J. Math. Soc. Japan
Vol. 50, No. 4, 1998

Small divisor problems and divergent formal power series solutions

By Masafumi YOSHINO

(Received Jan. 24, 1996)
(Revised Nov. 29, 1996)

1. Introduction.

In this note, we shall give realizations of divergent formal power series solutions of semilinear Goursat problems when small divisor difficulities occur without assuming any diophantine condition. As far as the author knows, small divisor phenomenon in Goursat problems was first noted by Leray for the following equation (cf. [**3**])

$$\varepsilon\partial_1\partial_2 u - \partial_1^2 u - \partial_2^2 u = f(x), \quad x = (x_1, x_2) \in \boldsymbol{R}^2 \tag{1.1}$$

with the boundary condition

$$u(x) = 0 \quad \text{if } x_1 = 0 \text{ or } x_2 = 0, \tag{1.2}$$

where $f(x)$ is analytic at the origin and $\partial_j = \partial/\partial x_j$, $j = 1, 2$. For the sake of brevity we denote the condition (1.2) by $u = O(x_1 x_2)$ and the problem (1.1)–(1.2) by (G). If we set $\partial_1\partial_2 u = v$ then, by (1.2), the problem (G) is equivalent to the equation $(\varepsilon - L)v = f(x)$, where $L = \partial_1^{-1}\partial_2 + \partial_1\partial_2^{-1}$ with $\partial_1^{-1}v(x_1, x_2) = \int_0^{x_1} v(t, x_2)\,dt$ and so on.

Leray showed that the eigenvalues of L have infinite dimensional eigenspaces and they form a dense subset of the closed interval $[-2, 2]$. Moreover, there exists a dense subset of $\varepsilon \in [-2, 2]$ which are not eigenvalues of L such that the equation $(\varepsilon - L)v = f(x)$ has divergent formal power series solutions with arbitrarily given growth order for an appropriate choice of f. He also proved the convergence of formal power series solutions under a certain diophantine condition. (cf. [**3**], [**11**] and Corollary 2.3 which follows.) One can easily prove that Leray's diophantine condition is weaker than the so-called Brjuno condition. (cf. [**4**] and [**10**]). The problems caused by such resonances and diophantine phenomena are typical small divisor problems which have been extensively studied by many authors. (cf. [**1**] and [**10**]).

The main point in this note is that we assume neither Brjuno condition nor Leray's diophantine condition. In spite of this we shall show that there exists a smooth solution which is asymptotically equal to a given divergent formal power series solution if the equations are nonresonant. (cf. (2.5) and a) of Remark 2.1.) This, in particular, implies the solvability of (G) in a class of smooth functions in a nonresonant case. We stress that because the linearized operator L has a dense eigenvalues in $[-2, 2]$ a

1991 *Mathematics Subject Classification*. primary 35C20 secondary 35A20, 35Q15.
Key words and phrases. small divisor, divergent formal power series, Goursat problem, Riemann-Hilbert factorization.
This research was partly supported by Grant-in-Aid for Scientific Research (No. 07640250), Ministry of Education, Science and Culture, Japan and by Chuo University Special Research Program. (1996)

generalized implicit function theorem no longer works in order to solve nonlinear problems with linear part L if $\varepsilon \in [-2,2]$ satisfies no diophantine condition. (cf. Nirenberg, [**8**]). On the other hand, our problem corresponds to an exceptional case from the viewpoint of the $\mathcal{E}$-wellposedness, i.e., unique solvability in a class of smooth functions. Indeed, in [**9**] it was proved that the problem (G) is $\mathcal{E}$-wellposed in some neighborhood of the origin if and only if $\varepsilon \in \boldsymbol{R}\backslash[-2,2]$, i.e., the equation is hyperbolic. Therefore our problem is illposed and we cannot expect the existence of a smooth solution in a neighborhood of the origin even for the linear problem (G). Finally, we remark another critical situation of our problem. In [**6**], it was proved that the unique solvability of a Goursat problem is equivalent to invertibility of a certain Toeplitz operator on a Hardy space $H^2(\boldsymbol{T}^2)$ on the torus $\boldsymbol{T}^2 = \boldsymbol{R}^2/2\pi\boldsymbol{Z}^2$. In case of the problem (G), we can easily see that the corresponding Toeplitz operator is invertible if and only if $\varepsilon \notin [-2,2]$. If $\varepsilon \in [-2,2]$, the Toeplitz operator is not a Fredholm operator due to the fact that the Toeplitz symbol is not Riemann-Hilbert factorizable. (cf. [**6**]).

We shall show that divergent formal solutions are asymptotic expansions of some smooth solutions to (G) in the sector $x_1 \geq 0$, $x_2 \geq 0$ without any diophantine conditions on ε. The result can be extended to semilinear equations. As an application of our theorem, we shall show a certain alternatives for the illposed problem (G). (cf. Theorem 2.4.)

