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The first derived limit and compactly F_σ sets

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For a sequence $a = \{a(n)\}$ of integers (a member of $\boldsymbol{N}^{\boldsymbol{N}}$), set

$$A_a = \bigoplus_{m=1}^{\infty} \bigoplus_{n=1}^{a(m)} \boldsymbol{Z}.$$

One can view A_a as the group of all functions from the set $D_a = \{(m,n) : n \leq a(m)\}$ in $\boldsymbol{Z}$ which are equal to 0 on all but finitely many pairs (m,n) in D_a. Considering $\boldsymbol{N}^{\boldsymbol{N}}$ as a directed set ordered by the product-ordering ($a \leq b$ iff $a(n) \leq b(n)$ for all n), we get an inverse system

$$\mathscr{A} = \langle A_a, \pi_a^b, \boldsymbol{N}^{\boldsymbol{N}} \rangle$$

of Abelian groups, where $\pi_a^b : A_b \longrightarrow A_a$ are the natural projections. The first derived limit

$$\underleftarrow{\lim}^{(1)} \mathscr{A}$$

of this inverse system is an object of considerable interest in several areas of mathematics ([**14**], [**4**], [**9**], [**10**], [**6**; §8]). The purpose of this short note is to connect it with yet another area, descriptive set theory. The problem we consider was originally asked by Jayne and Rogers and formulated in its present form by Fremlin ([**11**], [**1**; 230 (d)], [**2**; DI]). The original question of J. E. Jayne and C. A. Rogers states whether for a given Polish space M and analytic subset X of M which is not Borel there is always a compact subset K of M such that $X \cap K$ is not Borel. This of course leads to similar questions about other classes of sets of reals and the way they behave from 'the point of view of compact sets' (see [**11**], [**12**], [**18**]). For example, the role of Martin's axiom and the negation of the continuum hypothesis (or more precisely the role of the boundedness number of the ordering of eventual dominance in $\boldsymbol{N}^{\boldsymbol{N}}$) in finding positive answers to these kind of questions has been recognized very early ([**11**]). This was the motivation behind the problem (which we solve here) whether similar assumptions are also sufficient to answer the Jayne–Rogers question without the restriction that the set X is analytic ([**1**; 230 (d)], [**2**; DI]). In our proofs we shall use ideas from a few different subjects. For background on the homological algebra needed for this paper the reader is referred to [**4**]. The background on descriptive set theory can be found in [**17**], while the background on forcing axioms can be found in [**1**], [**5**], [**6**], [**7**], and [**15**]. The basic notions and facts from topology can be found in [**19**].

§1. A compactly-simple set.

THEOREM 1. *If* $\underleftarrow{\lim}^{(1)} \mathscr{A} \neq 0$ *then there is a subset* X *of* $\boldsymbol{R} \backslash \boldsymbol{Q}$ *which is not analytic but its intersection with every compact subset of* $\boldsymbol{R} \backslash \boldsymbol{Q}$ *is* F_σ.

PROOF: The derived functors $\varprojlim^{(n)}$ have their origin in the cohomology theory. In fact,

$$\varprojlim^{(n)} \mathscr{A}$$

can be viewed as Čech's cohomology groups of the space $\langle \boldsymbol{N}^{\boldsymbol{N}}, \tau\rangle$ where τ is the topology generated by sets of the form $\{a \in \boldsymbol{N}^{\boldsymbol{N}} : a \leq b\}$ for $b \in \boldsymbol{N}^{\boldsymbol{N}}$ (see [**4**]). This gives us a simple reformulation of the equality $\varprojlim^{(1)} \mathscr{A} = 0$ which one gets by considering the long exact sequence

$$0 \to \varprojlim \mathscr{A} \to \varprojlim \mathscr{B} \to \varprojlim \mathscr{B}/\mathscr{A} \to \varprojlim^{(1)} \mathscr{A} \to \varprojlim^{(1)} \mathscr{B} \to \dots$$

obtained from $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{B}/\mathscr{A} \to 0$, where $\mathscr{B} = \langle B_a, \pi_a^b, \boldsymbol{N}^{\boldsymbol{N}}\rangle$ is the inverse system of Abelian groups defined by

