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Jørgensen's inequality for classical Schottky groups of real type

Dedicated to Professor Mitsuru Nakai on his sixtieth birthday

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Abstract. In this paper we consider Jørgensen's inequalities for classical Schottky groups of the real type. The infimum of Jørgensen's numbers for groups of types II, V and VII are 16, $4(1+\sqrt{2})^2$ and $4(1+\sqrt{2})^2$, respectively, each of which is the best possible for Jørgensen's inequality.

0. Introduction.

Jørgensen's inequality gives a necessary condition for a non-elementary Möbius transformation group $G = \langle A_1, A_2 \rangle$ to be discrete (Jørgensen [**2**]): if $G = \langle A_1, A_2 \rangle$ is a non-elementary discrete group, then

$$J(G) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2| \geq 1,$$

where tr is the trace. The lower bound is the best possible. We call $J(G)$ *Jørgensen's number* for $G = \langle A_1, A_2 \rangle$.

With respect to Jørgensen's numbers it gives rise to the following problems: (1) Problem 1 is to find many non-Fuchsian extreme groups, where a two-generator group G is called *extreme* if $J(G) = 1$. (2) Problem 2 is to find the minimum values of Jørgensen's numbers for some subspaces of the Kleinian space of rank two, for example the Teichmuller space, the Schottky space, where the Kleinian space of rank two is the space of all two-generator Kleinian groups. For Problem 1, Jørgensen-Lascurain-Pignataro [**3**] and Sato-Yamada [**8**] gave uncountably many non-conjugate, non-Fuchsian extreme groups. For Problem 2, Gilman [**1**] and Sato [**7**] gave the best lower bound of Jørgensen's numbers for purely hyperbolic two-generator groups.

In this paper we will consider Jørgensen's numbers for classical Schottky groups of real type of genus two (see §1 for the definition). In Sato [**5**] we classified the groups into eight types. The groups of the first and the fourth types are called *Fuchsian Schottky groups*. In Gilman [**1**] and Sato [**7**] they gave Jørgensen's inequalities and the best lower bounds for Fuchsian Schottky groups:

(1) If $G = \langle A_1, A_2 \rangle$ is of the first type, that is, a Fuchsian Schottky group such that the axes of A_1 and A_2 are disjoint, then $J(G) > 16$. The lower bound is the best possible.

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(2) If $G = \langle A_1, A_2 \rangle$ is of the fourth type, that is, a Fuchsian Schottky group such that the axes of A_1 and A_2 intersect, then $J(G) > 4$. The lower bound is the best possible.

In this paper we will consider Jørgensen's numbers for three kinds of classical Schottky groups of real type, that is, the second, the fifth and the seventh types (see §1 for the definitions). The main results in this paper are as follows, which will be stated in §4.

(1) If $G = \langle A_1, A_2 \rangle$ is of the second type, then $J(G) > 16$.
(2) If $G = \langle A_1, A_2 \rangle$ is of the fifth type, then $J(G) > 4(1+\sqrt{2})^2$.
(3) If $G = \langle A_1, A_2 \rangle$ is of the seventh type, then $J(G) > 4(1+\sqrt{2})^2$.

All of the lower bounds in the above inequalities are the best possible.

Jørgensen's numbers for the rest types, that is, for classical Schottky groups of the third, the sixth and the eighth types will appear in the coming paper [**11**]. The results are as follows:

(1) If $G = \langle A_1, A_2 \rangle$ is of the third type, then $J(G) > 4$.
(2) If $G = \langle A_1, A_2 \rangle$ is of the sixth type, then $J(G) > 16$.
(3) If $G = \langle A_1, A_2 \rangle$ is of the eighth type, then $J(G) > 16$.

All of the lower bounds are the best possible.

In [**9**] we announced the main results in this paper and [**11**].

In §1 we will state notation and terminology, and state eight kinds of classical Schottky groups of real type of genus two. In §2 we will consider automorphisms of a free group on two generators, that is, Nielsen transformations and their properties. In §3 we will state fundamental regions for the Schottky modular groups of the second and the seventh types which are given in Sato [**6**]. In §4 we will state the main results in this paper. In §5 we will list properties of Jørgensen's numbers in a series of lemmas, which play important roles in the proofs of the main theorems. In §6 we will give a proof of the theorem on Jørgensen's numbers for classical Schottky groups of the second type. In §7 through §10 we will prove the main theorems on Jørgensen's numbers for the groups of the fifth and the seventh types. In §11 we will give some examples which guarantee that all of the lower bounds in the inequalities in the main theorems are the best possible.

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1. Notation and terminology.

Let $C_1, C_{g+1}; \dots; C_g, C_{2g}$ be a set of $2g$, $g \geq 1$, mutually disjoint Jordan curves on the Riemann sphere which comprise the boundary of a $2g$-ply connected region ω. Suppose there are g Möbius transformations $A_1, \dots, A_g$ which have the property that A_j maps C_j onto C_{g+j} and $A_j(\omega) \cap \omega = \emptyset$, $1 \leq j \leq g$. Then the g necessarily loxodromic transformations A_g generate a *marked Schottky group* $G = \langle A_1, \dots, A_g \rangle$ of genus g with ω as a fundamental region. In particular, if all C_j $(j = 1, 2, \dots, 2g)$ are circles, then we call $A_1, \dots, A_g$ a *set of classical generators of* G. A *classical Schottky group* is a Schottky group for which there exists some set of classical generator.

We denote by Möb the group of all Möbius transformation. We say two marked subgroups $G = \langle A_1, \dots, A_g \rangle$ and $\hat{G} = \langle \hat{A}_1, \dots, \hat{A}_g \rangle$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j = TA_jT^{-1}$ for $j = 1, 2, \dots, g$. The *Schottky space* (resp. *the classical Schottky space*) of genus g, denoted by $\boldsymbol{S}_g$ (resp. $\boldsymbol{S}_g^0$), is the set of all equivalence classes of marked Schottky groups (resp. marked classical Schottky groups) of genus $g \geq 1$.

We denote by $\boldsymbol{M}_2$ the set of all equivalence classes $[\langle A_1, A_2 \rangle]$ of marked groups $\langle A_1, A_2 \rangle$ generated by loxodromic transformations A_1 and A_2 whose fixed points are all distinct. Let $[\langle A_1, A_2 \rangle] \in \boldsymbol{M}_2$. For $j = 1, 2$, let λ_j ($|\lambda_j| > 1$), p_j and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j = 1/\lambda_j$. Thus $t_j \in D^* = \{z \mid 0 < |z| < 1\}$. We determine a Möbius transformation T by $T(p_1) = 0$, $T(p_3) = \infty$ and $T(p_2) = 1$, and define ρ by $\rho = T(p_4)$. Thus $\rho \in \boldsymbol{C} - \{0, 1\}$. We can define a mapping α of the space $\boldsymbol{M}_2$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by setting $\alpha([\langle A_1, A_2 \rangle]) = (t_1, t_2, \rho)$. Then we say $[\langle A_1, A_2 \rangle]$ represents (t_1, t_2, ρ) and (t_1, t_2, ρ) corresponds to $[\langle A_1, A_2 \rangle]$ or $\langle A_1, A_2 \rangle$. We write $t_1 = t_1(G)$, $t_2 = t_2(G)$ and $\rho = \rho(G)$. Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau = (t_1, t_2, \rho) \in (D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ under the normalization condition $p_1 = 0$, $p_3 = \infty$ and $p_2 = 1$; we define λ_j ($j = 1, 2$) and p_4 by setting $\lambda_j = 1/t_j$ and $p_4 = \rho$, respectively. We determine $A_1(z), A_2(z) \in$ Möb from τ as follows: the multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j, p_j and p_{2+j}, respectively. Thus we obtain a mapping β of $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ into $\boldsymbol{M}_2$ by setting $\beta(\tau) = [\langle A_1(z), A_2(z) \rangle]$. Then we note that $\beta\alpha = \alpha\beta = id$. Therefore we identify $\boldsymbol{M}_2$ with $\alpha(\boldsymbol{M}_2)$. Similarly we can define the mapping α^* of $\boldsymbol{S}_2$ or $\boldsymbol{S}_2^0$ into $(D^*)^2 \times (\boldsymbol{C} - \{0, 1\})$ by restricting α to this space, and identify $\boldsymbol{S}_2$ (resp. $\boldsymbol{S}_2^0$) with $\alpha^*(\boldsymbol{S}_2)$ (resp. $\alpha^*(\boldsymbol{S}_2^0)$). From now on we denote $\alpha(\boldsymbol{M}_2)$, $\alpha^*(\boldsymbol{S}_2)$ and $\alpha^*(\boldsymbol{S}_2^0)$ by $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_2^0$, respectively.

We call $G = \langle A_1, A_2 \rangle$ a marked group of real type if $(t_1, t_2, \rho) \in \boldsymbol{R} \cap \boldsymbol{M}_2$, that is, t_1, t_2 and ρ are all real numbers, where (t_1, t_2, ρ) corresponds to $G = \langle A_1, A_2 \rangle$. Then there are eight kinds of marked groups of real type as follows.

DEFINITION 1.1 (cf. [**5**]).
(1) G is of the first type (Type I) if $t_1 > 0$, $t_2 > 0$, $\rho > 0$.
(2) G is of the second type (Type II) if $t_1 > 0$, $t_2 < 0$, $\rho > 0$.
(3) G is of the third type (Type III) if $t_1 > 0$, $t_2 < 0$, $\rho < 0$.
(4) G is of the fourth type (Type IV) if $t_1 > 0$, $t_2 > 0$, $\rho < 0$.
(5) G is of the fifth type (Type V) if $t_1 < 0$, $t_2 > 0$, $\rho > 0$.
(6) G is of the sixth type (Type VI) if $t_1 < 0$, $t_2 < 0$, $\rho > 0$.
(7) G is of the seventh type (Type VII) if $t_1 < 0$, $t_2 < 0$, $\rho < 0$.
(8) G is of the eighth type (type VIII) if $t_1 < 0$, $t_2 > 0$, $\rho < 0$.

For each $k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII}$, we call the set of all equivalence classes of marked groups (resp. marked Schottky groups and marked classical Schottky groups) of Type k the real space (resp. the real Schottky space and the real classical Schottky space) of Type k, and denote them by $R_k\boldsymbol{M}_2$ (resp. $R_k\boldsymbol{S}_2$ and $R_k\boldsymbol{S}_2^0$).

2. The Nielsen transformations.

In this section we consider automorphisms of a free group on two generators, that is, Nielsen transformations, and their properties.

THEOREM A (Neumann [4]). *The group Φ_2 of automorphisms of $G = \langle A_1, A_2 \rangle$ has the following presentation*:

$$\Phi_2 = \langle N_1, N_2, N_3 \mid (N_2N_1N_2N_3)^2 = 1,$$
$$N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1 = 1, N_1N_3N_1N_3 = N_3N_1N_3N_1 \rangle,$$

where $N_1 : (A_1, A_2) \mapsto (A_1, A_2^{-1})$, $N_2 : (A_1, A_2) \mapsto (A_2, A_1)$ *and* $N_3 : (A_1, A_2) \mapsto (A_1, A_1A_2)$.

