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Third order ordinary differential equations and Legendre connections

By Hajime SATO*) and Atsuko Yamada YOSHIKAWA**)

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Abstract. We use N. Tanaka's theory of normal Cartan connections associated with simple graded Lie algebras to study Cartan's equivalence problem of single third order ordinary differential equations under contact transformations. As a result we obtain the complete structure equation with two differential invariants, which is applied on general Legendre Grassmann bundles of three-dimensional contact manifolds.

1. Introduction

The general equivalence problem is to recognize when two geometrical objects are mapped on each other by a certain class of diffeomorphisms. É. Cartan developed the general equivalence problem and provided a systematic procedure for determining the necessary and sufficient condition. In recent explorations on the equivalence problem of second order ordinary differential equations, N. Kamran, R. B. Gardner, P. J. Oliver, W. Shadwick, C. Grissom, G. Thompson, G. Wilkins have taken important steps in this direction ([4], [5]). In regard to the equivalence problem of third order ordinary differential equations, É. Cartan studied under point transformations ([2]), and in 1940 S. S. Chern turned his attention to the problem under contact transformations ([3]). We shall make several statements of S. S. Chern's important results, which are explained by R. B. Gardner in [4].

We let $y = y(x)$ be a function from $\boldsymbol{R}$ to $\boldsymbol{R}$, and give a third order ordinary differential equation

$$\frac{d^3y}{dx^3} = F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) \tag{1.1}$$

of normal type. We consider the equivalence problem for (1.1) under contact transformations on the second jet space $J^2 = J^2(\boldsymbol{R}, \boldsymbol{R})$ which are natural liftings of contact transformations on the first jet space $J^1 = J^1(\boldsymbol{R}, \boldsymbol{R})$. We give the following $\boldsymbol{R}^4$-valued one-form $\theta_F = {}^t(\theta_1, \theta_2, \theta_3, \theta_4)$ on J^2 with standard jet coordinates x, y, p, q:

$$\begin{aligned} \theta_1 &= dy - p\,dx, \\ \theta_2 &= dp - q\,dx, \end{aligned} \tag{1.2}$$

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$$\theta_3 = dq - F\,dx,$$

$$\theta_4 = dx.$$

For coframes $\theta_F, \theta_{\tilde{F}}$ corresponding to two third order ordinary differential equations, the equivalence problem resolves itself into the existence problem of a diffeomorphism $\phi : J^2 \to J^2$ satisfying $\phi^*\theta_F = g\theta_{\tilde{F}}$, where

$$g \in \hat{G} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ e & k & g & 0 \\ h & i & 0 & j \end{pmatrix} \in GL(4, \boldsymbol{R}) \right\}. \tag{1.3}$$

To begin with, we change horizontal directions of the connection of $\hat{G}$-structure, and reduce the group $\hat{G}$ to the isotropy subgroup $G_0^{\#}$ of simplified structure functions. Then we continue procedures of reduction. The last step of reductions, S. S. Chern obtained a structure function A for a five-dimensional structure group $\tilde{G}$:

$$A = \frac{c^3}{a^3}\left\{ -\frac{\partial F}{\partial y} - \frac{1}{3}\frac{\partial F}{\partial q}\frac{\partial F}{\partial p} - \frac{2}{27}\left(\frac{\partial F}{\partial q}\right)^3 + \frac{1}{2}\frac{d}{dx}\frac{\partial F}{\partial p} + \frac{1}{3}\frac{\partial F}{\partial q}\frac{d}{dx}\frac{\partial F}{\partial q} - \frac{1}{6}\frac{d^2}{dx^2}\frac{\partial F}{\partial q} \right\}. \tag{1.4}$$

When $A = 0$, we can not reduce the group $\tilde{G}$ any longer. Then we follow procedures for prolongation. Finally S. S. Chern obtained five structure functions. If all structure functions are equal to zero, then the third order ordinary differential equation is equivalent to $d^3y/dx^3 = 0$. The condition $A = 0$ is a necessary and sufficient condition that solution curves of the differential equation (1.1) on J^2 is equal to null-curves of $(2,1)$-conformal structure on the space of solution curves, which are given by a second order Monge equation. When $A \neq 0$, we reduce the structure group again. Then we follow procedures for prolongation. Finally S. S. Chern constructs a Cartan connection with values on a Lie algebra of a six-dimensional non-semisimple Lie group. This process of constructing structure equation with reductions and prolongations is very skillful, however, some problems still remain. For one thing, too many structure invariants remains. Structure functions can be simplified as Spencer cohomology elements. For example, when $A = 0$ we see that there is only one essential function from the calculation of cohomology ([**10**]), and we expect that the one can be expressed in terms of the function F and its derivatives. What is more, they change the structure groups according to cases.

To settle these problems, we shall discuss Cartan connections on J^2 through N. Tanaka's elaborate studies on connections associated with simple graded Lie algebras ([**9**]). Now we give some main points of his theory. Let $\mathfrak{G}$ be a simple graded Lie algebra $\mathfrak{g} = \sum_p \mathfrak{g}_p$, that is, $\mathfrak{g}$ is finite dimensional and simple, $\mathfrak{g}_{-1} \neq 0$ and $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$. Let G/H be a n-dimensional homogeneous space, where the Lie algebra of G is equal to $\mathfrak{g}$ and the Lie algebra of H to $\mathfrak{h} = \sum_{p\geq 0} \mathfrak{g}_p$. We represent the linear isotropy group $\tilde{G}$ of G/H on the vector space $\mathfrak{m}$, and extend the group $\tilde{G}$ to a linear Lie group $G_0^{\#} \subset GL(4, \boldsymbol{R})$. Assume that $\mathfrak{G}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. N. Tanaka constructs a unique normal Cartan connection (P, ω) of type $\mathfrak{G}$ on a n-dimensional manifold M from a $G_0^{\#}$-structure of type $\mathfrak{m}$ on M. Conversely every normal connections of type $\mathfrak{G}$ on M induce a $G_0^{\#}$-structures of type $\mathfrak{m}$ on M. Let

(P,ω) and (P',ω') be normal connections of type $\mathfrak{G}$ on M and M' respectively. Let $(P^{\#},\xi)$ and $(P'^{\#},\xi')$ be $G_0^{\#}$-structures of type $\mathfrak{m}$ on M and M' corresponding to (P,ω) and (P',ω') respectively. If $\varphi^{\#}:(P^{\#},\xi)\to(P'^{\#},\xi')$ is an isomorphism, then there exists a unique isomorphism $\varphi:(P,\omega)\to(P',\omega')$. Conversely each isomorphism $\varphi:(P,\omega)\to(P',\omega')$ induces an isomorphism $\varphi^{\#}:(P^{\#},\xi)\to(P'^{\#},\xi')$ ([**9**, Theorem 2.7]). The equivalence problem for third order ordinary differential equations under contact transformations resolves itself into the equivalence problem for $G_0^{\#}$-structures and G is equal to $Sp(2,\boldsymbol{R})$. Thus this problem is solved by getting invariant functions of structure equation of normal Cartan connections associated with $\mathfrak{sp}(2,\boldsymbol{R})$.

In this paper we construct concretely the structure equation with respect to third order ordinary differential equations. In Section 1, we outline the definition of normal Cartan connection in [**9**]. In Section 2, we summarize the result of S. S. Chern. In Section 3, we give a grading of Lie algebra $\mathfrak{G}=\mathfrak{sp}(2,\boldsymbol{R})$, and our main theorem (Theorem 4.1). Our main results are the following: For given third order ordinary differential equation (1.1), there exists a unique normal Cartan connection of type $\mathfrak{G}$ on J^2. There exists two essential invariant functions A and $\mathbf{b}$, and (1.1) is equivalent to $F=0$ under contact transformations if and only if $A=\mathbf{b}=0$. When $A=0$, the structure equation (4.6) coincide Chern's equation (3.12). In this section we show the existence of normal Cartan connection, and give expressions (4.7) and (4.8) to A and $\mathbf{b}$ using the function F. In Section 4 we construct the structure equation (4.6) in Theorem 4.1, and give some remarks on the Spencer cohomology of $\mathfrak{sp}(2,\boldsymbol{R})$. This results delete on Legendre Grassmann bundles $L(M)$ on three-dimensional contact manifold M. In Section 5, we define Legendre connections on $L(M)$. Then giving a third order ordinary differential equation on $L(M)$ is equal to giving a Legendre connection on $L(M)$. Thus we regard Theorem 3.1 as a theorem on $L(M)$ with given Legendre connection. For going over our calculations and checking our results, we carried out on computers using the symbolic manipulation program MAPLE.

In a forthcoming paper [**6**], we proceed to study a relation of structures in a twistor diagram through the result of this paper and give a foundation of projective contact geometry.

