space decomposition of V:

$$V = \bigoplus_{l \in \boldsymbol{Z}} V(l) \quad \text{with } V(l) := \{v \in V \mid \tau(H')v = lv\}. \tag{1.13}$$

Let

$$V = \bigoplus_{d \geq 0} [m_d] \cdot V_d \quad \text{with } [m_d] \cdot V \simeq V_d \oplus \cdots \oplus V_d \ (m_d\text{-copies}) \tag{1.14}$$

be the irreducible decomposition of the S-module V, where m_d denotes the multiplicity of simple S-module V_d (see 1.2) in V. Put

$$V^{(\kappa)} := \bigoplus_{d \in I(\kappa)} [m_d] \cdot V_d \subset V, \quad I(\kappa) := \{\kappa + 4n \mid n = 0, 1, \ldots\}, \tag{1.15}$$

for $\kappa = 0, 1, 2, 3$. Then $V^{(\kappa)}$ is the S-submodule of V generated by all the maximal weight vectors in V with weight $\lambda \equiv \kappa \pmod 4$. Clearly it holds that

$$V = \bigoplus_{\kappa=0}^{3} V^{(\kappa)} \quad \text{as } S\text{-modules}, \tag{1.16}$$

and that

$$V^{(\kappa)} = \bigoplus_{l \in \boldsymbol{Z}} V^{(\kappa)}(l) \quad \text{with } V^{(\kappa)}(l) := V^{(\kappa)} \cap V(l) \tag{1.17}$$

gives the weight space decomposition of $V^{(\kappa)}$, where $V^{(\kappa)}(l) = \{0\}$ if $\kappa - l \notin 2\boldsymbol{Z}$. Note that any S-submodule W of V decomposes as

$$W = \bigoplus_{\kappa=0}^{3} W \cap V^{(\kappa)}, \tag{1.18}$$

since each irreducible constituent of W with highest weight $d \in I(\kappa)$ is contained in $V^{(\kappa)}$.

Using the S-homomorphism J on V such that $J^4 = \tau(\varepsilon)^2 = \tau(1) = id_V$, we decompose the S-representation (τ, V) also as

$$V = \bigoplus_{\eta=0}^{3} V_{(\eta)} \quad \text{with } V_{(\eta)} := \{v \in V \mid Jv = i^{\eta} v\}. \tag{1.19}$$

Denote by $V_{(\eta)}(l) := V(l) \cap V_{(\eta)}$ the l-weight subspace of $V_{(\eta)}$. We observe that $V_{(\eta)}(l) = \{0\}$ if $\eta - l \notin 2\boldsymbol{Z}$, because $J^2 = \tau(\varepsilon) = \exp(\pi i \tau(H'))$ acts on $V(l)$ by the scalar $(-1)^l$.

Summarizing the above discussion, we immediately deduce the following lemma on the compatibility of two decompositions (1.16) and (1.19).

LEMMA 1.2. *(τ, V) admits the decomposition*:

$$V = \bigoplus_{\kappa, \eta=0}^{3} V_{\eta}^{\kappa} \quad \textit{with} \quad V_{\eta}^{\kappa} := V^{(\kappa)} \cap V_{(\eta)} \tag{1.20}$$

as S-modules, and V_{η}^{κ} equals $\{0\}$ if $\kappa - \eta \notin 2\boldsymbol{Z}$.

This lemma shows that the even part $V^{even} := \bigoplus_{l \in 2\boldsymbol{Z}} V(l)$ and the odd part $V^{odd} := \bigoplus_{l \in 2\boldsymbol{Z}+1} V(l)$ of V decompose respectively as

$$(1.21) \qquad \begin{cases} V^{even} = V^{(0)} \oplus V^{(2)} = V_{(0)} \oplus V_{(2)} = V_0^0 \oplus V_2^0 \oplus V_0^2 \oplus V_2^2, \\ V^{odd} = V^{(1)} \oplus V^{(3)} = V_{(1)} \oplus V_{(3)} = V_1^1 \oplus V_3^1 \oplus V_1^3 \oplus V_3^3. \end{cases}$$

We note that the involution $\tilde{\sigma}$ acts on V in the following way.

Lemma 1.3. *For $\kappa, \eta = 0, 1, 2, 3$, and $l \in \mathbf{Z}$, let $V_\eta^\kappa(l) := V_\eta^\kappa \cap V(l)$ denote the l-weight subspace of V_η^κ. Then it holds that*

$$(1.22) \qquad \tilde{\sigma} V_\eta^\kappa = V_\eta^\kappa, \quad \tilde{\sigma} V(l) = V(-l), \quad \text{and so } \tilde{\sigma} V_\eta^\kappa(l) = V_\eta^\kappa(-l).$$

Proof. It follows from (1.10) that, if W is any irreducible S-submodule of V, so is $\tilde{\sigma} W$ and $\dim W$ equals $\dim \tilde{\sigma} W$. This implies that each S-submodule V^κ is $\tilde{\sigma}$-stable. We get $\tilde{\sigma} V_{(\eta)} = V_{(\eta)}$ by virtue of the commutativity (1.11). Thus we get the first equality in (1.22). The second one follows from $\sigma H' = -H'$, and the third one is an immediate consequence of the former two. □

We now introduce a subspace $\mathbf{U}$ of V defined as follows:

$$(1.23) \qquad \mathbf{U} := (V_1^1(1) \oplus V_3^3(1)) \oplus \left(\bigoplus_{l \geq 2} V(l) \right).$$

Later in §3, this subspace $\mathbf{U}$ gives a nilpotent Lie subalgebra $\mathfrak{n}$ of a semisimple Lie algebra $\mathfrak{g}$ which admits locally free action on Harish-Chandra modules for $\mathfrak{g}$.

The following proposition is one of the main ingredients to establish our main result on locally free $U(\mathfrak{n})$-action on Harish-Chandra modules.

Proposition 1.2. *Let (τ, V) be a finite-dimensional S-module with $\tilde{\sigma}$, $J \in GL(V)$ as in* 1.3, *and let $\mathbf{U}$ be the subspace of V defined above. Then V is expressed as a sum of three subspaces as*

$$(1.24) \qquad V = V(\tilde{\sigma}, +1) + \operatorname{Ker} \tau(X) + \mathbf{U} = \tilde{\sigma}\mathbf{U} + \operatorname{Ker} \tau(X) + \mathbf{U},$$

where $V(\tilde{\sigma}, +1)$ is the subspace of $\tilde{\sigma}$-fixed vectors as in (1.12), *and* $\operatorname{Ker} \tau(X) = \{v \in V \mid \tau(X)v = 0\}$ *denotes the kernel of $\tau(X)$.*

Proof. Set $M := V(\tilde{\sigma}, +1) + \operatorname{Ker} \tau(X) + \mathbf{U}$. First we see easily from Lemma 1.3 together with the definition of $\mathbf{U}$ that the sum $V(\tilde{\sigma}, +1) + \mathbf{U}$ is decomposed as

$$\begin{aligned} V(\tilde{\sigma}, +1) + \mathbf{U} &= \mathbf{U} + \tilde{\sigma}\mathbf{U} + V(\tilde{\sigma}, +1) \\ &= \left\{ \bigoplus_{|l| \geq 2} V(l) \right\} \oplus \{V_1^1(1) \oplus V_1^1(-1)\} \oplus \{V_3^3(1) \oplus V_3^3(-1)\} \\ &\quad \oplus \{(V_1^3(1) \oplus V_1^3(-1)) \cap V(\tilde{\sigma}, +1)\} \oplus \{(V_3^1(1) \oplus V_3^1(-1)) \cap V(\tilde{\sigma}, +1)\} \\ &\quad \oplus \{V(0) \cap V(\tilde{\sigma}, +1)\}. \end{aligned}$$

Hence, for the proof of (1.24) it is enough to show that three subspaces:

$$(1.25) \qquad \begin{cases} Q_\eta^\kappa := (V_\eta^\kappa(1) \oplus V_\eta^\kappa(-1)) \cap V(\tilde{\sigma}; -1) \quad \text{with } (\kappa, \eta) = (1,3), (3,1), \quad \text{and} \\ R := V(0) \cap V(\tilde{\sigma}, -1), \end{cases}$$

are contained in M. Before proving this, we remark that

$$M \supset \operatorname{Ker} \tau(X) = (\exp i\tau(Y')) \cdot \operatorname{Ker} \tau(X') \tag{1.26}$$

by Lemma 1.1(3), and that the subspace $\operatorname{Ker} \tau(X')$ is exactly the linear span of all the maximal weight vectors in V with respect to $\tau(H')$.

Now let (κ, η) be $(1,3)$ or $(3,1)$, and let us show $Q_\eta^\kappa \subset M$. Consider the irreducible decomposition:

$$V_\eta^\kappa = \bigoplus_{p=1}^{s} V_{d_p} \tag{1.27}$$

as an S-module, where V_{d_p} is, as in 1.2, the irreducible S-module with highest weight $d_p \equiv \kappa \pmod 4$.

