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Gevrey regularizing effect for a nonlinear Schrödinger equation in one space dimension

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§0 Introduction.

In [3], [1] and [5] we have obtained the Gevrey regularizing effect for the solution of a non-linear Schrödinger equation

$$(1)\qquad \begin{cases} \partial_t u + i\Delta u = f(u), \\ u(0,x) = u_0(x). \end{cases}$$

In these papers, we treat Schrödinger equations whose non-linear terms depend only on the value of the unknown functions. In the present paper, we investigate Gevrey regularizing effect for the equation

$$(2)\qquad \begin{cases} Lu \equiv \partial_t u + i\partial_x^2 u = f(u, \partial_x u), \\ u(0,x) = u_0(x) \end{cases}$$

in one space dimension, whose non-linear term depends also on the derivatives of the unknown functions.

The existence of the solution for the equation (2) are obtained in [2] and [4] in case that $f(u,v)$ is a polynomial in the argument $(u,\bar{u},v,\bar{v})$. In the general case for $f(u,v)$, however, we can also obtain the existence of the solution as follows:

THEOREM 0. *Assume that a C^∞-function $f(u,v)$ satisfies $\partial_{\bar{v}} f(u,v) = 0$ and $f(0,0) = \partial_v f(0,0) = 0$. Then, for any R_0 there exists a constant $T \equiv T(R_0)$ such that, for any initial data $u_0 \equiv u_0(x)$ with $\|u_0\|_3 \le R_0$ and $\|xu_0\| \le R_0$, the solution $u(t,x)$ of (2) exists in $[0,T]$ and it belongs to $C([0,T];H^3) \cap C^1([0,T];H^1)$.*

Our concern is the Gevrey regularizing effect for the solution of (1). So, we omit the proof of Theorem 0 and, in the following, we treat only the Gevrey regularizing effect for the solution (2). The conditions of $f(u,v)$ are the following:

(A.0) $f(u,v)$ is a C^∞ complex valued function in $\boldsymbol{C}^2$, which is holomorphic with respect to v. Moreover, it satisfies $f(0,0) = \partial_v f(0,0) = 0$.

(A.1) Let s satisfy $s \ge 1$. For any positive number K, there exist constants $C = C(K)$ and $A = A(K)$ such that

$$|\partial_u^k \partial_{\bar{u}}^{k'} \partial_v^{k''} f(u,v)| \le CA^{k+k'+k''} k!^s k'!^s k''! \qquad \text{for } |u|, |v| \le K.$$

(A.2) For s in (A.1), the constant σ satisfies $\max(1, s/2) \leq \sigma \leq s$. Then, for any positve number K, there exist constants $C = C(K)$ and $A = A(K)$ such that

$$|\partial_u^k \partial_{\bar{u}}^{k'} \partial_v^{k''} f(u, v)| \leq C A^{k+k'+k''} k!^\sigma k'!^\sigma k''! \qquad \text{for } |u|, |v| \leq K.$$

Denote by $H^m = H^m(\mathbf{R}^1)$ a Sobolev space in $\mathbf{R}_x^1$ and $\|\cdot\|_m$ is its norm. We also denote $\|\cdot\| = \|\cdot\|_0$ (L^2-norm). Then, we can state our main theorems as follows:

Theorem 1. *Assume that* (A.0)–(A.1). *Let* $u(t,x) \in C([0,T]; H^3) \cap C^1([0,T]; H^1)$ *be a solution of* (2). *Then, if the initial value* $u_0(x)$ *satisfies*

$$\begin{cases} \|(x\partial_x)^l u_0\|_3 \leq C A^l l!^s, \\ \|x(x\partial_x)^l u_0\| \leq C A^l l!^s, \end{cases} \tag{3}$$

$u(t,x)$ satisfies

$$\|P^l u\|_3 \leq C A^l l!^s, \tag{4}$$

where $P = 2t\partial_t + x\partial_x$ is a dilation operator.

Theorem 2. *Let σ satisfy* $\max(s/2, 1) \leq \sigma \leq s$ *and assume* (A.0)–(A.2). *Then, under the condition* (3) *the solution* $u(t,x)$ *of* (2) *satisfies the following property: For any positive number R there exist constants $C = C_R$ and $A = A_R$ such that*

$$\|\partial_x^\alpha u(t,x)\|_{H^1(-R,R)} \leq C A^\alpha \alpha!^\sigma \quad \text{for } t \neq 0.$$

Remark. For the examples of $f(u)$ (in case $f(u,v)$ is independent of v) and of initial values $u_0(x)$, see [**5**].

The plan of this paper is as follows: In §1 we give preliminaries and in §2 and §3 we give proofs of Theorem 1 and Theorem 2.