2. Normal Cartan connection

Let G/H be a homogeneous space of a Lie group G and a closed subgroup H with dimension n, and $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Assume that $\mathfrak{g}$ is a graded Lie algebra with grading $\mathfrak{g}=\sum_{i=-\mu}^{\mu}\mathfrak{g}_i$, and that $\mathfrak{h}=\sum_{i=0}^{\mu}\mathfrak{g}_i$. By $\mathfrak{G}=(\mathfrak{g},\{\mathfrak{g}_i\}_{i=-\mu}^{\mu})$ we denote this graded Lie algebra $\mathfrak{g}$. Let M be an n-dimensional manifold, and (P,ω) a Cartan connection of type $\mathfrak{G}$ on M, where P is a principal H-bundle over M and ω is a $\mathfrak{g}$-valued one-form on P. The curvature form Ω of (P,ω) is a $\mathfrak{g}$-valued two-form on P defined by

$$\Omega=d\omega+\frac{1}{2}[\omega,\omega]. \tag{2.1}$$

Then there is a unique function $K:P\to\mathfrak{g}\otimes\bigwedge^2(\mathfrak{m}^*)$ such that

$$\Omega=\frac{1}{2}K(\omega_-\wedge\omega_-), \tag{2.2}$$

where ω_- is the $\mathfrak{m}$-component of ω with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. (See [**9**].) The space $\bigwedge^q(\mathfrak{m}^*)$ is decomposed to $\sum_{r_1,\dots,r_q<0} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q}$. We define a subspace $\bigwedge^q_p(\mathfrak{m}^*)$ of $\bigwedge^q(\mathfrak{m}^*)$ by

$$(2.3) \qquad \bigwedge\nolimits^q_p(\mathfrak{m}^*) = \sum_{\substack{r_1+\cdots+r_q=p,\\ r_1,\dots,r_q<0}} \mathfrak{g}^*_{r_1} \wedge \cdots \wedge \mathfrak{g}^*_{r_q},$$

and define subspaces $C^{p,q}(\mathfrak{G})$ of $\mathfrak{g} \otimes \bigwedge^q(\mathfrak{m}^*)$ by

$$(2.4) \qquad C^{p,q}(\mathfrak{G}) = \sum_k \mathfrak{g}_k \otimes \bigwedge\nolimits^q_{k-p-q+1}(\mathfrak{m}^*).$$

Let $e_1, \dots, e_r$ be a basis of $\mathfrak{g}$. The space $\sum_{i>0} \mathfrak{g}_i$ is regarded as the dual space $\mathfrak{m}^*$ of $\mathfrak{m} = \sum_{i<0} \mathfrak{g}_i$ and hence there is a basis $e^*_1, \dots, e^*_m$ of $\mathfrak{m}^*$ such that $B(e_i, e^*_j) = \delta_{ij}$, where B is the Killing form of $\mathfrak{g}$. Then there are operators $\partial : C^{p,q}(\mathfrak{G}) \to C^{p-1,q+1}(\mathfrak{G})$ and $\partial^* : C^{p,q}(\mathfrak{G}) \to C^{p+1,q-1}(\mathfrak{G})$ which are defined by

$$\begin{aligned}(2.5) \qquad &\partial c(X_1 \wedge \cdots \wedge X_{q+1}) \\ &= \sum_i (-1)^{i+1} [X_i, c(X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge X_{q+1})] \\ &+ \sum_{i<j} (-1)^{i+j} c([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \check{X}_i \wedge \cdots \wedge \check{X}_j \wedge \cdots \wedge X_{q+1}),\end{aligned}$$

$$\begin{aligned}(2.6) \qquad &\partial^* c(X_1 \wedge \cdots \wedge X_{q-1}) \\ &= \sum_i [e^*_i, c(e_i \wedge X_1 \wedge \cdots \wedge X_{q-1})] \\ &+ \frac{1}{2} \sum_{i,j} (-1)^{j+1} c([e^*_i, X_j]_- \wedge e_i \wedge X_1 \wedge \cdots \wedge \check{X}_j \wedge \cdots \wedge X_{q-1}),\end{aligned}$$

where $c \in C^{p,q}(\mathfrak{G})$, $X_1, \dots, X_{q+1} \in \mathfrak{m}$ and $[e^*_i, X_j]_-$ denotes the $\mathfrak{m}$-component of $[e^*_i, X_j]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. We call a form $c \in C^{p,q}(\mathfrak{G})$ harmonic if $(\partial^*\partial + \partial\partial^*)(c) = 0$. We denote by $H^{p,q}(\mathfrak{G})$ the space of all harmonic forms in $C^{p,q}(\mathfrak{G})$. Then the space $C^{p,q}(\mathfrak{G})$ is orthogonally decomposed as follows:

$$(2.7) \qquad C^{p,q}(\mathfrak{G}) = H^{p,q}(\mathfrak{G}) + (\partial^*\partial + \partial\partial^*) C^{p,q}(\mathfrak{G}).$$

Let K^p be the $C^{p,2}(\mathfrak{G})$-component of the curvature function K. We say that a Cartan connection (P, ω) of type $\mathfrak{G}$ on M is normal if K satisfies the following conditions:

$$(2.8) \qquad K^p = 0 \quad \text{for } p < 0,$$

$$(2.9) \qquad \partial^* K^p = 0 \quad \text{for } p \geq 0.$$

These simple conditions which are applicable to many problems are finally found by N. Tanaka in [**9**] after much work ([**7**, **8**]).

3. The equivalence problem of third order ordinary differential equations under contact transformations

Let $J^1 = J^1(\boldsymbol{R}, \boldsymbol{R})$ be the first jet space with standard jet coordinates x, y, p, where $p = dy/dx$, with contact form $\theta_1 = dy - p\,dx$. Let $\phi(x,y,p) = (X(x,y,p), Y(x,y,p), P(x,y,p))$ be a diffeomorphism from J^1 to J^1. We call ϕ be a contact transformation if it preserves the contact form up to multiple

$$\phi^*(\theta_1) = \rho\,\theta_1, \tag{3.1}$$

since ρ is defined by a diffeomorphism it is always non-zero function of x, y, p. To put it another way,

$$\frac{\partial Y}{\partial x} - P\frac{\partial X}{\partial x} = -p\rho, \quad \frac{\partial Y}{\partial y} - P\frac{\partial X}{\partial y} = \rho, \quad \frac{\partial Y}{\partial p} - P\frac{\partial X}{\partial p} = 0. \tag{3.2}$$

Let $J^2 = J^2(\boldsymbol{R}, \boldsymbol{R})$ be the second jet space with standard jet coordinates x, y, p, q, where $p = dy/dx$, $q = d^2y/dx^2$. We let $\phi^{(1)}(x,y,p,q) = (X(x,y,p), Y(x,y,p), P(x,y,p), Q(x,y,p,q))$ be the first prolongation of a contact transformation ϕ of J^1, that is

$$Q = \frac{\dfrac{\partial P}{\partial x} + p\dfrac{\partial P}{\partial y} + q\dfrac{\partial P}{\partial p}}{\dfrac{\partial X}{\partial x} + p\dfrac{\partial X}{\partial y} + q\dfrac{\partial X}{\partial p}}. \tag{3.3}$$

We also call $\phi^{(1)}$ a contact transformation of J^2. Let $F(x,y,p,q)$ be a real-valued function on an open set of J^2. We introduce an $\boldsymbol{R}^4$-valued one-form θ associated with F given by

$$\theta = \theta_F = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} dy - p\,dx \\ dp - q\,dx \\ dq - F\,dx \\ dx \end{pmatrix}. \tag{3.4}$$

The solutions of (1.1) are curves γ in J^2 which satisfy

$$\gamma^*\theta_1 = \gamma^*\theta_2 = \gamma^*\theta_3 = 0, \quad \gamma^*\theta_4 \neq 0. \tag{3.5}$$

We call two differential equations associated with F and $\tilde{F}$ equivalent if there exists a contact transformation $\phi^{(1)}$ which transforms $\theta_{\tilde{F}}$ to θ_F. This is the equivalence problem of third order differential equations under contact transformations treated in [**3**]. We recall S. S. Chern's procedure and interpret its main points.

Let $\hat{G}$ be a nine-dimensional closed subgroup of $G = Sp(2, \boldsymbol{R})$ given by

$$\hat{G} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ e & k & g & 0 \\ h & i & 0 & j \end{pmatrix} \right\}, \tag{3.6}$$

according to the notation of [**4**]. We begin constructing the structure equation of $\hat{G}$-structure on J^2. Then we absorb some terms of curvature part and reduce the structure group $\hat{G}$ to a seven-dimensional subgroup $G_0^{\#}$ given by

$$G_0^{\#} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ e & k & c^2/a & 0 \\ h & i & 0 & a/c \end{pmatrix} \right\}. \tag{3.7}$$

We continue procedures of absorption and reduction. The last step, we obtain the following reduced structure group $\tilde{G}$ and $\tilde{G}$-structure $\tilde{P}$ with an $\boldsymbol{R}^4$-valued one-form $\tilde{\omega}$ on $\tilde{P}$:

$$\tilde{G} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ b^2/2a & bc/a & c^2/a & 0 \\ h & i & 0 & a/c \end{pmatrix} \right\}. \tag{3.8}$$

$$\begin{aligned} \tilde{\omega} &= {}^t(\omega_1, \omega_2, \omega_3, \omega_4) \\ &= \begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ \dfrac{b^2}{2a} - \dfrac{c^2}{2a}\dfrac{\partial F}{\partial p} - \dfrac{c^2}{9a}\left(\dfrac{\partial F}{\partial q}\right)^2 + \dfrac{c^2}{6a}\dfrac{d}{dx}\dfrac{\partial F}{\partial q} & \dfrac{bc}{a} - \dfrac{c^2}{3a}\dfrac{\partial F}{\partial q} & \dfrac{c^2}{a} & 0 \\ h & i & 0 & \dfrac{a}{c} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix}, \end{aligned} \tag{3.9}$$

where $d/dx = \partial/\partial x + p\partial/\partial y + q\partial/\partial p + F\partial/\partial q$ is the total derivative. The structure equation of the $\tilde{G}$-structure $(\tilde{P}, \tilde{\omega})$ on J^2 is

$$d\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ 0 & \beta_1 & 2\beta_2 - \alpha & 0 \\ \eta_1 & \eta_2 & 0 & \alpha - \beta_2 \end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} + \begin{pmatrix} -\omega_2 \wedge \omega_4 \\ -\omega_3 \wedge \omega_4 \\ A\omega_1 \wedge \omega_4 \\ 0 \end{pmatrix}. \tag{3.10}$$