For a while we fix any $p \in \{1, 2, \dots, s\}$, and take a nonzero highest weight vector $v_{d_p}^{(d_p)} \in V_{d_p}(d_p) \subset \operatorname{Ker} \tau(X')$. Then one gets

$$(\exp i\tau(Y')) \cdot v_{d_p}^{(d_p)} = \sum_{j=0}^{d_p} i^j \cdot v_{d_p - 2j}^{(d_p)} \in M, \tag{1.28}$$

by (1.26), where $v_{d_p-2j}^{(d_p)} = \tau(Y')^j v_{d_p}^{(d_p)}/j! \in V_{d_p}$ is a weight vector with weight $d_p - 2j$. Since $\bigoplus_{|l| \geq 2} V(l) \subset M$, we find that $v_1^{(d_p)} + iv_{-1}^{(d_p)}$ lies in M, and this vector can be calculated as follows:

$$\begin{aligned} v_1^{(d_p)} + iv_{-1}^{(d_p)} &= v_1^{(d_p)} + (-1)^{(d_p+1)/2} i\tau(w) v_1^{(d_p)} \quad \text{by (1.9)} \\ &= v_1^{(d_p)} + (-1)^{(d_p+1)/2} i\tilde{\sigma} J v_1^{(d_p)} \quad \text{by (1.11)} \\ &= v_1^{(d_p)} + (-1)^{(\kappa+1)/2} i^{\eta+1} \tilde{\sigma} v_1^{(d_p)} \quad \text{since } v_1^{(d_p)} \in V_\eta^\kappa \\ &= v_1^{(d_p)} - \tilde{\sigma} v_1^{(d_p)} \end{aligned}$$

for $(\kappa, \eta) = (3,1)$ or $(1,3)$. We thus conclude $v_1^{(d_p)} - \tilde{\sigma} v_1^{(d_p)} \in M$.

Considering this inclusion for all $p = 1, 2, \dots, s$, we get $Q_\eta^\kappa \subset M$ since the vectors $v_1^{(d_p)} - \tilde{\sigma} v_1^{(d_p)}$ $(p = 1, 2, \dots, s)$ form a basis of Q_η^κ by (1.22).

The inclusion $R \subset M$ can be shown in the same (even easier) way. □

1.5. *S*-modules with *JS*-invariant form.

Let (τ, V) be, as in 1.3, a finite-dimensional S-module with extension $\tilde{\sigma} \in E_V$ and $J = \tilde{\sigma}\tau(w) \in F_V$. A bilinear form B on V is called J- and S-invariant, or *JS-invariant* for short, if it satisfies

$$B(Jv, Jv') = B(\tau(s)v, \tau(s)v') = B(v, v') \quad (s \in S), \tag{1.29}$$

or equivalently,

$$B(\tilde{\sigma}v, \tilde{\sigma}v') = B(v, v'), \quad \text{and} \quad B(\tau(Z)v, v') + B(v, \tau(Z)v') = 0 \quad (Z \in \mathfrak{s}) \tag{1.30}$$

for all $v, v' \in V$.

Now let us consider the subspaces $V(\tilde{\sigma}, \pm 1)$, $V(l)$ $(l \in \mathbf{Z})$ and the S-submodules $V^{(\kappa)}$, $V_{(\eta)}$ $(\kappa, \eta = 0, 1, 2, 3)$ of V defined in (1.12), (1.13), and (1.15), (1.19) respectively. If B is any JS-invariant bilinear form on V, these subspaces have the following orthogonality relations with respect to B:

(1.31) $$V(\tilde{\sigma}, \pm 1) \perp V(\tilde{\sigma}, \mp 1), \quad V(l) \perp V(l') \quad \text{if } l + l' \neq 0,$$

(1.32) $$V^{(\kappa)} \perp V^{(\kappa')} \text{ if } \kappa \neq \kappa', \quad V_{(\eta)} \perp V_{(\eta')} \quad \text{if } \eta + \eta' \neq 0 \text{ or } 4,$$

which can be checked easily by the JS-invariance of B. (For the third relation, use the fact that the Casimir element $H'^2 + 2(X'Y' + Y'X')$ for $\mathfrak{s}$ has distinct eigenvalues on each $V^{(\kappa)}$.) Here, for any subsets L_1 and L_2 of V, $L_1 \perp L_2$ stands for $B(v_1, v_2) = 0$ for every $v_1 \in L_1$ and $v_2 \in L_2$.

Now we get the following consequence of Proposition 1.2 for (τ, V) with JS-invariant form.

Theorem 1.1. *Assume that the S-module (τ, V) admits a JS-invariant, nondegenerate symmetric bilinear form B on V. Then we get the following.*

(1) $\dim \mathbf{U} = \dim \tau(X)V(\tilde{\sigma}, +1) = \dim \tau(X)V(\tilde{\sigma}, -1) = (\dim \tau(X)V)/2$.

(2) *B is nondegenerate on $\tau(X)V(\tilde{\sigma}, +1) \times \mathbf{U}$.*

(3) $V = (V(\tilde{\sigma}, +1) + \mathrm{Ker}\, \tau(X)) \oplus \mathbf{U}$ *(direct sum).*

Here $V(\tilde{\sigma}, \pm 1)$ and $\mathbf{U}$ are the subspaces of V defined by (1.12) *and in* (1.23), *respectively.*

Proof. (1) We first prove the equality $\dim \tau(X)V(\tilde{\sigma}, +1) = \dim \tau(X)V(\tilde{\sigma}, -1)$. This can be done just in the same way as in the proof of [**8**, Prop. 5].

In fact, set $(v, v')_X := B(\tau(X)v, v')$ $(v, v' \in V)$. Then we can see that $(\cdot, \cdot)_X$ gives a skew-symmetric bilinear form on V with kernel $\mathrm{Ker}\, \tau(X)$. Hence this $(\cdot, \cdot)_X$ naturally induces a nondegenerate, skew-symmetric bilinear form on $V/\mathrm{Ker}\, \tau(X)$ which we denote again by $(\cdot, \cdot)_X$. Note that the operator $\tau(X)$ sends $V(\tilde{\sigma}, \pm 1)$ to $V(\tilde{\sigma}, \mp 1)$ since $\sigma X = -X$. Hence we can identify $V/\mathrm{Ker}\, \tau(X)$ with the direct sum:

$$V(\tilde{\sigma}, +1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, +1)) \oplus V(\tilde{\sigma}, -1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, -1)),$$

in the canonical way, and each constituent $V(\tilde{\sigma}, \pm 1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, \pm 1))$ is totally isotropic with respect to $(\cdot, \cdot)_X$ by the first orthogonality in (1.31). This shows that the bilinear form $(\cdot, \cdot)_X$ gives a nondegenerate pairing on

$$V(\tilde{\sigma}, +1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, +1)) \times V(\tilde{\sigma}, -1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, -1)).$$

We thus obtain

$$\begin{aligned}\dim \tau(X)V(\tilde{\sigma}, +1) &= \dim V(\tilde{\sigma}, +1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, +1)) \\ &= \dim V(\tilde{\sigma}, -1)/(\mathrm{Ker}\, \tau(X) \cap V(\tilde{\sigma}, -1)) = \dim \tau(X)V(\tilde{\sigma}, -1),\end{aligned}$$

which is equal to $(\dim \tau(X)V)/2$.

Second, let us prove the first equality in (1), or equivalently $\dim \mathbf{U} = (\dim \tau(X)V)/2$ by the above result. Keeping (1.26) in mind, we can calculate $\dim \tau(X)V$ as

$$\begin{aligned}\dim \tau(X)V &= \dim V - \dim \mathrm{Ker}\, \tau(X') = \dim V - (\dim V(0) + \dim V(1)) \\ &= \dim V(1) + 2\sum_{l \geq 2} \dim V(l).\end{aligned}$$

Here we used the fact that $\dim \mathrm{Ker}\, \tau(X')$ coincides with the number of (linearly independent) maximal weight vectors in the S-module V, which is given by $\dim V(0) +$

$\dim V(1)$ (see 1.2). In view of the definition of $\mathbf{U}$, it is sufficient for us to show

$$\dim V_1^1(1) + \dim V_3^3(1) = \frac{1}{2} \dim V(1). \tag{1.33}$$

This is true because the bilinear form B gives nondegenerate pairings on

$$V_1^1(1) \times V_3^1(-1) \quad \text{and} \quad V_1^3(1) \times V_3^3(-1), \tag{1.34}$$

because of (1.31) and (1.32). Thus the proof of (1) is over.

