J. Math Soc. Japan
Vol. 50, No. 3, 1998

# On the Besov-Hankel spaces*

By Jorge J. BETANCOR and Lourdes RODRÍGUEZ-MESA

(Received Oct. 15, 1996)

## 1. Introduction and preliminaries.

Consider the Hankel transformation $h_\mu$ defined for suitable functions $\phi$ by

$$h_\mu(\phi)(x) = \int_0^\infty y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\phi(y)\,dy, \quad x \in (0,\infty),$$

where $J_\mu$ represents the Bessel function of the first kind and order $\mu$. Here and in the sequel $\mu$ is a real number greater than $-1/2$. The convolution for the transformation $h_\mu$ is defined through

$$(\phi \# \psi)(x) = \int_0^\infty (\tau_x\phi)(y)\psi(y)\,d\gamma(y), \quad x \in (0,\infty),$$

where the Hankel translation operator $\tau_x, x \in (0,\infty)$, is given by

$$(\tau_x\phi)(y) = \int_0^\infty D(x,y,z)\phi(z)\,d\gamma(z), \quad x, y \in (0,\infty),$$

being $d\gamma(x) = (x^{2\mu+1}/2^\mu\Gamma(\mu+1))\,dx$ and

$$D(x,y,z) = \frac{2^{3\mu-1}\Gamma(\mu+1)^2}{\Gamma(\mu+1/2)\sqrt{\pi}}(xyz)^{-2\mu}A(x,y,z)^{2\mu-1}, \quad x,y,z \in (0,\infty).$$

Here $A(x,y,z)$ is the area of a triangle with sides $x,y,z$ when such a triangle exists and $A(x,y,z) = 0$ otherwise.

In earlier papers ([**6**] and [**9**]) the $\#$-convolution have been investigated on the spaces $L^p_\mu$ defined for $1 \le p < \infty$ to consist of those complex-valued functions $\phi$, measurable on $(0,\infty)$ and such that $\|\phi\|_{p,\mu} < \infty$, where

$$\|\phi\|_{p,\mu} = \left\{\int_0^\infty |\phi(x)|^p x^{2\mu+1}\,dx\right\}^{1/p}.$$

By $L^\infty$ we denote as usual the space of essentially bounded measurable functions on $(0,\infty)$ and $\|\ \|_\infty$ represents the usual norm in $L^\infty$. The space of compactly supported continuous functions on $(0,\infty)$ is denoted by $C_0$.

Let $T \in (0,\infty)$. We define the Bochner-Riesz mean $\sigma_T^\beta(\phi)$ of a measurable function $\phi$ on $(0,\infty)$ by

A.M.S. Subject Classification: 46F12.
Key words and phrases: Hankel transform, $L^p$-spaces, Bochner-Riesz means.
*Partially supported by Consejería de Educación, Gobierno Autónomo de Canarias, Proyecto 967/15-9-95 and by DGICYT Grant PB 94-0591 (Spain).

$$\sigma_T^\beta(\phi)(x) = \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\left(1-\left(\frac{y}{T}\right)^2\right)^\beta h_\mu(\phi)(y)\,dy, \quad x\in(0,\infty).$$

According to (33) ∮ 8.5 [3] we can see ([1]) that when $1 \le p \le 2$ and $\beta > \mu + 1/2$

$$\sigma_T^\beta(\phi) = \phi_{T,\beta} \# \phi, \quad \phi \in L_\mu^p,$$

where $\phi_{T,\beta}(x) = 2^\beta\Gamma(\beta+1)T^{2(\mu+1)}(Tx)^{-\mu-\beta-1}J_{\mu+\beta+1}(Tx), T,\ x\in(0,\infty)$.

Moreover by virtue of Theorem 2.b [9] since $\phi_{T,\beta}\in L_\mu^1$ when $\beta > \mu + 1/2$

$$\|\phi_{T,\beta}\#\phi\|_{p,\mu} \le C\|\phi\|_{p,\mu}, \quad \phi\in L_\mu^p, \quad 1\le p\le\infty$$

for certain $C > 0$.

That suggests to define the operator $\sigma_T^\beta$ on $L_\mu^p$ by

$$\sigma_T^\beta(\phi) = \phi_{T,\beta}\#\phi,$$

when $1 \le p \le \infty$ and $\beta > \mu + 1/2$.

