J. Math. Soc. Japan
Vol. 50, No. 3, 1998

A closed form for unitons

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(Received Oct. 3, 1996)

1. Introduction

Harmonic maps between Riemannian manifolds M and N are critical values of an energy functional

$$\operatorname{energy}(S : M \to N) = \frac{1}{2}\int_M |dS|^2.$$

In the case of surfaces in $\mathrm{U}(N)$, with the standard (bi-invariant) metric, the energy takes the form

$$\operatorname{energy}(S) = \frac{1}{16\pi}\int_{\boldsymbol{R}^2}\left(|S^{-1}\frac{\partial}{\partial x}S|^2 + |S^{-1}\frac{\partial}{\partial y}S|^2\right) dx \wedge dy. \tag{1.1}$$

Unitons are harmonic maps $S : \boldsymbol{S}^2 \to \mathrm{U}(N)$. We write $\operatorname{Harm}(\boldsymbol{S}^2, \mathrm{U}(N))$ for the space of unitons. Some authors call them multi-unitons.

We are concerned with the based maps

$$\operatorname{Harm}_k^*(\boldsymbol{S}^2, \mathrm{U}(N)) \stackrel{\text{def}}{=} \{S \in \operatorname{Harm}(\boldsymbol{S}^2, \mathrm{U}(N)) : S(\infty) = \boldsymbol{I}, \operatorname{energy}(S) = k\}.$$

In [**Uhl**], Uhlenbeck showed that all unitons could be constructed from simpler unitons by 'adding a uniton'. This construction was investigated from different perspectives by Wood, Valli, Guest, Ohnita and Segal. We approach the question of constructing unitons via algebraic integration, using a twistor construction of Hitchin and Ward ([**Hi**], [**Wa**]).

We proved in [**An1**] that the based unitons, $\operatorname{Harm}^*(\boldsymbol{S}^2, \mathrm{U}(N))$, are isomorphic to uniton bundles, with energy corresponding to the bundles' second Chern class. In this paper we apply Horrocks' monad construction to the uniton bundles.

THEOREM A. *Based, rank-N unitons of energy $k/2$ are all of the form*

$$S = \boldsymbol{I} + a\alpha_2^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}b$$

for some choice of $N \times k$, $k \times N$, $k \times k$ and $k \times k$ matrices, a, b, α_1, α_2. (Multiplication is matrix multiplication.)

1991 *Mathematics Subject Classification.* 58E20 32L25 32L10 14F05.
Key words and phrases. uniton, harmonic map, chiral field, sigma model, monad.
Research supported by NSERC and FCAR scholarships and Professor J.C. Hurtubise.

This formula is the monad version of the 'monodromy' interpretation (§2.7) of Uhlenbeck's extended solution (see [**Uhl**]). The rest of the paper paints a Geometric Invariant Theory picture of the moduli space:

THEOREM B. 1. *The space of based unitons* $\mathrm{Harm}_k^*(\boldsymbol{S}^2, \mathrm{U}(N))$ *is isomorphic to the set of 5-tuples of matrices*

(monad data)
$$\gamma, \alpha_1', \delta \in \mathrm{gl}(k/2), \quad \gamma \textit{ nilpotent}$$
$$a' \in \mathrm{M}_{N,k/2}, \quad b' \in \mathrm{M}_{k/2,N}$$

satisfying
(nondegeneracy)
$$\mathrm{rank}\begin{pmatrix} \gamma \\ \alpha_1' + z \\ a' \end{pmatrix} = \mathrm{rank}(\gamma \quad \alpha_1' + z \quad b') = k/2 \qquad \forall z \in \boldsymbol{C}$$

(monad equation)
$$[\alpha_1', 2\gamma] + b'a' = 0$$

(time invariance)
$$[\delta, \gamma] = 0 \quad a'\delta = 0$$
$$[\delta, \alpha_1'] = \gamma \quad \delta b' = 0$$

quotiented by the action of $g \in \mathrm{Gl}(k/2)$
$$\gamma \mapsto g\gamma g^{-1} \quad \alpha_1' \mapsto g\alpha_1' g^{-1} \quad \delta \mapsto g\delta g^{-1}$$
$$a' \mapsto a'g^{-1} \quad b' \mapsto gb'.$$