(2) For any subset L of V, let $L^\perp$ denote the orthogonal of L in V with respect to B. Observe that $(\operatorname{Ker} \tau(X))^\perp = \tau(X)V$ and that $V(\tilde{\sigma}, \pm 1)^\perp = V(\tilde{\sigma}, \mp 1)$. Then we get

$$(V(\tilde{\sigma}, +1) + \operatorname{Ker} \tau(X))^\perp = \tau(X)V(\tilde{\sigma}, +1), \tag{1.35}$$

with $\tau(X)V(\tilde{\sigma}, \pm 1) \subset V(\tilde{\sigma}, \mp 1)$ in mind. Proposition 1.2 combined with this equality (1.35) shows that $\mathbf{U}^\perp \cap \tau(X)V(\tilde{\sigma}, +1) = \{0\}$. We thus get the assertion, because the subspaces $\mathbf{U}$ and $\tau(X)V(\tilde{\sigma}, +1)$ have the same dimension as is already shown in (1).

(3) Because of (1) and (1.35), it suffices to show that the intersection of the two direct summands of the right hand side is equal to zero. If v belongs to both summands, then it follows that $v \in \mathbf{U}$, that $v \in (\tau(X)V(\tilde{\sigma}, +1))^\perp$ by (1.35), and hence that $v = 0$ by (2). $\square$

Remark. Since $\tau(Y')V_\eta^\kappa(1) = V_\eta^\kappa(-1)$, it follows from (1.31) and (1.32) that

$$B(\tau(Y')v, v') = 0 \quad \text{for } v, v' \in \mathbf{U} \cap V(1) = V_1^1(1) \oplus V_3^3(1). \tag{1.36}$$

This combined with (1.33) shows that $\mathbf{U} \cap V(1)$ is a maximally totally isotropic subspace for the skew-symmetric bilinear form $V(1) \times V(1) \ni (v_1, v_2) \to B(\tau(Y')v_1, v_2) \in \boldsymbol{C}$ on $V(1)$.

1.6. An application of Theorem 1.1.

We conclude this section by an application of Theorem 1.1 to the case where $\mathfrak{g} = V$ is a semisimple Lie algebra and (τ, V) is the adjoint representation on $\mathfrak{g}$ of a Lie subalgebra $\mathfrak{s} \simeq \mathfrak{sl}(2, \boldsymbol{C}) \subset \mathfrak{g}$.

To be more precise, let $\mathfrak{g}$ be a complex semisimple Lie algebra, and θ be an involutive automorphism of $\mathfrak{g}$. We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the eigenspace decomposition of $\mathfrak{g}$ with respect to θ, where $\mathfrak{k} := \mathfrak{g}(\theta, +1)$ and $\mathfrak{p} := \mathfrak{g}(\theta, -1)$ are as in (1.12) with $\tilde{\sigma} = \theta$.

Let (X, H, Y) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with commutation relation (1.1). Such a triple is called *normal* (cf. [**8**]) if $\sigma = \theta$ acts on the elements X, H, and Y as in (1.4). Take an arbitrary normal $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}$, and set $\mathfrak{s} := \boldsymbol{C}X + \boldsymbol{C}H + \boldsymbol{C}Y = \boldsymbol{C}X' + \boldsymbol{C}H' + \boldsymbol{C}Y' \simeq \mathfrak{sl}(2, \boldsymbol{C})$. Here (X', H', Y') is the Cayley transform of (X, H, Y) defined by (1.2).

We consider $(\tau, V) = (\operatorname{ad} | \mathfrak{s}, \mathfrak{g})$, the adjoint representation of $\mathfrak{s}$ on $\mathfrak{g}$. Put $J := \theta \operatorname{Ad}(w)$, where w is defined by (1.5). Then the involution θ on $\mathfrak{g}$ is actually an extension of $\theta | \mathfrak{s}$ (= the restriction of θ to $\mathfrak{s}$) to $\mathfrak{g}$ in the sense of (1.10), and that the Killing form B of $\mathfrak{g}$ gives a nondegenerate JS-invariant form on $\mathfrak{g}$ (see 1.5 for the definition). Let

$$\mathfrak{n} = \mathfrak{n}_{\mathfrak{s}} := (\mathfrak{g}_1^1(1) \oplus \mathfrak{g}_3^3(1)) \oplus \left(\bigoplus_{l \geq 2} \mathfrak{g}(l) \right) \tag{1.37}$$

denote the subspace $\mathbf{U}$ of $\mathfrak{g}$ defined in (1.23) for $V = \mathfrak{g}$. Then it is easily seen that $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

Applying Theorem 1.1 to the above setting, we now get

THEOREM 1.2. *Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the symmetric decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to the involution θ of $\mathfrak{g}$, and let (X, H, Y) be a normal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Then one gets the following properties* (1)–(3) *for the nilpotent Lie subalgebra $\mathfrak{n} = \mathfrak{n}_{\mathfrak{s}}$ of $\mathfrak{g}$ given by* (1.37):

(1) $\dim \mathfrak{n} = \dim \operatorname{ad}(X)\mathfrak{k} = \dim \operatorname{ad}(X)\mathfrak{p} = (\dim \operatorname{ad}(X)\mathfrak{g})/2$,

(2) *the Killing form B of $\mathfrak{g}$ is nondegenerate on* $\operatorname{ad}(X)\mathfrak{k} \times \mathfrak{n}$,

(3) $\mathfrak{g} = (\mathfrak{k} + \mathfrak{z}(X)) \oplus \mathfrak{n}$ *as vector spaces.*

Here $\mathfrak{z}(X) := \operatorname{Ker} \operatorname{ad}(X)$ *denotes the centralizer of X in* $\mathfrak{g}$.

REMARKS. (1) Set $\tilde{\mathfrak{n}} := \bigoplus_{l \geq 1} \mathfrak{g}(l)$, then $\tilde{\mathfrak{n}}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}$ as its ideal. The Remark in 1.5 implies that our $\mathfrak{n}$ is a polarizing subalgebra (see e.g., [**1**, p. 28]) of $\tilde{\mathfrak{n}}$ for the linear form:

$$\xi_{Y'} : \tilde{\mathfrak{n}} \ni Z \mapsto B(Y', Z) \in \boldsymbol{C}$$

on $\tilde{\mathfrak{n}}$, defined by the nilpotent element $Y' \in \mathfrak{g}(-2)$ through the Killing form. In particular $\xi_{Y'}$ gives a one-dimensional representation of $\mathfrak{n}$.

(2) Let G_C be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and $N_C = \exp \mathfrak{n}$ be the analytic subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{n}$. Then, the character $\xi_{Y'}$ of $\mathfrak{n}$ gives rise to an induced G_C-module $\operatorname{Ind}_{N_C}^{G_C}(\exp \xi_{Y'})$, called the *generalized Gelfand-Graev representation* of G_C associated to the nilpotent orbit $\operatorname{Ad}(G_C)Y'$. ([**6**], see also [**14**, §1].)

2. Associated variety and a criterion for locally free $U(\mathfrak{n})$-action.

Let $\mathfrak{g}$ be any finite-dimensional complex Lie algebra, and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We now consider two Lie subalgebras $\mathfrak{k}$ and $\mathfrak{n}$ of $\mathfrak{g}$. In this section we give a simple criterion (Theorem 2.1) for a locally $U(\mathfrak{k})$-finite, irreducible $U(\mathfrak{g})$-module $\mathbf{X}$ to be a torsion free $U(\mathfrak{n})$-module. Our criterion is described by means of the Lie subalgebras $\mathfrak{k}$, $\mathfrak{n}$ and the associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$. It has, as we show in §3, an interesting application when $\mathbf{X}$ is a Harish-Chandra module for a semisimple Lie algebra $\mathfrak{g}$.

2.1. Associated variety for $U(\mathfrak{g})$-modules.

First let us review the definition and some fundamental properties of the associated variety for finitely generated $U(\mathfrak{g})$-modules, which is one of the principal objects in the present article. A basic reference is [**12**].

Denote by $(U_k(\mathfrak{g}))_{k=0,1,\dots}$ the natural increasing filtration of $U(\mathfrak{g})$, where $U_k(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ generated by elements $X_1 \cdots X_m$ ($m \leq k$) with $X_j \in \mathfrak{g}$ ($1 \leq j \leq m$). By the Poincaré-Birkhoff-Witt theorem, we can identify the associated graded ring

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{k \geq 0} U_k(\mathfrak{g})/U_{k-1}(\mathfrak{g}) \quad (U_{-1}(\mathfrak{g}) := (0))$$

with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k \geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of $S(\mathfrak{g})$ of degree k.