2. Notations and results.

For $d \geq 2$ we write the variable in $\boldsymbol{R}^d$ by $x = (x_1, x_2, \dots, x_d) = (x_1, x_2, x'') = (x', x'')$, and the differentiations by $D^\alpha = (-i\partial_1)^{\alpha_1} \cdots (-i\partial_d)^{\alpha_d}$ where $\partial_j = \partial/\partial x_j (j = 1, \dots, d)$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \boldsymbol{Z}_+^d$. We consider the following semilinear equation in $\boldsymbol{R}^d$

$$L(x,D)u - b(x,u) = 0, \quad L(x,D)u = \sum_{|\alpha| \leq 2} a_\alpha(x) D^\alpha u \tag{2.1}$$

with the boundary condition

$$u = 0 \quad \text{on} \quad x_1 = 0 \quad \text{and} \quad x_2 = 0, \tag{2.2}$$

where $a_\alpha(x)$ is real-valued and C^∞ in some neighborhood of the origin, and where $b(x,u)$ is smooth with respect to x and u in some neighborhood of the origin. Without loss of generality, we may assume that

$$(\partial b/\partial u)(x,0) \equiv 0. \tag{2.3}$$

We assume that (2.1) is elliptic at the origin, that is, there exists $K > 0$ such that

$$\sum_{|\alpha|=2} a_\alpha(0)\xi^\alpha \geq K|\xi|^2 \quad \text{for all } \xi \in \boldsymbol{R}^d. \tag{2.4}$$

Let $a_\alpha(x) \sim \sum_\eta a_\eta^\alpha(x'')x'^\eta$ and $b(x,u) \sim \sum_{\eta,j} b_{\eta,j}(x'')x'^\eta u^j$ be partial Taylor expansions of $a_\alpha(x)$ and $b(x,u)$ at $x = 0$ and $u = 0$, respectively. And, let $v(x) = x_1x_2 \sum_\eta v_\eta(x'')x'^\eta$ be a formal power series, where $v_\eta(x'')$ are smooth and real-valued in some neighborhood of the origin independent of η. We say that $v(x)$ is a formal power series solution of (2.1)–(2.2), if $v(x)$ formally satisfies (2.1) with $a_\alpha(x)$ and $b(x,u)$ replaced by their partial Taylor expansions in the above.

We set $\Omega = \{x = (x_1, x_2, x'') \in \boldsymbol{R}^d; x_1 \geq 0, x_2 \geq 0, |x| < \delta\}$, where we take $\delta > 0$ so small that L is strongly elliptic in Ω, that is, the condition (2.4) with $a_\alpha(0)$ replaced by $a_\alpha(x)$ is satisfied for each point $x \in \Omega$. We modify Ω in such a way that the boundary of Ω is smooth except for the edge $x_1 = x_2 = 0$ of some neighborhood of the origin. For the sake of simplicity, we denote the modified Ω by the same letter Ω. Following the terminology of [**2**], such a domain Ω is said to be a domain with 2-dimensional edges. (cf. (2.16) of [**2**]). Then we have

THEOREM 2.1 (Linear case). *Suppose* (2.4) *and that $b(x,u)$ is independent of u, i.e., $b(x,u) = f(x)$ for some smooth function $f(x)$. Moreover, assume that*

(2.5) *every formal power series solution w of the equation $L(0; \partial_1, \partial_2, 0)w = 0$ with the boundary condition* (1.2) *is unique.*

Suppose that (2.1)–(2.2) *has a formal power series solution $v(x) = x_1 x_2 \sum_\eta v_\eta(x'') x'^\eta$. Then, there exist a neighborhood V of the origin and a C^∞ function $u(x)$ in V which satisfies* (2.2) *and* (2.1) *in $V \cap \Omega$ such that*

$$u(x) - x_1 x_2 \sum_{|\eta| \leq n} v_\eta(x'') x'^\eta = O(|x'|^n), \quad n = 1, 2, \ldots \tag{2.6}$$

when $|x| \to 0$.

THEOREM 2.2 (Semilinear case). *Suppose that the conditions* (2.4), (2.5) *and*

$$a_0(0) > 0 \tag{2.7}$$

are satisfied. Moreover, assume that (2.1)–(2.2) *has a formal power series solution $v(x) = x_1 x_2 \sum_\eta v_\eta(x'') x'^\eta$. Then, there exist a neighborhood V of the origin and a smooth function $u(x)$ which satisfies* (2.1) *and* (2.2) *in $V \cap \Omega$ such that* (2.6) *is satisfied.*

REMARKS 2.1. a) The condition (2.5) corresponds to a nonresonant condition of the Goursat problem (2.1)–(2.2).

b) Theorem 2.2 holds, if we replace (2.2) with nonzero boundary conditions on two regular normally crossing hypersurfaces.