2. *These data determine the uniton bundle over a hemisphere. Reality determines it over the other hemisphere, giving monad data as in Theorem A as follows*:

(reality)
$$\alpha_1 = \begin{pmatrix} -\alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} -\boldsymbol{I} - 2\gamma^* & \\ & \boldsymbol{I} + 2\gamma \end{pmatrix}$$
$$a = (ib'^* \quad a') \qquad b = \begin{pmatrix} ia'^* \\ b' \end{pmatrix}$$

where ϕ_1 *and* ϕ_2 *are functions of* γ, a' *and* b' *determined by the big monad equation* $[\alpha_1, \alpha_2] + ba = 0$.

3. *The uniton number* (*see* §5) *is the smallest* $n \in \boldsymbol{Z}$ *such that* $\gamma^n = 0$, *i.e. it is* 1+ *the length of either polar jumping line.*

One geometric consequence of this picture is

COROLLARY C. *Two-unitons have normalized energy at least four. This bound is sharp.*

1.2. STRUCTURE OF THE PROOF. Except for §2, which lists notation and facts which must be taken on faith from [**An1**], the rest of the paper is concerned with the proofs.

In § 3 we show how a well-known monad representation for bundles on $\boldsymbol{P}^2$ can be made to apply, and how the reality and triviality properties can be interpreted, yielding a normalized monad. The uniton bundles are a subset thereof. The proofs of Theorems A and B are logically independent from this point on and we complete the proof of Theorem A in § 4. The proof of Theorem B naturally splits into three parts which are also logically independent. They are the interpretation of the uniton number in § 5; the proof that real bundles with the other triviality properties are necessarily trivial on real sections in § 6; and the interpretation of time-invariance in § 7. Corollary C is a simple consequence of the interpretation of time-invariance.

2. Prerequisites

2.1. Uniton Bundles. In [**An1**] we identified $\mathrm{Harm}^*(\boldsymbol{S}^2, \mathrm{U}(N))$ with a class of bundles on $\widetilde{\boldsymbol{TP}}^1 \stackrel{\text{def}}{=} \boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(2))$, the fibrewise compactification of the tangent bundle $\boldsymbol{TP}^1$ to the *complex* projective line.

Let (λ, η) and $(\hat{\lambda} = 1/\lambda, \hat{\eta} = \eta/\lambda^2)$ be coordinates on $\boldsymbol{TP}^1 \cong \mathcal{O}_{\boldsymbol{P}^1}(2)$, where λ is the usual coordinate on $\boldsymbol{P}^1$ and η is the coordinate associated to $d/d\lambda$. Meromorphic sections (s) of $\boldsymbol{TP}^1$ give all the holomorphic sections of $\widetilde{\boldsymbol{TP}}^1$ ($[s, 1]$ in projective coordinates on $\widetilde{\boldsymbol{TP}}^1$), save one. We fix notation for the lines on $\widetilde{\boldsymbol{TP}}^1$:

$$
(2.2)\qquad
\begin{aligned}
P_{\lambda_0} &= \{\lambda = \lambda_0\}, \\
G_0 &= \{(\lambda, [0, 1])\} = \text{zero section of } \boldsymbol{TP}^1, \\
G_\infty &= \{(\lambda, [1, 0])\} = \text{infinity section of } \widetilde{\boldsymbol{TP}}^1, \\
G_{\eta=s} &= \{(\lambda, [s(\lambda), 1])\}.
\end{aligned}
$$

If $y = (a, b, c) \in \boldsymbol{C}^3$, we will also write G_y for $G_{\eta=a-2b\lambda-c\lambda^2}$.