Let $\mathbf{X}$ be a finitely generated $U(\mathfrak{g})$-module. Take a finite-dimensional subspace $\mathbf{X}_0$ of $\mathbf{X}$ such that $\mathbf{X} = U(\mathfrak{g})\mathbf{X}_0$. Setting $\mathbf{X}_k = U_k(\mathfrak{g})\mathbf{X}_0$ $(k = 1, 2, \ldots)$, one gets an increasing filtration $(\mathbf{X}_k)_k$ of $\mathbf{X}$, and correspondingly a finitely generated, graded $S(\mathfrak{g})$-module

$$\mathbf{M} = \mathrm{gr}(\mathbf{X}; \mathbf{X}_0) := \bigoplus_{k \geq 0} \mathbf{M}_k \quad \text{with } \mathbf{M}_k = \mathbf{X}_k/\mathbf{X}_{k-1}. \tag{2.1}$$

The annihilator $\mathrm{Ann}_{S(\mathfrak{g})}\mathbf{M} := \{D \in S(\mathfrak{g}) \mid Dv = 0\ (\forall v \in \mathbf{M})\}$ of $\mathbf{M}$ is a graded ideal of $S(\mathfrak{g})$, and it defines an algebraic cone in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$:

$$\mathscr{V}(\mathbf{M}) := \{\lambda \in \mathfrak{g}^* \mid D(\lambda) = 0\ (\forall D \in \mathrm{Ann}_{S(\mathfrak{g})}\mathbf{M})\}, \tag{2.2}$$

as the set of common zeros of elements of $\mathrm{Ann}_{S(\mathfrak{g})}\mathbf{M}$. Here $S(\mathfrak{g})$ is regarded as the polynomial ring over $\mathfrak{g}^*$ in the canonical way. It is then easily seen that the variety $\mathscr{V}(\mathbf{M})$ does not depend on the choice of a generating subspace $\mathbf{X}_0$. Therefore we write $\mathscr{V}(\mathbf{X})$ for $\mathscr{V}(\mathbf{M})$.

DEFINITION. (Cf. [**12**], see also [**16**]) For a finitely generated $U(\mathfrak{g})$-module $\mathbf{X}$, the variety $\mathscr{V}(\mathbf{X}) \subset \mathfrak{g}^*$ and its dimension $d(\mathbf{X}) := \dim \mathscr{V}(\mathbf{X})$ are called respectively the *associated variety* and the *Gelfand-Kirillov dimension* of $\mathbf{X}$.

REMARK. By the Hilbert-Serre theorem (cf. [**16**, Th. 1.1]), the map $k \to \dim \mathbf{X}_k$ coincides with a polynomial in k of degree $d(\mathbf{X})$, for sufficiently large k.

Let $G_C^{ad} := \mathrm{Int}(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$. We write $I(\mathfrak{g})$ for the graded subalgebra of $S(\mathfrak{g})$ consisting of all G_C^{ad}-fixed elements in $S(\mathfrak{g})$. Then $I(\mathfrak{g})$ has a unique maximal graded ideal $I(\mathfrak{g})_+ := \bigoplus_{k>0} I(\mathfrak{g}) \cap S^k(\mathfrak{g})$.

Using the Schur lemma [**13**, Lemma 0.5.2] for irreducible $U(\mathfrak{g})$-modules, one can deduce the following.

LEMMA 2.1. (Cf. [**12**, Cor. 5.4]) *Suppose that $\mathbf{X}$ is a $U(\mathfrak{g})$-module of finite length. Then its associated variety $\mathscr{V}(\mathbf{X})$ is contained in the cone $\mathscr{N}^*$ defined by $I(\mathfrak{g})_+$:*

$$\mathscr{N}^* := \{\lambda \in \mathfrak{g}^* \mid D(\lambda) = 0 \quad (\forall D \in I(\mathfrak{g})_+)\}. \tag{2.3}$$

Notice that, if $\mathfrak{g}$ is semisimple, the cone $\mathscr{N}^*$ is the totality of nilpotent elements in $\mathfrak{g}$. Here $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ by the Killing form.

2.2. The variety $\mathscr{V}(\mathbf{X})$ for $(\mathfrak{g}, \mathfrak{k})$-module $\mathbf{X}$.

Now let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$. A $U(\mathfrak{g})$-module $\mathbf{X}$ is said to be locally $U(\mathfrak{k})$-finite if the $U(\mathfrak{k})$-submodule $U(\mathfrak{k})v$ is finite-dimensional for every $v \in \mathbf{X}$. By a $(\mathfrak{g}, \mathfrak{k})$-*module* is meant a locally $U(\mathfrak{k})$-finite, finitely generated $U(\mathfrak{g})$-module. Hereafter we exclusively consider such $(\mathfrak{g}, \mathfrak{k})$-modules.

Let $\tilde{K}_C$ denote the connected, simply connected Lie group with Lie algebra $\mathfrak{k}$. The natural inclusion $i : \mathfrak{k} \hookrightarrow \mathfrak{g}$ gives rise to a Lie group homomorphism:

$$\mathrm{Ad} : \tilde{K}_C \ni k \mapsto \mathrm{Ad}(k) \in G_C^{ad} \subset GL(\mathfrak{g}), \tag{2.4}$$

from $\tilde{K}_C$ into the group G_C^{ad} of all inner automorphisms of $\mathfrak{g}$, in the canonical way. We notice that, since $\tilde{K}_C$ is simply connected, any $(\mathfrak{g}, \mathfrak{k})$-module $\mathbf{X}$ admits a $\tilde{K}_C$-module structure compatible with the $U(\mathfrak{g})$-action in the following sense:

$$(\exp Z)\cdot v = \sum_{j=0}^{\infty}\frac{1}{j!}Z^j v \quad (Z \in \mathfrak{k}), \tag{2.5}$$

$$k\cdot(Dv) = (\mathrm{Ad}(k)D)\cdot kv \quad (D \in U(\mathfrak{g}),\ k \in \tilde{K}_C), \tag{2.6}$$

for every $v \in \mathbf{X}$. Here the sum in (2.5) converges because $Z^j v$ stay in a finite-dimensional subspace $U(\mathfrak{k})v$ for all $j \geq 0$.

By using this $\tilde{K}_C$-action, it is easy to deduce the following lemma on an orbital structure of the associated variety of a $(\mathfrak{g}, \mathfrak{k})$-module.

Lemma 2.2. (Cf. [**12**, Cor. 5.13]) *Let $\mathbf{X}$ be a $(\mathfrak{g}, \mathfrak{k})$-module. Then the associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$ is a union of K_C^{ad}-orbits contained in the orthogonal $\mathfrak{k}^{\perp} := \{\lambda \in \mathfrak{g}^* \mid \lambda(Z) = 0\ (\forall Z \in \mathfrak{k})\}$ of $\mathfrak{k}$ in $\mathfrak{g}^*$. Here $K_C^{ad} := \mathrm{Ad}(\tilde{K}_C) \subset G_C^{ad}$ denotes the analytic subgroup of G_C^{ad} with Lie algebra $\mathfrak{k}$, and it acts on $\mathfrak{g}^*$ through the coadjoint representation.*

2.3. A criterion for locally free $U(\mathfrak{n})$-action.

Let $\mathfrak{k}$ and K_C^{ad} be as in 2.2. Take another Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ (not necessarily the one given by (1.37)). We shall give a criterion for an irreducible $(\mathfrak{g}, \mathfrak{k})$-module to have locally free $U(\mathfrak{n})$-action.

To do this, let p^* be the surjective linear map from $\mathfrak{g}^*$ to $\mathfrak{n}^*$ defined by the restriction to $\mathfrak{n}$ of each linear form on $\mathfrak{g}$. We say that an element $\lambda \in \mathfrak{g}^*$ satisfies the condition $(P_{\mathfrak{k},\mathfrak{n}})$ if the projection p^* mapps the subspace $\mathrm{ad}^*(\mathfrak{k})\lambda := \{\mathrm{ad}^*(Z)\lambda \mid Z \in \mathfrak{k}\}$ *onto* $\mathfrak{n}^*$, i.e.,

$$p^*(\mathrm{ad}^*(\mathfrak{k})\lambda) = \mathfrak{n}^*. \tag{$P_{\mathfrak{k},\mathfrak{n}}$}$$

Here $\mathrm{ad}^*(Z)\lambda := (d/dt)(\exp tZ\cdot\lambda)|_{t=0}$, and $\mathrm{ad}^*(\mathfrak{k})\lambda$ can be identified naturally with the tangent space of K_C^{ad}-orbit $K_C^{ad}\cdot\lambda$ at the point λ.

This condition $(P_{\mathfrak{k},\mathfrak{n}})$ for λ has the following geometric interpretation.