Chern's invariant function A is

$$A = \frac{c^3}{a^3}\left\{ -\frac{\partial F}{\partial y} - \frac{1}{3}\frac{\partial F}{\partial q}\frac{\partial F}{\partial p} - \frac{2}{27}\left(\frac{\partial F}{\partial q}\right)^3 + \frac{1}{2}\frac{d}{dx}\frac{\partial F}{\partial p} + \frac{1}{3}\frac{\partial F}{\partial q}\frac{d}{dx}\frac{\partial F}{\partial q} - \frac{1}{6}\frac{d^2}{dx^2}\frac{\partial F}{\partial q} \right\}. \tag{3.11}$$

When $A = 0$, the isotropy subgroup remains unchanged, and we can not reduce the group $\tilde{G}$ any more. Then we follow procedures for prolongation of a $\tilde{G}$-structure and construct a Cartan connection (P, ω), $\omega = {}^t(\omega_1, \dots, \omega_{10})$, and obtain the following structure equation:

$$
\begin{aligned}
(3.12)\qquad d\omega_1 &= -\omega_1\wedge\omega_5 - \omega_2\wedge\omega_4,\\
d\omega_2 &= -\omega_1\wedge\omega_6 - \omega_2\wedge\omega_7 - \omega_3\wedge\omega_4,\\
d\omega_3 &= -\omega_2\wedge\omega_6 + \omega_3\wedge\omega_5 - 2\omega_3\wedge\omega_7,\\
d\omega_4 &= -\omega_1\wedge\omega_8 - \omega_2\wedge\omega_9 - \omega_4\wedge\omega_5 + \omega_4\wedge\omega_7,\\
d\omega_5 &= -2\omega_1\wedge\omega_{10} - \omega_2\wedge\omega_8 + \omega_4\wedge\omega_6,\\
d\omega_6 &= -\omega_2\wedge\omega_{10} - \omega_3\wedge\omega_8 - \omega_5\wedge\omega_6 - \omega_6\wedge\omega_7,\\
d\omega_7 &= -\omega_1\wedge\omega_{10} - \omega_2\wedge\omega_8 - \omega_3\wedge\omega_9,\\
d\omega_8 &= -\omega_4\wedge\omega_{10} - \omega_6\wedge\omega_9 - \omega_7\wedge\omega_8 + \mathbf{e}\omega_1\wedge\omega_2 + 2\mathbf{a}\omega_1\wedge\omega_3 - \mathbf{c}\omega_2\wedge\omega_3,\\
d\omega_9 &= -\omega_4\wedge\omega_8 + \omega_5\wedge\omega_9 - 2\omega_7\wedge\omega_9 + \mathbf{a}\omega_1\wedge\omega_2 - \mathbf{c}\omega_1\wedge\omega_3 + \mathbf{b}\omega_2\wedge\omega_3,\\
d\omega_{10} &= -\omega_5\wedge\omega_{10} - \omega_6\wedge\omega_8 + \mathbf{f}\omega_1\wedge\omega_2 + \mathbf{e}\omega_1\wedge\omega_3 + \mathbf{a}\omega_2\wedge\omega_3,
\end{aligned}
$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{f}$ coincide the functions a, b, c, e, f in [3] respectively.

When $A \neq 0$, we can reduce $\tilde{G}$-structure again. If we continue constructing a connection from (3.8), then we obtain curvature forms which are not only on the base forms $\omega_1, \omega_2, \omega_3, \omega_4$ but also on $\omega_1, \dots, \omega_{10}$. Thus S. S. Chern treated this problem separately in the case when $A = 0$ from in the case when $A \neq 0$, and he reduced the structure group when $A \neq 0$. However, we can absorb the torsion part of $d\omega_3$ of the structure equation (3.10) in a different way as follows:

$$
(3.13)\qquad d\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix} = \begin{pmatrix}\alpha & 0 & 0 & 0\\ \beta_1 & \beta_2 & 0 & 0\\ 0 & \beta_1 & 2\beta_2-\alpha & 0\\ \eta_1 & \eta_2 & 0 & \alpha-\beta_2\end{pmatrix} \wedge \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix} + \begin{pmatrix}-\omega_2\wedge\omega_4\\ -\omega_3\wedge\omega_4\\ B\omega_1\wedge\omega_2 + A\omega_1\wedge\omega_4\\ 0\end{pmatrix},
$$

and induces the function B giving rise to a function depending on A. In the sections that follow, we show that the structure equation satisfying Tanaka's normal condition for a Cartan connection (P, ω) associated with a simple graded Lie algebra $\mathfrak{sp}(2, \boldsymbol{R})$ induces this structure equation (3.13) adding the function B in surplus for a $\tilde{G}$-structure. The structure equation for (P, ω) covers both cases that $A = 0$ and $A \neq 0$.

4. Normal Cartan connection associated with $\mathfrak{sp}(2, \boldsymbol{R})$

From (3.12) we see that ω has its value on the Lie algebra $\mathfrak{sp}(2, \boldsymbol{R})$ of $Sp(2, \boldsymbol{R})$. We begin considering gradings of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2, \boldsymbol{R})$ of $G = Sp(2, \boldsymbol{R})$. In this section we define $\mathfrak{sp}(2, \boldsymbol{R})$ by

$$\mathfrak{g} = \mathfrak{sp}(2, \boldsymbol{R}) = \{X \in \mathfrak{gl}(4, \boldsymbol{R});\ {}^t XJ + JX = 0\}, \tag{4.1}$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.2}$$

We take a basis $e_1, \ldots, e_{10}$ of $\mathfrak{g}$ as follows:

$$\begin{aligned} &e_1 = E_{41}, \quad e_2 = \frac{1}{\sqrt{2}}(E_{31} + E_{42}), \\ &e_3 = E_{32}, \quad e_4 = \frac{1}{\sqrt{2}}(E_{21} - E_{43}), \\ &e_5 = \frac{1}{\sqrt{2}}(E_{11} - E_{44}), \quad e_6 = \frac{1}{\sqrt{2}}(E_{22} - E_{33}), \quad e_7 = \frac{1}{\sqrt{2}}(E_{12} - E_{34}), \\ &e_8 = E_{23}, \quad e_9 = \frac{1}{\sqrt{2}}(E_{13} + E_{24}), \quad e_{10} = E_{14}, \end{aligned} \tag{4.3}$$

where $E_{ij} \in \mathfrak{gl}(4, \boldsymbol{R})$ are matrices such that their (i,j)-component is equal to 1 and other components are 0. Lie brackets $[e_i, e_j]$ are presented in Table 1.

Table 1 tells us that $\mathfrak{g}$ has the following grading:

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3, \\ &\mathfrak{g}_{-3} = \{e_1\}, \quad \mathfrak{g}_{-2} = \{e_2\}, \quad \mathfrak{g}_{-1} = \{e_3, e_4\}, \quad \mathfrak{g}_0 = \{e_5, e_6\}, \\ &\mathfrak{g}_1 = \{e_7, e_8\}, \quad \mathfrak{g}_2 = \{e_9\}, \quad \mathfrak{g}_3 = \{e_{10}\}. \end{aligned} \tag{4.4}$$

We treat the grading Lie algebra $\mathfrak{G} = (\mathfrak{g}, \{\mathfrak{g}_i\}_{i=-3}^3)$ in (4.4), and decompose $\mathfrak{g}$ to $\mathfrak{m} + \mathfrak{h}$, where

$$\mathfrak{m} = \mathfrak{g}_{-3} + \mathfrak{g}_{-2} + \mathfrak{g}_{-1}, \quad \mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3. \tag{4.5}$$

By H we denote an isotropy subgroup of G of which Lie algebra is $\mathfrak{h}$. Our main theorem is the following:

THEOREM 4.1. *Let J^2 be the second jet space with standard jet coordinates x, y, p, q, and $\mathfrak{G}$ be a graded Lie algebra $(\mathfrak{sp}(2, \boldsymbol{R}), \{\mathfrak{g}_i\}_{i=-3}^3)$ given in* (4.4). *For given third order ordinary differential equation $d^3y/dx^3 = F(x, y, p, q)$, there exists a unique normal Cartan connection (P, ω), $\omega = {}^t(\omega_1, \ldots, \omega_{10})$ of type $\mathfrak{G}$ on J^2. The structure equation is*

$$\begin{aligned} d\omega_1 &= -\omega_1 \wedge \omega_5 - \omega_2 \wedge \omega_4, \\ d\omega_2 &= -\omega_1 \wedge \omega_6 - \omega_2 \wedge \omega_7 - \omega_3 \wedge \omega_4, \\ d\omega_3 &= -\omega_2 \wedge \omega_6 + \omega_3 \wedge \omega_5 - 2\omega_3 \wedge \omega_7 + B\omega_1 \wedge \omega_2 + A\omega_1 \wedge \omega_4, \\ d\omega_4 &= -\omega_1 \wedge \omega_8 - \omega_2 \wedge \omega_9 - \omega_4 \wedge \omega_5 + \omega_4 \wedge \omega_7, \end{aligned} \tag{4.6}$$