To encode unitarity, we need the real structure

$$
(2.3)\qquad \sigma(\lambda, \eta) = (1/\bar{\lambda}, -\bar{\lambda}^{-2}\bar{\eta}),
$$

which acts by

$$
\sigma(a, b, c) = (\bar{c}, -\bar{b}, \bar{a})
$$

on $\boldsymbol{C}^3 \cong H^0(\boldsymbol{P}^1, \mathcal{O}(2))$, the space of finite sections. We define 'time translation' as the one-parameter group of transformations:

$$
(2.4)\qquad
\begin{aligned}
\delta_t : (\lambda, \eta) &\mapsto (\lambda, \eta - 2t\lambda) \\
(a, b, c) &\mapsto (a, b + t, c).
\end{aligned}
$$

Definition 2.5. A rank N, or $\mathrm{U}(N)$, uniton bundle, $\mathscr{V}$, is a holomorphic rank N bundle on $\widetilde{\boldsymbol{TP}}^1$ which is a) trivial when restricted to the following curves in $\widetilde{\boldsymbol{TP}}^1$

1. the section at infinity,
2. nonpolar fibres (i.e. P_λ, $\lambda \in \boldsymbol{C}^*$),
3. real sections of $\boldsymbol{TP}^1$ (sections invariant under σ);

b) is equipped with bundle lifts

$$\begin{array}{ccccccc} \mathscr{V} & \xrightarrow{\tilde{\delta}_t} & \mathscr{V} & & \mathscr{V} & \xrightarrow{\tilde{\sigma}} & \mathscr{V}^* \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ \widetilde{TP}^1 & \xrightarrow{\delta_t} & \widetilde{TP}^1 & & \widetilde{TP}^1 & \xrightarrow{\sigma} & \widetilde{TP}^1 . \end{array}$$

1. $\tilde{\delta}_t$, a one-parameter family of holomorphic transformations fixing $\mathscr{V}$ above the section at infinity, and

2. $\tilde{\sigma}$, an antiholomorphic lift of σ such that the induced hermitian metric on $\mathscr{V}$ restricted to a fixed point of σ is positive definite; equivalently, such that the induced lift to the principal bundle of frames acts on fibres of fixed points of σ by $X \mapsto X^{*-1}$;

and c) has a framing, $\phi \in H^0(P_{-1}, \text{frames}(\mathscr{V}))$, of the bundle $\mathscr{V}$ restricted to the fibre $P_{-1} = \{\lambda = -1\} \subset \widetilde{TP}^1$ such that $\tilde{\sigma}(\phi) = \phi$.

REMARK 2.6. In §6 we will prove that condition a) 3. is unnecessary.

2.7. MONODROMY CONSTRUCTION. From [**An1**], we will also require the fact that Uhlenbeck's extended solution can be computed by composing the cycle of isomorphisms

$$\begin{array}{ccccc} \mathscr{V}_{\lambda,\infty} & \xleftarrow{\text{eval}} & H^0(G_\infty, \mathscr{V}) & \xrightarrow{\text{eval}} & \mathscr{V}_{-1,\infty} \\ \uparrow{\scriptstyle \text{eval}} & & & & \uparrow{\scriptstyle \text{eval}} \\ H^0(P_\lambda, \mathscr{V}) & & & & H^0(P_{-1}, \mathscr{V}) \\ \downarrow{\scriptstyle \text{eval}} & & & & \downarrow{\scriptstyle \text{eval}} \\ \mathscr{V}_{(\lambda, z-2t\lambda-\lambda^2\bar{z})} & \xleftarrow{\text{eval}} & H^0(G_{(z,t,\bar{z})}, \mathscr{V}) & \xrightarrow{\text{eval}} & \mathscr{V}_{(-1,z+2t-\bar{z})} \end{array} \tag{2.8}$$

counter clockwise, beginning at the top. The existence of the bundle isomorphism $\tilde{\delta}_t$ (time translation) ensures that the result doesn't depend on t. Finiteness, i.e. extension to S^2, follows from the compactness of $\widetilde{TP}^1$. Recall that the extended solution is the 'twistor lift' of the harmonic map, $S = E_1$. (Warning: the convention $S = E_{-1}$ is more common. For an explanation of extended solutions see [**Uhl**].)

