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The theorem of E. Hopf under uniform magnetic fields

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Introduction.

Let (M, g) be a complete Riemannian manifold, and let B be a closed 2-form on M. B may be regarded as a magnetic field on M. Let $\Omega : TM \to TM$ be a skewsymmetric matrix defined by $g_p(u, \Omega(v)) = B_p(u, v)$ $(u, v \in T_pM, p \in M)$. In [**8**], the Newtonian equation of a charged particle moving on M has been defined by

$$\frac{D}{dt}\dot{c} = \Omega(\dot{c}), \tag{1}$$

where $\dot{c}$ is the velocity vector field of the curve c and D/dt stands for the covariant derivative along c. *The magnetic flow* $\varphi_t : TM \to TM$ *associated with* B is defined by

$$\varphi_t(v) = \dot{c}_v(t),$$

where c_v is a solution curve of the equation (1) with $\dot{c}_v(0) = v \in TM$.

By using a geodesic flow, E. Hopf proved that *the total curvature of a compact surface without conjugate points is nonpositive, and vanishes if and only if the surface is flat* in [**5**]. L. W. Green extended the result of E. Hopf for a compact n-dimensional manifold in [**3**]. Recently, F. Guimarães ([**4**]) and N. Innami ([**6**]) have treated the noncompact case.

The non-existence of a pair of conjugate points along geodesics is equivalent to the non-existence of singular values of the exponential map. If there exists a magnetic field, then this equivalence no longer holds. From this fact, we find two concepts of non-conjugation for the magnetic flow, which are called *Jacobi field non-conjugation* and *exponential map non-conjugation*.

We call a magnetic field uniform if $\nabla B \equiv 0$, where ∇ is the Levi-Civita connection of M. In this paper, for each concept of non-conjugation, we will generalize E. Hopf's theorem with the help of the magnetic flow associated with a uniform magnetic field. In Section 3, the generalization for Jacobi field non-conjugation is treated. In Section 4, the following result will be proved as the generalization for exponential map non-conjugation.

THEOREM 1. *Let (M, g) be a compact orientable surface with a uniform magnetic field $B = b\,\mathrm{vol}_M$ $(b \in \boldsymbol{R})$ where vol_M is a canonical volume form of M, and let $\chi(M)$ denoted the Euler characteristic of M. Let $\exp^{\pm\Omega} : TM \to M$ be the exponential maps associated with B. Suppose that there exist no singular values of $\exp^{\pm\Omega}$. Then,*

$$-\frac{b^2}{2\pi}\mathrm{vol}(M) \geq \chi(M),$$

and the equality holds if and only if the curvature of M is constant $-b^2$.

See Section 4 for the definitions of $\exp^{\pm\Omega}$. This implies that if $\chi(M) \geq 0$ and $b \neq 0$, then there exists a singular value of $\exp^{\pm\Omega}$. The proof of Theorem 1 is carried out under more general situation, so to say, the case of a Kähler manifold with a Kähler magnetic field. In Section 5, the relation between Jacobi field non-conjugation and exponential map non-conjugation is discussed for two dimensional case.

1. Preliminaries.

Throughout this paper, we will assume $\nabla B \equiv 0$, or equivalently $\nabla\Omega \equiv 0$. We shall review some basic materials. See [**2**] for details.

DEFINITION 1.1. Let c_v be a solution curve of the equation (1) with $\dot{c}_v(0) = v \in TM$. Then, a vector field J along c_v is called *a Jacobi field under* B if it satisfies the Jacobi equation under B along c_v

$$\frac{D^2}{dt^2} J + R(\dot{c}_v, J)\dot{c}_v - \Omega\left(\frac{D}{dt} J\right) = 0, \tag{2}$$

where the curvature tensor R is defined by $R(X, Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$ for arbitrary vector fields X, Y, Z.

Let $\pi : TM \to M$ be the canonical projection. Let $v_1 = v/r$ where $r = \sqrt{g(v,v)}$, and let us choose $v_2, \dots, v_n \in T_{\pi(v)}M$ so that $\{v_1, v_2 \dots, v_n\}$ is an orthonormal basis in $T_{\pi(v)}M$. A vector field V_i $(i = 1, \dots, n)$ along c_v is defined by a solution of the equation

$$\frac{D}{dt} V_i - \Omega(V_i) = 0, \quad V_i(0) = v_i.$$

It is obvious that $V_1, \dots, V_n$ are orthonormal vector fields along c_v. In particular, $V_1 = \dot{c}_v/r$. If J is expressed by $J = \sum_{i=1}^n f_i V_i$ where each f_i is a smooth function along c_v, then the equation (2) is rewritten by the equation of the components $\boldsymbol{f} = (f_1, \dots, f_n)$

$$\ddot{\boldsymbol{f}} + \Omega_{\dot{c}_v}\dot{\boldsymbol{f}} + R_{\dot{c}_v}\boldsymbol{f} = 0, \tag{3}$$

where $\Omega_{\dot{c}_v}$ and $R_{\dot{c}_v}$ denote the matrixes $(g(V_i, \Omega(V_j)))$ and $(g(V_i, R(\dot{c}_v, V_j)\dot{c}_v))$ respectively.

DEFINITION 1.2. Let $v \in TM\backslash(0)$. A linear endomorphism $\tilde{R}_v$ of $T_{\pi(v)}M$ is defined by

$$\tilde{R}_v(w) = R(v, w)v + \frac{1}{g(v,v)} g(\Omega(v), w)\Omega(v),$$

where $w \in T_{\pi(v)} M$.