Also we consider the partial Hankel integral $s_T(\phi)$ of a measurable function $\phi$ on $(0,\infty)$ by

$$s_T(\phi)(x) = \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\phi)(y)\,dy, \quad x\in(0,\infty).$$

In [1] we establish that

$$s_T(\phi) = \varphi_T\#\phi, \quad \phi\in L_\mu^p,$$

when $1\le p\le 2$ and $\mu > -1/2$, where $\varphi_T(x) = T^{2(\mu+1)}(Tx)^{-\mu-1}J_{\mu+1}(Tx),\ T, x\in(0,\infty)$.

Moreover, according to Theorem 3 [7] and ∮5.1 (8) [13] (see also [14]) we can write for every $\phi\in C_0$,

$$\begin{aligned} s_T(\phi)(x) &= \int_0^\infty \phi(z)x^{-\mu}z^{\mu+1}\int_0^T J_\mu(zy)J_\mu(xy)y\,dy\,dz \\ &= T\Bigg(x^{-\mu}J_\mu(Tx)\int_0^\infty \frac{z^{\mu+2}}{z^2-x^2}J_{\mu+1}(Tz)\phi(z)\,dz \\ &\quad - x^{-\mu+1}J_{\mu+1}(Tx)\int_0^\infty \frac{z^{\mu+1}}{z^2-x^2}J_\mu(Tz)\phi(z)\,dz\Bigg) \\ &= T(x^{-\mu}J_\mu(Tx)H_-[z^{\mu+1}J_{\mu+1}(Tz)\phi(z)](x) \\ &\quad - x^{-\mu}J_{\mu+1}(Tx)H_+[z^{\mu+1}J_\mu(Tz)\phi(z)](x)), \quad x\in(0,\infty), \end{aligned} \tag{1}$$

where $H_-$ and $H_+$ represent the well-known odd and even Hilbert transforms.

By taking into account the behaviour of the Hilbert transforms on weighted $L^p$-spaces ([8]) it is easy to see that the last term in (1) defines a bounded linear operator from $L_\mu^p$ into itself when $4(\mu+1)/(2\mu+3) < p < 4(\mu+1)/(2\mu+1)$. Also, the boundedness of the operator $s_T$ (Corollary 1, [10]) allows to conclude that the equality in (1) holds for every $\phi\in L_\mu^p$, $4(\mu+1)/(2\mu+3) < p \le 2$.

In the sequel we define the operator $s_T$ on $L^p_\mu$ by the last term in (1) when $4(\mu+1)/(2\mu+3) < p < 4(\mu+1)/(2\mu+1)$. In [2] it was proved that $s_T(\phi)(x) \to \phi(x)$, as $T \to \infty$, almost everywhere $x \in (0,\infty)$, provided that $4(\mu+1)/(2\mu+3) < p < 4(\mu+1)/(2\mu+1)$. Also it is not hard to prove that $s_T(\phi) \to \phi$, as $T \to \infty$, in $L^p_\mu$ (see the proof of Theorem 2.2) when $4(\mu+1)/(2\mu+3) < p < 4(\mu+1)/(2\mu+1)$.

We introduce new function spaces that we call Besov-Hankel spaces as follows.

Let $\alpha > 0$ and $1 \le p, r < \infty$. We say that a measurable function $\phi$ on $(0,\infty)$ is in $BH^{p,r}_{\alpha,\mu}$ if $\phi \in L^p_\mu$ and

$$\int_0^\infty \left(\frac{w_{h,p}(\phi)(t)}{t^\alpha}\right)^r \frac{dt}{t} < \infty,$$

where $w_{h,p}(\phi)(t) = \|\tau_t\phi - \phi\|_{p,\mu}$, $t \in (0,\infty)$.

In this paper, inspired in the one due to D. V. Giang and F. Móricz [4], we obtain characterizations of the Besov-Hankel spaces involving the Bochner-Riesz means (Theorem 2.1) and the partial Hankel integrals (Theorem 2.2).

Throughout this paper $C$ will always denote a suitable positive constant that it is not necessarily the same in each occurrence. Also we represent by $p'$ the conjugate of $p$ (that is, $p' = p/(p-1)$, when $1 < p < \infty$ and $p' = \infty$, when $p = 1$).

## 2. Characterizations of Besov-Hankel spaces.

We now obtain characterizations of the Besov-Hankel spaces through the Bochner-Riesz means $\sigma^\beta_T$ and the partial Hankel integrals $s_T$. Previously we need establish some results.