$$B_a = \boldsymbol{Z}^{D_a}$$

for $a \in \boldsymbol{N}^{\boldsymbol{N}}$ (where π_a^b's are again the natural projections), and where $\mathscr{B}/\mathscr{A}$ is the inverse system of the corresponding quotients

$$\langle B_a/A_a, \pi_a^b, \boldsymbol{N}^{\boldsymbol{N}}\rangle.$$

It follows that $\varprojlim^{(1)} \mathscr{A} = 0$ iff the mapping

$$\phi : \varprojlim \mathscr{B} \to \varprojlim \mathscr{B}/\mathscr{A}$$

is onto (see [**4**; §1]). This is indeed a very useful reformulation which one can analyze independently of the Čech cohomology. In fact, questions of this sort have been long appearing in the area of combinatorial analysis of Hausdorff gaps of the structure $\mathscr{P}(\boldsymbol{N})/\mathrm{fin}$ (see [**8**], [**7**; p. 96], [**10**; §4], [**6**; 8.7], [**13**]). So, we can now start the proof of Theorem 1 by assuming that the mapping ϕ is not onto and use this to describe a compactly F_σ set $X \subseteq \boldsymbol{R}\backslash\boldsymbol{Q}$ which is not analytic. So let $\langle [f_a] : a \in \boldsymbol{N}^{\boldsymbol{N}}\rangle$ be a member of $\varprojlim \mathscr{B}/\mathscr{A}$ not in the range of this mapping. Thus, $[f_a]$ is the set of all $g : D_a \to \boldsymbol{Z}$ which agree with f_a on all but finitely many pairs (m, n) from D_a. Let

$$X = \bigcup_{a \in \boldsymbol{N}^{\boldsymbol{N}}} [f_a] \quad \text{and} \quad M = \bigcup_{a \in \boldsymbol{N}^{\boldsymbol{N}}} \boldsymbol{Z}^{D_a}.$$

For f and g in M let $\Delta(f, g)$ be the minimal $m \in \boldsymbol{N}$ for which there is $n \in \boldsymbol{N}$ such that either (m, n) belongs to the symmetric difference of $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ or $(m, n) \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $f(m, n) \neq g(m, n)$; if such an m does not exist (i.e. if $f = g$), we set $\Delta(f, g) = \infty$. Note that

$$\Delta(f, h) \geq \min\{\Delta(f, g), \Delta(g, h)\}$$

for every f, g and h from M. So, if we define

$$\rho(f, g) = 1/\Delta(f, g)$$

(with the convention that $1/\infty = 0$), we get an ultrametric on M which is easily seen to be complete. Note also that compact sets of (M, ρ) have empty interiors, so by

Mazurkiewicz's theorem ([**19**; §36 II]), M is homeomorphic to the irrationals. Every element f of M uniquely determines an $a \in \boldsymbol{N}^{\boldsymbol{N}}$ such that $D_a = \mathrm{dom}(f)$, so let $p(f)$ denote this a. This gives us a mapping

$$p : M \to \boldsymbol{N}^{\boldsymbol{N}}$$

which is easily seen to be continuous. We have two claims about the Polish space M and its subset X.

Claim 1. X is not analytic.

Proof: The key ingredient of the proof is the following regularity property of a given topological space Y introduced by the author in [**6**; §8] and denoted by OCA_Y: For every open symmetric relation $R \subseteq Y^2$ either

(1) There is a countable decomposition $Y = \bigcup_{n=1}^{\infty}, Y_n$ such that $Y_n^{[2]} \cap R = \emptyset$ for all n, or

(2) there is uncountable $Z \subseteq Y$ such that $Z^{[2]} \subseteq R$.