Lemma 2.3. *The image $p^*(K_C^{ad}\cdot\lambda)$ of K_C^{ad}-orbit $K_C^{ad}\cdot\lambda$ under p^* contains an open neighbourhood of $p^*(\lambda)$ in $\mathfrak{n}^*$ if and only if $\lambda \in \mathfrak{g}^*$ satisfies the condition $(P_{\mathfrak{k},\mathfrak{n}})$.*

Proof. The condition $(P_{\mathfrak{k},\mathfrak{n}})$ implies that the differential of

$$K_C^{ad} \ni k \mapsto p^*(k\cdot\lambda) \in \mathfrak{n}^*$$

is surjective at the origin $e \in K_C^{ad}$, and vice versa. This immediately proves the claim. □

Proposition 2.1. *Let $\mathbf{X}$ be a cyclic $(\mathfrak{g}, \mathfrak{k})$-module generated by a vector $v_0 \in \mathbf{X}$: $\mathbf{X} = U(\mathfrak{g})v_0$. For a Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, the annihilator $\mathrm{Ann}_{U(\mathfrak{n})}(v_0)$ vanishes if there exists an element $\lambda \in \mathscr{V}(\mathbf{X})$ satisfying the condition $(P_{\mathfrak{k},\mathfrak{n}})$.*

Proof. Let $\mathbf{M} = \mathrm{gr}(\mathbf{X}; \mathbf{X}_0) = \bigoplus_{k\geq 0}\mathbf{M}_k$, with $\mathbf{X}_0 = \boldsymbol{C}v_0$, be the graded $S(\mathfrak{g})$-module in (2.1). Write $\tilde{v}_0$ for the vector v_0 viewed as an element of $\mathbf{M}_0 \subset \mathbf{M}$. We remark that, since $\mathbf{M} = S(\mathfrak{g})\tilde{v}_0$, an element $D \in S(\mathfrak{g})$ lies in the annihilator $\mathrm{Ann}_{S(\mathfrak{g})}\mathbf{M}$ of $\mathbf{M}$ if and only if $D\tilde{v}_0 = 0$.

In order to prove the proposition, it suffices to show that the annihilator $\mathrm{Ann}_{S(\mathfrak{n})}(\tilde{v}_0)$ vanishes. This can be done as follows.

Suppose that $\lambda \in \mathscr{V}(\mathbf{X})$ satisfies the condition $(P_{\mathfrak{k},\mathfrak{n}})$, and that an element $D \in S(\mathfrak{n})$ annihilates the vector $\tilde{v}_0$. Then one sees from the above remark that

$$D(p^*(\mu)) = D(\mu) = 0 \quad \text{for all } \mu \in \mathscr{V}(\mathbf{X}).$$

This combined with Lemmas 2.2 and 2.3 shows that the function D on $\mathfrak{n}^*$ is identically zero on an open neighbourhood of $p^*(\lambda)$. We thus get $D = 0$ because $D \in S(\mathfrak{n})$ is a polynomial on $\mathfrak{n}^*$. $\square$

Now Proposition 2.1 yields the following criterion (sufficient condition) for the $U(\mathfrak{n})$-action on an irreducible module $\mathbf{X}$ to be locally free.

THEOREM 2.1. *Let $\mathfrak{k}$, $\mathfrak{n}$ be two Lie subalgebras of $\mathfrak{g}$, and let $\mathbf{X}$ be an irreducible $(\mathfrak{g},\mathfrak{k})$-module. Then, the action of the enveloping algebra $U(\mathfrak{n})$ on $\mathbf{X}$ is locally free, that is, $\mathbf{X}$ is a torsion free $U(\mathfrak{n})$-module, provided that the associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$ contains a point λ satisfying the condition $(P_{\mathfrak{k},\mathfrak{n}})$.*

REMARK. From the Remark in 2.1, it follows that

$$\dim \mathfrak{n} \le d(\mathbf{X}) = \dim \mathscr{V}(\mathbf{X}), \tag{2.7}$$

if a finitely generated $U(\mathfrak{g})$-module $\mathbf{X}$ has a locally free $U(\mathfrak{n})$-action.

3. Locally free $U(\mathfrak{n}(\mathscr{O}))$-action on Harish-Chandra modules.

From now on, let $\mathfrak{g}$ be a complex semisimple Lie algebra, and θ be an involutive automorphism of $\mathfrak{g}$. The associated symmetric decomposition of $\mathfrak{g}$ is denoted by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k} := \mathfrak{g}(\theta,+1)$ and $\mathfrak{p} := \mathfrak{g}(\theta,-1)$ as in 1.6. Then there exists a θ-stable real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that the restriction of θ to $\mathfrak{g}_0$ gives a Cartan involution of $\mathfrak{g}_0$ (see [**4**, Ch. III, Lemma 4.1]). We fix once and for all such a real form $\mathfrak{g}_0$, and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ denote the corresponding Cartan decomposition of $\mathfrak{g}_0$, where $\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0$ and $\mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{g}_0$.

By a *Harish-Chandra module*, we mean in this paper a $(\mathfrak{g},\mathfrak{k})$-module $\mathbf{X}$ (see 2.2) of finite length, associated with the symmetric pair $(\mathfrak{g},\mathfrak{k})$. As is shown by Harish-Chandra, the category of such $(\mathfrak{g},\mathfrak{k})$-modules plays an essential role in the study of representations of a real semisimple Lie group with Lie algebra $\mathfrak{g}_0$.

For each irreducible Harish-Chandra module $\mathbf{X}$, we construct in this section a family of nilpotent Lie subalgebras $\mathfrak{n}(\mathscr{O})$ of $\mathfrak{g}$ for which $\mathbf{X}$ is locally free as a $U(\mathfrak{n}(\mathscr{O}))$-module (see Theorem 3.2), by using the associated variety $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$ and the Cayley transformation of normal $\mathfrak{sl}_2$-triples. The Lie subalgebras $\mathfrak{n}(\mathscr{O})$ are parametrized by the K_C^{ad}-orbits $\mathscr{O}$ contained in $\mathscr{V}(\mathbf{X})$.

The main result of this paper is Theorem 3.2. The proof is carried out by combining Theorems 1.2 and 2.1. We shall give in 3.3 a concrete description of Lie subalgebras $\mathfrak{n}(\mathscr{O})$ associated to holomorphic orbits $\mathscr{O}$ when the real form $\mathfrak{g}_0$ is a simple Lie algebra of hermitian type.

3.1. Lie subalgebras $\mathfrak{n}(\mathscr{O})$ associated with a nilpotent K_C^{ad}-orbit $\mathscr{O}$.

We denote by $\mathscr{N}_{\mathfrak{p}}$ the totality of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{p}$. By [**8**, Th. 2], the variety $\mathscr{N}_{\mathfrak{p}}$ is a union of finitely many K_C^{ad}-orbits, where K_C^{ad} is as in 2.2 the connected Lie subgroup of $G_C^{ad} = \mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{k}$.

Let $\mathcal{O}$ be a K_C^{ad}-orbit in $\mathcal{N}_{\mathfrak{p}}$. Let us attach to $\mathcal{O}$ a K_C^{ad}-conjugacy class of nilpotent Lie subalgebras $\mathfrak{n}(\mathcal{O})$ of $\mathfrak{g}$. Suppose that $\mathcal{O} \neq \{0\}$, and take any element $X \in \mathcal{O}$. A strengthened version of the Jacobson-Morozov theorem [**8**, Prop.4] assures that X can be embedded to a unique, up to K_C^{ad}-conjugacy, normal $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}$ (see 1.6), where $H \in \mathfrak{k}$ and $X, Y \in \mathfrak{p}$. Set $\mathfrak{s} := \boldsymbol{C}X + \boldsymbol{C}H + \boldsymbol{C}Y \subset \mathfrak{g}$ and define a nilpotent Lie subalgebra $\mathfrak{n} = \mathfrak{n}_{\mathfrak{s}} = (\mathfrak{g}_1^1(1) \oplus \mathfrak{g}_3^3(1)) \oplus (\bigoplus_{l \geq 2} \mathfrak{g}(l))$ just as in (1.37), through the Cayley transform (X', H', Y') of (X, H, Y) defined by (1.2). Then it is immediate to check that, up to K_C^{ad}-conjugacy, the Lie subalgebra $\mathfrak{n}$ is uniquely determined by $\mathcal{O}$, independently of the choice of an X in $\mathcal{O}$ and of the choice of an $\mathfrak{sl}_2$-triple (X, H, Y). So we take up such $\mathfrak{n}$, and denote it by $\mathfrak{n}(\mathcal{O})$.

We attach $\mathfrak{n}(\mathcal{O}) = \{0\}$ for the zero orbit $\mathcal{O} = \{0\}$.

From Theorem 1.2(1), we get the following.

Lemma 3.1. *It holds that* $\dim \mathfrak{n}(\mathcal{O}) = \dim \mathcal{O} = (\dim \tilde{\mathcal{O}})/2$, *where* $\tilde{\mathcal{O}} := G_C^{ad} \cdot X$ *denotes the nilpotent* G_C^{ad}*-orbit in* $\mathfrak{g}$ *containing* $\mathcal{O}$.

Now let $\mathfrak{g} \ni Z \to \bar{Z} \in \mathfrak{g}$ be the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. Sekiguchi's result [**10**] enables us to choose a nice representative $\mathfrak{n}(\mathcal{O})$ which is compatible with this conjugation except the $\mathfrak{g}(1)$-part.