We shall apply Theorem 2.1 to (G) when $\varepsilon \in [-2, 2]$. Following Leray [**3**] (see also [**4**] and [**11**]), our problem is reduced to a division problem by the quantity ρ defined by

$$\rho = \liminf_{n \to \infty} |\sin n\pi t|^{1/n}, \quad \varepsilon = 2 \cos \pi t, \quad 0 \leq t \leq 1.$$

Namely, if $\rho > 0$, (G) has a unique analytic (local) solution for any analytic f. If $\rho = 0$ then two cases occur, i) t is an irrational number, ii) t is a rational number. In the case i), (G) has a unique formal solution for any analytic f, which is not necessarily convergent. In the case ii), (G) has a formal solution if and only if f satisfies an infinite number of compatibility conditions. Moreover, formal solutions are not unique, and they do not necessarily converge because of infinite dimensional kernel. (cf. [**3**] and [**11**]). We note that the latter case corresponds to a resonant one. Hence we are interested in the case i). For small $\delta > 0$, we set $\Omega_0 = \{x = (x_1, x_2); x_1 \geq 0, x_2 \geq 0, |x| < \delta\}$, where as in Theorem 2.1 we may assume that the boundary of Ω_0 is smooth except for the origin $x = 0$. Then we have

Corollary 2.3. *Suppose that* $\rho = 0$ *and* t *is irrational. Then every formal power series solution of* (G) *is an asymptotic expansion of some smooth solution of* (G) *in* Ω_0.

The following alternative shows rather simple structure of (G) in Ω despite the non wellposedness of (G) (cf. [**3**], [**9**]).

Theorem 2.4. *Suppose that* $\varepsilon \in [-2, 2]$. *Then, for every given function* f *being smooth in some neighborhood of the origin the following alternative holds; either* (G) *does not have a smooth solution in any neighborhood of the origin of* Ω_0 *or the solution is not unique.*

3. Proof of theorems.

3.1. Preliminary lemmas. In order to prove Theorem 2.1 we recall two lemmas of [**2**]. Let Ω be the domain given in Theorem 2.1. For $s \geq 0$, $H^s(\Omega)$ and $C_0^\infty(\Omega)$ denote the Sobolev space in Ω and the set of smooth functions with compact supports in Ω, respectively. We note that the space $H^0(\Omega)$ $(= L^2(\Omega))$ denotes the set of square integrable functions on Ω. The space $H_0^s(\Omega)$ denotes the closure of the set $C_0^\infty(\Omega)$ in $H^s(\Omega)$.

Let L be given by (2.1). Then according to [**2**] we say that L has a regularity property if $u \in H_0^1(\Omega)$ and $Lu \in H_0^{s-1}(\Omega)$ imply $u \in H^{s+1}(\Omega)$ for every s. We say that the operator $L : H_0^1(\Omega) \cap H^{s+1}(\Omega) \to H^{s-1}(\Omega)$ is a Fredholm operator, if the kernel of L is finite dimensional, the range *Rang* L of L is closed and the codimension of *Rang* L is finite.

Let A be the edge of Ω, $A = \{x \in \Omega; x_1 = x_2 = 0\}$. We set $x = (z, y)$ where $z = (x_1, x_2)$, $y = (x_3, \dots, x_d)$. For each $X = (0, X_y) \in A$, $X_y = (x_3, \dots, x_d)$ we denote by Γ_X, the cone $\{z \in \boldsymbol{R}^2; (z, X_y) \in \Omega\}$.

We denote by $\tilde{L}_x(D_y, D_z)$ the operator L frozen at X, namely, the one obtained by putting $x = X$ in L. We define the operator $\tilde{L}_X(0, D_z)$ by setting $D_y = 0$ in $\tilde{L}_X(D_y, D_z)$. The operator $L_X(D_z)$ denotes the principal part of $\tilde{L}_X(0, D_z)$. We take the polar coordinate (r, θ) in the z-plane with center at X. We define the class of functions $S^\lambda(\Gamma_X)$ by

$$S^\lambda(\Gamma_X) = \left\{ u; u = r^\lambda \sum_{q=0}^{Q} u_q(\theta)(\log r)^q, \quad u_q \in H_0^1((0, \pi/2)) \right\}. \tag{3.1}$$

Following [**2**] we say that L_X is injective modulo polynomials on $S^\lambda(\Gamma_X)$, if $u \in S^\lambda(\Gamma_X)$ and $L_X u \equiv 0$ modulo polynomials imply that $u \equiv 0$ modulo polynomials. Then we have

Lemma 3.1 ([**2**, Theorem 1.6]). *Let* $s > 0$, $s \neq 1/2$, *and assume that* L *is strongly elliptic. Then the operator,* $L : H_0^1(\Omega) \cap H^{s+1}(\Omega) \to H^{s-1}(\Omega)$ *is a Fredholm operator if and only if the following condition is satisfied.*

(3.2) *For any* $X \in A$ *and any* $\forall\lambda$, $\mathrm{Re}\lambda \in [0, s]$, L_X *is injective modulo polynomials on* $S^\lambda(\Gamma_X)$.

Lemma 3.2 ([**2**, Theorem 1.6]). *Let* $s > 0$, $s \neq 1/2$. *Then* L *has a regularity property if and only if* (3.2) *is satisfied. Moreover, if we assume* (3.2) *and the injectivity* *of L we have the estimate*

$$\exists c > 0, \forall u \in H^{s+1}(\Omega) \cap H_0^1(\Omega), \|u\|_{s+1,\Omega} \le c\|Lu\|_{s-1,\Omega} \text{ for every } s \ge 1.$$

The constant c in the above estimate is independent of δ with $0 < \delta < \delta_0$ for some $\delta_0 > 0$, where δ is a parameter in the definition of Ω.