Let pr_v be the projection map onto the normal subspace of v in $T_{\pi(v)}M$, and let us set $\Omega_{v,\perp} = \mathrm{pr}_v\Omega\mathrm{pr}_v$ and $\tilde{R}_{v,\perp} = \mathrm{pr}_v\tilde{R}_v\mathrm{pr}_v$. The equation (2) is split into the equations of the tangential and normal components of c_v by

$$\dot{f}_1 = \frac{1}{r} g(\Omega(\dot{c}_v), J) + \frac{C}{r}, \tag{4}$$

$$\ddot{\boldsymbol{f}}_\perp + \Omega_{\dot{c}_v,\perp}\dot{\boldsymbol{f}}_\perp + \tilde{R}_{\dot{c}_v,\perp}\boldsymbol{f}_\perp + \frac{C}{r^2}\Omega(\dot{c}_v) = 0, \tag{5}$$

where $\boldsymbol{f}_\perp = (f_2, \ldots, f_n)$ and $C \equiv g((D/dt)J, \dot{c}_v)$. For the case where (M, g) is an orientable surface with a uniform magnetic field $B = b\,\mathrm{vol}_M$ $(b \in \boldsymbol{R})$, let us set $V_2 = \Omega(V_1)/b$. Then, the equation (3) becomes

$$\ddot{\boldsymbol{f}} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \dot{\boldsymbol{f}} + \begin{pmatrix} 0 & 0 \\ 0 & r^2 R(c_v) \end{pmatrix} \boldsymbol{f} = 0,$$

where $R(c_v)$ stands for the curvature at c_v. The equations (4) and (5) are

$$\begin{cases} \dot{f}_1 = bf_2 + \dfrac{C}{r}, \\ \ddot{f}_2 + \{r^2 R(c_v) + b^2\} f_2 + \dfrac{C}{r} b = 0. \end{cases} \tag{6}$$

Definition 1.3. Let $p = c_v(\alpha)$ and $q = c_v(\beta)$ be two points on c_v with $\alpha \neq \beta$. p and q are *conjugate under B* along c_v if there exists a nonzero Jacobi field under B along c_v which vanishes for $t = \alpha$ and $t = \beta$.

Next, we shall recall a connection map $K : T(TM) \to TM$. Given a vector $\xi \in T_v(TM)$, let $Z_\xi : (-\varepsilon, \varepsilon) \to TM$ be a smooth curve with the initial condition ξ. Then, we define $K(\xi) = (D/dt)Z_\xi|_{t=0} \in T_{\pi(v)}M$ where D/dt stands for the covariant derivative along $\sigma_\xi = \pi(Z_\xi)$. $d\pi(\xi)$ is $(d/dt)\sigma_\xi|_{t=0}$ from the definition of $d\pi : T(TM) \to TM$. It is obvious that $d\pi(\xi)$ and $K(\xi)$ depend only on ξ. The kernels of $d\pi$ and K are called the vertical and horizontal subspaces of $T_v(TM)$ respectively, and the intersection of the vertical and horizontal subspaces is the zero vector. Since $\dim T_v(TM) = 2n$, $T_v(TM)$ is a direct sum of the horizontal and vertical subspaces. Therefore, we may consider $T_v(TM)$ as $T_{\pi(v)}M \oplus T_{\pi(v)}M$ by the correspondence

$$T_v(TM) \ni \xi \leftrightarrow (d\pi(\xi), K(\xi)) \in T_{\pi(v)}M \oplus T_{\pi(v)}M.$$

Given ξ, $\eta \in T_v(TM)$, we define the metric $\tilde{g}$ on TM by

$$\tilde{g}_v(\xi, \eta) = g_{\pi(v)}(d\pi(\xi), d\pi(\eta)) + g_{\pi(v)}(K(\xi), K(\eta)).$$

In order to prove that φ_t preserves the measure determined by $\tilde{g}$, we shall compute the divergence of the generating vector field of φ_t

$$\xi_v = \frac{d}{dt}\varphi_t(v)|_{t=0}.$$

Let $(\pi^{-1}(U); x_1, \ldots, x_n, v_1, \ldots, v_n)$ be a canonical coordinate system on TM, where $(U; x_1, \ldots, x_n)$ is a coodinate system on M and $v_i = dx_i(v)$ for $v \in \pi^{-1}(U)$. Using this coordinate system, we have

$$\xi_i(v) = dx_i(\xi_v) = v_i, \quad \xi_{n+i}(v) = dv_i(\xi_v) = -\sum_{j,k=1}^{n} \Gamma^i_{jk} v_j v_k + \sum_{j=1}^{n} \Omega^i_j v_j,$$

where Γ^i_{jk} are the Christoffel symbols associated with g and $\Omega^i_j = dx_i(\Omega(\partial/\partial x_j))$. From this, we get

$$\begin{aligned}\operatorname{div}(\xi)(v) &= \sum_{i=1}^{n}\left(\frac{\partial \xi_i}{\partial x_i}(v) + \xi_i(v)\sum_{\alpha,\beta=1}^{2n}\frac{1}{2}\tilde{g}^{\alpha\beta}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial x_i}\right)\\ &\quad + \sum_{i=1}^{n}\left(\frac{\partial \xi_{n+i}}{\partial v_i}(v) + \xi_{n+i}(v)\sum_{\alpha,\beta=1}^{2n}\frac{1}{2}\tilde{g}^{\alpha\beta}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial v_i}\right)\\ &= 2\sum_{i,j=1}^{n} v_i\Gamma_{ij}^{j} - 2\sum_{i,k=1}^{n}\Gamma_{ik}^{i}v_k + \sum_{i=1}^{n}\Omega_i^i = \sum_{i=1}^{n}\Omega_i^i,\end{aligned}$$

where $(\tilde{g}^{\alpha\beta}) = (\tilde{g}_{\alpha\beta})^{-1}$. In the above computation, we have used the identities in [7]

$$\sum_{\alpha,\beta=1}^{2n}\tilde{g}^{\alpha\beta}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial x_i} = 4\sum_{j=1}^{n}\Gamma_{ij}^{j}, \quad \sum_{\alpha,\beta=1}^{2n}\tilde{g}^{\alpha\beta}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial v_i} = 0.$$