More precisely, take a normal $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}$ with $X \in \mathcal{O}$. By virtue of [**10,** Lemma 1.4], there exists an element $k \in K_C^{ad}$ such that $(X_1, H_1, Y_1) := (k \cdot X, k \cdot H, k \cdot Y)$ is a *strictly* normal $\mathfrak{sl}_2$-triple in the following sense:

$$(3.1) \qquad \overline{X_1} = Y_1, \quad \overline{H_1} = -H_1, \quad \text{or equivalently } X_1 + Y_1,\ i(X_1 - Y_1) \in \mathfrak{p}_0, \quad iH_1 \in \mathfrak{k}_0.$$

Then, as is checked immediately, the Cayley transform (X_1', H_1', Y_1') of (X_1, H_1, Y_1) (see (1.2)) lies in $\mathfrak{g}_0$.

Theorem 3.1. (Kostant-Sekiguchi, see [**10**, Th. 1.9]) *Under the above notation, the assignment*

$$(3.2) \qquad \mathcal{O} = K_C^{ad} \cdot X \mapsto \mathcal{O}' := G^{ad} \cdot X_1'$$

gives a bijection (Kostant-Sekiguchi correspondence) between the set of nilpotent K_C^{ad}*-orbits in* $\mathfrak{p}$ *and that of nilpotent* G^{ad}*-orbits in* $\mathfrak{g}_0$. *Here* $G^{ad} \subset G_C^{ad}$ *denotes the adjoint group of* $\mathfrak{g}_0$.

Note that $\dim_{\boldsymbol{R}} \mathcal{O}' = \dim_{\boldsymbol{C}} \tilde{\mathcal{O}}$ is equal to 2 $\dim_{\boldsymbol{C}} \mathcal{O}$ by Lemma 3.1.

As for our Lie subalgebra $\mathfrak{n}(\mathcal{O})$, we have the following advantage of choosing a strictly normal $\mathfrak{sl}_2$-triple (X_1, H_1, Y_1).

Proposition 3.1. *Let* $\mathfrak{n}(\mathcal{O}) = (\mathfrak{g}_1^1(1) \oplus \mathfrak{g}_3^3(1)) \oplus (\bigoplus_{l \geq 2} \mathfrak{g}(l))$ *be a Lie subalgebra of* $\mathfrak{g}$ *constructed as above from a strictly normal* $\mathfrak{sl}_2$*-triple* (X_1, H_1, Y_1). *Then one has*

$$\overline{\mathfrak{g}(l)} = \mathfrak{g}(l) \quad (l \in \mathbf{Z}), \quad \mathfrak{g}(1) = \{\mathfrak{n}(\mathcal{O}) \cap \mathfrak{g}(1)\} \oplus \{\overline{\mathfrak{n}(\mathcal{O}) \cap \mathfrak{g}(1)}\},$$

where $\mathfrak{g}(l)$ *denotes as in 1.6 the l-eigensubspace of* $\mathfrak{g}$ *for* $\mathrm{ad}(H_1')$ *with* $H_1' = X_1 + Y_1 \in \mathfrak{p}_0$. *In particular,* $\mathfrak{n}(\mathcal{O})$ *is stable under the complex conjugation* $\bar{\cdot}$ *if and only if* $\mathfrak{g}(1) = \{0\}$, *i.e.,* $\mathcal{O}$ *is an even nilpotent orbit in* $\mathfrak{p}$.

PROOF. Note that the real form $\mathfrak{g}_0$ is stable under the linear operators $\mathrm{ad}(H_1')$ and stable also under $J = \theta\,\mathrm{Ad}(w)$ in 1.2 with $V = \mathfrak{g}$, $\tilde{\sigma} = \theta$. Here w is defined as in (1.5) with X' and Y' replaced by X_1' and Y_1' respectively. Then the claim immediately follows from (1.33) and from the definition of subspaces $\mathfrak{g}(l)$ and $\mathfrak{g}_\eta^\kappa(l)$ $(\kappa, \eta = 0, 1, 2, 3;\ l \in \boldsymbol{Z})$ given in 1.4. $\square$

3.2. Main result.

By virtue of Lemmas 2.1 and 2.2 we can see that the associated variety $\mathscr{V}(\mathbf{X})$ of each Harish-Chandra module $\mathbf{X}$ is a K_C^{ad}-stable algebraic cone in $\mathscr{N}_\mathfrak{p}$. Here we identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ itself through the Killing form B of $\mathfrak{g}$.

Now we are in a position to give a sufficient condition for the locally freeness of $U(\mathfrak{n}(\mathcal{O}))$-action on Harish-Chandra modules.

THEOREM 3.2. *Let* $\mathbf{X}$ *be an irreducible Harish-Chandra module. The action of enveloping algebra* $U(\mathfrak{n}(\mathcal{O}))$ *of* $\mathfrak{n}(\mathcal{O})$ *on* $\mathbf{X}$ *is locally free for every nilpotent* K_C^{ad}*-orbit* $\mathcal{O} \subset \mathfrak{p}$ *contained in the associated variety* $\mathscr{V}(\mathbf{X})$ of $\mathbf{X}$. *Here* $\mathfrak{n}(\mathcal{O})$ *is the nilpotent Lie subalgebra of* $\mathfrak{g}$ *constructed in* 3.1.

PROOF. Take an element $X \in \mathcal{O} \subset \mathscr{V}(\mathbf{X})$, and construct the Lie subalgebra $\mathfrak{n}(\mathcal{O})$ as in 3.1. By Theorem 1.2(2), the Killing form B of $\mathfrak{g}$ is nondegenerate on $[\mathfrak{k}, X] \times \mathfrak{n}(\mathcal{O})$. This shows that $X \in \mathscr{V}(\mathbf{X})$ satisfies the condition $(P_{\mathfrak{k},\mathfrak{n}})$ in 2.3 with $\mathfrak{n} = \mathfrak{n}(\mathcal{O})$. Hence the $U(\mathfrak{n}(\mathcal{O}))$-action on $\mathbf{X}$ is locally free by Theorem 2.1. Any K_C^{ad}-conjugate $k \cdot \mathfrak{n}(\mathcal{O})$ of $\mathfrak{n}(\mathcal{O})$ also has locally free action on $\mathbf{X}$, because the universal covering group $\tilde{K}_C$ of K_C^{ad} acts on $\mathbf{X}$ as in (2.5), (2.6). $\square$

We now deduce two important consequences of the above main result.

First, Theorem 3.2 together with the Remark in 2.3 allows us to derive the following theorem by considering a K_C^{ad}-orbit of $\mathscr{V}(\mathbf{X})$ of maximal dimension:

THEOREM 3.3. *Let* $\mathbf{X}$ *be as in Theorem* 3.2, *and let* $\mathcal{O}_{max}$ *be a nilpotent* K_C^{ad}*-orbit in* $\mathscr{V}(\mathbf{X})$ *of maximal dimension, that is,* $\dim \mathcal{O}_{max} = \dim \mathscr{V}(\mathbf{X})$. *Then the corresponding* $\mathfrak{n}(\mathcal{O}_{max})$ *is maximal* (*with respect to the inclusion relation*) *among the Lie subalgebras* $\mathfrak{n}$ *of* $\mathfrak{g}$ *whose enveloping algebras* $U(\mathfrak{n})$ *act on* $\mathbf{X}$ *locally freely.*

Second, let $\mathfrak{n}_{m,0}$ be a maximal nilpotent Lie subalgebra of the real form $\mathfrak{g}_0$ appearing in an Iwasawa decomposition of $\mathfrak{g}_0$. We see from Proposition 3.1 that, for every nilpotent K_C^{ad}-orbit $\mathcal{O}$ in $\mathfrak{p}$, $\mathfrak{n}(\mathcal{O})$ is conjugate to a Lie subalgebra of $\mathfrak{n}_m$ under the action of K_C^{ad} on $\mathfrak{g}$. Here $\mathfrak{n}_m$ denotes the complexification of $\mathfrak{n}_{m,0}$ in $\mathfrak{g}$. Note that $\dim \mathfrak{n}_m = \dim \mathscr{N}_\mathfrak{p}$. By Lemma 3.1 and Theorem 3.1, $\mathfrak{n}(\mathcal{O})$ equals the whole $\mathfrak{n}_m$ up to K_C^{ad}-conjugacy if and only if the orbit $\mathcal{O}$ is open in $\mathscr{N}_\mathfrak{p}$ (or equivalently, the corresponding G^{ad}-orbit $\mathcal{O}'$ is a principal nilpotent orbit in $\mathfrak{g}_0$).

A Harish-Chandra module $\mathbf{X}$ is called *large* if its associated variety $\mathscr{V}(\mathbf{X})$ contains an open K_C^{ad}-orbit in $\mathscr{N}_\mathfrak{p}$, or $\dim \mathscr{V}(\mathbf{X}) = \dim \mathscr{N}_\mathfrak{p}$. Our main result yields a characterization for an irreducible Harish-Chandra module to be large, as follows.