§1. Preliminaries.

In this section, we give several preparatory properties, whose proofs are omitted without mentioned.

Proposition 1.1. *Let $a(t,x)$, $a_1(t,x)$ and $a_2(t,x)$ be functions in $C([0,T]; \mathscr{B}^0)$ with the properties $\partial_x a, \partial_x a_1, \partial_x a_2 \in C([0,T]; H^{m-1})$ for $m \geq 1$, and let $w(t,x) \in C([0,T]; H^m) \cap C^1([0,T]; H^{m-2})$ be a solution of*

$$Lw = a\partial_x w + a_1 w + a_2 \bar{w} + f(t,x) \tag{1.1}$$

with $f \in C([0,T]; H^m)$. Assume

$$xa(t,x) \in C([0,T]; L^2). \tag{1.2}$$

Then, there exists a constant $C_1 \equiv C_{1,m}$ such that

$$\|w(t)\|_m \leq C_1 \left\{ \|w(0)\|_m + \int_0^t \|f(\tau)\|_m \, dt \right\} \tag{1.3}$$

holds.

REMARK. Takeuchi and Mizohata obtained in [6], [7], [8], [9] that for the L^2-well-posedness of the Cauchy problem of the Schrödinger equation (1.1) it is necessary and sufficient that the integral $\mathrm{Im}\int_{-\infty}^{x} a(t,y)\,dy$ is bounded in $\boldsymbol{R}_x^1$. We note that (1.2) guarantees the boundedness of $\mathrm{Im}\int_{-\infty}^{x} a(t,y)\,dy$.

PROOF. Set

$$\tilde{a}(t,x) = \exp\left(-\frac{1}{2}\mathrm{Im}\int_{-\infty}^{x} a(t,y)\,dy\right) \tag{1.4}$$

and set

$$|||w|||_m = \left[\|\tilde{a}\partial_x^m w\|^2 + \sum_{j=0}^{m-1}\|\partial_x^j w\|^2\right]^{1/2},$$

which is an equivalent norm of $\|w\|_m$ by means of (1.2). From (1.1) and (1.4) we have

$$\begin{aligned}\frac{d}{dt}|||w|||_m^2 &= 2\,\mathrm{Re}\left\{(\tilde{a}\partial_x^m\partial_t w, \tilde{a}\partial_x^m w) + \sum_{j=0}^{m-1}(\partial_x^j\partial_t w, \partial_x^j w)\right\}\\ &\le 2\,\mathrm{Re}(\tilde{a}^2\partial_x^m(-i\partial_x^2 w + a\partial_x w + a_1 w + a_2\bar{w} + f), \partial_x^m w)\\ &\quad + C_1'(|||w|||_m + \|f\|_m)|||w|||_m\\ &\le 2\,\mathrm{Re}\{i\|\tilde{a}\partial_x^{m+1}w\|^2 + ((2i\tilde{a}\tilde{a}' + \tilde{a}^2 a)\partial_x^{m+1}w, \partial_x^m w)\}\\ &\quad + C_1''(|||w|||_m + \|f\|_m)|||w|||_m\\ &\le C_1'''(|||w|||_m + \|f\|_m)|||w|||_m.\end{aligned}$$

Hence, we get

$$\frac{d}{dt}|||w|||_w \le \frac{1}{2}C_1'''(|||w|||_m + \|f\|_m)$$

and (1.3). Q.E.D.

LEMMA 1.2. *There exists a constant C_2 without depending on l such that*

$$\sum_{l'+l''=l}\frac{1}{(l'+1)^2(l''+2)^2} \le C_2\frac{1}{(l+1)^2}.$$

LEMMA 1.3. i) *Let $u \in H^1$ and $v \in H^m$, $m = 0,1$. Then, there exists an absolute constant C_3 such that*

$$\|uv\|_m \le C_3\|u\|_1\|v\|_m, \quad m = 0,1$$

hold.

ii) *Let $u \in H^1$ and $b \equiv b(t,x) \in \mathscr{B}^0$ satisfy $\partial_x b \in L_2$. Then, bu belongs to H^1 and satisfies*

$$\|bu\|_1 \le C_3\|u\|_1(|b|_{\mathscr{B}^0} + \|\partial_x b\|_{L^2}).$$

LEMMA 1.4. *Let $l \ge 2$ and assume $\|u\|_m \le C_0$ and*

$$\|P^k u\|_m \le C_0 A^{k-1} k!(k-1)!^{s-1}/(k+1)^2 \quad \textit{for } 1 \le k < l \tag{1.5}$$

for A satisfying $A \geq m$. Then, there exists a constant C_4 such that for κ with $1 \leq \kappa \leq m$ we have

$$\|(P^l \partial_x^\kappa - \partial_x^\kappa P^l)u\|_{m-\kappa} \leq C_4 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2. \tag{1.6}$$