The latter half of Lemma 3.2 follows from Peetre's lemma. (cf. Lemma 5.0 of [**2**] (5.2) of [**2**]). We can easily see that the constant c is independent of $\delta, 0 < \delta < \delta_0$ for some $\delta_0 > 0$, in view of the proofs of the theorems. (cf. Chapter 3 of [**2**]).

3.2. Proof of theorem 2.1. We divide the proof into three steps.

Step 1. We take $u_1 \in C_0^\infty$ such that, the partial Taylor expansions of u_1 with respect to x_1 and x_2 are equal to the formal solution v, and that $f - Lu_1 = O(|x'|^n)$ $(n = 1, 2, \ldots)$ as $|x'| \to 0$, $x' = (x_1, x_2)$. By multiplying $g \equiv f - Lu_1$ by a suitable cutoff function we may suppose that g is a smooth function on $\boldsymbol{R}^d$.

We shall choose an $h \in C_0^\infty(\Omega)$ such that the support of h does not contain the origin, and that the Dirichlet problem

$$Lu = g + h \quad \text{in } \Omega, \quad u \in H_0^1(\Omega) \tag{3.3}$$

has a smooth solution in Ω. We shall check the condition (3.2) for $s \ge 0$. By definition $L_X(D_z)$ is a homogeneous constant-coefficients second order operator with two independent variables, $L_X(D_z) = a\partial_1^2 + 2b\partial_1\partial_2 + c\partial_2^2$ for some real constants a, b and c. By a scale change of variables we may assume that $a = c = -1$. Moreover, by another scale change of variables and a rotation of the coordinates we may assume that $b = 0$, that is $L_X(D_z) = -\partial_1^2 - \partial_2^2$. We introduce the polar coordinate (r, θ) centered at X. Then we have

$$r^2 L_X(D_z) = -(r\partial_r)^2 - \partial_\theta^2 \tag{3.4}$$

where θ moves on some interval $\Omega_X = \{\theta; \theta_0 \le \theta \le \theta_0 + \pi/2\}$ for some θ_0 depending on a, b and c.

By comparing the powers of $\log r$ in the equation, $r^2 L_X(D_z)u = 0$ $(u = r^\lambda \sum_{0 \le q \le Q} u_q(\theta)(\log r)^q, u_q \in H_0^1(\Omega_X))$ we have the recurrence relations

$$(q+1)(q+2)u_{q+2}(\theta) + 2\lambda(q+1)u_{q+1}(\theta) + \lambda^2 u_q(\theta) + \partial_\theta^2 u_q(\theta) = 0, q = 0, 1, \ldots, Q. \tag{3.5}$$

By setting $q = Q$ in (3.5) we have

$$\partial_\theta^2 u_Q(\theta) + \lambda^2 u_Q(\theta) = 0, \quad u_Q \in H_0^1(\Omega_X). \tag{3.6}$$

The eigenvalues and the eigenfunctions of (3.6) are given by $\lambda = 2k$ and $\sin 2(\theta - \theta_0)k$ $(k = 1, 2, \ldots)$, respectively. We set $q = Q - 1$ in (3.5). Then we have

$$\partial_\theta^2 u_{Q-1}(\theta) + \lambda^2 u_{Q-1}(\theta) = -2\lambda Q u_Q(\theta), \quad u_{Q-1} \in H_0^1(\Omega_X). \tag{3.7}$$

If λ is not an eigenvalue of (3.6), then it follows from (3.6) and (3.7) that $u_Q = u_{Q-1} = 0$. Similarly, we have $u_q = 0$ $(q = 0, 1, \ldots, Q)$. Hence L_X is injective. On the other hand, if $\lambda = 2k$ and $u_Q = \sin 2(\theta - \theta_0)k$ then (3.7) does not have a solution.

Therefore, we have $Q = 0$. It follows that $u = r^{2k}\sin 2(\theta - \theta_0)k$. Hence u is a polynomial of x_1 and x_2. This proves (3.2).

Step 2. By Lemma 3.1 and Step 1 the operator $L : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ is a Fredholm operator. Let $\varphi_1, \dots, \varphi_N$ be an orthnormal basis of the orthogonal complement of the image of L in $L^2(\Omega)$. We want to choose $h \in C_0^\infty(\Omega)$ such that $\langle g + h, \varphi_j \rangle = 0$ for $j = 1, \dots, N$, where $\langle\ ,\ \rangle$ denotes the usual inner product in $L^2(\Omega)$.