TABLE 1

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	*	0	0	0	$\sqrt{2}e_1$	0	e_2	0	$-e_4$	$-\sqrt{2}e_5$
e_2	*	*	0	e_1	$\frac{\sqrt{2}}{2}e_2$	$\frac{\sqrt{2}}{2}e_2$	e_3	$-e_4$	$-\frac{\sqrt{2}}{2}e_5 - \frac{\sqrt{2}}{2}e_6$	$-e_7$
e_3	*	*	*	e_2	0	$\sqrt{2}e_3$	0	$-\sqrt{2}e_6$	$-e_7$	0
e_4	*	*	*	*	$\frac{\sqrt{2}}{2}e_4$	$-\frac{\sqrt{2}}{2}e_4$	$-\frac{\sqrt{2}}{2}e_5 + \frac{\sqrt{2}}{2}e_6$	0	e_8	e_9
e_5	*	*	*	*	*	0	$\frac{\sqrt{2}}{2}e_7$	0	$\frac{\sqrt{2}}{2}e_9$	$\sqrt{2}e_{10}$
e_6	*	*	*	*	*	*	$-\frac{\sqrt{2}}{2}e_7$	$\sqrt{2}e_8$	$\frac{\sqrt{2}}{2}e_9$	0
e_7	*	*	*	*	*	*	*	e_9	e_{10}	0
e_8	*	*	*	*	*	*	*	*	0	0
e_9	*	*	*	*	*	*	*	*	*	0
e_{10}	*	*	*	*	*	*	*	*	*	*

$$d\omega_5 = -2\omega_1 \wedge \omega_{10} - \omega_2 \wedge \omega_8 + \omega_4 \wedge \omega_6,$$

$$d\omega_6 = -\omega_2 \wedge \omega_{10} - \omega_3 \wedge \omega_8 - \omega_5 \wedge \omega_6 - \omega_6 \wedge \omega_7 + \mathbf{g}\,\omega_1 \wedge \omega_2 + \mathbf{h}\,\omega_1 \wedge \omega_4,$$

$$d\omega_7 = -\omega_1 \wedge \omega_{10} - \omega_2 \wedge \omega_8 - \omega_3 \wedge \omega_9 + \mathbf{i}\,\omega_1 \wedge \omega_2 - B\omega_1 \wedge \omega_4,$$

$$d\omega_8 = -\omega_4 \wedge \omega_{10} - \omega_6 \wedge \omega_9 - \omega_7 \wedge \omega_8 + \mathbf{e}\,\omega_1 \wedge \omega_2 + 2\mathbf{a}_1\,\omega_1 \wedge \omega_3 + \mathbf{g}\,\omega_1 \wedge \omega_4 - \mathbf{c}\,\omega_2 \wedge \omega_3,$$

$$d\omega_9 = -\omega_4 \wedge \omega_8 + \omega_5 \wedge \omega_9 - 2\omega_7 \wedge \omega_9 + \mathbf{a}_2\,\omega_1 \wedge \omega_2 - \mathbf{c}\,\omega_1 \wedge \omega_3 - \mathbf{i}\,\omega_1 \wedge \omega_4 + \mathbf{b}\,\omega_2 \wedge \omega_3,$$

$$d\omega_{10} = -\omega_5 \wedge \omega_{10} - \omega_6 \wedge \omega_8 + \mathbf{f}\,\omega_1 \wedge \omega_2 + \mathbf{e}\,\omega_1 \wedge \omega_3 + \mathbf{j}\,\omega_1 \wedge \omega_4 + \mathbf{a}_1\,\omega_2 \wedge \omega_3.$$

Furthermore, the invariant functions A and $\mathbf{b}$ induce the other invariant functions. The following are expressions for A and $\mathbf{b}$ using the function F on J^2, base coordinates x, y, p, q of J^2 and coordinates a, c of $H \supset \tilde{G}$.

$$(4.7) \quad A = \frac{c^3}{a^3}\left\{ -\frac{\partial F}{\partial y} - \frac{1}{3}\frac{\partial F}{\partial q}\frac{\partial F}{\partial p} - \frac{2}{27}\left(\frac{\partial F}{\partial q}\right)^3 + \frac{1}{2}\frac{d}{dx}\frac{\partial F}{\partial p} + \frac{1}{3}\frac{\partial F}{\partial q}\frac{d}{dx}\frac{\partial F}{\partial q} - \frac{1}{6}\frac{d^2}{dx^2}\frac{\partial F}{\partial q} \right\},$$

$$\mathbf{b} = \frac{a^2}{6c^5}\frac{\partial^4 F}{\partial q^4}. \tag{4.8}$$

When $A = 0$, the structure equation (4.6) coincide with the equation (3.12). Furthermore we obtain the following corollary:

COROLLARY 4.2. *The third order ordinary differential equation* (1.1) *is equivalent to* $F = 0$ *under contact transformations if and only if* $A = \mathbf{b} = 0$.

Now we prove Theorem 4.1. To begin with, we show that there exists a unique normal Cartan connection (P, ω) of type $\mathfrak{G}$ on J^2 inducing the $G_0^{\#}$-structure associated with the given third order ordinary differential equation. We consider the linear representation $\rho : H \to GL(4)$ and $d\rho : H \to \mathfrak{gl}(4)$. For

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{13} \\ a_{31} & a_{32} & -a_{22} & -a_{12} \\ a_{41} & a_{31} & -a_{21} & -a_{11} \end{pmatrix} \in \mathfrak{g}, \tag{4.9}$$

the assignment $X \mapsto \mathrm{ad}(X)$ gives an isomorphism of $\mathfrak{g}$ onto $\mathrm{ad}(\mathfrak{g})$. For $b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 \in \mathfrak{m}$, we have

$$\mathrm{ad}(X)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} -2a_{11} & -\sqrt{2}a_{21} & 0 & \sqrt{2}a_{31} \\ -\sqrt{2}a_{12} & -a_{11} - a_{22} & -\sqrt{2}a_{21} & a_{32} \\ 0 & -\sqrt{2}a_{12} & -2a_{22} & 0 \\ \sqrt{2}a_{13} & a_{23} & 0 & a_{22} - a_{11} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \tag{4.10}$$

thus

$$d\rho(X) = \begin{pmatrix} -2a_{11} & -\sqrt{2}a_{21} & 0 & \sqrt{2}a_{31} \\ -\sqrt{2}a_{12} & -a_{11} - a_{22} & -\sqrt{2}a_{21} & a_{32} \\ 0 & -\sqrt{2}a_{12} & -2a_{22} & 0 \\ \sqrt{2}a_{13} & a_{23} & 0 & a_{22} - a_{11} \end{pmatrix} \in \mathfrak{gl}(4). \tag{4.11}$$

The Lie algebra of $\mathrm{Aut}(\mathfrak{G}) = G_0$ is $\mathfrak{g}_0$. We will identify two groups G_0 and $\rho(G_0)$ through the isomorphism $\rho : G_0 \to \rho(G_0)$. From (4.11), we see that

$$d\rho(\mathfrak{g}_0) = \left\{ \begin{pmatrix} -2a_{11} & 0 & 0 & 0 \\ 0 & -a_{11} - a_{22} & 0 & 0 \\ 0 & 0 & -2a_{22} & 0 \\ 0 & 0 & 0 & a_{22} - a_{11} \end{pmatrix} \right\} \subset \mathfrak{gl}(4), \tag{4.12}$$

and

$$\rho(G_0) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c^2/a & 0 \\ 0 & 0 & 0 & a/c \end{pmatrix} \right\} \subset GL(4). \tag{4.13}$$

Let N^0 be a subgroup of $GL(4)$ consisting of all $a \in GL(4)$ such that $aY_p \equiv Y_p$

(mod $\sum_{j=p+1}^{-1} \mathfrak{g}_j$) for all $Y_p \in \mathfrak{g}_p$ and $p < 0$, then

$$N^0 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \right\}. \tag{4.14}$$

We denote by $G_0^{\#}$ the closed subgroup $G_0 \cdot N^0$ of $GL(4)$, and we see that this group $G_0^{\#}$ is equal to the group $G_0^{\#}$ given in (3.7). Let $\tilde{G}$ be the image $\rho(H)$ of H expressed as $\tilde{G} = G_0 \cdot \rho(\exp \mathfrak{g}_1) \cdot \rho(\exp \mathfrak{g}_2)$, which is coincide with the group $\tilde{G}$ given in (3.8). The graded Lie algebra $\mathfrak{G}$ is $(C_2, \{\alpha_1, \alpha_2\})$-type and $\mathfrak{G}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ ([**10**, Theorem 5.3]). Thus, from [**9**, Theorem 2.7], we prove that there exists a unique normal Cartan connection (P, ω) in Theorem 4.1.

The invariant function A is equal to Chern's function (3.11). Now we calculate **b**. From (3.4), (3.9) and (4.6), we give expressions to A and **b** using the function F on J^2, base coordinates x, y, p, q of J^2 and coordinates a, c of $H \supset \tilde{G}$.

From (3.4) we see that

$$\begin{aligned} &d\theta_1 = -\theta_2 \wedge \theta_4, \quad d\theta_2 = -\theta_3 \wedge \theta_4, \\ &d\theta_3 = -\frac{\partial F}{\partial y}\theta_1 \wedge \theta_4 - \frac{\partial F}{\partial p}\theta_2 \wedge \theta_4 - \frac{\partial F}{\partial q}\theta_3 \wedge \theta_4, \quad d\theta_4 = 0. \end{aligned} \tag{4.15}$$

From (4.15) and (3.9), we have

$$d(\omega_1) = da \wedge \theta_1 - a\theta_2 \wedge \theta_4. \tag{4.16}$$

From (4.16), (3.9) and (4.6), we have

$$\begin{aligned} 0 &= d\omega_1 + \omega_1 \wedge \omega_5 + \omega_2 \wedge \omega_4 \\ &= \theta_1 \wedge \left\{ a\omega_5 - da + (bi - ch)\theta_2 + \frac{ab}{c}\theta_4 \right\}, \end{aligned} \tag{4.17}$$

thus

$$\omega_5 = \frac{da}{a} + x_1\theta_1 + \frac{-bi + ch}{a}\theta_2 - \frac{b}{c}\theta_4. \tag{4.18}$$

Similarly from (3.4), (3.9) and (4.6), we obtain ω_6 and ω_7 by calculating $d\omega_2 + \omega_1 \wedge \omega_6 + \omega_2 \wedge \omega_7 + \omega_3 \wedge \omega_4$.

$$\begin{aligned} \omega_6 = {} & \frac{db}{a} + \frac{b}{ac}dc + x_3\theta_1 + \frac{1}{a}(-bx_2 + ay_1)\theta_2 + \frac{c}{a^2}(-bi + ch)\theta_3 \\ & + \frac{1}{ac^2}(abk - ace - b^2c)\theta_4, \end{aligned} \tag{4.19}$$

$$\omega_7 = \frac{dc}{c} + (ei - kh + ay_1)\theta_1 + x_2\theta_2 + \frac{ci}{a}\theta_3 + \frac{1}{3}\frac{\partial F}{\partial q}\theta_4, \tag{4.20}$$

where

$$k = \frac{bc}{a} - \frac{c^2}{3a}\frac{\partial F}{\partial q},$$
(4.21)
$$e = \frac{b^2}{2a} - \frac{c^2}{2a}\frac{\partial F}{\partial p} - \frac{c^2}{9a}\left(\frac{\partial F}{\partial q}\right)^2 + \frac{c^2}{6a}\frac{d}{dx}\frac{\partial F}{\partial q}.$$

By calculating the term of $\theta_1 \wedge \theta_4$ of $d\omega_3 + \omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5 + 2\omega_3 \wedge \omega_7 - A\omega_1 \wedge \omega_4$, we obtain the expression (4.7) of A. So far, our calculations agree with S. S. Chern.