Proof. From $P\partial_x = \partial_x(P-1)$ we have $P\partial_x^\kappa = \partial_x^\kappa(P-\kappa)$ and

$$P^l \partial_x^\kappa = \partial_x^\kappa (P-\kappa)^l,$$

which implies

$$P^l \partial_x^\kappa - \partial_x^\kappa P^l = \sum_{l'=0}^{l-1} \binom{l}{l'} (-\kappa)^{l-l'} \partial_x^\kappa P^{l'}. \tag{1.7}$$

Write $l'' = l - l'$. Then, since $1/l''! \leq 5/(l''+1)^2$, we have

$$\begin{aligned}\|(P^l \partial_x^\kappa - \partial_x^\kappa P^l)u\|_{m-\kappa} &\leq \sum_{l'=0}^{l-1} \binom{l}{l'} \kappa^{l-l'} \|P^{l'}u\|_{(m-\kappa)+\kappa} \\ &\leq 5\kappa^2 C_0 A^{l-2} l!(l-1)!^{s-1} \sum_{l'+l''=l} 1/(l'+1)^2(l''+1)^2 \\ &\leq 5\kappa^2 C_2 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.\end{aligned}$$

This proves the lemma. Q.E.D.

Lemma 1.5. *Let $l \geq 2$ and assume that $u(t,x)$ satisfies $\|u\|_1 \leq C_0$ and (1.5). Moreover, let $b(t,x) \in \mathscr{B}^0([0,T] \times \boldsymbol{R}^1)$ satisfy $\partial_x b(t,x) \in C([0,T]; L_2)$ and*

$$\begin{cases} |b|_{\mathscr{B}^0}, \|\partial_x b\| \leq C_5, \\ |P^k b|_{\mathscr{B}^0}, \|\partial_x P^k b\| \leq C_5 C_0 A^{k-1} k!(k-1)!^{s-1}/(k+1)^2 \quad \text{for } 1 \leq k < l, \\ |P^l b|_{\mathscr{B}^0}, \|\partial_x P^l b\| \leq C_5 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2. \end{cases} \tag{1.8}$$

Then, we have

$$\|P^l(bu) - bP^l u\|_1 \leq C_6 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2. \tag{1.9}$$

Proof. From Lemma 1.3-ii) we get

$$\begin{aligned}\|P^l(bu) - bP^l u\|_1 &= \left\| \sum_{l'=0}^{l-1} \binom{l}{l'} P^{l-l'} b \cdot P^{l'} u \right\|_1 \\ &\leq C_3 \sum_{l'=0}^{l-1} \binom{l}{l'} (|P^{l-l'} b|_{\mathscr{B}^0} + \|\partial_x P^{l-l'} b\|) \|P^{l'} u\|_1 \\ &\leq C_6 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.\end{aligned}$$

This proves the lemma. Q.E.D.

§2. Proof of Theorem 1.

By the usual method, we obtain $P^l u \in C([0,T];H_3)$ and $xP^l u \in C([0,T];H_2)$ under the condition (3). So, in order to prove Theorem 1, we have only to prove, by the induction on l

$$\|P^l u\|_3 \le C_0 A^{l-1} l!(l-1)!^{s-1}/(l+1)^2 \quad \textit{for } l \ge 1, \tag{2.1}$$

and, in addition, we will prove

$$\|xP^l u\|_2 \le C_0 A^{l-1} l!(l-1)!^{s-1}/(l+1)^2 \quad \textit{for } l \ge 1, \tag{2.2}$$

with appropriate constants C_0 and A. Now, let $l \ge 2$ and assume

$$\|P^k u\|_3 \le C_0 A^{k-1} k!(k-1)!^{s-1}/(k+1)^2 \quad \textit{for } 1 \le k < l, \tag{2.3}$$

$$\|xP^k u\|_2 \le C_0 A^{k-1} k!(k-1)!^{s-1}/(k+1)^2 \quad \textit{for } 1 \le k < l. \tag{2.4}$$

We also assume that the constant C_0 be taken such that

$$\begin{cases} \text{i)} & \|u\|_3 \le C_0, \\ \text{ii)} & \|xu\|_2 \le C_0. \end{cases} \tag{2.5}$$

In the following, we assume that the constant C_0 be taken such that (2.5) hold and

$$\begin{cases} \|Pu\|_3 \le C_0/4, \\ \|xPu\|_2 \le C_0/4, \end{cases}$$

which is the case of $k=1$ for (2.3)–(2.4). We fix such a constant C_0 throughout this section and take a constant A large enough according to the context.