We call the mappings N_1, N_2 and N_3 *Nielsen transformations.*

Let (t_1, t_2, ρ) be the point in $\boldsymbol{S}_2$ corresponding to a marked Schottky group $G = \langle A_1, A_2 \rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of (t_1, t_2, ρ) under the Nielsen transformations N_j $(j = 1, 2, 3)$. Then by straightforward calculations, we have the following.

LEMMA 2.1 (Sato [5, Lemma 2.1]). (i) $t_1(1) = t_1$, $t_2(1) = t_2$ *and* $\rho(1) = 1/\rho$. (ii) $t_1(2) = t_2$, $t_2(2) = t_1$ *and* $\rho(2) = \rho$. (iii) $t_1(3) = t_1$, $t_2(3) + (1/t_2(3)) = Q^2/t_1t_2(\rho - 1)^2 - 2$ *and* $\rho(3) + (1/\rho(3)) = P^2/t_1\rho(1 - t_2)^2 - 2$, *where* $P = \rho - t_2 - \rho t_1t_2 + t_1$ *and* $Q = \rho - t_2 + \rho t_1t_2 - t_1$.

We note that the spaces $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_2^0$ are all, Φ_2-invariant, that is $\phi(\boldsymbol{M}_2) = \boldsymbol{M}_2$, $\phi(\boldsymbol{S}_2) = \boldsymbol{S}_2$ and $\phi(\boldsymbol{S}_2^0) = \boldsymbol{S}_2^0$ for $\phi \in \Phi_2$. The following Propositions 2.1 and 2.2 are easily seen from Lemma 2.1 and the definitions of $R_k\boldsymbol{M}_2$, $R_k\boldsymbol{S}_2$ and $R_k\boldsymbol{S}_2^0$ $(k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII})$.

PROPOSITION 2.1. *Let X denote the spaces $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ or $\boldsymbol{S}_2^0$. Then $N_1(R_kX) = R_kX$ for $k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII}$.*

PROPOSITION 2.2. *Let X denote the spaces $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ or $\boldsymbol{S}_2^0$. Then*
(i) *$N_2(R_kX) = R_kX$ for $k = \mathrm{I}, \mathrm{IV}, \mathrm{VI}, \mathrm{VII}$.*
(ii) *$N_2(R_{\mathrm{II}}X) = R_{\mathrm{V}}X$ and $N_2(R_{\mathrm{V}}X) = R_{\mathrm{II}}X$.*
(iii) *$N_2(R_{\mathrm{III}}X) = R_{\mathrm{VIII}}X$ and $N_2(R_{\mathrm{VIII}}X) = R_{\mathrm{III}}X$.*

PROPOSITION 2.3. *Let X denote the spaces $\boldsymbol{M}_2$, $\boldsymbol{S}_2$ or $\boldsymbol{S}_2^0$. Then*
(i) *$N_3(R_kX) = R_kX$ for $k = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$.*
(ii) *$N_3(R_{\mathrm{V}}X) = R_{\mathrm{VII}}X$ and $N_3(R_{\mathrm{VII}}X) = R_{\mathrm{V}}X$.*
(iii) *$N_3(R_{\mathrm{VI}}X) = R_{\mathrm{VIII}}X$ and $N_3(R_{\mathrm{VIII}}X) = R_{\mathrm{VI}}X$.*

PROOF. For $k = \mathrm{I}, \mathrm{IV}$, see Sato [5] and for $k = \mathrm{II}$, V, VII, see Sato [6, p. 453]. Here we will only show this proposition for $k = \mathrm{III}$, VI, VIII. By Lemma 2.1 we easily see that $N_3(R_{\mathrm{III}}X) = R_{\mathrm{III}}X$. Let $\tau = (t_1, t_2, \rho) \in R_{\mathrm{VI}}X$. We set $N_3(\tau) = (t_1^*, t_2^*, \rho^*)$. Since $t_1^* = t_1$, we have $t_1^* < 0$. Let p and q be two solutions of the equation

$$t_1(1 - t_2)z^2 - (\rho - t_2 - \rho t_1t_2 + t_1)z + \rho(1 - t_2) = 0.$$

Then $pq = \rho(1 - t_2)/t_1(1 - t_2) = \rho/t_1 < 0$. Hence $\rho^* = q/p < 0$. Since

$$t_2^* + 1/t_2^* + 2 = (\rho - t_2 + t_1t_2\rho - t_1)^2/t_1t_2(\rho - 1) > 0,$$

we have $t_2^* > 0$. Hence $(t_1^*, t_2^*, \rho^*) \in R_{\mathrm{VIII}}X$. By the same way we have $N_3(R_{\mathrm{VIII}}X) = R_{\mathrm{VI}}X$. q.e.d.

3. Fundamental regions.

Let Φ_2 be the group of automorphisms of $G = \langle A_1, A_2 \rangle$ introduced in §2. Let $\phi_1, \phi_2 \in \Phi_2$. We say that ϕ_1 and ϕ_2 are *equivalent* if $\phi_1(G)$ is equivalent to $\phi_2(G)$, that is, if there exists $T \in$ Möb such that $\phi_2(G) = T\phi_1(G)T^{-1}$, and we denote it by $\phi_1 \sim \phi_2$. We denote by $[\phi]$ the equivalence class of $\phi \in \Phi_2$. The *Schottky modular group* of genus two, which is denoted by $\mathrm{Mod}(\boldsymbol{S}_2)$, is the set of all equivalence classes of orientation preservimg automorphisms of $\boldsymbol{S}_2$. We denote by $\mathrm{Mod}(R_k\boldsymbol{S}_2^0)$ the restriction of $\mathrm{Mod}(\boldsymbol{S}_2)$ to $R_k\boldsymbol{S}_2^0$ for $k = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VIII}$. We denote by $F_k(\mathrm{Mod}(\boldsymbol{S}_2^0))$ fundamental regions in $R_k\boldsymbol{S}_2^0$ for $\mathrm{Mod}(R_k\boldsymbol{S}_2^0)$.

Proposition 3.1 (Sato [6]).

$$F_{\mathrm{II}}(\mathrm{Mod}(\boldsymbol{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{II}}\boldsymbol{S}_2^0 \mid (1 + (t_1)^{1/2}t_2)/((t_1)^{1/2} + t_2) < \rho < ((1 - (t_1)^{1/2}t_2)/((t_1)^{1/2} - t_2))^2, -1 < t_2 < 0, 0 < t_1 < 1\}.$$

Proposition 3.2 (Sato [6]).

$$F_{\mathrm{VII}}(\mathrm{Mod}(\boldsymbol{S}_2^0)) = \{(t_1, t_2, \rho) \in R_{\mathrm{VII}}\boldsymbol{S}_2^0 \mid ((-t_1)^{1/2} + (-t_2)^{1/2})/(1 - (-t_1)^{1/2}(-t_2)^{1/2}) < (-\rho)^{1/2} < (1 - (-t_1)^{1/2}(-t_2)^{1/2})/((-t_1)^{1/2} + (-t_2)^{1/2}), t_2 < t_1, -1 < t_1 < 0\}.$$

Proposition 3.3 (Sato [6]). *The group* $\mathrm{Mod}(R_{\mathrm{VII}}\boldsymbol{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_1N_2]$.

4. Main theorems.

In this section the main theorems in this paper will be stated. The proofs of the theorems will be given in §6 through §10. Let G be a marked two-generator group generated by Möbius transformations A_1 and $A_2 : G = \langle A_1, A_2 \rangle$. The number

$$J(G) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2|$$

is called *Jørgensen's number* of $G = \langle A_1, A_2 \rangle$, where tr is the trace.

Theorem 1. *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{II}}\boldsymbol{S}_2^0$, *then* $J(G) > 16$. *The lower bound is the best possible.*

Theorem 2. *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{V}}\boldsymbol{S}_2^0$, *then* $J(G) > 4(1 + \sqrt{2})^2$. *The lower bound is the best possible.*

Theorem 3. *If* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0$, *then* $J(G) > 4(1 + \sqrt{2})^2$. *The lower bound is the best possible.*

5. Jørgensen's inequality.

In this section we will give some lemmas which are necessary to prove the main theorems stated in the previous section. We introduce the following two regions:

$$M_{\mathrm{II}} := \{\tau = (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = (t_1^{1/2}\rho^{1/2} - 1)/(\rho^{1/2} - t_1^{1/2}) < t_2 < 0, 1 < \rho < 1/t_1, 0 < t_1 < 1\}.$$

$$M_{\mathrm{VII}} := \{\tau = (t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid ((-t_1)^{1/2} + (-t_2)^{1/2})/\{(1 - (-t_1)^{1/2}(-t_2)^{1/2})\} < (-\rho)^{1/2}$$
$$< (1 - (-t_1)^{1/2}(-t_2)^{1/2})/((-t_1)^{1/2} + (-t_2)^{1/2}), -1 < t_2 < 0, -1 < t_1 < 0\}.$$

We easily see the following lemma.

LEMMA 5.1. *For each* $k = \mathrm{II}, \mathrm{VII}$

$$F_k(\mathrm{Mod}(\boldsymbol{S}_2^0)) \subseteq M_k \subseteq R_k\boldsymbol{S}_2^0.$$

THEOREM B (Jørgensen [**2**]). *Suppose that Möbius transformations* A_1 *and* A_2 *generate a non-elementary discrete group* G. *Then*

$$J(G) := |\mathrm{tr}^2(A_1) - 4| + |\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2| \geq 1.$$

The lower bound is the best possible.

Let $\tau = (t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2\rangle$. Then since $|\mathrm{tr}^2(A_1) - 4| = |1 - t_1|^2/|t_1|$ and $|\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2| = (|1 - t_1|^2/|t_1|) \cdot (|1 - t_2|^2/|t_2|) \cdot (|\rho|/|\rho - 1|^2)$, we have the following proposition by Theorem B.

PROPOSITION 5.1. *Let* $G = \langle A_1, A_2\rangle$ *be a non-elementary discrete group and let* $\tau = (t_1, t_2, \rho)$ *be the point corresponding to* $\langle A_1, A_2\rangle$. *Then*

$$J(\tau) := \frac{|1 - t_1|^2}{|t_1|} + \frac{|1 - t_1|^2|1 - t_2|^2|\rho|}{|t_1|\,|t_2|\,|\rho - 1|^2} \geq 1.$$

Let $\tau = (t_1, t_2, \rho)$ correspond to $G = \langle A_1, A_2\rangle$. We set

$$J_1(G) := |\mathrm{tr}(A)^2 - 4|, \quad J_1(\tau) := |1 - t_1|^2/|t_1|,$$
$$J_2(G) := |\mathrm{tr}(A_1A_2A_1^{-1}A_2^{-1}) - 2|$$

and

$$J_2(\tau) := \frac{|1 - t_1|^2|1 - t_2|^2|\rho|}{|t_1|\,|t_2|\,|\rho - 1|^2}.$$

Then $J_1(G) = J_1(\tau)$, $J_2(G) = J_2(\tau)$ and $J(G) = J(\tau)$

LEMMA 5.2. $J_2(G)$ *is* Φ_2*-invariant, that is,* $J_2(\phi(G)) = J_2(G)$ *for any* $\phi \in \Phi_2$.