Next, from (3.4), (3.9) and (4.6), we similarly obtain ω_8 and ω_9 by calculating $d\omega_4 + \omega_1 \wedge \omega_8 + \omega_2 \wedge \omega_9 + \omega_4 \wedge \omega_5 - \omega_4 \wedge \omega_7$.

(4.22)
$$\begin{aligned}\omega_8 = {} & \frac{dh}{a} - \frac{b}{ac}\, di + \frac{1}{a^2c}(bi - ch)\, da + \frac{1}{ac^2}(-bi + ch)\, dc + x_5\theta_1 \\ & + \frac{1}{ac}(-bcx_4 + aiy_1 + c^2y_2)\theta_2 + \frac{i}{a^2}(-bi + ch)\theta_3 \\ & + \frac{1}{ac^3}(-acei + abki - b^2ci + ac^2x_1 + abcx_2 - ac^2y_1)\theta_4,\end{aligned}$$

(4.23)
$$\begin{aligned}\omega_9 = {} & \frac{di}{c} - \frac{i}{ac}\, da + \frac{i}{c^2}\, dc \\ & + \frac{1}{ac^2}(aei^2 - akhi - bchi + c^2h^2 - acix_1 - achx_2 + 2a^2iy_1 + ac^2y_2)\theta_1 \\ & + x_4\theta_2 + \frac{i^2}{a}\theta_3 + \frac{1}{c^3}(-aki + bci + 2c^2h - acx_2)\theta_4.\end{aligned}$$

From (4.6), we see that

(4.24)
$$d\omega_7 + \omega_2 \wedge \omega_8 + \omega_3 \wedge \omega_9 \equiv 0 \mod \omega_1.$$

We calculate the term of $\theta_3 \wedge \theta_4$ of this expression (4.24) using (3.9), (4.6), (4.22) and (4.23):

(4.25)
$$\frac{1}{3a}\left(6ch - 2ci\frac{\partial F}{\partial q} + a\frac{\partial^2 F}{\partial q^2} - 6ax_2\right)\theta_3 \wedge \theta_4 = 0,$$

thus we see that

(4.26)
$$x_2 = \frac{ch}{a} - \frac{ci}{3a}\frac{\partial F}{\partial q} + \frac{1}{6}\frac{\partial^2 F}{\partial q^2}.$$

Using (4.26), we calculate (4.24) again and see the term of $\theta_2 \wedge \theta_3$;

(4.27)
$$\frac{1}{6a^2}\left(6c^2hi - 2c^2i^2\frac{\partial F}{\partial q} + 2aci\frac{\partial^2 F}{\partial q^2} - a^2\frac{\partial^3 F}{\partial q^3} - 6ac^2x_4\right)\theta_2 \wedge \theta_3 = 0,$$

thus we see that

(4.28)
$$x_4 = \frac{hi}{a} - \frac{i^2}{3a}\frac{\partial F}{\partial q} + \frac{i}{3c}\frac{\partial^2 F}{\partial q^2} - \frac{a}{6c^2}\frac{\partial^3 F}{\partial q^3}.$$

From (4.6), we see that

$$d\omega_9 + \omega_4 \wedge \omega_8 - \omega_5 \wedge \omega_9 + 2\omega_7 \wedge \omega_9 - \mathbf{b}\omega_2 \wedge \omega_3 \equiv 0 \mod \omega_1. \tag{4.29}$$

We calculate the term of $\theta_2 \wedge \theta_3$ of this expression (4.29) using (3.9), (4.6), (4.18), (4.20), (4.22), (4.23), (4.26) and (4.28):

$$\frac{1}{6ac^2}\left(a^2 \frac{\partial^4 F}{\partial q^4} - 6c^5 \mathbf{b}\right)\theta_2 \wedge \theta_3 = 0, \tag{4.30}$$

thus we see that

$$\mathbf{b} = \frac{a^2}{6c^5}\frac{\partial^4 F}{\partial q^4}. \tag{4.31}$$

It remains to construct the structure equation (4.6) of Theorem 4.1 and to prove Corollary 4.2. We prove the rest in Section 5.

It is known that if a second order ordinary differential equation $d^2y/dx^2 = F(x, y, dy/dx)$ can be transformed to $d^2y/dx^2 = 0$ under diffeomorphisms of (x, y)-plane, then F is a polynomial of p with at most degree three. (See [**1**].) In the case of third order ordinary differential equations, from (3.3) we calculate the function F satisfying

$$dQ \equiv g(dq - F\,dx) \; (\mathrm{mod}\, dy - p\,dx,\; dp - q\,dx). \tag{4.32}$$

Then we also see that F is a polynomial of q with at most degree three, that is $\mathbf{b} = 0$.

5. Construction of the structure equation

We will construct the structure equation for the normal Cartan connection (P, ω) of type $\mathfrak{G}$ on J^2 which induces the $G_0^{\#}$-structure on J^2. Let σ be an involutive automorphism of $\mathfrak{g}$ given by

$$\begin{aligned} &\sigma(e_1) = -e_{10}, \quad \sigma(e_2) = -e_9, \quad \sigma(e_3) = -e_8, \quad \sigma(e_4) = -e_7, \\ &\sigma(e_5) = -e_5, \quad \sigma(e_6) = -e_6, \end{aligned} \tag{5.1}$$

which has the following properties: (1) $\sigma\mathfrak{g}_p = \mathfrak{g}_{-p}$, (2) $B(X, \sigma X) < 0$ for $X \neq 0$. We define an inner product $\langle\,,\rangle$ in $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \sigma Y)$, for $X, Y \in \mathfrak{g}$. Then $e_1, \dots, e_{10}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\,,\rangle$. The basis $e_1^*, e_2^*, e_3^*, e_4^*$ of $\mathfrak{m}^*$ such that $B(e_i, e_j^*) = \delta_{ij}$ is

$$e_1^* = e_{10}, \quad e_2^* = e_9, \quad e_3^* = e_8, \quad e_4^* = e_7. \tag{5.2}$$

In the case of this graded Lie algebra $\mathfrak{G}$, from (2.6) and (4.4) we have

$$\begin{aligned} &\textstyle\bigwedge^1_{-3}(\mathfrak{m}^*) = \mathfrak{g}^*_{-3}, \quad \bigwedge^1_{-2}(\mathfrak{m}^*) = \mathfrak{g}^*_{-2}, \quad \bigwedge^1_{-1}(\mathfrak{m}^*) = \mathfrak{g}^*_{-1}, \\ &\textstyle\bigwedge^2_{-5}(\mathfrak{m}^*) = \mathfrak{g}^*_{-3} \wedge \mathfrak{g}^*_{-2}, \quad \bigwedge^2_{-4}(\mathfrak{m}^*) = \mathfrak{g}^*_{-3} \wedge \mathfrak{g}^*_{-1}, \\ &\textstyle\bigwedge^2_{-3}(\mathfrak{m}^*) = \mathfrak{g}^*_{-2} \wedge \mathfrak{g}^*_{-1}, \quad \bigwedge^2_{-2}(\mathfrak{m}^*) = \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1}. \end{aligned} \tag{5.3}$$

We recall that K^p is the $C^{p,2}(\mathfrak{G}) = \sum_k \mathfrak{g}_k \otimes \bigwedge^2_{k-p-1}(\mathfrak{m}^*)$-component of the curvature form K. Let $K_k^{i,j}$ $(1 \le i,j \le 4, 1 \le k \le 10)$ be functions such that

$$K(e_i \wedge e_j) = \sum_{k=1}^{10} K_k^{i,j} e_k. \tag{5.4}$$

We will express concretely $K^p(e_i \wedge e_j)$ for $-2 \le p \le 7$ and $1 \le i,\ j \le 4$. When $p = 0$ we have

$$\begin{aligned} K^0 \in C^{0,2}(\mathfrak{G}) &= \sum_k \mathfrak{g}_k \otimes \textstyle\bigwedge_{k-1}^2(\mathfrak{m}^*) \\ &= \mathfrak{g}_{-3} \otimes \textstyle\bigwedge_{-4}^2(\mathfrak{m}^*) + \mathfrak{g}_{-2} \otimes \textstyle\bigwedge_{-3}^2(\mathfrak{m}^*) + \mathfrak{g}_{-1} \otimes \textstyle\bigwedge_{-2}^2(\mathfrak{m}^*) \\ &= \mathfrak{g}_{-3} \otimes (\mathfrak{g}_{-3}^* \wedge \mathfrak{g}_{-1}^*) + \mathfrak{g}_{-2} \otimes (\mathfrak{g}_{-2}^* \wedge \mathfrak{g}_{-1}^*) + \mathfrak{g}_{-1} \otimes (\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^*). \end{aligned} \tag{5.5}$$

Then,

$$\begin{aligned} &K^0(e_1 \wedge e_2) = 0, \quad K^2(e_1 \wedge e_3) = K_1^{1,3} e_1, \\ &K^0(e_1 \wedge e_4) = K_1^{1,4} e_1, \quad K^0(e_2 \wedge e_3) = K_2^{2,3} e_2, \\ &K^0(e_2 \wedge e_4) = K_2^{2,4} e_2, \quad K^0(e_3 \wedge e_4) = K_3^{3,4} e_3 + K_4^{3,4} e_4. \end{aligned} \tag{5.6}$$

Similarly we obtain Table 2 which shows $K^p(e_i \wedge e_j)$.