Proposition 2.1. *Let $l \ge 2$. Then, if the solution $u(t,x)$ satisfies* (2.3) *and* (2.5)-i), *we have*

$$\|P^l u\|_2 \le C_7 C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2. \tag{2.6}$$

Proof. Using $LP = (P+2)L$ and the formula of the differentiation of composite function, we get

$$\begin{aligned} LP^l u &= (P+2)^l\{f(u,\partial_x u)\} \\ &= f_u P^l u + f_{\bar{u}} \overline{P^l u} + f_v \partial_x P^l u + g_l(t,x), \end{aligned} \tag{2.7}$$

$$\begin{aligned} g_l(t,x) &= 2^l f(u,\partial_x u) \\ &\quad + \sum_{l'=1}^{l-1} \sum_{k+k'+k''=1} \frac{l!}{(l-l')!} 2^{l-l'} \partial_u^k \partial_{\bar{u}}^{k'} \partial_v^{k''} f(u,\partial_x u)\, U_{l',k,k',k''} \\ &\quad + \sum_{l'=2}^{l} \sum_{2 \le k+k'+k'' \le l'} \frac{l!}{(l-l')!k!k'!k''!} 2^{l-l'} \partial_u^k \partial_{\bar{u}}^{k'} \partial_v^{k''} f(u,\partial_x u)\, U_{l',k,k',k''} \\ &\quad + f_v(P^l \partial_x - \partial_x P^l)u, \end{aligned} \tag{2.8}$$

$$(2.9)\qquad U_{l',k,k',k''}=\sum_{\substack{l_1+\cdots+l_{k+k'+k''}=l'\\ l_j\neq 0}}\prod_{j=1}^{k}\frac{1}{l_j!}P^{l_j}u\times\prod_{j=k+1}^{k+k'}\frac{1}{l_j!}P^{l_j}\bar{u}\prod_{j=k+k'+1}^{k+k'+k''}\frac{1}{l_j!}P^{l_j}\partial_x u.$$

From (2.3), (2.5)-i) and Lemma 1.4 we have

$$\begin{aligned}\|P^k\partial_x u\|_2 &\le \|(P^k\partial_x-\partial_x P^k)u\|_2+\|\partial_x P^k u\|_2\\ &\le C_8C_0A^{k-1}k!(k-1)!^{s-1}/(k+1)^2\quad\text{for } 1\le k<l\end{aligned}$$

and get

$$\|g_l\|_2\le C_9C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2$$

if we take A large enough. Combining this with

$$\|(x\partial_x)^l u_0\|_2\le C_{10}A^{l-2}l!^s/(l+1)^2,$$

which is derived from the assumption (3) of the initial condition, we get (2.6) from Proposition 1.1. Q.E.D.

PROPOSITION 2.2. *Let $l\ge 2$ and assume* (2.3)–(2.5) *and* (2.6). *Then, we have*

$$(2.10)\qquad \|xP^l u\|_1\le C_{11}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2$$

for a constant C_{11}.

PROOF. Set $w=xP^l u$. Then, we have from (2.7)

$$\begin{aligned}Lw&=xLP^l u+2i\partial_x P^l u\\ &=f_u w+f_{\bar{u}}\bar{w}+f_v\partial_x w-f_v P^l u+xg_l+2i\partial_x P^l u.\end{aligned}$$

Noting (2.8)–(2.9) and using (1.7) for $xP^k\partial_x u$ we get

$$(2.11)\qquad \|xg_l\|_1\le C_{12}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2$$

and from (2.6) we also have

$$(2.12)\qquad \|2i\partial_x P^l u\|_1\le C_{13}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2.$$

Moreover from (3) we get

$$(2.13)\qquad \|x(x\partial_x)^l u_0\|_2\le CA^{l-2}l!^s/(l+1)^2.$$

Combining this with (2.11)–(2.12), we get (2.10) by means of (1.3). Q.E.D.

Now, we define

$$(2.14)\qquad b_1(t,x)=\int_{-\infty}^{x}f_v(u(t,y),\partial_x u(t,y))\,dy$$

and

$$b(t,x) = \exp\left(\frac{i}{2}b_1(t,x)\right). \tag{2.15}$$

Since $f_v(0,0) = 0$, we can write

$$f_v(u,v) = f_{v,1}(u,v)u + f_{v,2}(u,v)\bar{u} + f_{v,3}(u,v)v \tag{2.16}$$

and

$$\begin{aligned} b_1(t,x) = {} & f_{v,3}(u,u')u + \int_{-\infty}^{x} \{\langle y\rangle^{-1} f_{v,1}\cdot(\langle y\rangle u) \\ & + \langle y\rangle^{-1} f_{v,2}\cdot(\langle y\rangle\bar{u}) - (\partial_x f_{v,3})u\}\, dy. \end{aligned} \tag{2.17}$$