Lemma 2.1. *Let $\beta > \mu + 1/2$ and $1 \le p < \infty$. Then for every $\phi \in L^p_\mu$*

$$\phi(x)\, ln 2 = \int_0^\infty [\sigma^\beta_{2T}(\phi)(x) - \sigma^\beta_T(\phi)(x)]\frac{dT}{T}, \quad a.e.\ x \in (0,\infty).$$

Proof. Let $\phi \in L^p_\mu$. For every $T > 0$ we can write

$$\sigma^\beta_{2T}(\phi)(x) - \sigma^\beta_T(\phi)(x) = \int_T^{2T} \frac{d}{dt}\sigma^\beta_t(\phi)(x)\,dt, \quad a.e.\ x \in (0,\infty). \tag{2}$$

Note that according to Theorem 2.b [9] $(d/dt)\sigma^\beta_t(\phi) \in L^p_\mu$, for every $t \in (0,\infty)$.

By virtue of Corollary 2 [11] $\sigma^\beta_T(\phi)(x) \to \phi(x)$, as $T \to \infty$, almost everywhere $x \in (0,\infty)$. Moreover $\sigma^\beta_T(\phi)(x) \to 0$, as $T \to 0^+$, uniformly in $x \in (0,\infty)$. Indeed, according to Theorem 2.b [9] we can write

$$|\sigma^\beta_T(\phi)(x)| \le \|\phi_{T,\beta}\|_{p',\mu}\,\|\phi\|_{p,\mu} = CT^{2(\mu+1)/p}\|\phi\|_{p,\mu}, \quad T, x \in (0,\infty).$$

Hence $\sigma^\beta_T(\phi)(x) \to 0$, as $T \to 0^+$, uniformly in $x \in (0,\infty)$.

By integrating both of the sides in (2) one obtains

$$\int_0^\infty [\sigma^\beta_{2T}(\phi)(x) - \sigma^\beta_T(\phi)(x)]\frac{dT}{T} = \int_0^\infty \frac{d}{dt}\sigma^\beta_t(\phi)(x)\int_{t/2}^t \frac{dT}{T}\,dt = (ln 2)\phi(x), \quad a.e.\ x \in (0,\infty).$$

■

LEMMA 2.2. *Let* $\alpha, \beta \in \boldsymbol{R}, -1/2 < \mu < \beta - \alpha - 1/2$ *and* $1 \le p < \infty$. *Let* $\phi$ *be a locally integrable function on* $(0, \infty)$. *Then*

$$\left\{\int_0^\infty \left| t^{\alpha+\mu-\beta+1/2} \int_{1/t}^\infty z^{\mu-\beta-1/2}\phi(z)\,dz\right|^p \frac{dt}{t}\right\}^{1/p} \le C\left\{\int_0^\infty |t^{-\alpha}\phi(t)|^p \frac{dt}{t}\right\}^{1/p}. \tag{3}$$

PROOF. It is easy to see that

$$G(t) = t^{\alpha+\mu-\beta+1/2} \int_{1/t}^\infty z^{\mu-\beta-1/2}\phi(z)\,dz = t^\alpha \int_1^\infty z^{\mu-\beta-1/2}\phi\left(\frac{z}{t}\right) dz, \quad t \in (0, \infty).$$

Let $1 < p < \infty$. Hence by using Fubini Theorem and Hölder inequality, we obtain

$$\begin{aligned}\int_0^\infty |G(t)|^p \frac{dt}{t} &\le \int_0^\infty |G(t)|^{p-1} \int_1^\infty t^{\alpha-1} z^{\mu-\beta-1/2}\left|\phi\left(\frac{z}{t}\right)\right| dz\,dt \\ &= \int_1^\infty \int_0^\infty |G(t)|^{p-1}\, t^{\alpha-1} z^{\mu-\beta-1/2}\left|\phi\left(\frac{z}{t}\right)\right| dt\,dz \\ &\le \int_1^\infty z^{\mu-\beta-1/2}\left\{\int_0^\infty |G(t)|^{(p-1)p'} \frac{dt}{t}\right\}^{1/p'} \left\{\int_0^\infty \left|t^\alpha\phi\left(\frac{z}{t}\right)\right|^p \frac{dt}{t}\right\}^{1/p} dz.\end{aligned}$$

A straightforward manipulation leads to

$$\begin{aligned}\left\{\int_0^\infty |G(t)|^p \frac{dt}{t}\right\}^{1/p} &\le \int_1^\infty z^{\mu-\beta-1/2+\alpha}\,dz \left\{\int_0^\infty |t^{-\alpha}\phi(t)|^p \frac{dt}{t}\right\}^{1/p} \\ &\le C\left\{\int_0^\infty |t^{-\alpha}\phi(t)|^p \frac{dt}{t}\right\}^{1/p}.\end{aligned}$$

If $p = 1$ (3) follows immediately from Fubini Theorem. ■

In the following we characterize the Besov-Hankel space through the Bochner-Riesz mean $\sigma_T^\beta$.