Alternatively, we can use the language of moving frames. Define a meromorphic frame g which agrees with the fixed frame and is holomorphic on $G_\infty \cup TC^*$ (remember that $\mathscr{V}|_{G_\infty \cup TC^*}$ is trivial); and a family of frames f_y of $\mathscr{V}|_{G_y}$ (also trivial) which agree with the fixed frame of $\mathscr{V}|_{P_{-1}}$. The triviality properties make the frames unique. The extended solution is the change of frame

$$E_\lambda(z) = g^{-1} \cdot f_\lambda|_{(\lambda, \eta = z - \bar{z}\lambda^2)}. \tag{2.9}$$

3. Uniton bundles and $\boldsymbol{P}^2$ monads

The triviality properties of uniton bundles allow us to identify them with bundles on $\boldsymbol{P}^2$ which have a well-known monad representation. These monads have two essential features:

1. they are self-dual, i.e. the transposed monad is a monad of the same form representing the dual bundle, and
2. we know when the represented bundle is trivial on hyperplanes. In the usual notation (described below), $\mathscr{V} \to \boldsymbol{P}^2$ is trivial on a hyperplane $L = \overline{p_1 p_2} \Leftrightarrow \det(K_{p_2} \circ J_{p_1}) \neq 0$, and any choice of spanning representatives of $\ker K_{p_i}/\operatorname{im} J_{p_i}$ frames $\mathscr{V}|_L$ canonically, in particular a basis of $\ker K_{p_1} \cap \ker K_{p_2}$ does. (See [**Do**].)

3.1. The birational equivalence. If X, Y and W are homogeneous coordinates on $\boldsymbol{P}^2$, and λ and η affine base and fibre coordinates on $\widetilde{T\boldsymbol{P}}^1$,

$$\{X = \lambda Y, (X+Y)W = \eta Y^2\} \subset \widetilde{T\boldsymbol{P}}^1 \times \boldsymbol{P}^2$$

is the graph of the birational equivalence ρ, which comes from

1. blowing up the point $(\lambda = -1, \eta = 0)$,
2. blowing down $\tilde{P}_{-1}$ (the proper transform of the fibre $\{\lambda = -1\}$), and
3. blowing down the image of G_∞.

Under the birational equivalence ρ, the ruling $\{P_\lambda : \lambda \in \boldsymbol{P}^1\}$ of $\widetilde{T\boldsymbol{P}}^1$ is mapped to the pencil of lines $\{X = \lambda Y\}$ on $\boldsymbol{P}^2$, except for the fibre P_{-1}, which is mapped to a point; the line $X + Y = 0$ is the exceptional divisor of the blowup. The curve G_∞ becomes the point $[0,0,1] \in \boldsymbol{P}^2$. Since push forward gives an isomorphism between bundles on $\widetilde{T\boldsymbol{P}}^1$ and bundles on $\boldsymbol{P}^2$ which are trivial on $G_\infty \cup P_{-1}$ and $\{X+Y\}$ respectively, we will use the same letter for a bundle and its push forward.

Assume now that $\mathscr{V}$ is a uniton bundle. Since $\mathscr{V}$ is trivial on generic lines, $\mathscr{V}$ admits a monad representation [**OSS**, example 3, p249], $\mathscr{V} = \ker K/\operatorname{im} J$, where

$$0 \to \mathscr{O}_{\boldsymbol{P}^2}(-1)^k \xrightarrow{J} \mathscr{O}_{\boldsymbol{P}^2}^{2k+N} \xrightarrow{K} \mathscr{O}_{\boldsymbol{P}^2}(1)^k \to 0 \tag{3.2}$$

is a complex of linear maps, such that $K \circ J = 0$, and on each fibre J is injective and K surjective, with $k = c_2(\mathscr{V})$.