This implies the well-definedness of $b(t,x)$. Now, note

$$Pb_1 = \int_{-\infty}^{x} (P+1)f_v\, dy. \tag{2.18}$$

Then, using (2.18) we get

Lemma 2.3. *Assume* (2.3)–(2.5), (2.6) *and* (2.10). *Then,* $b_1(t,x)$ *satisfies*

$$\begin{cases} |b_1|_{\mathscr{B}^0}, \|\partial_x b_1\| \le C_{14}, \\ |P^k b_1|_{\mathscr{B}^0}, \|\partial_x P^k b_1\| \le C_{14}C_0A^{k-1}k!(k-1)!^{s-1}/(k+1)^2 \quad \text{for } 1 \le k < l, \\ |P^l b_1|_{\mathscr{B}^0}, \|\partial_x P^l b_1\| \le C_{14}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2, \end{cases} \tag{2.19}$$

and the weight function $b(t,x)$ *of* (2.15) *satisfies* (1.8).

Proof. From (2.18) we have

$$\begin{aligned} P^k b_1 &= \int_{-\infty}^{x} (P+1)^k \{f_v(u,\partial_x u)\}\, dy \\ &= \int_{-\infty}^{x} \left[f_v + \sum_{k'=1}^{k} \sum_{1\le \nu+\nu'+\nu''\le k'} \frac{k!}{(k-k')!\nu!\nu'!\nu''!} (\partial_u^{\nu}\partial_{\bar{u}}^{\nu'}\partial_v^{\nu''+1} f)\, U_{k',\nu,\nu',\nu''} \right] dy. \end{aligned} \tag{2.20}$$

Hence, using the same discussion of (2.16)–(2.17) to the first term of the last member of (2.20) and noting that $U_{k',\nu,\nu',\nu''}$ is an L^1-function from (2.4) and (2.10), we get the estimate of $|P^k b_1|_{\mathscr{B}^0}$ and $|P^l b_1|_{\mathscr{B}^0}$. Differentiating (2.20) with respect to x, we also get the estimate of $\|\partial_x P^k b_1\|$ and $\|\partial_x P^l b_1\|$. Finally, we get (1.8) for $b(t,x)$ of (2.15) from (2.19). Q.E.D.

Lemma 2.4. *Assume* (2.3)–(2.5), (2.6) *and* (2.10). *Then, we have*

$$\begin{cases} |Lb|_{\mathscr{B}^0}, \|\partial_x Lb\| \le C_{15}, \\ |P^k Lb|_{\mathscr{B}^0}, \|\partial_x P^k Lb\| \le C_{15}C_0A^{k-1}k!(k-1)!^{s-1}/(k+1)^2 \quad \text{for } 1 \le k < l, \\ |P^l Lb|_{\mathscr{B}^0}, \|\partial_x P^l Lb\| \le C_{15}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2. \end{cases} \tag{2.21}$$

PROOF. From $Lu=f$, we have

$$\begin{aligned} L\{f_v\} &= f_{vu}f + f_{v\bar{u}}\bar{f} + f_{vv}\partial_x f + 2if_{v\bar{u}}\bar{u}'' \\ &\quad + i\{f_{vuu}(u')^2 + f_{v\bar{u}\bar{u}}(\bar{u}')^2 + f_{vvv}(u'')^2 \\ &\quad + 2f_{vu\bar{u}}u'\bar{u}' + 2f_{vuv}u'u'' + 2f_{v\bar{u}v}\bar{u}'u''\}. \end{aligned}$$

Since $Lb_1 = \int_{-\infty}^x L\{f_v\}\,dy$, we get

$$\begin{aligned} Lb_1 &= f_{vv}f + 2if_{v\bar{u}}\bar{u}' \\ &\quad + \int_{-\infty}^x [f_{vu}f + f_{v\bar{u}}\bar{f} - \partial_x\{f_{vv}\}f - 2i\partial_x\{f_{v\bar{u}}\}\bar{u}' \\ &\quad + i\{f_{vuu}(u')^2 + f_{v\bar{u}\bar{u}}(\bar{u}')^2 + f_{vvv}(u'')^2 \\ &\quad + 2f_{vu\bar{u}}u'\bar{u}' + 2f_{vuv}u'u'' + 2f_{v\bar{u}v}\bar{u}'u''\}]\,dy \end{aligned}$$

and

$$\partial_x(Lb_1) = L\{f_v\}.$$

This shows that we can operate L to b_1 and get

$$|Lb_1|_{\mathscr{B}^0},\ \|\partial_x Lb_1\| \le C_{16}. \tag{2.22}$$

Moreover, from (2.3)–(2.5), (2.6) and (2.10) we also get

$$\begin{cases} |P^k Lb_1|_{\mathscr{B}^0},\ \|\partial_x P^k Lb_1\| \le C_{16}C_0A^{k-1}k!(k-1)!^{s-1}/(k+1)^2 & \textit{for } 1\le k<l, \\ |P^l Lb_1|_{\mathscr{B}^0},\ \|\partial_x P^l Lb_1\| \le C_{16}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2. \end{cases} \tag{2.23}$$