Since $\varphi_1, \dots, \varphi_N$ are linearly independent, there exist points $t_1, \dots, t_N \in \Omega$ such that

$$\begin{vmatrix} \varphi_1(t_1) & \cdots & \varphi_1(t_N) \\ \vdots & & \vdots \\ \varphi_N(t_1) & \cdots & \varphi_N(t_N) \end{vmatrix} \neq 0. \tag{3.8}$$

Indeed, if otherwise then the vector ${}^t(\varphi_1(t), \dots, \varphi_N(t))$ lies in a plane for all t. This contradicts the linearly independentness of φ_j $(j = 1, \dots, N)$. We note that we can take the points $t_1, \dots, t_N$ from the interior of Ω.

On the other hand, φ_j is smooth in the interior of Ω, because $L^*\varphi_j = 0$ in the interior of Ω, where L^* is a formal adjoint of L. Hence we can choose $\psi_k \in C_0^\infty(\Omega)$ $(k = 1, \dots, N)$ with supports contained in a small neighborhood of t_k such that $\langle \varphi_j, \psi_k \rangle - \varphi_j(t_k)$ are sufficiently small for $j, k = 1, \dots, N$. Hence we have $\det(\langle \varphi_j, \psi_k \rangle)_{j,k} \neq 0$. We set $h = \sum_{k=1} c_k \psi_k$. Then the equations $\langle g + h, \varphi_j, \rangle = 0$ $(j = 1, \dots, N)$ give rise to the equations $\langle g, \varphi_j \rangle = -\sum_k c_k \langle \varphi_j, \psi_k \rangle$ $(j = 1, \dots, N)$, which uniquely determine c_k. We note that $g + h \equiv g$ in some neighborhood of the origin by the definition of t_k. The equation (3.3) has a solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$. It follows from Lemma 3.2 and Step 2 that $u \in H^\infty(\Omega)$. Hence, by Sobolev embedding theorem u is smooth up to the boundary of Ω.

Step 3. We shall show that all derivatives of u with respect to x' vanish at the origin. Let $u = \sum_\eta u_\eta(x'')x'^\eta$ be a Taylor expansion of u at the origin. We substitute u into (3.3) and (2.2), and we compare the coefficients of x'^η. Because all derivatives of $g + h$ with respect to x_1 and x_2 vanish on $x_1 = x_2 = 0$, we have

$$A_k(x'')U_k(x'') + (\cdots) = 0, \quad k = 1, 2, \dots, \tag{3.9}$$

where $A_k(x'')$ is a Toeplitz matrix given by

$$A_k(x'') = (c_{i-j})_{i,j},$$

$$c_{-1} = a_{(2,0;0)}(x''), c_0 = a_{(1,1;0)}(x''), c_1 = a_{(0,2;0)}(x''), c_n = 0 \text{ (otherwise)},$$

with $a_\alpha(x)$ being given by (2.1). The vector $U_k(x')$ is defined by an appropriate ordering of $u_\eta(x'')$ for $|\eta| = k + 2$. Here the dots in (3.9) denote the terms determined by $u_\eta(x'')$, $|\eta| \leq k + 1$. By the assumption (2.5) we see that $\det A_k(0) \neq 0$ for all k. Hence, by indection on k every $U_k(x'')$ vanishes in some neighborhood of the origin. In view of the definition of u_1, $u + u_1$ gives the desired solution. ■

Proof of theorem 2.4. By substituting the formal power series expansions of u into (1.1) we can easily prove: If t is a rational number, then the problem (G) has a smooth solution only if the Taylor coefficients of f at the origin satisfy infinite number of compatibility conditions. Moreover, in view of the recurrence relations for the partial Taylor expansions of the solution similar to (3.9) we see that there is a nontrivial kernel of (G). This shows that the theorem is true for rational t.

In case t is irrational, there exists a formal solution to (G). Hence (G) has a smooth solution in some Ω_0 by Theorem 2.1. In view of the proof of Theorem 2.1 a smooth solution is not unique by the arbitrariness of h in (3.3). We note that these solutions have the same Taylor coefficients at the origin. ■

In order to prove Theorem 2.2 we first prepare a lemma.

LEMMA 3.3 ([7, Theorem 7.1]). *Let Ω be a domain as in Theorem 2.2 and let l and s be integers such that $l \geq 1$ and $s > d/2$. Assume that $u_1, \ldots, u_l \in H^s(\Omega)$. Then, if $\nu_1, \ldots, \nu_l \in \boldsymbol{N}^d$ and $\sum_{j=1}^{l} |\nu_j| \leq s$ it holds that*

$$\partial^{\nu_1} u_1 \partial^{\nu_2} u_2 \cdots \partial^{\nu_l} u_l \in L^2(\Omega)$$

and we have the estimate

$$\|\partial^{\nu_1} u_1 \partial^{\nu_2} u_2 \cdots \partial^{\nu_l} u_l\|_{L^2} \leq C \prod_{j=1}^{l} \|u_j\|_s,$$

where the constant $C > 0$ depends only on d, $\nu_1, \ldots, \nu_l$ and does not depend on $u_1, \ldots, u_l$.