PROOF. Since $J_2(N_j(G)) = J_2(G)$ $(j = 1, 2, 3)$, the desired result follows from Theorem A. q.e.d.

We can easily see the following lemma.

LEMMA 5.3. $J_1(G)$ *and* $J(G)$ *are invariant under the Nielsen transformations* N_1 *and* N_3, *that is,*
(i) $J_1(N_1(G)) = J_1(G)$ *and* $J_1(N_3(G)) = J_1(G)$.
(ii) $J(N_1(G)) = J(G)$ *and* $J(N_3(G)) = J(G)$.

By Lemma 5.1 we have the following.

Proposition 5.2. *For $k = \mathrm{II}, \mathrm{VII}$*

$$\inf\{J(G) \mid G \in F_k(\mathrm{Mod}\, S_2^0)\} \geq \inf\{J(G) \mid G \in M_k\} \geq \inf\{J(G) \mid G \in R_k S_2^0\}.$$

Lemma 5.4. *For each $k = \mathrm{II}, \mathrm{VII}$, if $\tau = (t_1, t_2, \rho) \in M_k$ and $\tau_0 = (t_1, t_{20}, \rho) \in \partial M_k$ $(t_{20} \neq 0)$, then $J(\tau_0) < J(\tau)$.*

Proof. Since the function $(1 - t_2)^2/t_2$ is negative and monotonously decreasing in the interval $-1 < t_2 < 0$, the desired result follows from the definition of the space M_k. q.e.d.

Corollary. *For each $k = \mathrm{II}, \mathrm{VII}$, $\inf\{J(G) \mid G \in \partial_0 M_k\} \leq \inf\{J(G) \mid G \in M_k\}$, where $\partial_0 M_k = \{G \in \partial M_k \mid t_2(G) \neq 0\}$ and $(t_1(G), t_2(G), \rho(G))$ corresponds to G.*

6. Proof of Theorem 1.

Lemma 6.1. *Let*

$$f(x,y) = \frac{(1-x^2)^2(1-x)^2y^2}{x^2(1-y)^2(1-xy)(y-x)}.$$

Then $f(x,y) > 16$ for $1 < y < 1/x$ and $0 < x < 1$.

Proof. We set $X = x + 1/x$ and $Y = y + 1/y$. Then we have

$$\begin{aligned} f(x,y) &= (1-x^2)^2(1-x)^2y^2/x^2(1-y)^2(1-xy)(y-x) \\ &= (X^2-4)(X-2)/(X-Y)(Y-2). \end{aligned}$$

We write $g(X,Y)$ for the last term of the above equation. We will show $g(X,Y) > 16$. Since $X > 2$, $Y > 2$ and $X - Y = (y-x)(1-xy)/xy > 0$, it suffices to show

$$(X^2-4)(X-2) > 16(X-Y)(Y-2).$$

Noting that $X > 2$, we have the desired result, since

$$\begin{aligned} (X^2-4)(X-2) - 16(X-Y)(Y-2) &= 16Y^2 - 16Y(X+2) + (X^2-4)(X-2) + 32X \\ &= 16\{Y - (X+2)/2\}^2 + (X-2)^3. \end{aligned}$$

q.e.d.

Corollary. *If $\tau = (t_1, t_2, \rho)$ is a point on the boundary of M_{II} defined by the equation*

$$t_2 = (t_1^{1/2}\rho^{1/2} - 1)/(\rho^{1/2} - t_1^{1/2}) \quad (0 < \rho < 1/t_1, 0 < t_1 < 1),$$

then $J_2(\tau) > 16$.

Proof. We set $x = t_1^{1/2}$ and $y = \rho^{1/2}$. By substituting $t_2 = (t_1^{1/2}\rho^{1/2} - 1)/(\rho^{1/2} - t_1^{1/2})$ for the defining equation of $J_2(\tau)$,

$$J_2(\tau) = \frac{|1-t_1|^2|1-t_2|^2|\rho|}{|t_1|\,|t_2|\,|\rho-1|^2},$$

we have

$$J_2(\tau) = \frac{(1-x^2)^2(1-x)^2y^2}{x^2(1-y)^2(1-xy)(y-x)}.$$

By Lemma 6.1 we have the desired result. q.e.d.

LEMMA 6.2. *Let* $\tau = (t_1, t_2, \rho) \in M_{\mathrm{II}}$ *and* $\tau_0 = (t_1, t_{20}, \rho) \in \partial M_{\mathrm{II}}$ $(t_{20} \neq 0)$. *Then* $J_2(\tau) > J_2(\tau_0)$.

PROOF. Since $-1 < t_{20} < t_2 < 0$, we have $(1-t_{20})^2/|t_{20}| < (1-t_2)^2/|t_2|$. Hence $J(\tau) > J(\tau_0)$. q.e.d.

The following corollary follows from Corollary to Lemma 6.1 and Lemma 6.2.

COROLLARY. *Let* $\tau = (t_1, t_2, \rho) \in M_{\mathrm{II}}$. *Then* $J_2(\tau) > 16$.

PROOF OF THEOREM 2. By Proposition 3.1 we have that for any $\tau \in R_{\mathrm{II}}\boldsymbol{S}_2^0$ there exists $\phi \in \mathrm{Mod}(R_{\mathrm{II}}\boldsymbol{S}_2^0)$ such that $\phi(\tau) \in M_{\mathrm{II}}$. Then by Lemma 5.2, Proposition 5.2 and Corollary to Lemma 6.2, we have $J(\tau) = J_1(\tau) + J_2(\tau) \geq J_2(\tau) = J_2(\phi(\tau)) > 16$. q.e.d.

7. Proof of Theorem 3: Part 1.

In this section through §10 we will prove Theorem 3. Throughout this section let N_1, N_2 and N_3 be the Nielsen transformations defined in §2, and let $\varphi = N_3^2$ and $\chi = N_1N_2$. Let M_{VII} be the set introduced in §5. We easily see the following two lemmas. We omit the proofs.

LEMMA 7.1. (i) $N_1^2 = 1$, $N_2^2 = 1$, $N_1N_2 \sim N_2N_1$ *and* $N_1N_2N_1N_2 \sim 1$.

(ii) $\chi^n \sim \begin{cases} N_1N_2 & \textit{if } n \textit{ is odd}. \\ 1 & \textit{if } n \textit{ is even}. \end{cases}$

(iii) $\chi^{-1} \sim \chi$.

(iv) $\chi N_1 = N_1\chi^{-1}$.

(v) $\varphi N_1 = N_1\varphi^{-1}$ *and* $N_1\varphi = \varphi^{-1}N_1$.

LEMMA 7.2.

(i) $N_1(M_{\mathrm{VII}}) = M_{\mathrm{VII}}$ *and* $N_2(M_{\mathrm{VII}}) = M_{\mathrm{VII}}$.

(ii) $\chi^n(M_{\mathrm{VII}}) = M_{\mathrm{VII}}$ $(n \in \boldsymbol{Z})$.

Let $J(G)$ be the Jørgensen number for $G = \langle A_1, A_2 \rangle$. By Lemmas 5.3 and 7.2, we have the following lemmas.

LEMMA 7.3. *Let* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0$. *Then*

(i) $J(\varphi^m(G)) = J(G)$ $(m \in \boldsymbol{Z})$.

(ii) $J(\chi(G)) = J(N_2(G))$.

LEMMA 7.4.

(i) $\inf\{J(N_1(G)) \mid G \in M_{\mathrm{VII}}\} = \inf\{J(G) \mid G \in M_{\mathrm{VII}}\}$.

(ii) $\inf\{J(N_2(G)) \mid G \in M_{\mathrm{VII}}\} = \inf\{J(G) \mid G \in M_{\mathrm{VII}}\}$.

(iii) $\inf\{J(\chi(G)) \mid G \in M_{\mathrm{VII}}\} = \inf\{J(G) \mid G \in M_{\mathrm{VII}}\}$.

(iv) $\inf\{J(\varphi^m(G)) \mid G \in M_{\mathrm{VII}}\} = \inf\{J(G) \mid G \in M_{\mathrm{VII}}\}$ $(m \in \boldsymbol{Z})$.

Noting that $\chi^2 \sim 1$ (Lemma 7.1), we can classify elements $\phi \in \mathrm{Mod}(R_{\mathrm{VII}}\boldsymbol{S}_2^0)$ into the following four types by Proposition 3.3:

(T_1) $\phi = \varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}$,

(T_2) $\phi = \chi\varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}$,

(T_3) $\phi = \varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}\chi$,

(T_4) $\phi = \chi\varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}\chi$,

where $m(k) = \pm 1, \pm 2, \pm 3, \ldots$.

We define marked groups G_j and G_j^* $(j = 1, 2, 3, \ldots)$ by setting

$$G_{2k-1} = G_{2k-1}(G_0) := \varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0),$$
$$G_{2k} = G_{2k}(G_0) := \chi\varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0),$$
$$G_{2k}^* = G_{2k}^*(G_0) := \varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}\chi(G_0),$$
$$G_{2k+1}^* = G_{2k+1}^*(G_0) := \chi\varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}\chi(G_0)$$

for $G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}$.

Then we have $G_{2k} = \chi G_{2k-1}$ and $G_{2k+1}^* = \chi G_{2k}^*$.

LEMMA 7.5. *Let* G_{2k-1}, G_{2k}, G_{2k}^* *and* G_{2k+1}^* *be the groups defined in the above. Then*

(i) $J(G_{2k-1}) = J(G_{2k-2})$ $(k = 1, 2, 3, \ldots)$.

(ii) $J(G_{2k}^*) = J(G_{2k-1}^*)$ $(k = 1, 2, 3, \ldots)$.

(iii) $\inf\{J(G_{2k}^*) \mid G_0 \in M_{\mathrm{VII}}\} = \inf\{J(G_{2k-1}) \mid G_0 \in M_{\mathrm{VII}}\}$ $(k = 1, 2, 3, \ldots)$.

(iv) $\inf\{J(G_{2k+1}^*) \mid G_0 \in M_{\mathrm{VII}}\} = \inf\{J(G_{2k}) \mid G_0 \in M_{\mathrm{VII}}\}$ $(k = 1, 2, 3, \ldots)$.

PROOF. (i) and (ii) follow from Lemma 7.3. (iii) and (iv) follow from Lemma 7.2. q.e.d.