(Notation: $S^{[2]}$ denotes the set of all $(x, y) \in S^2$ such that $x \neq y$.) Let OCA_Y^* denote this statement when in (2) we require Z to be homeomorphic to the Cantor set rather than just to be uncountable. It is easily seen that OCA_Y implies OCA_Z for every continuous image Z of Y and that same is true about the stronger version OCA_Y^*. It is also easily seen that OCA_Y^* is true for some of the standard spaces like $Y = \boldsymbol{R}$ or $Y = \boldsymbol{R} \backslash \boldsymbol{Q}$. It follows that OCA_Y^* is true for every analytic subset Y of some Polish space P. In fact, it is proved in [**6**; §6] that one of the standard forcing axioms, PFA, implies OCA_Y for every second countable space Y. It is also shown in [**6**; 8.7] that if OCA_Y is true for every separable metric space Y then $\underleftarrow{\lim}^{(1)} \mathscr{A} = 0$, and it was this result (and its proof) that led us in discovering Theorem 1, the main result of this note. This result also explains why PFA implies $\underleftarrow{\lim}^{(1)} \mathscr{A} = 0$ a fact first proved in [**10**] and a fact which according to the referee 'explains the meaning of Theorem 2' below. The Principle of Open Colouring continues to be a rich source of quite diverse applications and the reader is referred to [**6**], [**15**], [**16**] for an introduction to this area.

Consider the set R of all $(f, g) \in X$ such that $f(n, m) \neq g(n, m)$ for some $(n, m) \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Clearly, this is an open symmetric relation on X, so if X is analytic, the Principle of Open Colouring would apply to it. So let us examine the two alternatives of OCA_X^* (see [**6**; 8.7]):

Case 1. $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n^{[2]} \cap R = \emptyset$ for all n. Then there must be n such that

$$K_n = \{a \in \boldsymbol{N}^{\boldsymbol{N}} : \mathrm{dom}(f) = D_a \text{ for some } f \in X_n\}$$

is cofinal in $\boldsymbol{N}^{\boldsymbol{N}}$ under the ordering of eventual dominance. Fixing such n, note that, by the definition of R, the union of X_n is a partial function from $\boldsymbol{N} \times \boldsymbol{N}$ into $\boldsymbol{Z}$. Let $g : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{Z}$ be an arbitrary extension of this function. Then $g \in \underleftarrow{\lim}\, \mathscr{B}$ and

$$g \upharpoonright D_a \in [f_a] \quad \text{for all } a \in \boldsymbol{N}^{\boldsymbol{N}},$$

i.e., $\langle [f_a] : a \in \boldsymbol{N}^{\boldsymbol{N}} \rangle = \phi(g)$, a contradiction.

CASE 2. There is a perfect set $P \subseteq X$ such that $P^{[2]} \subseteq R$. Note that

$$K = \{a \in N^N : \mathrm{dom}(f) = D_a \text{ for some } f \in P\},$$

being the image of P under the continuous map $p : M \to N^N$, is a compact subset of N^N, so we can find a $b \in N^N$ such that $a \leq b$ for all $a \in K$. It follows that every function from P is a subfunction of some function from the countable set $[f_b]$. Since P is uncountable there exist two distinct $f, g \in P$ which are induced by the same member of $[f_b]$. So, in particular, $f \cup g$ is a function which is in direct contradiction of the fact that (f, g) belongs to R.

CLAIM 2. $X \cap K$ is F_σ for every compact subset K of M.

PROOF: Since the image of K under the continuous mapping $p : M \to N^N$ is a compact subset of N^N there is $b \in N^N$ which bounds (everywhere) all a's for which D_a appears as the domain of some function from K. It follows that every member from $X \cap K$ is a subfunction of some function from the countable set $[f_b]$. But for a fixed $f \in [f_b]$ the set

$$X_f = \{f \restriction D_a : a \in N^N, a \leq b\}$$

is compact in M and it is a subset of X. Hence $X \cap K$, being equal to the union of the countable family $X_f \cap K$ $(f \in [f_b])$ of compact sets, is a σ-compact subset of M.

§2. A nonempty derived limit.

THEOREM 2. (PFA) *There is a σ-closed poset which preserves the power-set of ω_1 and which forces* $\underleftarrow{\lim}^{(1)} \mathscr{A} \neq 0$.

COROLLARY 3. $\underleftarrow{\lim}^{(1)} \mathscr{A} \neq 0$ *is consistent with Martin's axiom and the negation of CH.*

REMARK. Note that this answers another question found in the literature, the question of S. Kamo [**13**; p. 358].