Finally, we note

$$Lb = \frac{i}{2}bLb_1 - \frac{i}{4}bb_1'^2.$$

Then, we get (2.21) by (2.22)–(2.23), (2.18) and (1.8) for b. Q.E.D.

LEMMA 2.5. *Assume* (2.3)–(2.5), (2.6) *and* (2.10). *Then, we have*

$$\|P^l\partial_x^2 u\|_1 \le C_{17}C_0A^{l-2}l!(l-1)!^{s-1}/(l+1)^2. \tag{2.24}$$

PROOF. From $Lu=f(u,u')$ we obtain

$$Lu'' = f_{2,1} + f_{2,2}u'' + f_{vv}u''^2 + f_{\bar{u}}\bar{u}'' + f_v\partial_x u''$$

with

$$\begin{cases} f_{2,1}(u,u') = f_{uu}(u,u')u'^2 + 2f_{u\bar{u}}(u,u')u'\bar{u}' + f_{\bar{u}\bar{u}}(u,u')\bar{u}'^2 \\ f_{2,2}(u,u') = 2f_{uv}(u,u')u' + 2f_{\bar{u}v}(u,u')\bar{u}' + f_u(u,u'). \end{cases}$$

Set $w=bu''$ with $b=b(t,x)$ in (2.15). Then, we have

$$Lw = f_{2,1}b + \{f_{2,2} + (Lb)b^{-1}\}w + f_{vv}b^{-1}w^2 + f_{\bar{u}}b\bar{b}^{-1}\bar{w}. \tag{2.25}$$

For k with $k < l$, we have from Lemma 1.5

$$\begin{aligned}\|P^k w\|_1 &= \|P^k(bu'')\|_1 \leq \|bP^k u''\|_1 + \|P^k(bu'') - bP^k u''\|_1 \\ &\leq C_{18} C_0 A^{k-1} k!(k-1)!^{s-1}/(k+1)^2,\end{aligned}$$

since $b(t,x)$ satisfies (1.8). Now, we apply the method of proving Proposition 2.1. Then, from Proposition 1.1 we get

$$\|P^l w\|_1 \leq C_{19} C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.$$

Finally, we apply Lemma 1.5 again to estimate

$$\|bP^l u''\|_1 \leq \|P^l w\|_1 + \|bP^l u'' - P^l(bu'')\|_1.$$

Then, we get (2.24). Q.E.D.

Next, we use the discussion of proving Proposition 2.2 to the equation (2.25) with the initial condition

$$\|(x(x\partial_x)^l w)(0,\cdot)\| \leq C_{20} A^{l-2} l!^s/(l+1)^2,$$

which is guaranteed from (2.13) and (1.8) for b. Then, we get

LEMMA 2.6. *Assume* (2.3)–(2.5), (2.6), (2.10) *and* (2.24). *Then, we have*

$$\|xP^l \partial_x^2 u\| \leq C_{21} C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2. \tag{2.26}$$

Now, we are prepared to prove Theorem 1.

PROOF OF THEOREM 1. From Lemma 2.5 and Lemma 1.4 we have

$$\begin{aligned}\|\partial_x^2 P^l u\|_1 &\leq \|P^l \partial_x^2 u\|_1 + \|(\partial_x^2 P^l - P^l \partial_x^2)u\|_1 \\ &\leq (C_{17} + C_4) C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.\end{aligned}$$

Combining this with (2.6) we get

$$\|P^l u\|_3 \leq C_{22} C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.$$

Similarly, we get from (2.10) and (2.26)

$$\|xP^l u\|_2 \leq C_{23} C_0 A^{l-2} l!(l-1)!^{s-1}/(l+1)^2.$$

Finally, we take the constant A satisfying $A \geq C_{22}$ and $A \geq C_{23}$. Then, we get (2.1) and (2.2) for l. Q.E.D

§3. Local Gevrey regularity.