THEOREM 2.1. *Let* $\alpha > 0, -1/2 < \mu < \beta - \alpha - 1/2, 1 \le p, r < \infty$ *and* $\phi \in L_\mu^p$. *The following three properties are equivalent.*

(i) $\phi \in BH_{\alpha,\mu}^{p,r}$.

(ii) $T^\alpha \|\sigma_T^\beta(\phi) - \phi\|_{p,\mu} \in L^r\left((0\,\infty), \frac{dT}{T}\right)$.

(iii) $T^\alpha \|\sigma_{2T}^\beta(\phi) - \sigma_T^\beta(\phi)\|_{p,\mu} \in L^r\left((0\,\infty), \frac{dT}{T}\right)$.

PROOF. (i) $\Rightarrow$ (ii). Let $\phi \in BH_{\alpha,\mu}^{p,r}$. By using the generalized Minkowski inequality and by taking into account well-known boundedness properties of the Bessel function we obtain

$$\begin{aligned}&\|\sigma_T^\beta(\phi) - \phi\|_{p,\mu} \\ &\quad \le \int_0^{1/T} |\phi_{T,\beta}(z)|\, w_{h,p}(\phi)(z)\,d\gamma(z) + \int_{1/T}^\infty |\phi_{T,\beta}(z)|\, w_{h,p}(\phi)(z)\,d\gamma(z)\end{aligned}$$

$$\leq C\left(T^{2\mu+2}\int_0^{1/T} w_{h,p}(\phi)(z)z^{2\mu+1}\,dz + T^{\mu-\beta+1/2}\int_{1/T}^{\infty} z^{\mu-\beta-1/2}w_{h,p}(\phi)(z)\,dz\right)$$

$$\leq C\left(T\int_0^{1/T} w_{h,p}(\phi)(z)\,dz + T^{\mu-\beta+1/2}\int_{1/T}^{\infty} z^{\mu-\beta-1/2}w_{h,p}(\phi)(z)\,dz\right), \quad T\in(0,\infty).$$

According to Lemma 6 [**4**] and Lemma 2.2 it follows

$$\left\{\int_0^\infty [T^\alpha\|\sigma_T^\beta(\phi)-\phi\|_{p,\mu}]^r\frac{dT}{T}\right\}^{1/r} \leq C\left(\left\{\int_0^\infty\left[T^{\alpha+1}\int_0^{1/T} w_{h,p}(\phi)(z)\,dz\right]^r\frac{dT}{T}\right\}^{1/r}\right.$$

$$\left. + \left\{\int_0^\infty [T^{\alpha+\mu-\beta+1/2}\int_{1/T}^\infty z^{\mu-\beta-1/2}w_{h,p}(\phi)(z)\,dz]^r\frac{dT}{T}\right\}^{1/r}\right)$$

$$\leq C\left\{\int_0^\infty\left(\frac{w_{h,p}(\phi)(z)}{z^\alpha}\right)^r\frac{dz}{z}\right\}^{1/r} < \infty.$$

Thus (ii) is established.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) We define the operator $\Delta$ as follows

$$\Delta(\phi,x,t) = (\tau_t\phi)(x) - \phi(x), \quad x,t\in(0,\infty).$$

Since $\tau_t$ is a bounded operator in $L^p_\mu$ for every $t\in(0,\infty)$ ([**12**], p. 16), $\tau_t\phi\in L^p_\mu$, $t\in(0,\infty)$, and according to Lemma 2.1 we can write

$$\Delta(\phi,x,t)\,ln\,2 = \int_0^\infty [(\phi_{2T,\beta}-\phi_{T,\beta})\#(\tau_t\phi-\phi)](x)\frac{dT}{T}, \quad x,t\in(0,\infty). \tag{4}$$

Moreover if $\psi\in L^1_\mu$ and $\varphi\in L^p_\mu$ then

$$\tau_t(\psi\#\varphi) = \psi\#(\tau_t\varphi) = (\tau_t\psi)\#\varphi, \quad t\in(0,\infty). \tag{5}$$

To see (5) it is sufficient to note that each of the terms define bounded bilinear operators from $L^p_\mu$ into itself (Theorem 2.b [**9**] and p. 16 [**12**]) and that (5) holds when $\psi$ and $\varphi$ belong to $C_0$.