PROPOSITION 7.1.

$$\inf\{J(G) \mid G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0\} = \inf\{J(G) \mid G = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}\}.$$

Proposition 7.1 follows from Proposition 5.2 if the following proposition is shown.

PROPOSITION 7.2.

$$\inf\{J(G) \mid G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0\} \geq \inf\{J(G) \mid G = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}\}.$$

This proposition follows from the following proposition and Lemma 7.5.

PROPOSITION 7.3. *Let G_{2k} $(k = 0, 1, 2, \ldots)$ be the groups as in the above. Then*

$$\inf\{J(G_{2k}) \mid G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}\} > \inf\{J(G_{2k-2}) \mid G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}\} \quad (k = 1, 2, 3, \ldots).$$

In this section we will only prove Proposition 7.3 for the case of $k = 1$, that is, the following Proposition 7.4, and we will prove the proposition for the general case by induction in sections 8 and 9.

PROPOSITION 7.4. *Let $G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}$ and let $G_2 = \chi\varphi^m(G_0)$ $(m \in \boldsymbol{Z})$. Then*

$$\inf\{J(G_2) \mid G_0 \in M_{\mathrm{VII}}\} > \inf\{J(G_0) \mid G_0 \in M_{\mathrm{VII}}\}.$$

We easily see the following lemma by Lemma 5.2 and the defining equation of Jørgensen's number, and so we omit the proof.

LEMMA 7.6. *Let $G = \langle A_1, A_2 \rangle$ and $G^* = \langle A_1^*, A_2^* \rangle$ be in $R_{\mathrm{VII}}\boldsymbol{S}_2^0$. Let $(t_1(G), t_2(G), \rho(G))$ and $(t_1(G^*), t_2(G^*), \rho(G^*))$ correspond to G and G^*, respectively. Then the following four inequalities are equivalents*:

(i) $J(G^*) > J(G)$.

(ii) $J_1(G^*) > J_1(G)$.

(iii) $-t_1(G^*) < -t_1(G)$.

(iv) $\mathrm{tr}^2(A_1^*) < \mathrm{tr}^2(A_1) < 0$.

Let $G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}$ and let (t_1, t_2, ρ) correspond to G_0. Then $G_2 = \chi\varphi^m(G_0) = \langle A_1^{2m} A_2, A_1^{-1} \rangle$, where $m = m(1) \in \boldsymbol{Z} \backslash \{0\}$. We consider Proposition 7.4 for the following four cases:

(C_1) $\rho \leq -1$ and $m \geq 1$.

(C_2) $\rho \leq -1$ and $m \leq -1$.

(C_3) $-1 < \rho < 0$ and $m \geq 1$.

(C_4) $-1 < \rho < 0$ and $m \leq -1$.

LEMMA 7.7. *Let $G_0 = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}$. Then*

(i) *$J(\chi\varphi^m(G_0)) \geq J(G_0)$ for G_0 and G_2 in case (C_1) if and only if $J(\chi\varphi^m(G_0)) \geq J(G_0)$ for G_0 and G_2 in case (C_4).*

(ii) *$J(\chi\varphi^m(G_0)) \geq J(G_0)$ for G_0 and G_2 in case (C_2) if and only if $J(\chi\varphi^m(G_0)) \geq J(G_0)$ for G_0 and G_2 in case (C_3).*

PROOF. (i) We assume that $J(\chi\varphi^m(G_0)) \geq J(G_0)$ for G_0 and G_2 in case (C_1). Let (t_1, t_2, ρ) correspond to G_0 in case (C_4), that is, $-1 < \rho < 0$ and let $m \leq -1$. Then $\rho(N_1(G_0)) < -1$ and $-m \geq 1$. Thus $J(\chi\varphi^{-m}(N_1(G_0))) \geq J(N_1(G_0))$ by the assumption.

Hence by Lemmas 5.3 and 7.1 we have

$$\begin{aligned} J(\chi\varphi^m(G_0)) &= J(N_1N_2\varphi^m(G_0)) = J(N_1N_2\varphi^m(N_1N_1(G_0))) \\ &= J(N_1N_2N_1\varphi^{-m}(N_1(G_0))) = J(\chi\varphi^{-m}(N_1(G_0))) \\ &\geq J(N_1(G_0)) = J(G_0). \end{aligned}$$

Conversely, suppose the inequality holds for G in case (C_4). By the same method as above we have the inequality for G in case (C_1).

We also obtain (ii) by the same way as in (i). q.e.d.

In order to prove Proposition 7.4 it suffices to show Proposition 7.3 for G in cases (C_1) and (C_3) by Lemma 7.7.

LEMMA 7.8. *Let $G = \langle A_1, A_2 \rangle \in M_{\mathrm{VII}}$ and let (t_1, t_2, ρ) correspond to G. If $\rho \leq -1$ and $m \geq 1$, then $J(\chi\varphi^m(G)) > J(G)$.*

PROOF. Since $\chi\varphi^m(G) = \langle A_1^{2m}A_2, A_1^{-1}\rangle$, it suffices to show by Lemma 7.6 that $\mathrm{tr}^2(A_1^{2m}A_2) < \mathrm{tr}^2(A_1)$. Since

$$\mathrm{tr}^2(A_1^{2m}A_2) = \{(\rho - t_2) + t_1^{2m}(t_2\rho - 1)\}^2/t_1^{2m}t_2(\rho-1)^2$$

and $\mathrm{tr}^2 A_1 = (1+t_1)^2/t_1$, it suffices to show

$$\{(\rho - t_2) + t_1^{2m}(t_2\rho - 1)\}^2 > t_1^{2m-1}t_2(\rho-1)^2(1+t_1)^2.$$

We set $x = -t_1$, $y = -t_2$ and $z = -\rho$. Then $0 < x < 1$, $0 < y < 1$ and $z > 1$. We set

$$\begin{aligned} I &:= \{(\rho - t_2) + t_1^{2m}(t_2\rho - 1)\}^2 - t_1^{2m-1}t_2(\rho-1)^2(1+t_1)^2 \\ &= \{(y-z) + x^{2m}(yz-1)\}^2 - x^{2m-1}y(z+1)^2(1-x)^2. \end{aligned}$$

Furthermore, we set

$$I_1 := \{(y-z) + x^{2m}(yz-1)\} - x^{m-1/2}y^{1/2}(z+1)(1-x)$$

and

$$I_2 := \{(y-z) + x^{2m}(yz-1)\} + x^{m-1/2}y^{1/2}(z+1)(1-x).$$

Then $I = I_1I_2$. We will show $I_1 < 0$ and $I_2 < 0$. We note that $yz < 1$ for $G \in M_{\mathrm{VII}}$. It is easy to see $I_1 < 0$, because $y - z < 0$ and $x^{2m}(yz-1) < 0$. Next we will show $I_2 < 0$. Since $G \in M_{\mathrm{VII}}$, we have

$$1 < (-\rho)^{1/2} < \{1 - (-t_1)^{1/2}(-t_2)^{1/2}\}/\{(-t_1)^{1/2} + (-t_2)^{1/2}\},$$

that is,

$$1 < z^{1/2} < (1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2}).$$

It suffices to show

$$f(z) := z\{1 - x^{2m}y - x^{m-1/2}y^{1/2}(1-x)\} - \{y - x^{2m} + x^{m-1/2}y^{1/2}(1-x)\} > 0$$

in the interval $1 < z < \{(1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2})\}^2$.

Since

$$f'(z) = 1 - x^{2m}y - x^{m-1/2}y^{1/2}(1-x)$$
$$= (1 - x^{m-1/2}y^{1/2})(1 + x^{m+1/2}y^{1/2}) > 0,$$

it suffices to show $f(1) > 0$, that is,

$$g(y) := \{1 - x^{2m}y - x^{m-1/2}y^{1/2}(1-x)\} - \{y - x^{2m} + x^{m-1/2}y^{1/2}(1-x)\} > 0$$

in the interval $0 < y < \{(1-x^{1/2})/(1+x^{1/2})\}^2$.

Set $Y = y^{1/2}$. Then

$$g(Y) = -Y^2(1+x^{2m}) - 2x^{m-1/2}(1-x)Y + (1+x^{2m}).$$

Since $g(0) = (1+x^{2m}) > 0$ and $g(\{(1-x^{1/2})/(1+x^{1/2})\}^2) > 0$, we have $g(Y) > 0$ in the interval $0 < Y < (1-x^{1/2})/(1+x^{1/2})$. q.e.d.

LEMMA 7.9. *Let $G = \langle A_1, A_2\rangle \in M_{\mathrm{VII}}$ and let (t_1, t_2, ρ) correspond to G. If $-1 < \rho < 0$ and $m \geq 1$, then $J(\chi\varphi^m(G)) > J(G)$.*

PROOF. It suffices to show $\mathrm{tr}^2(A_1^{2m}A_2) < \mathrm{tr}^2(A_1)$ by Lemma 7.6. We use the same notation I, I_1 and I_2 as in the proof of Lemma 7.8. We set $x = -t_1$, $y = -t_2$ and $z = -\rho$. Since $\{(-t_1)^{1/2} + (-t_2)^{1/2}\}/\{1 - (-t_1)^{1/2}(-t_2)^{1/2}\} < (-\rho)^{1/2} < 1$, that is, $(x^{1/2} + y^{1/2})/(1 - x^{1/2}y^{1/2}) < z^{1/2} < 1$, we have

$$y - z + x^{2m}(yz - 1) = -z(1 - x^{2m}y) + y - x^{2m}$$
$$< -\{(x^{1/2}+y^{1/2})/(1-x^{1/2}y^{1/2})\}^2(1-x^{2m}y) + (y - x^{2m}) < 0.$$

Hence $I_1 < 0$.

Next we will show $I_2 < 0$. We set $f(z) = I_2$. Since

$$f(z) = z\{-1 + x^{2m}y + x^{m-1/2}y^{1/2}(1-x)\} + \{y - x^{2m} + x^{m-1/2}y^{1/2}(1-x)\}$$

and

$$-1 + x^{2m}y + x^{m-1/2}y^{1/2}(1-x) = -(1 - x^{m-1/2}y^{1/2})(1 + x^{m+1/2}y^{1/2}) < 0,$$

it suffices to show $f(z) < 0$ for $z = \{(x^{1/2}+y^{1/2})/(1-x^{1/2}y^{1/2})\}^2$. Since

$$f(\{(x^{1/2}+y^{1/2})/(1-x^{1/2}y^{1/2})\}^2)$$
$$= -\{(x^{1/2}+y^{1/2})^2(1 - x^{2m}y - x^{m-1/2}y^{1/2} + x^{m+1/2}y^{1/2})$$
$$+ (1 - x^{1/2}y^{1/2})^2(y - x^{2m} + x^{m-1/2}y^{1/2} - x^{m+1/2}y^{1/2})\}/(1-x^{1/2}y^{1/2})^2,$$

it suffices to show

$$g(z) := (x^{1/2}+y^{1/2})^2(1 - x^{2m}y - x^{m-1/2}y^{1/2} + x^{m+1/2}y^{1/2})$$
$$- (1 - x^{1/2}y^{1/2})^2(y - x^{2m} + x^{m-1/2}y^{1/2} - x^{m+1/2}y^{1/2}) > 0.$$

By straightforward calculations, we have

$$g(z) = (1+y)[x(1+x^{2m-1})(1-y) + x^{1/2}y^{1/2}\{2(1-x^{2m}) - x^{m-1}(1-x^2)\}].$$

Since

$$2(1-x^{2m})-x^{m-1}(1-x^2)=(1-x^{2m})+(1-x^{m-1})+x^{m+1}(1-x^{m-1})>0,$$

we have $g(z)>0$. Hence $I_2<0$ and so $I>0$. q.e.d.