TABLE 2

	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_1 \wedge e_4$	$e_2 \wedge e_3$	$e_2 \wedge e_4$	$e_3 \wedge e_4$
K^{-2}	0	0	0	0	0	$K_1^{3,4} e_1$
K^{-1}	0	0	0	$K_1^{2,3} e_1$	$K_1^{2,4} e_1$	$K_2^{3,4} e_2$
K^0	0	$K_1^{1,3} e_1$	$K_1^{1,4} e_1$	$K_2^{2,3} e_2$	$K_2^{2,4} e_2$	$K_3^{3,4} e_3 + K_4^{3,4} e_4$
K^1	$K_1^{1,2} e_1$	$K_2^{1,3} e_2$	$K_2^{1,4} e_2$	$K_3^{2,3} e_3 + K_4^{2,3} e_4$	$K_3^{2,4} e_3 + K_4^{2,4} e_4$	$K_5^{3,4} e_5 + K_6^{3,4} e_6$
K^2	$K_2^{1,2} e_2$	$K_3^{1,3} e_3 + K_4^{1,3} e_4$	$K_3^{1,4} e_3 + K_4^{1,4} e_4$	$K_5^{2,3} e_5 + K_6^{2,3} e_6$	$K_5^{2,4} e_5 + K_6^{2,4} e_6$	$K_7^{3,4} e_7 + K_8^{3,4} e_8$
K^3	$K_3^{1,2} e_3 + K_4^{1,2} e_4$	$K_5^{1,3} e_5 + K_6^{1,3} e_6$	$K_5^{1,4} e_5 + K_6^{1,4} e_6$	$K_7^{2,3} e_7 + K_8^{2,3} e_8$	$K_7^{2,4} e_7 + K_8^{2,4} e_8$	$K_9^{3,4} e_9$
K^4	$K_5^{1,2} e_5 + K_6^{1,2} e_6$	$K_7^{1,3} e_7 + K_8^{1,3} e_8$	$K_7^{1,4} e_7 + K_8^{1,4} e_8$	$K_9^{2,3} e_9$	$K_9^{2,4} e_9$	$K_{10}^{3,4} e_{10}$
K^5	$K_7^{1,2} e_7 + K_8^{1,2} e_8$	$K_9^{1,3} e_9$	$K_9^{1,4} e_9$	$K_{10}^{2,3} e_{10}$	$K_{10}^{2,4} e_{10}$	0
K^6	$K_9^{1,2} e_9$	$K_{10}^{1,3} e_{10}$	$K_{10}^{1,4} e_{10}$	0	0	0
K^7	$K_{10}^{1,2} e_{10}$	0	0	0	0	0

From Table 1, (2.6), (4.3) and (5.2), we have

$$(5.7)\quad \begin{aligned}\partial^*K(e_1) &= [e_2^*, K(e_2\wedge e_1)] + [e_3^*, K(e_3\wedge e_1)] + [e_4^*, K(e_4\wedge e_1)]\\ &\quad + \frac{1}{2}\{K([e_2^*, e_1]\wedge e_2) + K([e_4^*, e_1]\wedge e_4)\}\\ &= [e_9, K(e_2\wedge e_1)] + [e_8, K(e_3\wedge e_1)] + [e_7, K(e_4\wedge e_1)] - K(e_2\wedge e_4).\end{aligned}$$

By similar calculation we have

$$(5.8)\quad \begin{aligned}\partial^*K(e_2) &= [e_{10}, K(e_1\wedge e_2)] + [e_8, K(e_3\wedge e_2)] + [e_7, K(e_4\wedge e_2)] - K(e_3\wedge e_4),\\ \partial^*K(e_3) &= [e_{10}, K(e_1\wedge e_3)] + [e_9, K(e_2\wedge e_3)] + [e_7, K(e_4\wedge e_3)],\\ \partial^*K(e_4) &= [e_{10}, K(e_1\wedge e_4)] + [e_9, K(e_2\wedge e_4)] + [e_8, K(e_3\wedge e_4)].\end{aligned}$$

The normal condition (2.9) are

$$(5.9)\quad \begin{aligned} p = -2 &: K_1^{3,4} = 0\\ p = -1 &: K_1^{2,3} = K_1^{2,4} = K_2^{3,4} = 0.\end{aligned}$$

When $p = 0$, the normal conditions (2.9) is equal to the following:

$$(5.10)\quad \begin{aligned}\partial^*K^0(e_1) &= [e_9, K^0(e_2\wedge e_1)] + [e_8, K^0(e_3\wedge e_1)] + [e_7, K^0(e_4\wedge e_1)] - K^0(e_2\wedge e_4),\\ &= [e_9, 0] + [e_8, -K_1^{1,3}e_1] + [e_7, -K_1^{1,4}e_1] - K_2^{2,4}e_2,\\ &= (K_1^{1,4} - K_2^{2,4})e_2 = 0,\\ \partial^*K^0(e_2) &= [e_{10}, K^0(e_1\wedge e_2)] + [e_8, K^0(e_3\wedge e_2)] + [e_7, K^0(e_4\wedge e_2)] - K^0(e_3\wedge e_4)\\ &= [e_{10}, 0] + [e_8, -K_2^{2,3}e_2] + [e_7, -K_2^{2,4}e_2] - K_3^{3,4}e_3 - K_4^{3,4}e_4\\ &= (K_2^{2,4} - K_3^{3,4})e_3 - (K_2^{2,3} - K_4^{3,4})e_4 = 0,\\ \partial^*K^0(e_3) &= [e_{10}, K^0(e_1\wedge e_3)] + [e_9, K^0(e_2\wedge e_3)] + [e_7, K^0(e_4\wedge e_3)]\\ &= [e_{10}, K_1^{1,3}e_1] + [e_9, K_2^{2,3}e_2] + [e_7, -K_3^{3,4}e_3 - K_4^{3,4}e_4]\\ &= \frac{1}{\sqrt{2}}(2K_1^{1,3} + K_2^{2,3} - K_4^{3,4})e_5 + \frac{1}{\sqrt{2}}(K_2^{2,3} + K_4^{3,4})e_6 = 0\\ \partial^*K^0(e_4) &= [e_{10}, K^0(e_1\wedge e_4)] + [e_9, K^0(e_2\wedge e_4)] + [e_8, K^0(e_3\wedge e_4)]\\ &= [e_{10}, K_1^{1,4}e_1] + [e_9, K_2^{2,4}e_2] + [e_8, K_3^{3,4}e_3 + K_4^{3,4}e_4]\\ &= \frac{1}{\sqrt{2}}(2K_1^{1,4} + K_2^{2,4})e_5 + \frac{1}{\sqrt{2}}(K_2^{2,4} + 2K_3^{3,4})e_6 = 0,\end{aligned}$$

That is to say,

$$(5.11)\qquad p = 0:\quad K_1^{1,3} = -K_2^{2,3} = K_4^{3,4},\quad K_1^{1,4} = K_2^{2,4} = K_3^{3,4} = 0.$$

By a similar calculation we have normal conditions for $p=1,\dots,5$.

(5.12)
$$p=1:\quad K_2^{1,4}=K_4^{2,3}=K_3^{2,4}=0,$$
$$K_3^{2,3}=K_4^{2,4}=-K_1^{1,2}-K_2^{1,3},$$
$$K_5^{3,4}=\frac{3\sqrt{2}}{2}K_1^{1,2}+\frac{\sqrt{2}}{2}K_2^{1,3},$$
$$K_6^{3,4}=\frac{\sqrt{2}}{2}K_1^{1,2}+\frac{\sqrt{2}}{2}K_2^{1,3},$$
$$p=2:\quad K_6^{2,3}=-\frac{\sqrt{2}}{2}K_4^{1,3}-\frac{1}{2}K_5^{2,3},$$
$$K_5^{2,4}=-\frac{\sqrt{2}}{2}K_2^{1,2}-\frac{\sqrt{2}}{2}K_4^{1,4},$$
$$K_6^{2,4}=-\frac{\sqrt{2}}{2}K_2^{1,2}-\sqrt{2}K_3^{1,3}+\frac{\sqrt{2}}{2}K_4^{1,4},$$
$$K_7^{3,4}=K_2^{1,2}+K_3^{1,3}-K_4^{1,4},$$
$$K_8^{3,4}=-\frac{1}{2}K_4^{1,3}-\frac{\sqrt{2}}{4}K_5^{2,3},$$
$$p=3:\quad K_6^{1,4}=-\sqrt{2}K_3^{1,2}+3K_5^{1,4},$$
$$K_7^{2,3}=K_4^{1,2}-\frac{\sqrt{2}}{2}K_5^{1,3}+\frac{\sqrt{2}}{2}K_6^{1,3},$$
$$K_7^{2,4}=-\sqrt{2}K_5^{1,4},\quad K_8^{2,4}=K_4^{1,2}+\sqrt{2}K_6^{1,3},$$
$$K_9^{3,4}=-K_4^{1,2}-\frac{\sqrt{2}}{2}K_5^{1,3}-\frac{\sqrt{2}}{2}K_6^{1,3},$$
$$p=4:\quad K_9^{2,4}=\frac{\sqrt{2}}{2}K_5^{1,2}+\frac{\sqrt{2}}{2}K_6^{1,2}+K_7^{1,3}-K_8^{1,4},$$
$$K_{10}^{3,4}=-\frac{3\sqrt{2}}{2}K_5^{1,2}-\frac{\sqrt{2}}{2}K_6^{1,2}-K_7^{1,3}+K_8^{1,4},$$
$$p=5:\quad K_{10}^{2,4}=K_7^{1,2}-K_9^{1,4}.$$

When $p=6$ or 7, the spaces $C^{7,1}(\mathfrak{G})$ and $C^{8,1}(\mathfrak{G})$ are equal to 0, and hence there are no normal conditions.