Hence from (4) and (5) we deduce

$$\Delta(\phi,x,t)\,ln\,2 = \int_0^\infty \Delta(\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi),x,t)\frac{dT}{T}, \quad x,t\in(0,\infty). \tag{6}$$

Since $\tau_t$ is a contractive operator on $L^p_\mu$, for every $t\in(0,\infty)$, we can write

$$\|\Delta(\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi),\cdot,t)\|_{p,\mu} \leq 2\,\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu}, \quad t,T\in(0,\infty). \tag{7}$$

Also,

$$\|\Delta(\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi),\cdot,t)\|_{p,\mu} \leq CtT\,\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu}, \quad t,T\in(0,\infty). \tag{8}$$

Indeed, since $C_0$ is a dense subset of $L^p_\mu$, there exists a sequence $(\phi_n)_{n=1}^\infty$ contained in $C_0$ such that $\phi_n\to\phi$, as $n\to\infty$, in $L^p_\mu$.

Hence, according to Theorem 2.b [9], $\phi_{T,\beta}\#\phi_n \to \phi_{T,\beta}\#\phi$, as $n\to\infty$, in $L^p_\mu$, for each $T\in(0,\infty)$. Then $\tau_t(\phi_{T,\beta}\#\phi_n)\to\tau_t(\phi_{T,\beta}\#\phi)$, as $n\to\infty$, in $L^p_\mu$, for every $t,T\in(0,\infty)$.

As in Corollary 2.2 [5] we choose a smooth function $\xi$ on $(0,\infty)$ such that $\xi(y)=1$, for every $y\in(0,1]$ and $\xi(y)=0$ for every $y\geq 2$. Denote by $g=h_\mu(\xi)$ and $g_\varepsilon(y)=\varepsilon^{2\mu+2}g(\varepsilon y)$, $\varepsilon,y\in(0,\infty)$. We have

$$\begin{aligned}\|\Delta(\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi),\cdot,t)\|_{p,\mu} &= \lim_{n\to\infty}\|\Delta(\sigma^\beta_{2T}(\phi_n)-\sigma^\beta_T(\phi_n),\cdot,t)\|_{p,\mu}\\ &= \lim_{n\to\infty}\|g_{2T}\#\Delta(\sigma^\beta_{2T}(\phi_n)-\sigma^\beta_T(\phi_n),\cdot,t)\|_{p,\mu}\\ &= \lim_{n\to\infty}\|(\tau_t g_{2T}-g_{2T})\#(\phi_{2T,\beta}\#\phi_n-\phi_{T,\beta}\#\phi_n)\|_{p,\mu}\\ &\leq C\|\tau_t g_{2T}-g_{2T}\|_{1,\mu}\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu}, \quad t,T\in(0,\infty).\end{aligned}$$

Thus (8) is established.

By combining (6), (7) and (8), according to generalized Minkowski inequality, we conclude that

$$w_{h,p}(\phi)(t)\leq C\left\{\int_0^{1/t} t\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu}\,dT+\int_{1/t}^\infty\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu}\frac{dT}{T}\right\},\quad t\in(0,\infty).$$

From Lemma 4 [4] it deduces

$$\left\{\int_0^\infty\left(\frac{w_{h,p}(\phi)}{t^\alpha}\right)^r\frac{dt}{t}\right\}^{1/r}\leq C\left\{\int_0^\infty(T^\alpha\|\sigma^\beta_{2T}(\phi)-\sigma^\beta_T(\phi)\|_{p,\mu})^r\frac{dT}{T}\right\}^{1/r},$$

and (i) is proved. ∎

Next, Besov-Hankel spaces are characterized through the partial Hankel integral $s_T$.

Theorem 2.2. *Let $\alpha>0$, $\mu>-1/2$, $4(\mu+1)/(2\mu+3)<p<4(\mu+1)/(2\mu+1)$, $1\leq r<\infty$ and $\phi\in L^p_\mu$. Then the following three statements are equivalent.*

(i) $\phi\in BH^{p,r}_{\alpha,\mu}$.

(ii) $T^\alpha\|s_T(\phi)-\phi\|_{p,\mu}\in L^r\left((0,\infty),\frac{dT}{T}\right)$.

(iii) $T^\alpha\|s_{2T}(\phi)-s_T(\phi)\|_{p,\mu}\in L^r\left((0,\infty),\frac{dT}{T}\right)$.