In this section, we prove Theorem 2. For a positive constant R we take a C^∞-function $\chi(x)$ satisfying

$$\chi(x) = \begin{cases} 1 & for\ |x| \leq R, \\ 0 & for\ |x| \geq R+1. \end{cases}$$

Then, in order to prove Theorem 2, we have only to prove for $t > 0$

$$(3.1)_\alpha \qquad \|\chi(x)^\alpha \partial_x^\alpha P^l u\|_1 \le C_0' A^{\alpha+l-2} t^{-\alpha+1} (\alpha+l-3)!^\sigma l!^{s-\sigma} \quad \textit{for any } l$$

holds for any α with $\alpha \ge 3$.

LEMMA 3.1. *There exists a constant C_0' such that an estmate $(3.1)_\alpha$ with $\alpha = 3$ holds.*

PROOF. From Theorem 1 we have

$$(3.2) \qquad \|\chi^\alpha \partial_x^\alpha P^l u\|_1 \le C_0' A^{(\alpha+l-2)_+} (\alpha+l-3)_+!^\sigma l!^{s-\sigma} \quad \text{for } \alpha = 0, 1, 2$$

hold, where $m_+ = \max(m, 0)$. Now, using

$$(3.3) \qquad \partial_x^2 = \frac{1}{i} L - \frac{1}{2ti}(P - x\partial_x)$$

and $LP^l = (P+2)^l L$, we write

$$\partial_x^3 P^l u = \frac{1}{i} \partial_x (P+2)^l \{f(u, u')\} - \frac{1}{2ti} \partial_x P^{l+1} u + \frac{1}{2ti} \partial_x x \partial_x P^l u.$$

Then, we get $(3.1)_3$ from (3.2). Q.E.D.

Now, we prove $(3.1)_\alpha$ for $\alpha \ge 4$ by the induction. Set $\alpha' = \alpha - 2$ and write

$$(3.4) \qquad \chi^\alpha \partial_x^\alpha P^l u = \chi^\alpha \partial_x^{\alpha'} \partial_x^2 P^l u$$
$$= \frac{1}{i} \chi^\alpha \partial_x^{\alpha'} (P+2)^l \{f(u, u')\} - \frac{1}{2ti} \chi^\alpha \partial_x^{\alpha'} P^{l+1} u + \frac{1}{2ti} \chi^\alpha \partial_x^{\alpha'} x \partial_x P^l u.$$

Then, we have as in [5]

LEMMA 3.2 (cf. Theorem 4.4 and Theorem 4.5 of [5]). *Assume $(3.1)_\beta$ for $\beta < \alpha$ and (3.3) hold. Then, there exist constants C_{24} and C_{25} such that the inequalities*

$$(3.5) \qquad \|\chi^\alpha \partial_x^{\alpha'} P^{l+1} u\|_1 \le C_{24} C_0' A^{\alpha'+l-1} t^{-\alpha'+1} (\alpha'+l-1)!^\sigma l!^{s-\sigma},$$

$$(3.6) \qquad \|\chi^\alpha \partial_x^{\alpha'} x \partial_x P^l u\|_1 \le C_{25} C_0' A^{\alpha'+l-1} t^{-\alpha'} (\alpha'+l-1)!^\sigma l!^{s-\sigma}$$

hold.

Now, we estimate $\chi^\alpha \partial_x^{\alpha'} (P+2)^l \{f(u, u')\}$.

LEMMA 3.3. *Assume $(3.1)_\beta$ for $\beta < \alpha$ and (3.3) hold. Then, there exists a constant C_{26} such that*

$$(3.7) \qquad \|\chi^\alpha \partial_x^{\alpha'} (P+2)^l \{f(u, u')\}\|_1 \le C_{26} C_0' A^{\alpha'+l-1} t^{-\alpha'} (\alpha'+l-1)!^\sigma l!^{s-\sigma}$$

holds for $t > 0$.

PROOF. From the assumption $(3.1)_\alpha$ of the induction we have for β with $3 \le \beta < \alpha$

$$(3.8) \qquad \|\chi^\beta \partial_x^\beta P^l u\|_1 \le C_{27} C_0' A^{(\beta+l-2)_+} t^{-\beta+1}$$
$$\times (\beta+l-1)!(\beta+l-2)_+!^{\sigma-1} l!^{s-\sigma}/(\beta+l+1)^2$$

and we note, from (3.2), the above inequalities (3.8) hold also for $\beta = 0, 1, 2$ with $\beta + l \geq 1$. Now, we follow the discussions of proving Lemma 4.6 in [5]. Then, we have