§8. Proof of Theorem 3: Part 2.

Throughout this section let N_1, N_2 and N_3 be the Nielsen transformations and let $\varphi=N_3^2$ and $\chi=N_1N_2$. Let $G_0=\langle A_{10},A_{20}\rangle\in M_{\mathrm{VII}}$ and set

$$G_{2k}=\chi\varphi^{m(k)}\chi\varphi^{m(k-1)}\cdots\chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0)\in R_{\mathrm{VII}}\boldsymbol{S}_2^0\quad(k=1,2,3,\dots)$$

as in §7. In this and next sections we will show the following proposition.

PROPOSITION 8.1.

$$\inf\{J(G_{2k})\mid G_{2k}\in R_{\mathrm{VII}}\boldsymbol{S}_2^0\}>\inf\{J(G_0)\mid G_0\in M_{\mathrm{VII}}\}$$

for $k=2,3,4,\dots$.

Let $G_{2k-2}=\langle A_1,A_2\rangle\in R_{\mathrm{VII}}\boldsymbol{S}_2^0$ and let (t_1,t_2,ρ) correspond to G_{2k-2}. Then we have $G_{2k}=\chi\varphi^m(G_{2k-2})=\langle A_1^{2m}A_2,A_2^{-1}\rangle$ with $m=m(k+1)$. As in the previous section, we consider the following four cases:

(C_1) $\rho\le -1$ and $m\ge 1$.

(C_2) $\rho\le -1$ and $m\le -1$.

(C_3) $-1<\rho<0$ and $m\ge 1$.

(C_4) $-1<\rho<0$ and $m\le -1$.

By the same way as in the proof of Lemma 7.7, we have the following.

LEMMA 8.1. *Let* $G=\langle A_1,A_2\rangle\in R_{\mathrm{VII}}\boldsymbol{S}_2^0$ *and let* (t_1,t_2,ρ) *correspond to* G.

(i) *The inequality*

$$(*)\qquad J(G)<J(\chi\varphi^m(G))$$

holds for G *in case* (C_1) *if and only if the inequality* $(*)$ *holds for* G *in case* (C_4).

(ii) *The inequality* $(*)$ *holds for* G *in case* (C_2) *if and only if the inequality* $(*)$ *holds for* G *in case* (C_3).

In this section, from now on let $G=\langle A_1,A_2\rangle\in R_{\mathrm{VII}}\boldsymbol{S}_2^0\backslash M_{\mathrm{VII}}$ and let (t_1,t_2,ρ) correspond to G. Throughout this section, let

$$(**)\qquad A_1=1/t_1^{1/2}\begin{pmatrix}1&0\\0&t_1\end{pmatrix}\quad\text{and}\quad A_2=1/t_2^{1/2}(\rho-1)\begin{pmatrix}\rho-t_2&\rho(t_2-1)\\1-t_2&t_2\rho-1\end{pmatrix}.$$

Then we have $\operatorname{tr}A_1=(1+t_1)/t_1^{1/2}$ and $\operatorname{tr}A_2=\{(\rho-t_2)+(t_2\rho-1)\}/t_2^{1/2}(\rho-1)$. Furthermore we set $x=-t_1$, $y=-t_2$ and $z=-\rho$. In this section we consider the case of $\rho<-1$, that is, the case of $z>1$.

LEMMA 8.2. *Let* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}} S_2^0 \backslash M_{\mathrm{VII}}$ *and let* (t_1, t_2, ρ) *correspond to* G. *Set* $x = -t_1$, $y = -t_2$ *and* $z = -\rho$. *If* $z > 1$, *then*

$$z - y + x(yz - 1) > 0.$$

PROOF. Since $G \in R_{\mathrm{VII}} S_2^0 \backslash M_{\mathrm{VII}}$, we have

$$z^{1/2} > (1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2}) > 1. \tag{1}$$

From (1), we have

$$1 - x^{1/2}y^{1/2} > x^{1/2} + y^{1/2} \tag{2}$$

and

$$1 > x^{1/2} + y^{1/2}. \tag{3}$$

By (1), (2) and (3) we have

$$\begin{aligned} z - y + x(yz - 1) &= z(1 + xy) - (x + y) \\ &> \{(1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2})\}^2(1 + xy) - (x + y) \\ &= (1 + x)(1 + y)\{(1 + xy) - (x^{1/2} + y^{1/2})^2\}/(x^{1/2} + y^{1/2})^2 \\ &> (1 + x)(1 + y)\{(1 + xy) - x^{1/2} - y^{1/2}\}/(x^{1/2} + y^{1/2})^2 \\ &> (1 + x)(1 + y)(1 - x^{1/2}y^{1/2} - x^{1/2} - y^{1/2})/(x^{1/2} + y^{1/2})^2 > 0. \end{aligned}$$

q.e.d.

LEMMA 8.3. *Under the same assumption as Lemma* 8.2, *for* $m = 1, 2, 3, \ldots$

$$\frac{(z - y) + x^{2m-1}(yz - 1)}{x^{m-1/2}y^{1/2}(z + 1)} > \frac{(z - y) + x(yz - 1)}{x^{1/2}y^{1/2}(z + 1)} \tag{$*$}$$

PROOF. By simple calculation we can see that if we show the inequality

$$(z - y) - x^m(yz - 1) > 0$$

holds, then we have the desired inequality $(*)$.

Since $z > \{(1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2})\}^2$, we have

$$(z - y) - x^m(yz - 1) > (1 + y)\{(1 - y)(1 + x^{m+1}) - 2x^{1/2}y^{1/2}(1 - x^m)\}/(x^{1/2} + y^{1/2})^2.$$

Since $1 > x^{1/2}y^{1/2} + x^{1/2} + y^{1/2}$, we have

$$\begin{aligned} &(1 - y)(1 + x^{m+1}) - 2x^{1/2}y^{1/2}(1 - x^m) \\ &\qquad > (1 - y^{1/2})(x^{1/2} + y^{1/2}) + (1 - y)x^{m+1} + 2x^{m+1/2}y^{1/2} > 0. \end{aligned}$$

q.e.d.

REMARK. Lemma 8.3 means $\mathrm{tr}(A_1^{2m-1}A_2) \leq \mathrm{tr}(A_1A_2) < 0$.

COROLLARY. *Let* $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}} \backslash M_{\mathrm{VII}}$ *and let* A_1 *and* A_2 *be the matrices as in* $(**)$. *Then for* $m = 1, 2, 3, \ldots$

(i) $\mathrm{tr}(A_1^{4m-1}A_2) > 2$.

(ii) $\mathrm{tr}(A_1^{4m-3}A_2) < -2$.

PROOF. By Lemmas 8.2 and 8.3, we have

$$\begin{aligned}\mathrm{tr}(A_1^{4m-1}A_2) &= \{(\rho - t_2) + t_1^{4m-1}(t_2\rho - 1)\}/t_1^{(4m-1)/2}t_2^{1/2}(\rho - 1)\\ &= \{(z-y) + x^{4m-1}(yz-1)\}/x^{(4m-1)/2}y^{1/2}(z+1)\\ &> \{(z-y) + x(yz-1)\}/x^{1/2}y^{1/2}(z+1) > 0.\end{aligned}$$

Since $A_1^{4m-1}A_2$ is a hyperbolic transformation, we have $\mathrm{tr}(A_1^{4m-1}A_2) > 2$.

(ii) By Lemmas 8.2 and 8.3, we have

$$\begin{aligned}\mathrm{tr}(A_1^{4m-3}A_2) &= \{(\rho - t_2) + t_1^{4m-3}(t_2\rho - 1)\}/t_1^{(4m-3)/2}t_2^{1/2}(\rho - 1)\\ &= -\{(z-y) + x^{4m-3}(yz-1)\}/x^{(4m-3)/2}y^{1/2}(z+1)\\ &< -\{(z-y) + x(yz-1)\}/x^{1/2}y^{1/2}(z+1) < 0.\end{aligned}$$

Since $A_1^{4m-3}A_2$ is a hyperbolic transformation, we have $\mathrm{tr}(A_1^{4m-3}A_2) < -2$. q.e.d.

LEMMA 8.4. *Under the same assumption as Lemma* 8.2

$$(z-y) - x^2(yz-1) > 0.$$

PROOF. Since $z > \{(1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2})\}^2$, we have

$$\begin{aligned}(z-y) - x^2(yz - x^2) &> \{(1 - x^{1/2}y^{1/2})/(x^{1/2} + y^{1/2})\}^2(1 - x^2y) - (y - x^2)\\ &= \{(1 - x^{1/2}y^{1/2})^2(1 - x^2y) - (x^{1/2} + y^{1/2})^2(y - x^2)\}/(x^{1/2} + y^{1/2})^2.\end{aligned}$$

Since $1 > x^{1/2}y^{1/2} + x^{1/2} + y^{1/2}$, we have

$$\begin{aligned}&(1 - x^{1/2}y^{1/2})^2(1 - x^2y) - (x^{1/2} + y^{1/2})^2(y - x^2)\\ &\quad = (1+y)(1 + x^3 - y - yx^3 - 2x^{1/2}y^{1/2} + 2x^2x^{1/2}y^{1/2})\\ &\quad > (1+y)\{(1 - y^{1/2})(x^{1/2} + y^{1/2}) + x^3(1-y) + 2x^2x^{1/2}y^{1/2}\} > 0.\end{aligned}$$

q.e.d.

LEMMA 8.5. *Under the same assumption as Lemma* 8.2,

$$\frac{(z-y) - x^{2m}(yz-1)}{x^m y^{1/2}(z+1)} \geq \frac{(z-y) - x^2(yz-1)}{xy^{1/2}(z+1)}.$$

for $m = 1, 2, 3, \ldots$.