COROLLARY 4. *Martin's axiom together with the negation of CH does not imply that a subset of $\boldsymbol{R} \backslash \boldsymbol{Q}$ is Borel if its intersection with every compact subset of $\boldsymbol{R} \backslash \boldsymbol{Q}$ is Borel.*

PROOF OF THEOREM 2: Let $\mathscr{U}$ be a fixed ultrafilter on N generated by a $\subseteq^*$-decreasing ω_2-sequence of its elements. (Under our assumption Martin's axiom holds and the continuum is equal to ω_2, so a straightforward diagonalization gives us such an ultrafilter; see for example [**7**; Chapter 3].) Let $\mathscr{I}$ be the set of all $X \subseteq N \times N$ such that for every $m \in N$ the set $(X)_m = \{n \in N : \langle m, n \rangle \in X\}$ does not belong to $\mathscr{U}$. Finally, let

$$\mathscr{P} = \bigcup_{X \in \mathscr{I}} \{0,1\}^X$$

ordered by $\subseteq^*$. Clearly, $\mathscr{P}$ is a σ-closed poset. We shall show that it satisfies the conclusions of Theorem 2. Note that for every $a \in N^N$ the set D_a belongs to $\mathscr{I}$ so the generic filter of $\mathscr{P}$ contains a function $f_a : D_a \to \{0, 1\}$. Moreover, it is easily seen that so obtained $\langle [f_a] : a \in N^N \rangle$ is an element of $\varprojlim \mathscr{B}/\mathscr{A}$ not in the range of the mapping

$$\phi : \varprojlim \mathscr{B} \to \varprojlim \mathscr{B}/\mathscr{A}.$$

Hence, the proof of Theorem 2 is finished once we show that $\mathscr{P}$ is sufficiently distributive so that no new subsets of ω_1 are added and therefore that $\mathfrak{c} = \omega_2$ and MA get preserved. So, let $\mathscr{D}_\alpha(\alpha < \omega_1)$ be a given $\subseteq$-decreasing sequence of dense open subsets of $\mathscr{P}$. We shall show that $\bigcap_{\alpha<\omega_1} \mathscr{D}_\alpha$ is nonempty and, therefore, dense by the homogeneity of $\mathscr{P}$. Let

$$\mathscr{R} = \bigcup_{X \in \mathscr{I}} \{0,1\}^X$$

but with $\subseteq$ as the ordering rather than $\subseteq^*$. Then $\mathscr{R}$ is the so-called Grigorieff poset associated to the ideal $\mathscr{I}$. Now, we know that $\mathscr{I}$ is a second category P-ideal and so the poset $\mathscr{R}$ is proper (see [3], [5; pp. 214–221]). Note that each $\mathscr{D}_\alpha$ is also dense in $\mathscr{R}$, so an application of PFA would give us a sequence $p_\alpha \in \mathscr{D}_\alpha$ $(\alpha < \omega_1)$ such that $p_\alpha \cup p_\beta$ is a function for all α and β in ω_1. Let $X_\alpha = \mathrm{dom}(p_\alpha)$ for $\alpha < \omega_1$. By our assumption on $\mathscr{U}$ for each $m \in N$ we can find $Y_m \notin \mathscr{U}$ such that

$$(X_\alpha)_m \subseteq^* Y_m \quad \text{for all} \quad \alpha < \omega_1.$$

For each α, fix $b_\alpha \in N^N$ such that

$$(X_\alpha)_m \subseteq Y_m \cup \{1, \dots, b_\alpha(m)\} \quad \text{for all} \quad m \in N.$$

Then again by PFA, we can find $c \in N^N$ and uncountable $\Omega \subseteq \omega_1$ such that $b_\alpha(m) \le c(m)$ for all $\alpha \in \Omega$ and all $m \in N$. Let

$$X = \bigcup_{m=1}^{\infty} \{m\} \times (Y_m \cup \{1, \dots, c(m)\}).$$

Then $X \in \mathscr{I}$ and $X_\alpha \subseteq X$ for all $\alpha \in \Omega$. Choose $p : X \to \{0,1\}$ arbitrarily extending

$$\bigcup_{\alpha \in \Omega} p_\alpha.$$

Then $p \in \mathscr{P}$ and $p_\alpha \subseteq p$ for all $\alpha \in \Omega$, i.e., $p \in \mathscr{D}_\alpha$ for all $\alpha \in \Omega$. Since $\mathscr{D}_\alpha$'s are decreasing, it follows that p is a member of the intersection $\bigcap_{\alpha<\omega_1} \mathscr{D}_\alpha$.

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