PROOF. Because s is an integer such that $s > d/2$ it follows that $k := s - [d/2] - 1 \geq 0$, where $[d/2]$ denotes the largest integer which does not exceed $d/2$. We set $\tilde{\nu}_j = \nu_j - k$ $(1 \leq j \leq l)$. If $\tilde{\nu}_j < 0$, it follows that $\nu_j < k$ and $\partial^{\nu_j} u_j \in H^{s-k} = H^{[d/2]+1}$. Hence, the Sobolev embedding theorem implies that $\partial^{\nu_j} u_j$ is continuous, and its supremum norm is bounded by $\|u_j\|_s$. Therefore, $\partial^{\nu_j} u_j$ can be omitted from the beginning. By the same reason, we may assume that $\nu_j \geq 1$.

Because $\sum_{j=1}^{l} |\nu_j| \leq s$, it follows that

$$\sum_{j=1}^{l} |\tilde{\nu}_j| \leq s - kl \leq s - k = [d/2] + 1,$$

and $\partial^{\nu_j} u_j = \partial^{\tilde{\nu}_j} \partial^k u_j$, $\partial^k u_j \in H^{s-k} = H^{[d/2]+1}$. Therefore, by replacing u_j with $\partial^k u_j$ if necessary we may assume that $s = [d/2] + 1$. Moreover, the lemma is clear in case $l = 1$. Under these conditions, our lemma is reduced to Theorem 7.1 of [7]. For the sake of completeness, we will give the proof.

By Sobolev's embedding theorem and the boundedness of the domain, we see that $\partial^{\nu_j} u_j \in L^{p_j}$ $(1 \leq j \leq l)$, where p_j is any number satisfying

$$\frac{|\nu_j|}{d} - \frac{1}{d} \leq \frac{1}{p_j} \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{p_j} \neq 0 \quad \text{if } d \text{ is even},$$

$$\frac{|\nu_j|}{d} - \frac{1}{2d} \leq \frac{1}{p_j} \leq \frac{1}{2} \quad \text{if } d \text{ is odd}.$$

We have, if d is even,

$$\sum_{j=1}^{l} \left(\frac{|\nu_j|}{d} - \frac{1}{d} \right) \leq \frac{1}{d} \left(\frac{d}{2} + 1 - l \right) = \frac{1}{2} + \frac{1-l}{d} \leq \frac{1}{2} - \frac{1}{d} < \frac{1}{2},$$

since $l \geq 2$. Similarly, if d is odd we have $\sum_{j=1}^{l}(|\nu_j|/d - 1/(2d)) \leq 1/2 - 1/(2d)$. Therefore, one can choose $p_j > 1$ satisfying the above inequality and $\sum_{j=1}^{l} 1/p_j = 1/2$. By Hölder's inequality, we have

$$\int |\partial^{\nu_1} u_1 \cdots \partial^{\nu_l} u_l|^2 \, dx \leq \prod_j \left(\int |\partial^{\nu_j} u_j|^{2 \cdot p_j/2} \, dx \right)^{2/p_j}$$
$$= \prod_j \|\partial^{\nu_j} u_j\|_{L^{p_j}}^2 \leq C \prod_j \|u_j\|_{[d/2]+1}^2,$$

for some $C > 0$ independent of j. This ends the proof of the lemma. ∎

3.3. PROOF OF THEOREM 2.2. We devide the proof into 5 steps.

Step 1. We take the domain Ω in Theorem 2.2 so small that $a_0(x) > 0$ in Ω. Let k be an integer and let $v(x) = x_1 x_2 \sum_\eta v_\eta(x'')x'^\eta$ be a formal solution of (2.1)–(2.2). We set $v_k(x) = x_1 x_2 \sum_{|\eta| \leq k} v_\eta(x'')x'^\eta$. We take a C^∞ function u_0 whose partial Taylor expansion with respect to $x' = (x_1, x_2)$ at $x' = 0$ is equal to $v(x)$. Then we have

$$Lu_0 - b(x, u_0) = (Lv_k - b(x, v_k)) + L(u_0 - v_k) + (b(x, v_k) - b(x, u_0)). \tag{3.10}$$

By the definition of the formal solution, the first term in the right-hand side of (3.10) is $O(|x'|^{k-2})$. The second term is $O(|x'|^{k-1})$ because $u_0 - v_k$ is $O(|x'|^{k+1})$ and L is a second order operator. In view of the assumption (2.3) the third term is $O(|x'|^{k+1})$. Since k is arbitrary we see that $Lu_0 - b(x, u_0)$ is flat at $x' = 0$.

Step 2. We set $u = u_0 + w$. Then (2.1) is equivalent to

$$Lw = b(x, u_0) - Lu_0 + b(x, u_0 + w) - b(x, u_0) \equiv F(w). \tag{3.11}$$

We shall solve (3.11) with the boundary condition $w = O(x_1 x_2)$ by iteration. We set $W_{-1} = 0$ and we want to determine w_n $(n = 0, 1, 2 \ldots)$ by the relations $w_n = O(x_1 x_2)$ and

$$Lw_0 = F(0), Lw_{n+1} = F(W_n) - F(W_{n-1}), W_n = w_0 + \cdots + w_n, n = 1, 2, \ldots \tag{3.12}$$

Step 3. Let $s > d/2 \geq 1$ be an integer and let $t > 0$ be a small number chosen later. We want to show that w_n $(n = 0, 1, 2, \ldots)$ are well-defined as smooth functions in some neighborhood of the origin of Ω, and they satisfy for some $C > 0$ independent of n and t that $\|w_n\|_s \leq Ct^{1+n/2}$ for $n = 0, 1, 2, \ldots$.