PROOF. The inequality in this lemma is equivalent to the following one:

$$(*) \qquad (1 - x^{m-1})\{(z-y) + (yz-1)x^{m+1}\} \geq 0.$$

If $yz - 1 \geq 0$, then it is obvious that the inequality $(*)$ holds. Therefore we assume $yz - 1 < 0$. Then since $1 > x^{1/2}y^{1/2} + x^{1/2} + y^{1/2}$, we have

$$\begin{aligned}(z-y) + x^{m+1}(yz-1) &> (z-y) - x(1 - yz) > z - y - x > 1 - y - x\\ &> x^{1/2}y^{1/2} + x^{1/2} + y^{1/2} - x - y\\ &= x^{1/2}(1 - x^{1/2}) + y^{1/2}(1 - y^{1/2}) + x^{1/2}y^{1/2} > 0.\end{aligned}$$

q.e.d.

REMARK. Let A_1 and A_2 be the matrice (**). Then the inequality in Lemma 8.5 means $i\,\mathrm{tr}(A_1^{2m}A_2) \le i\,\mathrm{tr}(A_1^2 A_2)$.

LEMMA 8.6. *Let* $G = \langle A_1, A_2\rangle$ *be as in Lemma* 8.2 *and let* A_1 *and* A_2 *be the matrices* (**). *Then*

(i) $i\,\mathrm{tr}(A_1^{4m-2}A_2) < 0 \quad (m = 1,2,3,\dots)$.

(ii) $i\,\mathrm{tr}(A_1^{4m}A_2) > 0 \quad (m = 1,2,3,\dots)$.

PROOF. (i) Let x, y and z be as in Lemma 8.2. By Lemmas 8.4 and 8.5, we have

$$\begin{aligned} -i\,\mathrm{tr}(A_1^{4m-2}A_2) &= -\{(z-y) - x^{4m-2}(yz-1)\}/x^{2m-1}y^{1/2}(z+1) \\ &\le -\{(z-y) - x^2(yz-1)\}/xy^{1/2}(z+1) < 0. \end{aligned}$$

(ii) By Lemmas 8.4 and 8.5, we have

$$\begin{aligned} i\,\mathrm{tr}(A_1^{4m}A_2) &= \{(z-y) - x^{4m}(yz-1)\}/x^{2m}y^{1/2}(z+1) \\ &> \{(z-y) - x^2(yz-1)\}/xy^{1/2}(z+1) > 0. \end{aligned}$$ q.e.d.

The following lemma is easily seen and so the proof is omitted.

LEMMA 8.7. *Let*

$$A = 1/t_1^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix}.$$

Then for $m = 1,2,3,\dots,$

(i) $i\,\mathrm{tr}(A^{4m-3}) > 0$.

(ii) $\mathrm{tr}(A^{4m-2}) < -2$.

(iii) $i\,\mathrm{tr}(A^{4m-1}) < 0$.

(iv) $\mathrm{tr}(A^{4m}) > 2$.

PROPOSITION 8.2. *Let* $G_0 = \langle A_{10}, A_{20}\rangle \in M_{\mathrm{VII}}$ *and let* $G_{2j} = \chi\varphi^{m(j)}\chi\varphi^{m(j-1)}\dots\chi\varphi^{m(2)}\chi\varphi^{m(1)}(G_0)$ *with* $m(l) \in \mathbf{Z}\backslash\{0\}$ $(l = 1,2,3,\dots)$. *Suppose that* $\rho(G_{2k}) \le -1$ *and* $m(k+1) \ge 1$ $(k = 1,2,3,\dots)$. *If* $J(G_{2k-2}) < J(G_{2k})$, *then* $J(G_{2k}) < J(G_{2k+2})$ $(k = 1,2,3,\dots)$.

REMARK. This proposition means that $J(G_{2k}) < J(G_{2k+2})$ $(k = 1,2,3,\dots)$ holds in case (C_1).

PROOF OF PROPOSITION 8.2. We set $G_{2k} = \langle A_{1,2k}, A_{2,2k}\rangle = \langle B_1, B_2\rangle$. Then $J(G_{2k-2}) < J(G_{2k})$ if and only if $\mathrm{tr}^2(B_1) < \mathrm{tr}^2(B_2) < 0$. We have $G_{2k+2} = \langle B_1^{2m}B_2, B_1^{-1}\rangle$, where $m = m(k+1)$. We set $Y := \mathrm{tr}^2(B_1) - \mathrm{tr}^2(B_1^{2m}B_2)$. Then

$$Y = (i\,\mathrm{tr}(B_1^{2m}B_2) + i\,\mathrm{tr}(B_1))(i\,\mathrm{tr}(B_1^{2m}B_2) - i\,\mathrm{tr}(B_1)).$$

We note that

$$(*) \qquad i\,\mathrm{tr}(B_1^{2m}B_2) = i\,\mathrm{tr}(B_1^m)\mathrm{tr}(B_1^m B_2) - i\,\mathrm{tr}(B_2).$$

Then we may assume that

$$(**)\qquad B_1 = 1/t_1^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad B_2 = 1/t_2^{1/2}(\rho-1)\begin{pmatrix} \rho - t_2 & \rho(t_2-1) \\ 1-t_2 & t_2\rho - 1 \end{pmatrix}.$$

Then we have $\operatorname{tr}(B_1) = (1-t_1)/t_1^{1/2}$ and $\operatorname{tr}(B_2) = \{(\rho - t_2) + (t_2\rho - 1)\}/t_2(\rho - 1)$.

(i) The case of $m = 4n-1$ $(n = 1,2,3,\ldots)$.

By corollary to Lemma 8.3, we have $\operatorname{tr}(B_1^m B_2) > 2$. Since $i\operatorname{tr}(B_2) > 0$ and $i\operatorname{tr}(B_1^m) < 0$ by Lemma 8.7, we have $i\operatorname{tr}(B_1^{2m}B_2) < 2i\operatorname{tr}(B_1^m) - i\operatorname{tr}(B_2) < 0$. Hence $i\operatorname{tr}(B_1^{2m}B_2) - i\operatorname{tr}(B_1) < 0$. Therefore if we show $i\operatorname{tr}(B_1^{2m}B_2) + i\operatorname{tr}(B_1) < 0$, then we have the desired inequality $Y > 0$. By $(*)$ and corollary to Lemma 8.3, we have

$$\begin{aligned} i\operatorname{tr}(B_1^{2m}B_2) + i\operatorname{tr}(B_1) &= i\operatorname{tr}(B_1^m)\operatorname{tr}(B_1^m B_2) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) \\ &< 2i\operatorname{tr}(B_1^{4n-1}) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) \\ &< -2i\operatorname{tr}(B_1) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) < 0. \end{aligned}$$

for $m = 1,2,3,\ldots$.

(ii) The case of $m = 4n-3$ $(n = 1,2,3,\ldots)$.

By corollary to Lemma 8.3, we have $\operatorname{tr}(B_1^m B_2) = \operatorname{tr}(B_1^{4n-3}B_2) < -2$. By $(*)$ and Lemma 8.7 we have

$$\begin{aligned} i\operatorname{tr}(B_1^{2m}B_2) &= i\operatorname{tr}(B_1^m)\operatorname{tr}(B_1^m B_2) - i\operatorname{tr}(B_2) \\ &< -2i\operatorname{tr}(B_1^m) - i\operatorname{tr}(B_2) < 0. \end{aligned}$$

Hence $i\operatorname{tr}(B_1^{2m}B_2) - i\operatorname{tr}(B_1) < 0$. Therefore if we show $i\operatorname{tr}(B_1^{2m}B_2) + i\operatorname{tr}(B_1) < 0$, then we have the desired inequality $Y > 0$.

By $(*)$ and corollary to Lemma 8.3, we have

$$\begin{aligned} i\operatorname{tr}(B_1^{2m}B_2) + i\operatorname{tr}(B_1) &= i\operatorname{tr}(B_1^m)\operatorname{tr}(B_1^m B_2) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) \\ &< -2i\operatorname{tr}(B_1^m) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) \\ &< -2i\operatorname{tr}(B_1) - i\operatorname{tr}(B_2) + i\operatorname{tr}(B_1) < 0 \end{aligned}$$

for $m = 1,2,3,\ldots$.

(iii) The case of $m = 4n-2$ $(n = 1,2,3,\ldots)$.

In this case we have

$$Y = (i\operatorname{tr}(B_1^{8n-4}B_2) + i\operatorname{tr}(B_1))(i\operatorname{tr}(B_1^{8n-4}B_2) - i\operatorname{tr}(B_1)).$$

1) The case of $n = 2l$ $(l = 1,2,3,\ldots)$.

By corollary to Lemma 8.3, $\operatorname{tr}(B_1^{2n-1}B_2) = \operatorname{tr}(B_1^{4l-1}B_2) > 2$. Hence by $(*)$, Lemma 8.7 and corollary to Lemma 8.3, we have $i\operatorname{tr}(B_1^{4n-2}B_2) + i\operatorname{tr}(B_1) < 0$. Thus $i\operatorname{tr}(B_1^{4n-2}B_2) < -i\operatorname{tr}(B_1) < 0$. By using this inequality and the hypothesis of induction $i\operatorname{tr}(B_1) > i\operatorname{tr}(B_2)$, we have

$$\begin{aligned} i\operatorname{tr}(B_1^{8n-4}B_2) - i\operatorname{tr}(B_1) &= i\operatorname{tr}(B_1^{4n-2}B_2)\operatorname{tr}B_1^{4n-2} - i\operatorname{tr}(B_2) - i\operatorname{tr}(B_1) \\ &> -i\operatorname{tr}(B_1)\operatorname{tr}(B_1^{4n-2}) - i\operatorname{tr}(B_2) - i\operatorname{tr}(B_1) \\ &> 2i\operatorname{tr}(B_1) - i\operatorname{tr}(B_2) - i\operatorname{tr}(B_1) > 0. \end{aligned}$$

Furthermore, since $i\,\mathrm{tr}(B_1) > 0$, we have

$$i\,\mathrm{tr}(B_1^{8n-4}B_2) + i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_1^{8n-4}B_2) - i\,\mathrm{tr}(B_1) + 2i\,\mathrm{tr}(B_1)$$
$$> 2i\,\mathrm{tr}(B_1) > 0.$$

Therefore we have $Y > 0$.

2) The case of $n = 2l - 1$ $(l = 1, 2, 3, \ldots)$.

By corollary to Lemma 8.3, we have

$$\mathrm{tr}(B_1^{2n-1}B_2) = \mathrm{tr}(B_1^{4l-3}B_2) < -2.$$

Hence

$$\begin{aligned} i\,\mathrm{tr}(B_1^{4n-2}B_2) + i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{2n-1}B_2)\mathrm{tr}(B_1^{2n-1}) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< -2i\,\mathrm{tr}(B_1^{2n-1}) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< -2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &= -i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) < 0. \end{aligned}$$

Thus we have

$$i\,\mathrm{tr}(B_1^{4n-2}B_2) < -i\,\mathrm{tr}(B_1) < 0.$$

Hence by Lemma 8.7 and the hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{8n-4}B_2) - i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{4n-2}B_2)\mathrm{tr}(B_1^{4n-2}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> -i\,\mathrm{tr}(B_1)\mathrm{tr}(B_1^{4n-2}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) > 0. \end{aligned}$$

Furthermore, since $i\,\mathrm{tr}(B_1) > 0$, we have

$$i\,\mathrm{tr}(B_1^{8n-4}B_2) + i\,\mathrm{tr}(B_1) = i\,\mathrm{tr}(B_1^{8n-4}B_2) - i\,\mathrm{tr}(B_1) + 2i\,\mathrm{tr}(B_1)$$
$$> 2i\,\mathrm{tr}(B_1) > 0.$$

Therefore we have $Y > 0$.