$$\begin{aligned}
&\|\chi^{\alpha}\partial_x^{\alpha'}(P+2)^l\{f(u,u')\}\|_1 \\
&\leq \sum_{l'=0}^{l}\sum_{1\leq k+k'+k''\leq \alpha'+l'}\frac{\alpha'!l!}{(l-l')!k!k'!k''!}2^{l-l'}|\partial_u^k\partial_{\bar{u}}^{k'}\partial_v^{k''}f(u,u')| \\
&\quad\times\sum_{\substack{v_1+\cdots+v_{k+k'+k''}=\alpha'+l'\\ v_j\neq 0}}\prod_{j=1}^{k+k'}\sum_{\alpha_j+l_j=v_j}\frac{1}{\alpha_j!l_j!}\|\chi^{\alpha_j}\partial^{\alpha_j}P^{l_j}u_j\|_1 \\
&\quad\times\prod_{j=1+k+k'}^{k+k'+k''}\sum_{\alpha_j+l_j=v_j}\frac{1}{\alpha_j!l_j!}\|\chi^{\alpha_j}\partial^{\alpha_j+1}P^{l_j}u\|_1,
\end{aligned}$$

where

$$u_j = \begin{cases} u & \text{for } j\leq k, \\ \bar{u} & \text{for } k<j\leq k+k'. \end{cases}$$

Hence, we get

$$\begin{aligned}
&\|\chi^{\alpha}\partial_x^{\alpha'}(P+2)^l\{f(u,u')\}\|_1 \\
&\leq \sum_{l'=0}^{l}\sum_{1\leq k+k'+k''\leq \alpha'+l'}\frac{l!}{l'!}\frac{2^{l-l'}}{(l-l')!}C_{28}^{1+k+k'+k''}(k+k'+k''-1)!^{\sigma-1} \\
&\quad\times\sum_{\substack{v_1+\cdots+v_{k+k'+k''}=\alpha'+l'\\ v_j\neq 0}}\alpha'!l'!\prod_{j=1}^{k+k'+k''}\sum_{\alpha_j+l_j=v_j}\frac{(\alpha_j+l_j)!}{\alpha_j!l_j!} \\
&\quad\times C_{27}C_0'A^{\alpha_j+l_j-1}t^{-\alpha_j}(\alpha_j+l_j-1)!^{\sigma-1}l_j!^{s-\sigma}/(\alpha_j+l_j+1)^2 \\
&\leq C_{28}^2C_{27}C_0'A^{\alpha'+l-1}t^{-\alpha'}(\alpha'+l-1)!^{\sigma-1}l!^{s-\sigma}\sum_{l'=0}^{l}\frac{l!(\alpha'+l')!}{l'!}\left(\frac{2}{A}\right)^{l-l'} \\
&\quad\times\sum_{1\leq k+k'+k''\leq\alpha'+l'}\left(\frac{C_{28}C_{27}C_0'}{A}\right)^{k+k'+k''-1} \\
&\quad\times\sum_{\substack{v_1+\cdots+v_{k+k'+k''}=\alpha'+l'\\ v_j\neq 0}}\prod_{j=1}^{k+k'+k''}\frac{1}{(v_j+1)^2} \\
&\leq C_{28}^2C_{27}C_0'A^{\alpha'+l-1}t^{-\alpha'}(\alpha'+l-1)!^{\sigma-1}l!^{s-\sigma}\sum_{l'=0}^{l}\frac{l!(\alpha'+l')!}{l'!(\alpha'+l'+1)^2}\left(\frac{2}{A}\right)^{l-l'} \\
&\quad\times\sum_{1\leq k+k'+k''\leq\alpha'+l'}\left(\frac{C_2C_{28}C_{27}C_0'}{A}\right)^{k+k'+k''-1} \\
&\leq C_{26}C_0'A^{\alpha'+l-1}t^{-\alpha'}(\alpha'+l-1)!^{\sigma}l!^{s-\sigma},
\end{aligned}$$

if we take a constant A such that $A \geq 4$ and $A \geq C_2 C_{28} C_{27} C_0'$. This proves the lemma. Q.E.D.

Now, we are prepared to prove Theorem 2.

PROOF OF THEOREM 2. From (3.5)–(3.7) we have

$$\|\chi(x)^\alpha \partial_x^\alpha P^l u\|_1 \leq (C_{24}T + C_{25} + C_{26}T) C_0' A^{\alpha'+l-1} t^{-\alpha'-1} (\alpha' + l - 1)!^\sigma l!^{s-\sigma}$$
$$= (C_{24}T + C_{25} + C_{26}T) C_0' A^{\alpha+l-3} t^{-\alpha+1} (\alpha + l - 3)!^\sigma l!^{s-\sigma}.$$

Now, we take A such that $C_{24}T + C_{25} + C_{26}T \leq A$. Then, we get $(3.1)_\alpha$. This proves Theorem 2. Q.E.D.

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