By (2.7) and the argument of Step 2 of Theorem 2.1 we can uniquely determine w_0 as a smooth function up to the boundary of Ω. It follows from Lemma 3.2 that

$$\|w_0\|_s \leq c\|F(0)\|_{s-2} \leq c\|F(0)\|_s.$$

Because $F(0)$ is flat at the origin, for any $t < 1/4$ we can take Ω such that $\|F(0)\|_s < t$. Hence we have $\|w_0\|_s \leq ct$.

Step 4. Suppose now that w_k, $(0 \leq k \leq n)$ are defined as smooth functions in some neighborhood of the origin of Ω so that $\|w_k\|_s \leq Ct^{1+k/2}$ for $k = 0, 1, 2, \ldots, n$.

By Lemma 3.2 and (2.7) we have the estimate, for $n = 1, 2, \ldots,$

$$\|w_{n+1}\|_s \leq c\|F(W_n) - F(W_{n-1})\|_{s-2} = c\|b(x, u_0 + W_n) - b(x, u_0 + W_{n-1})\|_{s-2} \tag{3.13}$$
$$= c\left\|w_n \int_0^1 b_u'(x, v) \, d\theta\right\|_{s-2},$$

where $v = u_0 + W_{n-1} + \theta w_n$. Hence it is sufficient to estimate $\|w_n b'_u(x,v)\|_{s-2}$, where $0 \le \theta \le 1$.

Because $u_0 = O(x_1 x_2)$ we take Ω so small that $\|u_0\|_s \le t$. By Sobolev embedding theorem and $s > d/2$, we have that

$$|v|_\infty \le (|u_0|_\infty + |W_{n-1}|_\infty + |w_n|_\infty) \le (t + Ct + \ldots + Ct^{1+n/2}).$$

Hence $b'_u(x,v)$ is well-defined for sufficiently small t. Moreover, it follows from (2.3) that $|b'_u(x,v)|_\infty \le Mt$ for some $M > 0$ independent of t.

Because s is an integer, it follows that

$$\|w_n b'_u(x,v)\|_{s-2}^2 = \sum_{|\alpha| \le s-2} \|D_x^\alpha (w_n b'_u(x,v))\|_{L^2}^2.$$

On the other hand we have

$$D_x^\alpha (w_n b'_u(x,v)) = \sum_{k+j=\alpha} D^k w_n D_x^j (b'_u(x,v)) \binom{\alpha}{k},$$

$$D_x^j (b'_u(x,v)) = \sum_{\gamma+\delta=j} \binom{j}{\gamma} D_x^\delta ((D_x^\gamma b'_u)(\cdot, v)),$$

where the differentiation D_x^δ applies to $F(x) := (D_x^\gamma b'_u)(\cdot, v(x))$. By the repeated use of the Leibniz rule, we have

$$D_x^\delta ((D_x^\gamma b'_u)(\cdot, v)) = \sum_{\substack{\delta = \nu_1 \eta^1 + \cdots + \nu_l \eta^l \\ l \ge 1, \nu_i \in N, \eta^j \in N^d}} C_{\eta^1 \ldots \eta^l, \nu, \delta} (D_x^{\eta^1} v)^{\nu_1} \cdots (D_x^{\eta^l} v)^{\nu_l} \times D_u^{|\nu|} (D_x^\gamma b'_u)(\cdot, v),$$

where $|\nu| = \nu_1 + \cdots + \nu_l$, $\nu = (\nu_1, \cdots, \nu_l)$, and the constants $C_{\eta^1 \ldots \eta^l, \nu, \delta}$ can be bounded by a constant depending only on s and the dimension d. Therefore we have

$$D_x^j (b'_u(x,v)) = \sum_{\gamma+\delta=j} \binom{j}{\gamma} \sum C_{\eta^1 \ldots \eta^l, \nu, \delta} D_u^{|\nu|} (D_x^\gamma b'_u)(\cdot, v) (D_x^{\eta^1} v)^{\nu_1} \cdots (D_x^{\eta^l} v)^{\nu_l}.$$

It follows that

$$D_x^\alpha (w_n b'_u(x,v)) = \sum_{k+j=\alpha} \binom{\alpha}{k} D^k w_n D_x^j (b'_u)(x,v)$$

$$= \sum_{k+j=\alpha} \binom{\alpha}{k} D_x^k w_n \sum_{\gamma+\delta=j} \binom{j}{\gamma} \sum C_{\eta^1 \ldots \eta^l, \nu, \delta} D_u^{|\nu|} (D_x^\gamma b'_u)(\cdot, v) (D_x^{\eta^1} v)^{\nu_1} \cdots (D_x^{\eta^l} v)^{\nu_l}.$$