(iv) The case of $m = 4n$ $(n = 1, 2, 3, \ldots)$.

In this case we have

$$Y = (i\,\mathrm{tr}(B_1^{8n}B_2) + i\,\mathrm{tr}(B_1))(i\,\mathrm{tr}(B_1^{8n}B_2) - i\,\mathrm{tr}(B_1)).$$

By $(*)$ we have the following:

$$i\,\mathrm{tr}(B_1^{8n}B_2) = i\,\mathrm{tr}(B_1^{4n}B_2)\,\mathrm{tr}(B_1^{4n}) - i\,\mathrm{tr}(B_2),$$
$$i\,\mathrm{tr}(B_1^{4n}B_2) = i\,\mathrm{tr}(B_1^{2n}B_2)\,\mathrm{tr}(B_1^{2n}) - i\,\mathrm{tr}(B_2),$$

and

$$i\,\mathrm{tr}(B_1^{2n}B_2) = i\,\mathrm{tr}(B_1^{n}B_2)\,\mathrm{tr}(B_1^{n}) - i\,\mathrm{tr}(B_2).$$

1) The case of $n = 4l - 1$ $(l = 1, 2, 3, \ldots)$.

By corollary to Lemma 8.3, $\mathrm{tr}(B_1^n B_2) = \mathrm{tr}(B_1^{4l-1}B_2) > 2$. Therefore by Lemma 8.7 we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{2n}B_2) + i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^n B_2)\mathrm{tr}(B_1^n) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< 2i\,\mathrm{tr}(B_1^n) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< -2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &= -i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) < 0. \end{aligned}$$

Hence $i\,\mathrm{tr}(B_1^{2n}B_2) < -i\,\mathrm{tr}(B_1)$. By using this inequality and the hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{4n}B_2) - i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{2n}B_2)\mathrm{tr}(B_1^{2n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> -2i\,\mathrm{tr}(B_1^{2n}B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) > 0. \end{aligned}$$

Thus $i\,\mathrm{tr}(B_1^{4n}B_2) > i\,\mathrm{tr}(B_1)$. By using this inequality and the hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{8n}B_2) - i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{4n}B_2)\mathrm{tr}(B_1^{4n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> i\,\mathrm{tr}(B_1)\mathrm{tr}(B_1^{4n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) > 0. \end{aligned}$$

Furthermore since $i\,\mathrm{tr}(B_1) > 0$, we have

$$i\,\mathrm{tr}(B_1^{8n}B_2) + i\,\mathrm{tr}(B_1) = i\,\mathrm{tr}(B_1^{8n}B_2) - i\,\mathrm{tr}(B_1) + 2i\,\mathrm{tr}(B_1) > 0.$$

Therefore we have the desired inequality $Y > 0$.

2) The case of $n = 4l - 3$ $(l = 1, 2, 3, \dots)$.

By corollary to Lemma 8.3, we have $\mathrm{tr}(B_1^n B_2) = \mathrm{tr}(B_1^{4l-3}B_2) < -2$. By using this inequality we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{2n}B_2) + i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^n)\mathrm{tr}(B_1^n B_2) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< -2i\,\mathrm{tr}(B_1^n) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &< -2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) < 0. \end{aligned}$$

Hence $i\,\mathrm{tr}(B_1^{2n}B_2) < -i\,\mathrm{tr}(B_1) < 0$. By hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$ and Lemma 8.7, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{4n}B_2) - i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{2n}B_2)\mathrm{tr}(B_1^{2n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> -i\,\mathrm{tr}(B_1)\mathrm{tr}(B_1^{2n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) > 0. \end{aligned}$$

Hence $i\,\mathrm{tr}(B_1^{4n}B_2) > i\,\mathrm{tr}(B_1) > 0$. By this inequality and the hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{8n}B_2) - i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{4n}B_2)\mathrm{tr}(B_1^{4n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> i\,\mathrm{tr}(B_1)\mathrm{tr}(B_1^{4n}) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) - i\,\mathrm{tr}(B_1) > 0. \end{aligned}$$

Furthermore noting $i\,\mathrm{tr}(B_1) > 0$, we have $i\,\mathrm{tr}(B_1^{8n}B_2) + i\,\mathrm{tr}(B_1) > 0$ and so $Y > 0$.

3) The case of $n = 4l - 2$ $(l = 1,2,3,\ldots)$.

This case is similarly treated to the case of (ii), that is, the case of $m = 4n - 2$.

4) The case of $n = 4l$ $(l = 1,2,3,\ldots)$.

By Lemma 8.7 and the same way as the case of $m = 4n$, $n = 2l$, we have

$$\begin{aligned} i\,\mathrm{tr}(B_1^{16l}B_2) + i\,\mathrm{tr}(B_1) &= i\,\mathrm{tr}(B_1^{8l}B_2)\mathrm{tr}(B_1^{8l}) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1) \\ &> 2i\,\mathrm{tr}(B_1) - i\,\mathrm{tr}(B_2) + i\,\mathrm{tr}(B_1). \end{aligned}$$

By the hypothesis of induction $i\,\mathrm{tr}(B_1) > i\,\mathrm{tr}(B_2)$, we have the desired inequality $i\,\mathrm{tr}(B_1^{16l}B_2) > i\,\mathrm{tr}(B_1)$. q.e.d.

§9. Proof of Theorem 3: Part 3.

Let (t_1, t_2, ρ) correspond to $G = \langle A_1, A_2 \rangle$. Then we write $t_1 = t_1(G)$, $t_2 = t_2(G)$ and $\rho = \rho(G)$, respectively. In this section we will show the following proposition.

PROPOSITION 9.1. *Let $G_0 \in M_{\mathrm{VII}}$ and set $G_{2j} = \chi\varphi^{m(j)}\chi \cdots \varphi^{m(2)}\chi\varphi^{m(1)}(G_0)$ with $m(l) \in \mathbf{Z}\backslash\{0\}$ $(l = 1,2,3,\ldots)$. Suppose $\rho(G_{2k}) < -1$ and $m(k+1) \le -1$. If $J(G_{2k-2}) < J(G_{2k})$, then $J(G_{2k}) < J(G_{2k+2})$ $(k = 1,2,3,\ldots)$.*

REMARK. This proposition means that $J(G_{2k}) < J(G_{2k+2})$ $(k = 1,2,3,\ldots)$ holds in case (C_2) in §8.

LEMMA 9.1. *Let $G_0 = \langle A_{10}, A_{20} \rangle \in M_{\mathrm{VII}}$ and $G_2 = \chi\varphi^{-1}(G_0) = \langle B_1, B_2 \rangle$. Let (t_1, t_2, ρ) correspond to G_2 and set $x = -t_1$, $y = -t_2$ and $z = -\rho$. Then*

$$x(z - y) + (yz - 1) > 0,$$

that is, $\mathrm{tr}(B_1^{-1}B_2) > 0$, *where*

$$B_1 = 1/t_1^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & t_1 \end{pmatrix} \quad \text{and} \quad B_2 = 1/t_2^{1/2}(\rho - 1)\begin{pmatrix} \rho - t_2 & \rho(t_2 - 1) \\ 1 - t_2 & t_2\rho - 1 \end{pmatrix}.$$

PROOF. We note $z^{1/2} > (1 + x^{1/2}y^{1/2})/(y^{1/2} - x^{1/2})$. By this inequality, we have

$$\begin{aligned} x(z - y) + (yz - 1) &= z(x + y) - (1 + xy) \\ &> \{(1 + x^{1/2}y^{1/2})/(y^{1/2} - x^{1/2})\}^2(x + y) - (1 + xy) \\ &= 2x^{1/2}y^{1/2}(1 + x)(1 + y)/(y^{1/2} - x^{1/2})^2 > 0. \end{aligned}$$

q.e.d.

LEMMA 9.2. *Let $G = \langle A_1, A_2 \rangle \in R_{\mathrm{VII}}S_2^0\backslash M_{\mathrm{VII}}$ and set $H = \chi\varphi^{-1}(G) = \langle B_1, B_2 \rangle$. Let $x = -t_1(G)$, $y = -t_2(G)$, $z = -\rho(G)$; $x_1 = -t_1(H)$, $y_1 = -t_2(H)$, $z_1 = -\rho(H)$. If $x(z - y) + (yz - 1) > 0$, then $x_1(z_1 - y_1) + (y_1z_1 - 1) > 0$.*

REMARK. Since $\operatorname{tr}(A_1^{-1}A_2) = \{x(z-y)+(yz-1)\}/x^{1/2}y^{1/2}(z+1)$ and $\operatorname{tr}(B_1^{-1}B_2) = \{x_1(z_1-y_1)+(y_1z_1-1)\}/x_1^{1/2}y_1^{1/2}(z_1+1)$, this lemma means that if $\operatorname{tr}(A_1^{-1}A_2)>0$, then $\operatorname{tr}(B_1^{-1}B_2)>0$.

PROOF OF LEMMA 9.2. Since $B_1 = A_1^{-2}A_2$ and $B_2 = A_1^{-1}$, we have $\operatorname{tr}(B_1^{-1}B_2) = \operatorname{tr}((A_1^{-2}A_2)^{-1}A_1^{-1}) = \operatorname{tr}(A_1^{-1}A_2) > 0$. q.e.d.

LEMMA 9.3. *If* $-1<\rho(G)<0$, *then* $\rho(\chi(G))<-1$.

PROOF. This lemma follows from $\rho(\chi(G)) = 1/\rho(G)$. q.e.d.

By straightforward calculations we have the following lemma.

LEMMA 9.4. *If* $-1<\rho(G)<0$, *then* $-1<\rho(\varphi^{-1}\chi(G))<0$.

LEMMA 9.5. *If* $-1<\rho(\varphi^{-1}\chi(G))<0$, *then*

$$J(\chi\varphi^{-1}\chi(G)) < J(\chi\varphi^{-m}\chi(G)) \quad (m\geq 2).$$

PROOF. By Lemma 7.1, we have

$$\begin{aligned} J(\chi\varphi^{-m}\chi(G)) &= J(N_1N_2\varphi^{-m}\chi(G)) = J(N_2N_1\varphi^{-m}\chi(G)) \\ &= J(N_2\varphi^{m}N_1\chi(G)) = J(N_1N_2\varphi^{m}N_1\chi(G)) = J(\chi\varphi^{m}N_1\chi(G)). \end{aligned}$$

for $m=1,2,3,\dots$. Since $\rho(\varphi N_1\chi(G))<-1$ by Lemmas 7.1 and 9.4, we have $J(\chi\varphi N_1\chi(G)) < J(\chi\varphi^m N_1\chi(G))$ $(m\geq 2)$ by the same way as in §8. Therefore we have the desired inequality, $J(\chi\varphi^{-1}\chi(G)) < J(\chi\varphi^{-m}\chi(G))$ for $m\geq 2$. q.e.d.