From Table 1, (5.4) and normal conditions (5.9), (5.11), (5.12), we construct the structure equation of normal Cartan connection of type $\mathfrak{G}$ with respect to the basis $(e_1,\dots,e_{10})$:

(5.13)
$$de_1=-\sqrt{2}e_1\wedge e_5-e_2\wedge e_4,$$
$$de_2=-e_1\wedge e_7-\frac{\sqrt{2}}{2}e_2\wedge e_5-\frac{\sqrt{2}}{2}e_2\wedge e_6-e_3\wedge e_4,$$

$$de_3 = -e_2 \wedge e_7 - \sqrt{2}e_3 \wedge e_6 + K_3^{1,2} e_1 \wedge e_2 + K_3^{1,4} e_1 \wedge e_4,$$

$$de_4 = e_1 \wedge e_9 + e_2 \wedge e_8 - \frac{\sqrt{2}}{2} e_4 \wedge e_5 + \frac{\sqrt{2}}{2} e_4 \wedge e_6,$$

$$de_5 = \sqrt{2}e_1 \wedge e_{10} + \frac{\sqrt{2}}{2} e_2 \wedge e_9 + \frac{\sqrt{2}}{2} e_4 \wedge e_7,$$

$$de_6 = \frac{\sqrt{2}}{2} e_2 \wedge e_9 + \sqrt{2}e_3 \wedge e_8 - \frac{\sqrt{2}}{2} e_4 \wedge e_7 + K_6^{1,2} e_1 \wedge e_2 - \sqrt{2}K_3^{1,2} e_1 \wedge e_4,$$

$$de_7 = e_2 \wedge e_{10} + e_3 \wedge e_9 - \frac{\sqrt{2}}{2} e_5 \wedge e_7 + \frac{\sqrt{2}}{2} e_6 \wedge e_7 + K_7^{1,2} e_1 \wedge e_2 + K_7^{1,4} e_1 \wedge e_4,$$

$$de_8 = -e_4 \wedge e_9 - \sqrt{2}e_6 \wedge e_8 + K_8^{1,2} e_1 \wedge e_2 + K_8^{1,3} e_1 \wedge e_3 + \frac{\sqrt{2}}{2} K_6^{1,2} e_1 \wedge e_4 + K_8^{2,3} e_2 \wedge e_3,$$

$$de_9 = -e_4 \wedge e_{10} - \frac{\sqrt{2}}{2} e_5 \wedge e_9 - \frac{\sqrt{2}}{2} e_6 \wedge e_9 - e_7 \wedge e_8 + K_9^{1,2} e_1 \wedge e_2 + K_9^{1,3} e_1 \wedge e_3 + K_7^{1,2} e_1 \wedge e_4 + K_8^{1,3} e_2 \wedge e_3,$$

$$de_{10} = -\sqrt{2}e_5 \wedge e_{10} - e_7 \wedge e_9 + K_{10}^{1,2} e_1 \wedge e_2 + K_{10}^{1,3} e_1 \wedge e_3 + K_{10}^{1,4} e_1 \wedge e_4 + \frac{1}{2} K_9^{1,3} e_2 \wedge e_3.$$

Now we change the basis $(e_1, \dots, e_{10})$ for $(\omega_1, \dots, \omega_{10})$ by the following relations:

$$e_1 = \omega_1, \quad e_2 = \frac{\sqrt{2}}{2}\omega_2, \quad e_3 = \frac{1}{2}\omega_3, \quad e_4 = \sqrt{2}\omega_4,$$

$$(5.14) \qquad e_5 = \frac{\sqrt{2}}{2}\omega_5, \quad e_6 = -\frac{\sqrt{2}}{2}\omega_5 + \sqrt{2}\omega_7, \quad e_7 = \frac{\sqrt{2}}{2}\omega_6,$$

$$e_8 = -2\omega_9, \quad e_9 = -\sqrt{2}\omega_8, \quad e_{10} = -\omega_{10}.$$

Thus we obtain the structure equation (4.6) in Theorem 4.1. The invariant functions A and $\mathbf{b}$ in Theorem 4.1 are

$$(5.15) \qquad A = 2\sqrt{2}K_3^{1,4}, \quad \mathbf{b} = -\frac{\sqrt{2}}{8} K_8^{2,3}.$$

There are many invariant functions $K_k^{i,j}$; nevertheless, we will show that the essential ones are only $K_3^{1,4}$ and $K_8^{2,3}$. From the structure equation (5.13), we have

$$(5.16) \qquad \begin{aligned} 0 &= d^2 e_3 \\ &= -de_2 \wedge e_7 + e_2 \wedge de_7 - \sqrt{2}de_3 \wedge e_6 + \sqrt{2}e_3 \wedge de_6 \\ &\quad + dK_3^{1,2} \wedge e_1 \wedge e_2 + K_3^{1,2} de_1 \wedge e_2 - K_3^{1,2} e_1 \wedge de_2 \\ &\quad + dK_3^{1,4} \wedge e_1 \wedge e_4 + K_3^{1,4} de_1 \wedge e_4 - K_3^{1,4} e_1 \wedge de_4. \end{aligned}$$

By substituting (5.13) for (5.16), we obtain the following equation:

(5.17) $$\begin{aligned}0 &= d^2e_3\\ &= (dK_3^{1,2} + \sqrt{2}K_6^{1,2}e_3 + \frac{3\sqrt{2}}{2}K_3^{1,2}e_5 - \frac{\sqrt{2}}{2}K_3^{1,2}e_6 - K_3^{1,4}e_8) \wedge e_1 \wedge e_2\\ &\quad + (dK_3^{1,4} - 3K_3^{1,2}e_3 + \frac{3\sqrt{2}}{2}K_3^{1,4}e_5 - \frac{3\sqrt{2}}{2}K_3^{1,4}e_6) \wedge e_1 \wedge e_4\\ &\quad - K_7^{1,4}e_1 \wedge e_2 \wedge e_4.\end{aligned}$$

Thus we obtain conditions for $K_3^{1,2}$ and $K_3^{1,4}$ by looking at coefficients of $e_1 \wedge e_2$ and $e_1 \wedge e_4$.

(5.18) $$\begin{aligned}dK_3^{1,2} &\equiv -\sqrt{2}K_6^{1,2}e_3 - \frac{3\sqrt{2}}{2}K_3^{1,2}e_5 + \frac{\sqrt{2}}{2}K_3^{1,2}e_6 + K_3^{1,4}e_8\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_4),\\ dK_3^{1,4} &\equiv 3K_3^{1,2}e_3 - \frac{3\sqrt{2}}{2}K_3^{1,4}e_5 + \frac{3\sqrt{2}}{2}K_3^{1,4}e_6\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_4).\end{aligned}$$

Using the structure equation (5.13) and (5.18) similarly, we calculate d^2e_6, then we obtain the following condition for $dK_6^{1,2}$.

(5.19) $$\begin{aligned}dK_6^{1,2} &\equiv (\sqrt{2}K_8^{1,2} + \frac{\sqrt{2}}{2}K_9^{1,3})e_3 - \frac{3\sqrt{2}}{2}K_6^{1,2}e_5 - \frac{\sqrt{2}}{2}K_6^{1,2}e_6 - 2\sqrt{2}K_3^{1,2}e_8\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_4).\end{aligned}$$

We continue calculating d^2e_k for $k = 7, \ldots 10$, then we obtain the following conditions:

(5.20) $$\begin{aligned}dK_7^{1,2} &\equiv (K_9^{1,2} - K_{10}^{1,3})e_3 - 2\sqrt{2}K_7^{1,2}e_5 - \frac{\sqrt{2}}{2}K_6^{1,2}e_7 + K_7^{1,4}e_8 - K_3^{1,2}e_9\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_4),\\ dK_7^{1,4} &\equiv 2K_7^{1,2}e_3 - 2\sqrt{2}K_7^{1,4}e_5 + \sqrt{2}K_7^{1,4}e_6 + K_3^{1,2}e_7 - K_3^{1,4}e_9\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_4),\\ dK_8^{1,2} &\equiv -\frac{3\sqrt{2}}{2}K_8^{1,2}e_5 - \frac{3\sqrt{2}}{2}K_8^{1,2}e_6 - K_8^{1,3}e_7 + \frac{3\sqrt{2}}{2}K_8^{1,2}e_8\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_3, e_4),\\ dK_8^{1,3} &\equiv -\frac{3}{2}K_9^{1,3}e_4 - \sqrt{2}K_8^{1,3}e_5 - 2\sqrt{2}K_8^{1,3}e_6 - K_8^{2,3}e_7\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_3),\\ dK_8^{2,3} &\equiv -2K_8^{1,3}e_4 - \frac{\sqrt{2}}{2}K_8^{2,3}e_5 - \frac{5\sqrt{2}}{2}K_8^{2,3}e_6\\ &\qquad (\mathrm{mod}\ e_1, e_2, e_3),\end{aligned}$$