In the above expression, the term which corresponds to $k = \alpha$ is $D_x^\alpha w_n b'_u(x,v)$. This term can be estimated in the following way;

$$\|D_x^\alpha w_n b'_u(x,v)\|_{L^2} \le \|w_n\|_{s-2} |b'_u(x,v)|_\infty \le Mt \|w_n\|_{s-2} \le MCt^{2+n/2}.$$

In order to estimate the general case $k \ne \alpha$ we shall estimate the term

$$\|(D_x^k w_n)(D_x^{\eta^1} v)^{\nu_1} \cdots (D_x^{\eta^l} v)^{\nu_l}\|_{L^2},$$

because the term $|D_u^{|\gamma|}(D_x^\gamma b_u')(\cdot,v)|_\infty$ is bounded by some constant. By Lemma 3.3 and the inductive assumption we have

$$\left\|(D_x^k w_n)\prod_{i=1}^{l}(D_x^{\eta^i}v)^{\nu^i}\right\|_{L^2} \le Ct^{1+n/2}\prod_{i=1}^{l}(Ct)^{\nu_i}.$$

Summing up the above estimates we obtain that

$$\|w_n b_u'(\cdot,v)\|_{s-2} \le C_1 t^{1+(n+1)/2} t^{1/2},$$

for some $C_1 > 0$ independent of n. We take t sufficiently small that $C_1 t^{1/2} \le 1$. Then we have that

$$\|w_{n+1}\|_s \le c\|w_n b_u'(\cdot,v)\|_{s-2} \le C_3 t^{1+(n+1)/2},$$

for some $C_3 > 0$ independent of n. This proves the desired estimates for w_{n+1}. We note that, by Lemma 3.2, these arguments also show that w_n's $(n = 0, 1, 2, \ldots)$ are defined as a smooth functions up to the boundary of Ω.

Step 5. By the estimates for w_k we can easily see that the series $\sum_{n=0}^\infty w_n$ converges to some w in H^s. The limit $w \in H_0^1 \cap H^s$ is a solution of (3.11) such that $w = O(x_1x_2)$. The regularity of w is a standard argument if we note Lemma 3.2. Indeed, the above arguments shows that $\|F(w)\|_s < \infty$ if $\|w\|_s < \infty$ because $s > d/2$. By Lemma 3.2, we see that $\|w\|_{s+2} < \infty$, which implies that w is smooth.

We shall show that w is flat on $x_1 = x_2 = 0$. We follow the arguments of Step 3 of the proof of Theorem 2.1. By the conditions (2.3) and $w = O(x_1x_2)$, $u_0 = O(x_1x_2)$, the contributions from the nonlinear terms of the equation (3.11) appear only in the dotted part in (3.9), which proves the assertion for w. This proves Theorem 2.2. ■

Acknowledgement The author would like to express sincere thanks to the referee who pointed out a gap in the proof of Theorem 2.2.

References

[1] V. I. Arnold, Small divisors and stability problems in classical and Celestial Mechanics, Uspekhi Mat. Nauk USSR **18** (**6**) (1963), 81–192.
[2] M. Dauge, Elliptic boundary value problems on corner domains, Lec. Note Math. **1341**, Springer Verlag, Berlin 1988.
[3] J. Leray, Caractère non Fredholmien du problème de Goursat, J. Math. Pures Appl. **53** (1974), 133–136.
[4] J. Leray and C. Pisot, Une fonction de la théorie des nombres, J. Math. Pures Appl. **53** (1974), 137–145.
[5] M. Miyake, An operator $L = aI - D_t^j D_x^{-j-\alpha} - D_t^{-j} D_x^{j+\alpha}$ and its nature in Gevrey functions, Tsukuba J. Math. **17** (**1**) (1993), 85–98.
[6] M. Miyake and M. Yoshino, Wiener-Hopf equation and Fredholm property of the Goursat problem in Gevrey space, Nagoya Math. J. **135** (1994), 165–196.
[7] S. Mizohata, Henbibun houteisikiron (in Japanese), Iwanami shoten, Tokyo, 1976.
[8] L. Nirenberg, An abstract form of the nonlinear Cauchy-Kowalevski theorem, J. Diff. Geometry **6** (1972), 561–576.
[9] T. Nishitani, On the $\mathscr{E}$-wellposedness for the Goursat problem with constant coefficients, J. Math. Kyoto Univ. **20** (1980), 179–190.
[10] J. C. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes de $(\boldsymbol{C}, 0)$, C. R. Acad. Sci. Paris **306** (1988), 55–58.

[11] M. Yoshino, Spectral property of Goursat problems, Tokyo J. Math. **4** (1981), 55–71.
[12] ———, Divergent formal solutions to Fuchsian partial differential equations, J. Math. Soc. Japan **40** (**1**) (1988), 139–150.

Masafumi YOSHINO
Faculty of Economics, Chuo University,
Higashinakano Hachioji Tokyo 192-03
Japan