If $B = \sum_{i<j} b_{ij}\,dx_i \wedge dx_j$ in U, then we have $\Omega_j^i = \sum_{k=1}^{n} g^{ik}b_{kj}$ where $(g^{ij}) = (g_{ij})^{-1}$. Because of $b_{ij} = -b_{ji}$ for all $1 \le i,j \le n$, we find

$$\sum_{i=1}^{n}\Omega_i^i = \sum_{i,k=1}^{n} g^{ik}b_{ki} = -\sum_{i,k=1}^{n} g^{ki}b_{ik} = -\sum_{k=1}^{n}\Omega_k^k.$$

Therefore, $\operatorname{div}(\xi)$ vanishes on TM. This means that φ_t is a measure preserving transformation. Since φ_t leaves the tangent sphere bundle $S_rM = \{v \in TM; g(v,v) = r^2\}$ invariant for all $t \in \boldsymbol{R}$, we restrict φ_t to S_1M. Let N denote the unit normal vector field of S_1M in TM with $d\pi(N_v) = 0$ and $K(N_v) = v$ ($v \in S_1M$). By the definition of the divergence, we derive the identity

$$\operatorname{div}(\xi \mid S_1M) = \operatorname{div}(\xi) - \tilde{g}(\tilde{\nabla}_N(\xi \mid S_1M), N) = \operatorname{div}(\xi) + \tilde{g}(\xi \mid S_1M, \tilde{\nabla}_N N),$$

where $\tilde{\nabla}$ is the Levi-Civita connection associated with $\tilde{g}$. For each $v \in S_1M$, let us set a curve $\varsigma_v(s) = (x_1(\pi(v)), \dots, x_n(\pi(v)), (s+1)v_1, \dots, (s+1)v_n) \subset T_{\pi(v)}M$. By the direct computation, we find

$$\frac{\tilde{D}}{ds}\dot{\varsigma}_v(0) = \sum_{i,j=1}^{n} v_i v_j \tilde{\nabla}_{\partial/\partial v_i}\frac{\partial}{\partial v_j} = \tilde{\nabla}_{N_v}N_v - N_v,$$

where $\tilde{D}/ds$ stands for the covariant derivative along ς_v. Since ς_v is a geodesic in TM, we have $\tilde{\nabla}_{N_v}N_v = N_v$ for all $v \in S_1M$. Because of $\tilde{g}(\xi \mid S_1M, N) = 0$, it follows that $\operatorname{div}(\xi \mid S_1M) = 0$ on S_1M. Therefore, we conclude that $\varphi_t \mid S_1M$ preserves the measure determined by $\tilde{g} \mid S_1M$.

2. Matrix differential equations.

We shall study the real $m \times m$ matrix differential equation on $\boldsymbol{R}$

$$\ddot{X}(t) + P\dot{X}(t) + Q(t)X(t) = 0, \tag{7}$$

where P is a constant skewsymmetric matrix and $Q(t)$ is a smooth symmetric matrix on $\boldsymbol{R}$.

Let $X_\nu(t)$ be a solution of the equation (7) with $X_\nu(\nu) = 0$ and $\dot{X}_\nu(\nu) = I_m$ for all $\nu \in \boldsymbol{R}$. We shall assume that det $X_\nu(t) \neq 0$ for all $t \neq \nu$. Substituting $Y(t) = e^{(t/2)P}X(t)$ into the equation (7), we obtain equation

$$\ddot{Y}(t) + e^{(t/2)P}(Q(t) + \frac{1}{4}P^\dagger P)e^{-(t/2)P}Y(t) = 0, \tag{8}$$

where the dagger denotes the transpose operation. We should note that $e^{(t/2)P}(Q(t) + P^\dagger P/4)e^{-(t/2)P}$ is a symmetric matrix. It is obvious that $Y_\nu(t) = e^{(t/2)P}X_\nu(t)e^{-(\nu/2)P}$ is a solution of the equation (8) with $Y_\nu(\nu) = 0$ and $\dot{Y}_\nu(\nu) = I_m$. From the above assumption, it is obvious that det $Y_\nu(t) \neq 0$ for all $t \neq \nu$. Therefore, we may apply a useful method of L. W. Green [3] to the equation (8). For all $\tau \neq \nu$, let $Y_\nu(t;\tau)$ denote a unique solution of the equation (8) with $Y_\nu(\nu;\tau) = I_m$ and $Y_\nu(\tau;\tau) = 0$. We find that $Y_\nu(t;\infty) = \lim_{\tau\to\infty} Y_\nu(t;\tau)$ exists and that det $Y_\nu(t;\infty) \neq 0$ for all $t \in \boldsymbol{R}$.

Let us set $U_\nu(t) = \dot{Y}_\nu(t;\infty)Y_\nu(t;\infty)^{-1}$. Then, $U_\nu(t)$ is a symmetric solution of the Riccati equation

$$\dot{U}_\nu(t) + U_\nu^2(t) + e^{(t/2)P}(Q(t) + \frac{1}{4}P^\dagger P)e^{-(t/2)P} = 0.$$

Moreover, the construction of $U_\nu(t)$ is independent of ν. Indeed, because of $Y_\nu(t;\tau) = Y_{\bar{\nu}}(t;\tau)Y_{\bar{\nu}}(\nu;\tau)^{-1}$ for all $\bar{\nu} \neq \nu$ by the uniqueness of $Y_\nu(t;\tau)$, we have

$$Y_\nu(t;\infty) = Y_{\bar{\nu}}(t;\infty)Y_{\bar{\nu}}(\nu;\infty)^{-1}.$$

Therefore, $U_\nu(t) = U_{\bar{\nu}}(t)$. Let us express this by $U(t)$.