The monad representation (3.2) is not unique; $\mathrm{Gl}(k) \times \mathrm{Gl}(2k+N) \times \mathrm{Gl}(k)$ acts on the vector spaces linearly, inducing monad equivalences. Given that $\mathscr{V}|_{\{X+Y\}}$ is trivial, we can assume that the monad has the block form

$$J \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{I} \\ 0 \\ 0 \end{pmatrix} W + \begin{pmatrix} 0 \\ \boldsymbol{I} \\ 0 \end{pmatrix} (X - Y) + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix} (X + Y), \tag{3.3}$$

$$K \stackrel{\text{def}}{=} (0 \quad \boldsymbol{I} \quad 0) W + (-\boldsymbol{I} \quad 0 \quad 0)(X - Y) + (-\alpha_2 \quad \alpha_1 \quad b)(X + Y), \tag{3.4}$$

where the α_i, a and b are $k \times k$, $N \times k$ and $k \times N$ matrices, respectively. This form is stabilised by an action of $\mathrm{Gl}(k) \times \mathrm{Gl}(N)$. The $\mathrm{Gl}(N)$ action corresponds to changes of frame. Since uniton bundles are framed it does not act.

The monad equation $K \circ J = 0$ is

$$[\alpha_1, \alpha_2] + ba = 0 \tag{3.5}$$

and nondegeneracy says that

$$\begin{pmatrix} \alpha_1 + u \\ \alpha_2 + v \\ a \end{pmatrix} \quad \text{and} \quad (-\alpha_2 - v \quad \alpha_1 + u \quad b)$$

have full rank for all $u, v \in \boldsymbol{C}$. Since

$$K|_{[0,0,1]} \circ J|_{[\lambda,1,0]} = \text{const}(\alpha_2(\lambda + 1) + (\lambda - 1)),$$

$\mathscr{V}|_{\{X=\lambda Y\}}$ is holomorphically nontrivial iff $(1-\lambda)/(1+\lambda)$ is an eigenvalue of α_2.

Since only $\mathscr{V}|_{\{X=0\}}$ and $\mathscr{V}|_{\{Y=0\}}$ are nontrivial,

$$\alpha_2 = \begin{pmatrix} -\boldsymbol{I} - 2\gamma' & \\ & \boldsymbol{I} + 2\gamma \end{pmatrix}$$

for some nilpotent matrices γ, γ'. If we correspondingly decompose

$$\alpha_1 = \begin{pmatrix} \alpha_1'' & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix}$$

the monad equation (3.5) implies that ϕ_1 and ϕ_2 are determined by γ, γ', a and b.

REMARK 3.6. We know from [**Hu**] that bundles on $\boldsymbol{P}^2$ can be profitably studied by decomposing them into a collection of framed local jumps. Such local data can be uniquely glued back into a trivial bundle on the complement of the jumping lines by using the framing along a fixed section which is part of the data.

This technique has a direct translation into the language of monads. The eigenvalues of α_2 correspond to the jumping lines in the pencil of lines through $\{X = 0 = Y\}$. Decomposing the space on which α_2 acts

$$\boldsymbol{C}^k = \oplus V_\mu; \quad V_\mu \stackrel{\text{def}}{=} \ker(\alpha_2 - \mu)^k$$

and defining projections $\pi_\mu : \boldsymbol{C}^k \to V_\mu$, and injections $\iota_\mu : V_\mu \to \boldsymbol{C}^k$, the monad data

$$\pi_\mu \alpha_2 \iota_\mu, \pi_\mu \alpha_1 \iota_\mu, \pi_\mu a, b\iota_\mu$$

represent the bundle $\mathscr{V}_\mu$ formed by gluing the single framed local jump at $\{X = \mu Y\}$ into the trivial bundle on its complement. Concretely, we are putting α_2 into block diagonal form and throwing out the off-diagonal blocks of the corresponding decomposition of α_1. The 'diagonal blocks' of the monad equation are monad equations for the decomposition products, while the off-diagonal pieces give nondegenerate linear equations which determine the off-diagonal blocks of α_1.