$$dK_9^{1,2} \equiv -2\sqrt{2}K_9^{1,2}e_5 - \sqrt{2}K_9^{1,2}e_6 - (K_8^{1,2} + K_9^{1,3})e_7 + 2K_7^{1,2}e_8 + \frac{\sqrt{2}}{2}K_6^{1,2}e_9$$
$$(\mathrm{mod}\ e_1, e_2, e_3, e_4),$$
$$dK_9^{1,3} \equiv -2K_{10}^{1,3}e_4 - \frac{3\sqrt{2}}{2}K_9^{1,3}e_5 - \frac{3\sqrt{2}}{2}K_9^{1,3}e_6 - 2K_8^{1,3}e_7$$
$$(\mathrm{mod}\ e_1, e_2, e_3),$$
$$dK_{10}^{1,2} \equiv -\frac{5\sqrt{2}}{2}K_{10}^{1,2}e_5 - \frac{\sqrt{2}}{2}K_{10}^{1,2}e_6 - (K_9^{1,2} + K_{10}^{1,3})e_7 + K_{10}^{1,4}e_8 + K_7^{1,2}e_9$$
$$(\mathrm{mod}\ e_1, e_2, e_3, e_4),$$
$$dK_{10}^{1,3} \equiv -2\sqrt{2}K_{10}^{1,3}e_5 - \sqrt{2}K_{10}^{1,3}e_6 - \frac{3}{2}K_9^{1,3}e_7$$
$$(\mathrm{mod}\ e_1, e_2, e_3, e_4),$$
$$dK_{10}^{1,4} \equiv -\frac{5\sqrt{2}}{2}K_{10}^{1,4}e_5 + \frac{\sqrt{2}}{2}K_{10}^{1,4}e_6 - K_7^{1,2}e_7 + K_7^{1,4}e_9$$
$$(\mathrm{mod}\ e_1, e_2, e_3, e_4).$$

From conditions (5.18), (5.19) and (5.20), we see that $K_3^{1,4}$ and $K_8^{2,3}$ derive the other functions $K_k^{i,j}$. If $K_3^{1,4} = 0$, then

$$K_4^{1,2} = 0, \quad K_6^{1,2} = 0, \quad K_7^{1,2} = 0, \quad K_7^{1,4} = 0, \quad K_8^{1,2} = -2\sqrt{2}\mathbf{a},$$
$$K_8^{1,3} = 4\mathbf{c}, \quad K_8^{2,3} = -4\sqrt{2}\mathbf{b}, \quad K_9^{1,2} = -2\mathbf{e}, \quad K_9^{1,3} = -4\sqrt{2}\mathbf{a},$$
$$K_{10}^{1,2} = -\sqrt{2}\mathbf{f}, \quad K_{10}^{1,3} = -2\mathbf{e}, \quad K_{10}^{1,4} = 0, \tag{5.21}$$

and hence the structure equation (4.6) in Theorem 4.1 coincide (3.10). Furthermore if $K_8^{2,3} = 0$, then all of $K_k^{i,j}$ are equal to 0. Thus we have proven Corollary 4.2.

Now we recall that the orthogonal decomposition (2.7) of the space $C^{p,2}(\mathfrak{G})$. The harmonic part $H(K)$ of the curvature K with respect to the decomposition (2.7) gives a fundamental system of invariants of connection (P, ω), that is to say, K vanishes if and only if $H(K)$ vanishes ([**9**, Theorem 2.9]). From [**10**, Proposition 5.5 (III) (5)] we see that there are two non-vanishing parts of $H^{p,2}(\mathfrak{G})$, and we compare with the essential invariants A and $\mathbf{b}$.

$$(C_2, \{\alpha_1, \alpha_2\})\text{-type} : A \in \mathfrak{g}_{-1} \otimes \textstyle\bigwedge^2_{-4}(\mathfrak{m}^*) \subset K^2,$$
$$\mathbf{b} \in \mathfrak{g}_1 \otimes \textstyle\bigwedge^2_{-3}(\mathfrak{m}^*) \subset K^3. \tag{5.22}$$

6. Legendre connections on Legendre Grassmann bundles

In this section we define Legendre connections on Legendre Grassmann bundles on three-dimensional contact manifolds, and describe a relation between Legendre connections and third order ordinary differential equations.

Let M be a three-dimensional contact manifold with contact form ω_1' given by

$$\omega_1' = dy - p\,dx \tag{6.1}$$

where (x, y, p) is a local coordinate of M. For $m \in M$, let u_m be a one-dimensional subspace of T_mM generated by a vector $X \in T_mM$ such that $(\omega_1')_m(X) = 0$. We call u_m a Legendre subspace at m. By $L_m(M)$ we denote the space of all Legendre subspaces on m, and let $L(M) = \bigcup_{m \in M} L_m(M)$. The vector X generating u_m is expressed by $\alpha(\partial/\partial x + p\partial/\partial y) + \beta\partial/\partial p$ for some α and β. Thus u_m is a one-dimensional subspace of two-dimensional space, and hence $L_m(M) \cong \mathrm{Gr}(2, 1)$. Consequently $L(M)$ is a principal S^1-bundle on M, and called the Legendre Grassmann bundle of M. By $\pi : L(M) \to M$ we denote the bundle projection.

We take a local coordinate (x, y, p, q) of $L(M)$ such that $\pi(x, y, p, q) = (x, y, p)$, and let ω_1 and ω_2 be one-forms on $L(M)$ given by

$$\omega_1 = dy - p\,dx, \quad \omega_2 = dp - q\,dx. \tag{6.2}$$

Let $D = D(L(M) : \omega_1, \omega_2) = \bigcup_{u \in L(M)} D_u$ be a global two-dimensional tautological contact distribution on $L(M)$ defined by

$$\begin{aligned} D_u &= \{X \in T_u(L(M)) \,|\, \pi_*(X) \in u\} \\ &= \{X \in T_u(L(M)) \,|\, (\omega_1)_u(X) = (\omega_2)_u(X) = 0\}. \end{aligned} \tag{6.3}$$

Then we see that D_u is a two-dimensional vector subspace of $T_u(L(M))$ generated by $(\partial/\partial x)_u + p(\partial/\partial y)_u + q(\partial/\partial p)_u$ and $(\partial/\partial q)_u$. We define the derived system ∂D of D by

$$\partial D = D + [D, D]. \tag{6.4}$$

Since $[q\partial/\partial p, \partial/\partial q] = \partial/\partial p$, the vector subspace ∂D_u of $T_u(L(M))$ is a three-dimensional space generated by $(\partial/\partial x)_u + p(\partial/\partial y)_u$, $(\partial/\partial p)_u$ and $(\partial/\partial q)_u$. Thus,

$$\partial D_u = \{X \in T_u(L(M)) \,|\, (\omega_1)_u(X) = 0\}. \tag{6.5}$$

Let $\mathrm{Ch}(\partial D)$ be the Cauchy characteristics of $(\partial D, \omega_1)$, that is

$$\mathrm{Ch}(\partial D) = \{X \in \partial D \,|\, X \rfloor d\omega_1 \equiv 0 \bmod \omega_1\}. \tag{6.6}$$

Then we see that $\mathrm{Ch}(\partial D_u)$ is a one-dimensional vector subspace of ∂D_u generated by $(\partial/\partial q)_u$. As a result, $\mathrm{Ch}(\partial D_u)$ is a vertical subspace of D_u with respect to the bundle projection π. Now we define Legendre connections of $L(M)$ as follows:

DEFINITION 6.1. Suppose that we give a decomposition

$$D_u = \mathrm{Ch}(\partial D_u) + E_u, \tag{6.7}$$

where E_u is a one-dimensional subspace of D_u smoothly depending on $u \in L(M)$. Then we say that we give a *Legendre connection* E on $L(M)$.

Assume that a third order ordinary differential equation $d^3y/dx^3 = F(x, y, p, q)$ is given on $L(M)$. We let ω_3 be a one-form on $L(M)$ such that

$$\omega_3 = dq - F\,dx, \tag{6.8}$$

and give a global one-dimensional distribution E on $L(M)$ by

(6.9) $$E_u = \{X \in T_u(L(M)) \,|\, (\omega_1)_u(X) = (\omega_2)_u(X) = (\omega_3)_u(X) = 0\}.$$

Then we see that E_u is a one-dimensional vector space generated by $(\partial/\partial x)_u + p(\partial/\partial y)_u + q(\partial/\partial p)_u + F(\partial/\partial q)_u$, and that $\mathrm{Ch}(\partial D) \cap E = \{0\}$. Thus we obtain a Legendre connection E. Conversely assume that a Legendre connection E is given on $L(M)$. If a vector $\alpha_u((\partial/\partial x)_u + p(\partial/\partial y)_u + q(\partial/\partial p)_u) + \beta_u(\partial/\partial q)_u \in D_u$ is an element of E_u, then there exists an F_u such that $\beta_u = F_u\alpha_u$. Thus we obtain a function F on $L(M)$. Consequently giving a third order ordinary differential equation on $L(M)$ is equivalent to giving a Legendre connection on $L(M)$. As a result we obtain the following Corollary from Theorem 4.1:

COROLLARY 6.1. *Let M be a three-dimensional contact manifold. By $\mathfrak{G}$ we denote the graded Lie algebra* $(\mathfrak{sp}(2,\boldsymbol{R}), \{\mathfrak{g}_i\}_{i=-3}^{3})$ *given in* (4.4). *Suppose that a Legendre connection is given on the Legendre Grassmann bundle $L(M)$ of M. Then there exists a unique normal Cartan connection of type $\mathfrak{G}$ on $L(M)$. The structure equation is given in* (4.6).

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H. SATO
Graduate school of Mathematics
Nagoya University,
Chikusa-ku, Nagoya 464-8602,
Japan
E-mail address: hsato@math.nagoya-u.ac.jp

A. Y. YOSHIKAWA
Graduate school of Mathematics,
Nagoya University,
Chikusa-ku, Nagoya 464-8602,
Japan
E-mail address: ayamada@math.nagoya-u.ac.jp