3. Jacobi field non-conjugation.

In this section, we shall assume that there exists no pair of conjugate points under B along an arbitrary solution curve of the equation (1) whose velocity is 1, what is called, *Jacobi field non-conjugation*.

From the equation (3), the real $n \times n$ matrix differential equation along c_v is derived by

$$\ddot{X} + \Omega_{\dot{c}_v}\dot{X} + R_{\dot{c}_v}X = 0. \tag{9}$$

Note that $\Omega_{\dot{c}_v} \equiv \Omega_v$ on c_v because of $\nabla\Omega \equiv 0$. Indeed, since Ω is skewsymmetric, we have

$$\frac{d}{dt}g(V_i, \Omega(V_j)) = g(\Omega(V_i), \Omega(V_j)) + g(V_i, \Omega^2(V_j)) = 0.$$

Let $X_{v,\nu}(t)$ be a solution of the equation (9) with $X_{v,\nu}(\nu) = 0$ and $\dot{X}_{v,\nu}(\nu) = I_n$ for all $\nu \in \boldsymbol{R}$. The above assumption implies that det $X_{v,\nu}(t) \neq 0$ for all $t \neq \nu$ and $v \in S_1M$. Thus, the results in Section 2 hold for the equation (9) along c_v for all $v \in S_1M$. Let $U_v(t)$ be the matrix along c_v which corresponds to $U(t)$ in Section 2.

DEFINITION 3.1. For each $v \in S_1M$, a linear endomorphism $\hat{K}_v$ of $T_{\pi(v)}M$ is defined by

$$\hat{K}_v(w) = R(v, w)v + \frac{1}{4}\Omega^\dagger\Omega(w),$$

where $w \in T_{\pi(v)}M$.

The equations that $U_v(t)$ should satisfy are

$$\dot{U}_v(t) + U_v^2(t) + e^{(t/2)\Omega_v}\hat{K}_{\varphi_t(v)}e^{-(t/2)\Omega v} = 0, \quad U_v(t)^\dagger = U_v(t). \tag{10}$$

LEMMA 3.2. *We have the identity*

$$U_{\varphi_s(v)}(t) = e^{-(s/2)\Omega_v}U_v(s+t)e^{(s/2)\Omega_v}$$

for all $s, t \in \boldsymbol{R}$ *and* $v \in S_1M$.

PROOF. For all $v \in S_1M$, let $Y_{v,v}(t)$ and $Y_{v,v}(t;\tau)$ denote the matrixes along c_v which correspond to $Y_v(t)$ and $Y_v(t;\tau)$ in Section 2 respectively. Because of $\Omega_{\varphi_s(v)} = \Omega_v$ and $R_{\dot{c}_{\varphi_s}(v)}(t) = R_{\dot{c}_v}(t+s)$ for all $s \in \boldsymbol{R}$, we have $X_{\varphi_s(v),v}(t) = X_{v,s+v}(t+s)$. For this reason, we find the identities

$$Y_{\varphi_s(v),v}(t) = e^{-(s/2)\Omega_v}Y_{v,s+v}(t+s)e^{(s/2)\Omega_v},$$
$$Y_{\varphi_s(v),v}(t;\tau) = e^{-(s/2)\Omega_v}Y_{v,s+v}(t+s;s+\tau)e^{(s/2)\Omega_v}.$$

Therefore, we obtain

$$Y_{\varphi_s(v),v}(t;\infty) = e^{-(s/2)\Omega_v}Y_{v,s+v}(t+s;\infty)e^{(s/2)\Omega_v},$$

which completes the proof. □

Let tr denote the trace. Let us set $F(v) = \operatorname{tr} U_v(0)$. Since $\operatorname{tr} U_v(0)$ is independent of the choice of orthonormal vector fields in $T_{\pi(v)}M$, $F(v)$ is a well defined function on S_1M. By the same reason, both $G(v) = \operatorname{tr}\dot{U}_v(0)$ and $H(v) = \operatorname{tr} U_v^2(0)$ are well defined on S_1M. By Lemma 3.2, we find

$$F(\varphi_t(v)) = \operatorname{tr} U_{\varphi_t(v)}(0) = \operatorname{tr} U_v(t),$$
$$G(\varphi_t(v)) = \operatorname{tr}\dot{U}_{\varphi_t(v)}(0) = \operatorname{tr}\dot{U}_v(t) = \dot{F}(\varphi_t(v))$$

along c_v ($v \in S_1M$). From the equation (10), the functions $F(\varphi_t(v))$ and $H(\varphi_t(v))$ satisfy the equation

$$\dot{F}(\varphi_t(v)) + H(\varphi_t(v)) + \operatorname{tr}\hat{K}_{\varphi_t(v)} = 0.$$

Integrating the both sides with respect to t, we have

$$F(\varphi_1(v)) - F(v) + \int_0^1 H(\varphi_s(v))\,ds + \int_0^1 \operatorname{tr}\hat{K}_{\varphi_s(v)}\,ds = 0.$$

Let dV_{S_1M} be the volume element on S_1M determined by $\tilde{g}\,|\,S_1M$. Then, $dV_{S_1M} = d\omega dV_M$ where $d\omega$ is the measure on the unit $(n-1)$-sphere S^{n-1} and dV_M is the volume element on M. Since M is compact, $F(v)$, $H(v)$, and $\operatorname{tr}\hat{K}_v$ are integrable on S_1M. Integrate the both sides with respect to dV_{S_1M} over all of S_1M, and use the fact that dV_{S_1M} is invariant with respect to the magnetic flow $\varphi_t : S_1M \to S_1M$. Then, we get

$$0 = \int_{S_1M}\int_0^1 H(\varphi_t(v))\,ds\,dV_{S_1M} + \int_{S_1M}\int_0^1 \operatorname{tr}\hat{K}_{\varphi_t(v)}\,ds\,dV_{S_1M}$$
$$= \int_{S_1M} H(v)\,dV_{S_1M} + \int_{S_1M}\operatorname{tr}\hat{K}_v\,dV_{S_1M}.$$