Proof. Let $\beta>\mu+\alpha+1/2$.

(i) ⇒ (ii). Let $T\in(0,\infty)$. Assume that $\psi\in C_0$. Then by Theorem 2.d [9] we can write

$$h_\mu(\sigma^\beta_T(\psi))=h_\mu(\phi_{T,\beta})h_\mu(\psi).$$

Hence according to (33) ∮8.5 [3] it follows that $s_T(\sigma^\beta_T(\psi))=\sigma^\beta_T(\psi)$.

Moreover both of the members of the last equality define bounded linear operators from $L^p_\mu$ into itself. Since $C_0$ is a dense subset of $L^p_\mu$ we conclude that

$$s_T(\sigma^\beta_T(\phi))=\sigma^\beta_T(\phi). \tag{9}$$

By taking into account again that $\{s_T\}_{T>0}$ is a uniformly bounded family of operators from $L^p_\mu$ into itself (Corollary 1, [**10**]) and by (9) one has

$$\|s_T(\phi) - \phi\|_{p,\mu} \leq \|s_T(\sigma^\beta_T(\phi) - \phi)\|_{p,\mu} + \|\sigma^\beta_T(\phi) - \phi\|_{p,\mu} \leq C\|\sigma^\beta_T(\phi) - \phi\|_{p,\mu}.$$

Hence (ii) can be deduced now from Theorem 2.1.
(ii) $\Rightarrow$ (iii). It is clear.
(iii) $\Rightarrow$ (i). Firstly we prove that

$$\frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t s_t(\phi)(x)\,dt = \sigma^\beta_T(\phi)(x), \quad T \in (0,\infty) \text{ and } a.e.\ x \in (0,\infty). \tag{10}$$

If $\psi \in C_0$ Fubini Theorem leads to

$$\begin{aligned}
&\frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t s_t(\psi)(x)\,dt \\
&= \frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t \int_0^t y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\psi)(y)\,dy\,dt \\
&= \frac{2\beta}{T^{2\beta}}\int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\psi)(y)\int_y^T (T^2 - t^2)^{\beta-1}\,t\,dt\,dy \\
&= \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\left(1 - \left(\frac{y}{T}\right)^2\right)^\beta h_\mu(\psi)(y)\,dy = \sigma^\beta_T(\psi)(x), \quad T, x \in (0,\infty).
\end{aligned}$$

Moreover the left hand side of (10) defines a bounded operator from $L^p_\mu$ into itself. Indeed, from generalized Minkowski inequality we deduce

$$\left\|\frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t s_t(\phi)(x)\,dt\right\|_{p,\mu} \leq \frac{2\beta}{T^{2\beta}}\int_0^T (T^2 - t^2)^{\beta-1} t\,\|s_t(\phi)\|_{p,\mu}\,dt \leq C\|\phi\|_{p,\mu}.$$

Hence, since $C_0$ is a dense subset of $L^p_\mu$, (10) holds.

According to again generalized Minkowski inequality, from (10) and Lemma 5 [**4**] it infers

$$\begin{aligned}
&\left\{\int_0^\infty [T^\alpha\|\sigma^\beta_{2T}(\phi) - \sigma^\beta_T(\phi)\|_{p,\mu}]^r \frac{dT}{T}\right\}^{1/r} \\
&\leq 2\beta\left\{\int_0^\infty \left[T^{\alpha-2\beta}\int_0^T (T^2 - t^2)^{\beta-1} t\|s_{2t}(\phi) - s_t(\phi)\|_{p,\mu}\,dt\right]^r \frac{dT}{T}\right\}^{1/r} \\
&\leq 2\beta\left\{\int_0^\infty \left[T^{\alpha-1}\int_0^T \|s_{2t}(\phi) - s_t(\phi)\|_{p,\mu}\,dt\right]^r \frac{dT}{T}\right\}^{1/r} \\
&\leq C\left\{\int_0^\infty [T^\alpha\|s_{2T}(\phi) - s_T(\phi)\|_{p,\mu}]^r \frac{dT}{T}\right\}^{1/r}.
\end{aligned}$$

By invoking now Theorem 2.1 the proof is finished. ∎

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Jorge J. BETANCOR and Lourdes RODRÍGUEZ-MESA
Departamento de Análisis Matemático.
Universidad de La Laguna
38271 - La Laguna. Tenerife
Islas Canarias. España.
E-mail: jbetanco@ull.es