LEMMA 9.6. *Let* $G=\langle A_1,A_2\rangle \in R_{\mathrm{VII}}S_2^0\backslash M_{\mathrm{VII}}$, $H_1=\chi\varphi^{-1}(G)$ *and* $H_m=\chi\varphi^{-m}(G)$ $(m\geq 2)$. *If* $-1<\rho(\varphi^{-1}(G))<0$, *then* $J(H_1)<J(H_m)$.

PROOF. We consider G in this lemma as $\chi(G)$ in Lemma 9.5. By Lemma 9.5, we have $J(H_1) = J(\chi\varphi^{-1}(G)) < J(\chi\varphi^{-m}(G)) = J(H_m)$. q.e.d.

LEMMA 9.7. *Let* $G=\langle A_1,A_2\rangle \in R_{\mathrm{VII}}S_2^0\backslash M_{\mathrm{VII}}$ *with* $-1<\rho(G)<0$. *Let* $H^* = \chi(G) = \langle B_1,B_2\rangle$. *Then* $J(H^*) < J(\chi\varphi^{-1}(H^*))$.

PROOF. By Lemmas 9.1 and 9.2, we have $\operatorname{tr}(A_1^{-1}A_2)>0$. Hence by Lemma 9.2, we have $\operatorname{tr}(B_1^{-1}B_2)>0$. Since $\chi\varphi^{-1}(H^*) = \langle B_1^{-2}B_2, B_1^{-1}\rangle$, by Lemma 7.4 we have that $J(H^*) < J(\chi\varphi^{-1}(H^*))$ if and only if $\operatorname{tr}^2(B_1^{-2}B_2) < \operatorname{tr}^2(B_1) < 0$. Hence it suffices to show that $Y := i^2\operatorname{tr}^2(B_1^{-2}B_2) - i^2\operatorname{tr}^2(B_1) > 0$. By induction we will show $Y>0$, that is, we will show that if $i\operatorname{tr}(B_1) > i\operatorname{tr}(B_2)$, then $i\operatorname{tr}(B_1^{-2}B_2) > i\operatorname{tr}(B_1)$. Since $\operatorname{tr}(B_1^{-1}B_2)>2$ by Lemmas 9.1 and 9.2, we have

$$\begin{aligned} i\operatorname{tr}(B_1^{-2}B_2) - i\operatorname{tr}(B_1) &= i\operatorname{tr}(B_1^{-1}B_2)\operatorname{tr}(B_1^{-1}) - i\operatorname{tr}(B_2) - i\operatorname{tr}(B_1) \\ &> 2i\operatorname{tr}(B_1^{-1}) - i\operatorname{tr}(B_2) - i\operatorname{tr}(B_1) > 0. \end{aligned}$$

Since $i\operatorname{tr}(B_1)>0$, we have

$$i\operatorname{tr}(B_1^{-2}B_2) + i\operatorname{tr}(B_1) = i\operatorname{tr}(B_1^{-2}B_2) - i\operatorname{tr}(B_1) + 2i\operatorname{tr}(B_1) > 0.$$

Therefore we have the desired inequality $Y>0$. q.e.d.

COROLLARY. *Let* $G = \langle A_1, A_2\rangle \in R_{\mathrm{VII}}S_2^0\backslash M_{\mathrm{VII}}$ *with* $-1 < \rho(G) < 0$. *Let* $H^* = \chi(G)$. *Then* $J(H^*) < J(\chi\varphi^{-m}(H^*))$ $(m = 1, 2, 3, \ldots)$.

PROOF. By Lemma 9.7, we have $J(H^*) < J(\chi\varphi^{-1}(H^*))$. Since $-1 < \rho(\varphi^{-1}\chi(G)) < 0$ by Lemma 9.4, we have $J(\chi\varphi^{-1}(H^*)) \le J(\chi\varphi^{-m}(H^*))$ $(m = 1, 2, 3, \ldots)$ by Lemma 9.5. Hence we have the desired inequality. q.e.d.

Proposition 9.1 follows from this corollary. Proposition 7.3 follows from Propositions 7.4, 8.2 and 9.1. We have Proposition 7.2 by Proposition 7.3 and Lemma 7.5.

§10. Proof of Theorem 3: Part 4 and Proof of Theorem 2.

In this section we will finish the proof of Theorem 3 and give a proof of Theorem 2. First we will show the following proposition.

PROPOSITION 10.1. *Let* $G = \langle A_1, A_2\rangle \in M_{\mathrm{VII}}$. *Then* $J(G) > 4(1+\sqrt{2})^2$. *The lower bound is the best possible.*

We set

$$\partial^+ M_{\mathrm{VII}} = \{(t_1, t_2, \rho) \mid (-\rho)^{1/2} = \{1 - (-t_1)^{1/2}(-t_2)^{1/2}\}/\{(-t_1)^{1/2} + (-t_2)^{1/2}\}, \\ -1 < t_2 < 0, -1 < t_1 < 0\}$$

and

$$\partial^- M_{\mathrm{VII}} = \{(t_1, t_2, \rho) \mid (-\rho)^{1/2} = \{(-t_1)^{1/2} + (-t_2)^{1/2}\}/\{1 - (-t_1)^{1/2}(-t_2)^{1/2}\}, \\ -1 < t_2 < 0, -1 < t_1 < 0\}.$$

LEMMA 10.1.

$$\inf\{J(G) \mid G \in \partial^+ M_{\mathrm{VII}}\} = \inf\{J(G) \mid G \in \partial^- M_{\mathrm{VII}}\}.$$

PROOF. Noting that $N_1(\partial^+ M_{\mathrm{VII}}) = \partial^- M_{\mathrm{VII}}$, we have the desired result by Lemma 5.3. q.e.d.

LEMMA 10.2. $J(G) \ge 4(1+\sqrt{2})^2$ *on* $\partial^+ M_{\mathrm{VII}}$. *The lower bound is the best possible.*

PROOF. Let $\tau = (t_1, t_2, \rho) \in \partial^+ M_{\mathrm{VII}}$. We set $X = (-t_1)^{1/2}$ and $Z = (-\rho)^{1/2}$. Then the equation

$$(-t_2)^{1/2} = \{1 - (-\rho)^{1/2}(-t_1)^{1/2}\}/\{(-\rho)^{1/2} + (-t_1)^{1/2}\}$$

turns into

$$t_2 = -\{(1 - XZ)/(X + Z)\}^2.$$

By substituting $t_1 = -X^2$, $\rho = -Z^2$ and $t_2 = -\{(1 - XZ)/(X + Z)\}^2$ for

$$J(\tau) = \frac{|1 - t_1|^2}{|t_1|} + \frac{|1 - t_1|^2|1 - t_2|^2|\rho|}{|t_1|\,|t_2|\,|\rho - 1|^2},$$

we have

$$J(\tau) = \frac{(1+X^2)^2}{X^2} + \frac{(1+X^2)^4 Z^2}{X^2(X+Z)^2(1-XZ)^2}.$$

By calculus we have that $J(\tau)$ attains the minimum value $4(1+\sqrt{2})^2$ at the point $(t_1, t_2, \rho) = (t_{10}, t_{20}, -1)$, where $t_{10} = -(1+\sqrt{2}) + \sqrt{2+2\sqrt{2}}$ and $t_{20} = -\{(1-(-t_{10})^{1/2})/(1+(-t_{10})^{1/2})\}^2$. q.e.d.

By corollary to Lemma 5.4, Lemmas 10.1 and 10.2 we have Proposition 10.1.

Proof of Theorem 3. We can prove Theorem 3 by Proposition 7.1 and 10.1. We can see by Example 2 in §11 that the lower bound $4(1+\sqrt{2})^2$ is the best possible. q.e.d.

Proof of Theorem 2. Theorem 2 follows from Theorem 3, Proposition 2.3 and Lemma 5.3. Example 3 in §11 shows that the lower bound $4(1+\sqrt{2})^2$ is the best possible. q.e.d.

§11. Examples.

Let $\tau_n = \{(t_{1n}, t_{2n}, \rho)\}$ $(n = 1, 2, 3, \ldots)$ be a sequence of points in $\boldsymbol{R}^3 \cap \boldsymbol{M}_2$ and let $G_n = \langle A_{1n}, A_{2n} \rangle$ be the groups representing τ_n. In this section we will give sequences of classical Schottky groups $\{G_n\}$ whose Jørgensen's numbers $J(G_n)$ tend to the lower bound in the inequalities in Theorems 1, 2 and 3.

Example 1 (Type II). Let $t_{1n} = (1-1/n)^2$, $t_{2n} = -(2-\sqrt{3}) + (3-\sqrt{3})/2(n+5)$ and $\rho_n = 2/\sqrt{3}n + 1$ $(n = 1, 2, 3, \ldots)$. Then (i) $G_n \in R_{\mathrm{II}}\boldsymbol{S}_2^0$ for all sufficiently large integers n and (ii) $\lim_{n\to\infty} J(G_n) = 16$.

Example 2 (Type VII). Let $t_{1n} = -(\sqrt{-t_{10}} - 1/n)^2$, $t_{2n} = t_{20}$ and $\rho_n = -1$ $(n = 1, 2, 3, \ldots)$, where $t_{10} = -(1+\sqrt{2}) + \sqrt{2} + \sqrt{2+2\sqrt{2}}$ and $t_{20} = -\{(1-(-t_{10})^{1/2})/(1+(-t_{10})^{1/2}\}^2$. Then (i) $G_n \in R_{\mathrm{VII}}\boldsymbol{S}_2^0$ for all sufficiently large integers n and (ii) $\lim_{n\to\infty} J(G_n) = 4(1+\sqrt{2})^2$.

Example 3 (Type V). Let $\tau_n = (t_{1n}, t_{2n}, \rho)$ $(n = 1, 2, 3, \ldots)$ be as in Example 2, and let $G_n = \langle A_{1n}, A_{2n} \rangle \in R_{\mathrm{VII}}\boldsymbol{S}_2^0$ represent τ. We set $G_n' = N_3(G_n)$, where N_3 is the Nielsen transformation defined in §2. By Proposition 2.3 and Lemma 5.3, we have (i) $G_n' \in R_{\mathrm{V}}\boldsymbol{S}_2^0$ for all sufficiently large integer n and (ii) $\lim_{n\to\infty} J(G_n') = 4(1+\sqrt{2})^2$.

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