Since $\operatorname{tr}\Omega^{\dagger}\Omega$ is constant on M, we may compute the last integral as follows:

$$\begin{aligned}\int_{S_1M} \operatorname{tr}\hat{K}_v\, dV_{S_1M} &= \int_{S_1M} \operatorname{Ric}(v,v)\, dV_{S_1M} + \frac{1}{4}\int_{S_1M} \operatorname{tr}\Omega^{\dagger}\Omega\, dV_{S_1M} \\ &= \frac{\omega_{n-1}}{n}\int_M S(p)\, dV_M + \frac{\omega_{n-1}}{4}\operatorname{vol}(M)\operatorname{tr}\Omega^{\dagger}\Omega,\end{aligned}$$

where $S(p)$ stands for the scalar curvature at $p \in M$ and $\operatorname{vol}(M)$ is the volume of M. Since $H(v) = \operatorname{tr} U_v^2(0)$ is non-negative on S_1M, we have

$$\frac{1}{\operatorname{vol}(M)}\int_M S(p)\, dV_m + \frac{n}{4}\operatorname{tr}\Omega^{\dagger}\Omega = -\frac{n}{\operatorname{vol}(S_1M)}\int_{S_1M} H(v)\, dV_{S_1M} \le 0,$$

where $\operatorname{vol}(S_1M)$ is the volume of S_1M. The equality holds if and only if $H(v) = 0$ for all $v \in S_1M$, that is to say, $U_v(0) = 0$ for all $v \in S_1M$. By Lemma 3.2, we find that $U_v(t) \equiv 0$ along c_v for all $v \in S_1M$. This means that $\hat{K}_v = 0$ for all $v \in S_1M$. It follows that

$$g(R(v,w)v,w) = \frac{1}{4}g(\Omega(w),\Omega(w))$$

for all $w \in T_{\pi(v)}M$ and $v \in S_1M$. Since $g(\Omega(v),\Omega(v)) = -4g(R(v,v)v,v) = 0$ for all $v \in S_1M$, we find $\Omega \equiv 0$ on TM. Moreover, we get $R \equiv 0$ on M. Therefore, the following result has been proved.

THEOREM 3.3. *Let (M,g) be a compact Riemannian manifold with a uniform magnetic field B. Suppose that there exists no pair of conjugate points under B along an arbitrary solution curve of the equation* (1) *whose velocity is* 1. *Then,*

$$\frac{1}{\operatorname{vol}(M)}\int_M S(p)\, dV_M \le -\frac{n}{4}\operatorname{tr}\Omega^{\dagger}\Omega,$$

and the equality holds if and only if (M,g) is flat and $B \equiv 0$ on M.

For the case where (M,g) is a compact orientable surface with a uniform magnetic field $B = b\operatorname{vol}_M$ ($b \in \boldsymbol{R}$), the above inequality becomes

$$\frac{1}{\operatorname{vol}(M)}\int_M R(p)\, dV_M \le -\frac{1}{2}b^2,$$

and the equality holds if and only if (M,g) is flat and $b = 0$. Theorem 3.3 contains Hopf's result in the special case where $b = 0$. However, we would like to look for a geometric inequality which is sharp in the case of $b \neq 0$. In the next section, we will derive such a geometric inequality from exponential map non-conjugation.

4. Exponential map non-conjugation.

In this section, we will introduce the exponential maps associated with B and find the other geometric inequality by assuming that there exist no singular values of the exponential maps associated with B, what is called, *exponential map non-conjugation*.

DEFINITION 4.1. Let $w \in TM$. Then, the exponential maps $\exp^{\pm\Omega} : TM \to M$ associated with B are respectively defined as

$$\exp^{\Omega}(w) = c_{v(w)}(\sqrt{g(w,w)}),$$
$$\exp^{-\Omega}(w) = c_{v(w)}(-\sqrt{g(w,w)})$$

where $v(w) = w/\sqrt{g(w,w)} \in S_1 M$.

We investigate the geometrical meaning of exponential map non-conjugation. We shall look at the real $(n-1)\times(n-1)$ matrix differential equation along c_v

$$\ddot{\mathscr{X}} + \Omega_{\dot{c}_v,\perp}\dot{\mathscr{X}} + \tilde{R}_{\dot{c}_v,\perp}\mathscr{X} = 0. \tag{11}$$

This is the equation of the normal components of a Jacobi field under B along c_v with $g((D/dt)J, \dot{c}_v) \equiv 0$. Note that $\Omega_{\dot{c}_v,\perp} \equiv \Omega_{v,\perp}$ on c_v and that $\tilde{R}_{\dot{c}_v,\perp}$ is symmetric on c_v. Let $\mathscr{X}_{v,v}(t)$ be a solution of the equation (11) with $\mathscr{X}_{v,v}(v) = 0$ and $\dot{\mathscr{X}}_{v,v}(v) = I_{n-1}$.

LEMMA 4.2. *Let $d_w \exp^{\pm\Omega}$ denote the differentials of $\exp^{\pm\Omega}$ at $w \in TM$ respectively. Then,*

1. $\det(d_w \exp^{\Omega}) = g(w,w)^{-(n-1)/2} \det \mathscr{X}_{v(w),0}(\sqrt{g(w,w)})$,
2. $\det(d_w \exp^{-\Omega}) = -g(w,w)^{-(n-1)/2} \det \mathscr{X}_{v(w),0}(-\sqrt{g(w,w)})$.