THEOREM 3.4. *An irreducible Harish-Chandra module* $\mathbf{X}$ *is large if and only if* $\mathbf{X}$ *is a locally free* $U(\mathfrak{n}_m)$*-module.*

Proof. Assume that an irreducible Harish-Chandra module $\mathbf{X}$ is large, and take a K_C^{ad}-orbit $\mathcal{O}_{max}$ in $\mathscr{V}(\mathbf{X})$ of maximal dimension. As we have observed above, the Lie subalgebra $\mathfrak{n}(\mathcal{O}_{max})$ is conjugate to $\mathfrak{n}_m$ under the action of K_C^{ad} on $\mathfrak{g}$. Thus Theorem 3.2 yields the locally freeness of the $U(\mathfrak{n}_m)$-action on $\mathbf{X}$. The converse follows immediately from the Remark in 2.3. $\square$

Remark. The largeness of an irreducible Harish-Chandra module $\mathbf{X}$ is characterized also by the existence of Whittaker vectors for $\mathbf{X}$. See for example [**7**, Th. K] and [**9**, Cor. 2.2].

3.3. Lie subalgebras $\mathfrak{n}(\mathcal{O}_t)$ for holomorphic orbits $\mathcal{O}_t$.

Now suppose that $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a noncompact real simple Lie algebra of hermitian type. We denote by ω the unique (up to sign) $\mathfrak{k}_0$-invariant complex structure on $\mathfrak{p}_0$. Extending ω to $\mathfrak{p}$ by complex linearity, one gets a triangular decomposition

$$\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{with } \mathfrak{p}_\pm := \{Z \in \mathfrak{p} \mid \omega Z = \pm iZ\}, \tag{3.3}$$

of $\mathfrak{g}$ such that

$$[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}. \tag{3.4}$$

It then follows that the subspaces $\mathfrak{p}_\pm$ are included in the nilpotent variety $\mathscr{N}_\mathfrak{p}$ of $\mathfrak{p}$, since $(\operatorname{ad} Z)^3 = 0$ for every $Z \in \mathfrak{p}_\pm$. A K_C^{ad}-orbit $\mathcal{O}$ contained in $\mathfrak{p}_+$ is called *holomorphic*, as $\mathfrak{p}_+$ is naturally identified with the holomorphic tangent space at the origin of the hermitian symmetric space G/K with $\mathfrak{g}_0 = \operatorname{Lie}(G)$ and $\mathfrak{k}_0 = \operatorname{Lie}(K)$.

We conclude this article by describing the nilpotent Lie subalgebras $\mathfrak{n}(\mathcal{O})$ of $\mathfrak{g}$ associated with holomorphic K_C^{ad}-orbits $\mathcal{O}$.

3.3.1. In order to do this, we prepare after [**14**, 3.1] and [**15**, 9.1] refined structure theorems for $\mathfrak{g}$, originally due to Harish-Chandra and Moore. Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ which is contained in $\mathfrak{k}_0$. We denote by Δ the root system of $\mathfrak{g}$ with respect to the complexification $\mathfrak{t}$ of $\mathfrak{t}_0$. For $\gamma \in \Delta$, the corresponding root subspace is denoted by $\mathfrak{g}(\mathfrak{t}; \gamma)$. A root $\gamma \in \Delta$ is called compact (resp. noncompact) if $\mathfrak{g}(\mathfrak{t}; \gamma) \subset \mathfrak{k}$ (resp. $\mathfrak{g}(\mathfrak{t}; \gamma) \subset \mathfrak{p}$), and Δ_c (resp. Δ_n) stands for the set of all compact (resp. noncompact) roots. To each $\gamma \in \Delta$ we attach a nonzero vector $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ satisfying

$$X_\gamma - X_{-\gamma}, \quad i(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + i\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma. \tag{3.5}$$

Here H_γ is the element of $i\mathfrak{t}_0$ corresponding to the coroot $\gamma^\vee := 2\gamma/(\gamma, \gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B.

Take a positive system Δ^+ of Δ compatible with the decomposition (3.3):

$$\mathfrak{p}_\pm = \bigoplus_{\gamma \in \Delta_n^+} \mathfrak{g}(\mathfrak{t}; \pm\gamma) \quad \text{with } \Delta_n^+ := \Delta^+ \cap \Delta_n,$$

and fix a lexicographic order on $i\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \dots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \dots, \gamma_r$.

Now, put $\mathfrak{t}^- := \sum_{k=1}^{r} \mathbf{C}H_{\gamma_k} \subset \mathfrak{t}$, and denote by $\pi(\gamma) \in (\mathfrak{t}^-)^*$ the restriction to $\mathfrak{t}^-$ of a linear form $\gamma \in \mathfrak{t}^*$. For integers k, m with $1 \leq m < k \leq r$, we define subsets P_{km}, P_k, P_0 of Δ_n^+ and subsets C_{km}, C_k, C_0 of Δ_c^+ respectively by

$$P_{km} := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \pi(\gamma) = \frac{\pi(\gamma_k) + \pi(\gamma_m)}{2} \right\}, \tag{3.6}$$

$$C_{km} := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \pi(\gamma) = \frac{\pi(\gamma_k) - \pi(\gamma_m)}{2} \right\}, \tag{3.7}$$

$$P_k := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \pi(\gamma) = \frac{\pi(\gamma_k)}{2} \right\}, \quad C_k := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \pi(\gamma) = \frac{\pi(\gamma_k)}{2} \right\}, \tag{3.8}$$

$$P_0 := \{\gamma_1, \gamma_2, \ldots, \gamma_r\}, \quad C_0 := \{\gamma \in \Delta_c^+ \mid \pi(\gamma) = 0\}. \tag{3.9}$$

Then, by Harish-Chandra the subsets Δ_n^+ and Δ_c^+ are expressed as

$$\Delta_n^+ = \left(\bigcup_{1 \leq k \leq r} P_k \right) \cup P_0 \cup \left(\bigcup_{1 \leq m < k \leq r} P_{km} \right), \tag{3.10}$$

$$\Delta_c^+ = C_0 \cup \left(\bigcup_{1 \leq k \leq r} C_k \right) \cup \left(\bigcup_{1 \leq m < k \leq r} C_{km} \right), \tag{3.11}$$

(3.12)

where the unions are disjoint.

We set $H_k := X_{\gamma_k} + X_{-\gamma_k} \in \mathfrak{p}_0$ for $1 \leq k \leq r$. Then

$$\mathfrak{a}_{p,0} := \sum_{k=1}^{r} \boldsymbol{R} H_k \tag{3.13}$$

turns out to be a maximal abelian subspace of $\mathfrak{p}_0$. Let Ψ denote the root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_{p,0}$, and for each k let $\psi_k \in \mathfrak{a}_{p,0}^*$ be the linear form on $\mathfrak{a}_{p,0}$ defined by $\psi_k(H_m) = 2\delta_{km}$ ($m = 1, \ldots, r$; with Kronecker's δ_{km}). Moore's restricted root theorem describes Ψ as follows.

THEOREM 3.5. (Moore) *The elements ψ_k $(1 \leq k \leq r)$ form a basis of $\mathfrak{a}_{p,0}^*$, and there exist only two possibilities for the root system Ψ:*

$$\Psi = \left\{ \pm\left(\frac{\psi_k - \psi_m}{2}\right) \,\middle|\, 1 \leq m < k \leq r \right\} \cup \left\{ \pm\left(\frac{\psi_k + \psi_m}{2}\right) \,\middle|\, 1 \leq m \leq k \leq r \right\},$$

if the subsets P_k and C_k are empty for every k, or otherwise

$$\Psi = \left\{ \pm\left(\frac{\psi_k - \psi_m}{2}\right) \,\middle|\, 1 \leq m < k \leq r \right\} \cup \left\{ \pm\left(\frac{\psi_k + \psi_m}{2}\right) \,\middle|\, 1 \leq m \leq k \leq r \right\}$$
$$\cup \left\{ \pm\frac{\psi_k}{2} \,\middle|\, 1 \leq k \leq r \right\}.$$

The former possibility occurs exactly when the corresponding hermitian symmetric space is analytically equivalent to a tube domain.

3.3.2. For each restricted root $\psi \in \Psi$, let $\mathfrak{g}(\mathfrak{a}_p; \psi)$ denote the complexified root subspace of $\mathfrak{g}$ corresponding to ψ. We can now write down a basis of each $\mathfrak{g}(\mathfrak{a}_p; \psi)$ by means of the vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ $(\gamma \in \Delta)$ defined in (3.5), as follows.

Proposition 3.2. (Hashizume, cf [**15**, Lemmas 9.1 and 9.2]) (1) *For* $1 \le m < k \le r$, *the vectors*

$$E_\gamma^{\pm} := X_\gamma + [X_{-\gamma_k}, X_\gamma] \pm [X_{-\gamma_m}, X_\gamma] \pm [X_{-\gamma_m}, [X_{-\gamma_k}, X_\gamma]] \tag{3.14}$$

form a basis of the root subspace $\mathfrak{g}(\mathfrak{a}_p; (\psi_k \pm \psi_m)/2)$, *where* γ *runs over the elements of* P_{km} *in* (3.6).