PROOF. Let $V_1, \dots, V_n$ be orthonormal vector fields along $c_{v(w)}$ defined in Section 1. Note that $(v(w), \dots, v_n)$ is an orthonormal basis in $T_{\pi(w)}M$. First,

$$\begin{aligned} d_w \exp^{\Omega}(v(w)) &= \frac{d}{ds}\exp^{\Omega}(w + sv(w))|_{s=0} \\ &= \frac{d}{ds} c_{v(w)}(\sqrt{g(w,w)} + s)|_{s=0} \\ &= \dot{c}_{v(w)}(\sqrt{g(w,w)}). \end{aligned}$$

Next, let us set $v_i(w;\theta) = v(w)\cos\theta + v_i \sin\theta \in S_1 M$ $(i = 2, \dots, n)$. Then,

$$\begin{aligned} d_w \exp^{\Omega}(v_i) &= \frac{1}{\sqrt{g(w,w)}}\frac{d}{d\theta}\exp^{\Omega}(\sqrt{g(w,w)}v_i(w;\theta))|_{\theta=0} \\ &= \frac{1}{\sqrt{g(w,w)}}\frac{d}{d\theta} c_{v_i(w;\theta)}(\sqrt{g(w,w)})|_{\theta=0} \\ &= \frac{1}{\sqrt{g(w,w)}} J_i(\sqrt{g(w,w)}), \end{aligned}$$

where $J_i = (d/d\theta)c_{v_i(w;\theta)}|_{\theta=0}$ is a Jacobi field under B along $c_{v(w)}$ with $J_i(0) = 0$ and $(D/dt)J_i(0) = v_i$. Note that $g((D/dt)J_i, \dot{c}_{v(w)}) \equiv g(v_i, v(w)) = 0$. If $J_i = \sum_{j=1}^n f_{i,j}V_j$, then $\boldsymbol{f}_{i,\perp} = (f_{i,2}, \dots, f_{i,n})$ is a solution of the equation (5) with $\boldsymbol{f}_{i,\perp}(0) = 0$ and $\dot{\boldsymbol{f}}_{i,\perp}(0) = \boldsymbol{e}_{i-1}$ where $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})$ is a canonical orthonormal basis in $\boldsymbol{R}^{n-1}$. Therefore, $\boldsymbol{f}_{i,\perp} = \mathscr{X}_{v(w),0}\boldsymbol{e}_{i-1}$. This implies the first identity. In the same way, the second identity is shown. □

By Lemma 4.2 we see that exponential map non-conjugation is equivalent to the condition that $\det \mathscr{X}_{v,0}(t) \neq 0$ for all $t \neq 0$ and $v \in S_1 M$, namely, that for all $v \in S_1 M$, the normal components of a nonzero Jacobi field J under B along c_v with $g((D/dt)J, \dot{c}_v) = 0$ vanishes in at most one point.

Since $\Omega_{v,\perp} = \Omega_{\varphi_v(v),\perp}$ and $\tilde{R}_{\dot{c}_v,\perp}(t) = \tilde{R}_{\dot{c}_{\varphi_v(v)},\perp}(t-v)$ for all $v \in \boldsymbol{R}$, we have $\mathscr{X}_{v,v}(t) = \mathscr{X}_{\varphi_v(v),0}(t-v)$ for all $v \in \boldsymbol{R}$. After all, exponential map non-conjugation implies that $\det \mathscr{X}_{v,v}(t) \neq 0$ for all $t \neq v$ and $v \in S_1M$. Therefore, we may apply the results in Section 2 to the equation (11) along c_v for all $v \in S_1M$. Let $\mathscr{U}_v(t)$ denote the matrix along c_v which corresponds to $U(t)$ in Section 2.

Definition 4.3. For each $v \in S_1M$, a linear endomorphism $\tilde{K}_v$ of $T_{\pi(v)}M$ is defined by

$$\tilde{K}_v(w) = R(v,w)v + \frac{1}{4}\Omega^\dagger\Omega(w) + \frac{3}{4}g(\Omega(v),w)\Omega(v),$$

where $w \in T_{\pi(v)}M$.

For all $v \in S_1M$ and $w \in T_{\pi(v)}M$, we have

$$\tilde{R}_{v,\perp}(w) + \frac{1}{4}\Omega^\dagger_{v,\perp}\Omega_{v,\perp}(w) = \tilde{K}_{v,\perp}(w),$$

where $\tilde{K}_{v,\perp} = \mathrm{pr}_v \tilde{K}_v \mathrm{pr}_v$. See [**2**] for details. Therefore, the equations which $\mathscr{U}_v(t)$ should satisfy are

$$\dot{\mathscr{U}}_v(t) + \mathscr{U}_v^2(t) + e^{(t/2)\Omega_{v,\perp}}\tilde{K}_{\varphi_t(v),\perp}e^{-(t/2)\Omega_{v,\perp}} = 0, \quad \mathscr{U}_v(t)^\dagger = \mathscr{U}_v(t). \tag{12}$$

In the same way as the proof of Lemma 3.2, we have

$$\mathscr{U}_{\varphi_t(v)}(s) = e^{-(t/2)\Omega_{v,\perp}}\mathscr{U}_v(s+t)e^{(t/2)\Omega_{v,\perp}}$$

for all $s, t \in \boldsymbol{R}$ and $v \in S_1M$.

By taking the trace, we have functions $\mathscr{F}(v) = \mathrm{tr}\,\mathscr{U}_v(0)$, $\mathscr{G}(v) = \mathrm{tr}\,\dot{\mathscr{U}}_v(0)$ and $\mathscr{H}(v) = \mathrm{tr}\,\mathscr{U}_v^2(0)$ on S_1M. From the equation (12), the functions $\mathscr{F}(\varphi_t(v))$ and $\mathscr{H}(\varphi_t(v))$ satisfy the equation

$$\dot{\mathscr{F}}(\varphi_t(v)) + \mathscr{H}(\varphi_t(v)) + \mathrm{tr}\,\tilde{K}_{\varphi_t(v),\perp} = 0,$$

where we have used the identity $\mathscr{G}(\varphi_t(v)) = \dot{\mathscr{F}}(\varphi_t(v))$. By the same argument as that in Section 3, we have

$$\int_{S_1M} \mathrm{tr}\,\tilde{K}_{v,\perp}\,dV_{S_1M} = -\int_{S_1M} \mathscr{H}(v)\,dV_{S_1M} \leq 0.$$

Now, let us add the assumption that $n = 2m$ and all eigenvalues of $\Omega^\dagger\Omega$ are b^2 ($b \in \boldsymbol{R}$) in order to compute the lefthand side of the above identity. Such an example is a Kähler manifold with a Kähler magnetic field. See [**1**] for a Kähler manifold with a Kähler magnetic field. In particular, note that if $n = 2$, this assumption is always satisfied. Then, because of $\mathrm{tr}(\Omega^\dagger\Omega)_{v,\perp} = (2m-1)b^2$ for all $v \in S_1M$, we find

$$\int_{S_1M} \mathrm{tr}\,\tilde{K}_{v,\perp}\,dV_{S_1M} = \int_{S_1M} Ric(v,v)\,dV_{S_1M} + \frac{1}{4}\int_{S_1M} \mathrm{tr}(\Omega^\dagger\Omega)_{v,\perp}\,dV_{S_1M}$$
$$+ \frac{3}{4}\int_{S_1M} g(\Omega(v),\Omega(v))\,dV_{S_1M}$$

$$= \frac{\omega_{2m-1}}{2m} \int_M S(p)\,dV_M + \frac{(2m-1)b^2}{4}\,\omega_{2m-1}\operatorname{vol}(M) + \frac{3b^2}{4}\,\omega_{2m-1}\operatorname{vol}(M)$$

$$= \frac{\omega_{2m-1}}{2m} \int_M S(p)\,dV_M + \frac{(m+1)b^2}{2}\,\omega_{2m-1}\operatorname{vol}(M).$$

For this reason, we derive

$$\frac{1}{\operatorname{vol}(M)} \int_M S(p)\,dV_M + m(m+1)b^2 = -\frac{2m}{\operatorname{vol}(S_1M)} \int_{S_1M} \mathscr{H}(v)\,dV_{S_1M} \le 0.$$

The equality holds if and only if $\mathscr{H}(v) = 0$ for all $v \in S_1M$, that is to say, $\mathscr{U}_v(t) \equiv 0$ along c_v for all $v \in S_1M$. This means that $\tilde{K}_{v,\perp} = 0$ for all $v \in S_1M$. It follows that

$$\begin{aligned} g(R(v, \mathrm{pr}_v(w))v, \mathrm{pr}_v(w)) &= -\frac{1}{4} g(\Omega(\mathrm{pr}_v(w)), \Omega(\mathrm{pr}_v(w))) - \frac{3}{4} g(\Omega(v), \mathrm{pr}_v(w))^2 \\ &= -\frac{b^2}{4}\, g(\mathrm{pr}_v(w), \mathrm{pr}_v(w)) - \frac{3}{4} g(\Omega(v), \mathrm{pr}_v(w))^2 \end{aligned}$$

for all $w \in T_{\pi(v)}M$ and $v \in S_1M$. Therefore, the following results are obtained.

THEOREM 4.4. *Let (M, g) be an even dimensional compact Riemannian manifold with a uniform magnetic field B. Suppose that all eigenvalues of $\Omega^\dagger\Omega$ are b^2 $(b \in \boldsymbol{R})$ and that there exist no singular values of* $\exp^{\pm\Omega}$. *Then,*

$$\frac{1}{\operatorname{vol}(M)} \int_M S(p)\,dV_M \le -\frac{n(n+2)}{4}\, b^2,$$

and the equality holds if and only if $\tilde{K}_{v,\perp} = 0$ for all $v \in S_1M$.

COROLLARY 4.5. *Let (M, g) be a compact Kähler manifold with a Kähler magnetic field $B = bB_M$ $(b \in \boldsymbol{R})$ where B_M denotes the Kähler form. Suppose that there exist no singular values of* $\exp^{\pm\Omega}$. *Then,*

$$\frac{1}{\operatorname{vol}(M)} \int_M S(p)\,dV_M \le -\frac{n(n+2)}{4}\, b^2,$$

and the equality holds if and only if (M, g) is a compact Kähler manifold of constant holomorphic sectional curvature $-b^2$.

For the case where (M, g) is a compact orientable surface with a uniform magnetic field $b\operatorname{vol}_M$ $(b \in \boldsymbol{R})$, the above inequality becomes

$$\frac{1}{\operatorname{vol}(M)} \int_M R(p)\,dV_M \le -b^2,$$

and the equality holds if and only if the curvature of M is constant $-b^2$. Thanks to Gauss-Bonnet formula, this is expressed as

$$-\frac{b^2}{2\pi} \operatorname{vol}(M) \ge \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M. Therefore, the proof of Theorem 1 is completed.

REMARK. In this paper, we have assumed that M is compact so that functions on S_1M are integrable. We expect that the noncompact case is treated. Refer to [4], [6].

5. Relation between two non-conjugation.

By comparing Theorem 3.3 with Theorem 4.4, we may conjecture that exponential map non-conjugation is stronger than Jacobi field non-conjugation. For the case where (M,g) is a compact orientable surface with a uniform magnetic field $\boldsymbol{B} = b\,\mathrm{vol}_M$ ($b \in \boldsymbol{R}$), we will show that this conjecture is true.