(2) *The element*

$$E_k := i(H_{\gamma_k} - X_{\gamma_k} + X_{-\gamma_k})/2 \tag{3.15}$$

lies in $\mathfrak{g}(\mathfrak{a}_p; \psi_k)$ *and it holds that* $\dim \mathfrak{g}(\mathfrak{a}_p; \psi_k) = 1$ *for every* $1 \le k \le r$.

(3) *The subspace* $\mathfrak{g}(\mathfrak{a}_p; \psi_k/2)$ *has a basis*:

$$E_\gamma^1 := X_\gamma + [X_{-\gamma_k}, X_\gamma] \quad (\gamma \in P_k \cup C_k) \tag{3.16}$$

for every $1 \le k \le r$, *where* P_k *and* C_k *are as in* (3.8).

3.3.3. Set $X(t) := \sum_{t<k\le r} X_{\gamma_k} \in \mathfrak{p}_+ (X(r) := 0)$ for $0 \le t \le r$, and let $\mathcal{O}_t \subset \mathfrak{p}_+$ be the holomorphic K_C^{ad}-orbit through $X(t)$. The following well-known proposition parametrizes such K_C^{ad}-orbits in $\mathcal{N}_\mathfrak{p}$.

Proposition 3.3. *The subspace* $\mathfrak{p}_+$ *splits into a disjoint union of* $r+1$ *number of* K_C^{ad}-*orbits* $\mathcal{O}_t$ $(0 \le t \le r) : \mathfrak{p}_+ = \coprod_{0 \le t \le r} \mathcal{O}_t$, *and the closure* $\overline{\mathcal{O}_t}$ *of orbit* $\mathcal{O}_t$ *is equal to* $\bigcup_{s \ge t} \mathcal{O}_s$ *for every* t.

Remark. When $\mathfrak{g}_0 = \mathfrak{su}(l,n)$ $(l \ge n)$ or $\mathfrak{sp}(n, \boldsymbol{R})$, the real rank r of $\mathfrak{g}_0$ is equal to n, and $\mathcal{O}_t$ consists of all matrices in $\mathfrak{p}_+$ of rank $r - t$ (cf. [**2**, Prop. 12.1]).

Suggested by the above proposition, we want to describe the nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O}_t)$ in terms of root vectors $E_\gamma^{\pm}$, E_k and E_γ^1 in Proposition 3.2, for every $0 \le t \le r$.

This is achieved in the following way. Put

$$H(t) := \sum_{t<k\le r} H_{\gamma_k}, \quad Y(t) := \sum_{t<k\le r} X_{-\gamma_k}. \tag{3.17}$$

Then it follows from (3.5) together with the strong orthogonality of γ_k's that $(X(t), H(t), Y(t))$ is a strictly normal $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We denote by $(X'(t), H'(t), Y'(t))$ the Cayley transform of $(X(t), H(t), Y(t))$ defined in (1.2). Noting that $H'(t) = \sum_{t<k\le r} H_k$, one deduces from Theorem 3.5 the following.

Lemma 3.2. *The Lie algebra* $\mathfrak{g}$ *decomposes into a direct sum of the eigensubspaces for* $\operatorname{ad} H'(t)$ *as*:

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2), \tag{3.18}$$

where

$$\mathfrak{g}(\pm 2) = \bigoplus_{t<m\leq k\leq r} \mathfrak{g}\left(\mathfrak{a}_p;\ \pm\left(\frac{\psi_k+\psi_m}{2}\right)\right),$$

$$\mathfrak{g}(\pm 1) = \bigoplus_{1\leq m\leq t<k\leq r}\left(\mathfrak{g}\left(\mathfrak{a}_p;\ \pm\left(\frac{\psi_k+\psi_m}{2}\right)\right)\oplus\mathfrak{g}\left(\mathfrak{a}_p;\ \pm\left(\frac{\psi_k-\psi_m}{2}\right)\right)\right)\oplus\left(\bigoplus_{t<k\leq r}\mathfrak{g}\left(\mathfrak{a}_p;\ \pm\frac{\psi_k}{2}\right)\right),$$

$$\mathfrak{g}(0) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus \left(\bigoplus_{1\leq m\leq k\leq t}\left(\mathfrak{g}\left(\mathfrak{a}_p;\frac{\psi_k+\psi_m}{2}\right)\oplus\mathfrak{g}\left(\mathfrak{a}_p;\frac{\psi_k-\psi_m}{2}\right)\right)\right)$$
$$\oplus\left(\bigoplus_{1\leq m\leq k\leq t}\left(\mathfrak{g}\left(\mathfrak{a}_p;-\frac{\psi_k+\psi_m}{2}\right)\oplus\mathfrak{g}\left(\mathfrak{a}_p;-\frac{\psi_k-\psi_m}{2}\right)\right)\right),$$

and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ *denotes the centralizer of* $\mathfrak{a}$ *in* $\mathfrak{g}$. *In particular,* $\mathfrak{g}^3_\eta(l) = \{0\}$ *for all* η *and* l.

By using Proposition 3.2 and Lemma 3.2, we obtain the following complete description of Lie subalgebra $\mathfrak{n}(\mathcal{O}_t)$ associated to the orbit $\mathcal{O}_t$.

THEOREM 3.6. *Let* $\mathcal{O}_t = K_C^{ad}\cdot X(t)$ *with* $0\leq t\leq r$ *be a holomorphic* K_C^{ad}*-orbit in* $\mathfrak{p}_+$, *and let* $\mathfrak{n}(\mathcal{O}_t)$ *be the Lie subalgebra of* $\mathfrak{g}$ *constructed as in* 3.1 *from the Cayley transform of* $(X(t), H(t), Y(t))$. *Then* $\mathfrak{n}(\mathcal{O}_t)$ *is expressed as*

$$\mathfrak{n}(\mathcal{O}_t) = \mathfrak{g}^1_1(1)\oplus\mathfrak{g}(2), \tag{3.19}$$

with $\mathfrak{g}(2)$ *as in Lemma* 3.2, *and* $\mathfrak{g}^1_1(1)$ *is the subspace of* $\mathfrak{g}(1)$ *having a basis*:

$$E^+_\gamma - E^-_\gamma\ (\gamma\in P_{km};\ 1\leq m\leq t<k\leq r),\quad E^1_\gamma(\gamma\in C_k;\ t<k\leq r). \tag{3.20}$$

Here $E^\pm_\gamma$ *and* E^1_γ *are as in Proposition* 3.2.

PROOF. The first claim (3.19) follows immediately from Lemma 3.2 since $\mathfrak{g}(j) = \{Z\in\mathfrak{g}\,|\,[H'(t),Z] = jZ\}$ equals $\{0\}$ for $|j|\geq 3$. Notice that the subspace $\mathfrak{g}^1_1(1)$ of $\mathfrak{g}(1)$ is given as

$$\mathfrak{g}^1_1(1) = \{Z\in\mathfrak{g}(1)\,|\,JZ = iZ\}, \tag{3.21}$$

where $J = \theta w\in GL(\mathfrak{g})$ with

$$w = \exp\frac{\pi}{2}(X'(t)-Y'(t)) = \exp\frac{\pi i}{2}H(t)\in G_C^{ad}, \tag{3.22}$$

by definition (see 1.4 and 1.6). Then one finds by direct computation that the operator J acts on vectors $E^\pm_\gamma$, E^1_γ respectively as

$$JE^\pm_\gamma = -iE^\mp_\gamma \quad \text{if } \gamma\in P_{km} \quad \text{with } k>t\geq m, \tag{3.23}$$

$$JE^1_\gamma = -iE^1_\gamma \quad \text{if } \gamma\in P_k \quad \text{with } k>t, \tag{3.24}$$

$$JE^1_\gamma = iE^1_\gamma \quad \text{if } \gamma\in C_k \quad \text{with } k>t. \tag{3.25}$$

This combined with Proposition 3.2 and Lemma 3.2 shows that the vectors in (3.20) form a basis of the subspace $\mathfrak{g}_1^1(1)$. $\square$

Theorem 3.2 implies that the (at most) two-step nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O}_t)$ acts locally freely on the Harish-Chandra module of a holomorphic discrete series for every t, because its associated variety coincides with the whole $\mathfrak{p}_+$ (cf. [**17**]). More generally, the associated variety $\mathscr{V}(\mathbf{X})$ of any irreducible highest weight Harish-Chandra module $\mathbf{X}$ is contained in $\mathfrak{p}_+$. By Theorem 3.2, Proposition 3.3 and the Remark in 2.3, the $U(\mathfrak{n}(\mathcal{O}_t))$-action on $\mathbf{X}$ is locally free if and only if $t \geq t_X$, where $\mathcal{O}_{t_X}$ is the unique open orbit in $\mathscr{V}(\mathbf{X})$. The integer $l_X := r - t_X$ gives a kind of *rank* for highest weight module $\mathbf{X}$ (cf. [**5**, p. 136]; see also the Remark succeeding Proposition 3.3).

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