Let $v \in S_1M$. Let $(\alpha_v(t), \beta_v(t))$ be a solution of the equation (6) along c_v with $(\alpha_v(0), \beta_v(0)) = (0,0)$ and $(\dot{\alpha}_v(0), \dot{\beta}_v(0)) = (1,0)$. Then,

$$\begin{cases} \dot{\alpha}_v = b\beta_v + 1, \\ \ddot{\beta}_v + \{R(c_v) + b^2\}\beta_v + b = 0. \end{cases}$$

Let $(\gamma_v(t), \delta_v(t))$ be a solution of the equation (6) along c_v with $(\gamma_v(0), \delta_v(0)) = (0,0)$ and $(\dot{\gamma}_v(0), \dot{\delta}_v(0)) = (0,1)$. Then,

$$\begin{cases} \dot{\gamma}_v = b\delta_v, \\ \ddot{\delta}_v + \{R(c_v) + b^2\}\delta_v = 0. \end{cases}$$

Note that $\mathscr{X}_{v,0}(t) = \delta_v(t)$ and

$$X_{v,0}(t) = \begin{pmatrix} \alpha_v(t) & \gamma_v(t) \\ \beta_v(t) & \delta_v(t) \end{pmatrix}.$$

Let $b > 0$ for the sake of simplicity.

LEMMA 5.1. *Suppose that $\delta_v(t) \neq 0$ for all $t \neq 0$. Then, $\gamma_v(t) > 0$ for all $t \neq 0$.*

LEMMA 5.2. *Suppose that $\delta_v(t) \neq 0$ for all $t \neq 0$. Then, β_v/δ_v is a monotone decreasing function of C^1-class on c_v.*

PROOF. Since

$$\lim_{t\to 0} \frac{\beta_v}{\delta_v}(t) = \frac{\dot{\beta}_v}{\dot{\delta}_v}(0) = 0,$$

β_v/δ_v is well defined and continuous at $t = 0$. If $t \neq 0$, we have

$$\frac{d}{dt}\frac{\beta_v}{\delta_v} = \frac{\dot{\beta}_v\delta_v - \beta_v\dot{\delta}_v}{\delta_v^2}.$$

Since

$$\frac{d}{dt}(\dot{\beta}_v\delta_v - \beta_v\dot{\delta}_v) = \ddot{\beta}_v\delta_v - \beta_v\ddot{\delta}_v = -b\delta_v = -\dot{\gamma}_v,$$

we find

$$\frac{d}{dt}\frac{\beta_v}{\delta_v} = -\frac{\gamma_v}{\delta_v^2} < 0$$

for all $t \neq 0$. Moreover, we obtain

$$\lim_{t\to 0}\frac{d}{dt}\frac{\beta_v}{\delta_v}(t) = -\lim_{t\to 0}\frac{\gamma_v}{\delta_v^2}(t) = -\frac{b\dot{\delta}_v}{2(\ddot{\delta}_v\delta_v + \dot{\delta}_v^2)}(0) = -\frac{b}{2} < 0.$$

Therefore, $(d/dt)(\beta_v/\delta_v)$ is well defined and continuous at $t = 0$. □

LEMMA 5.3. *Suppose that $\delta_v(t) \neq 0$ for all $t \neq 0$. Then,*
1. *if $t > 0$,*

$$\alpha_v(t) > \frac{\beta_v}{\delta_v}(t)\gamma_v(t) + t,$$

2. *if $t < 0$,*

$$\alpha_v(t) < \frac{\beta_v}{\delta_v}(t)\gamma_v(t) + t.$$

PROOF. Let $0 < s < t$. By Lemma 5.2, we have

$$\frac{\beta_v}{\delta_v}(s) > \frac{\beta_v}{\delta_v}(t).$$

Because of $\delta_v(s) > 0$, we find

$$\beta_v(s) > \frac{\beta_v}{\delta_v}(t)\delta_v(s).$$

Integrating the both sides with respect to s, we obtain

$$\alpha_v(t) = b\int_0^t \beta_v(s)\,ds + t > \frac{\beta_v}{\delta_v}(t)b\int_0^t \delta_v(s)\,ds + t = \frac{\beta_v}{\delta_v}(t)\gamma_v(t) + t,$$

which implies the first inequality. The second inequality is proved in the same way. □

COROLLARY 5.4. *Suppose that $\delta_v(t) \neq 0$ for all $t \neq 0$. Then,*

$$(\alpha_v\delta_v - \beta_v\gamma_v)(t) > t\delta_v(t) > 0$$

for all $t \neq 0$.

Let us note that Lemma 5.3 and Corollary 5.4 are satisfied for $b < 0$. Because of $X_{v,\nu}(t) = X_{\varphi_\nu(v),0}(t-\nu)$ and $\delta_{v,\nu}(t) = \delta_{\varphi_\nu(v),0}(t-\nu)$ for all $\nu \in \boldsymbol{R}$. Corollary 5.4 implies that if $\delta_{v,\nu}(t) \neq 0$ for all $t \neq \nu$, then

$$\det X_{v,\nu}(t) > (t-\nu)\delta_{v,\nu}(t) > 0$$

for all $t \neq \nu$. Therefore, we obtain the following result.

THEOREM 5.5. *Let (M, g) be a compact orientable surface with a uniform magnetic field $B = b\,\mathrm{vol}_M$ ($b \in \boldsymbol{R}$). If there exist no singular values of $\exp^{\pm\Omega}$, then there exists no* *pair of conjugate points under B along an arbitrary solution curve of the equation* (1) *whose velocity is* 1.

COROLLARY 5.6. *Let* (M, g) *be a compact orientable surface with a uniform magnetic field* $B = b\,\mathrm{vol}_M$ $(b \in \boldsymbol{R})$, *and let* $\kappa_{\max}(M)$ *denote the maximum of curvature of* M. *If* $\kappa_{\max}(M) + b^2 \leq 0$, *then there exists no pair of conjugate points under B along an arbitrary solution curve of the equation* (1) *whose velocity is* 1.

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