In our case, there are two pieces, resulting in two bundles $\mathscr{V}_0$ and $\mathscr{V}_\infty$. The real structure interchanges them, so

$$\operatorname{rank} \boldsymbol{I} + \gamma = \dim V_0 = c_2(\mathscr{V}_0) = c_2(\mathscr{V}_\infty) = \dim V_\infty = \operatorname{rank} \boldsymbol{I} + \gamma'.$$

in particular, k is even.

3.7. Reality. On the monad level, the real structure is represented by a monad map

$$(J, K) \to (\overline{\sigma^* K^t}, \overline{\sigma^* J^t}).$$

Motivated by the previous remark, it is not hard to show that real bundles have real monads with the block form

$$\alpha_2 = \begin{pmatrix} -\boldsymbol{I} - 2\gamma^* & \\ & \boldsymbol{I} + 2\gamma \end{pmatrix}, \tag{3.8a}$$

$$\alpha_1 = \begin{pmatrix} -\alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix}, \tag{3.8b}$$

$$b = \begin{pmatrix} ia'^* \\ b' \end{pmatrix}, \tag{3.8c}$$

$$a = (ib'^* \quad a'). \tag{3.8d}$$

The off-diagonal blocks can be calculated explicitly to be

$$\begin{aligned} \phi_1 &= -\frac{i}{2} \sum_{j\geq 1} \gamma^{*(j-1)} (\boldsymbol{I} + \gamma^*)^{-j} a'^* a' (\boldsymbol{I} + \gamma)^{-j} \gamma^{j-1}, \\ \phi_2 &= \frac{i}{2} \sum_{j\geq 1} \gamma^{j-1} (\boldsymbol{I} + \gamma)^{-j} b' b'^* (\boldsymbol{I} + \gamma^*)^{-j} \gamma^{*(j-1)}. \end{aligned} \tag{3.9}$$

4. The closed form

We will calculate the monodromy interpretation of the uniton (2.8) in terms of bases for the vector spaces given by the monad data α_1, α_2, a, b.

Under the birational equivalence ρ, the lines G_∞ and P_{-1} are identified to a point. The line P_λ corresponds to $\{X = \lambda Y\}$, and the lines $G_{(z,(\bar{z}-z)/2,\bar{z})}$ are sent to the degenerate quadrics $(\bar{z}X - zY + W)(X + Y) = 0$. Since the bundle is time-translation invariant it suffices to parametrise the sections of $\mathscr{V}|_{\{X=\lambda Y\}}$ and of $\mathscr{V}|_{\{\bar{z}X - zY + W = 0\}}$.

Sections of $\mathscr{V}|_{\{\bar{z}X - zY + W = 0\}}$ are parametrised by

$$\ker K_{p_1} \cap \ker K_{p_2},$$

where $\overline{p_1 p_2} = \{\bar{z}X - zY + W = 0\}$. Taking $p_1 = [1, 1, z - \bar{z}]$ and $p_2 = [1, -1, -z - \bar{z}]$,

$$H^0(\mathscr{V}|_{\overline{p_1 p_2}}) \cong \ker \begin{pmatrix} K_{p_2} \\ K_{p_1} \end{pmatrix} = \ker \begin{pmatrix} -\boldsymbol{I} & -x & 0 \\ -\alpha_2 & \alpha_1 + iy & a \end{pmatrix}.$$

Real triviality is equivalent to

$$\begin{aligned}(4.1)\qquad 0 &\neq \det 4((z-\bar{z})J_W + 2J_{X+Y})((-z-\bar{z})K_W + 2K_{X-Y}) \\ &= \det(\alpha_1 + x\alpha_2 + iy\boldsymbol{I}),\end{aligned}$$

which implies that the kernel is

$$(4.2)\qquad \left\{ \begin{pmatrix} x(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1} bs \\ -(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1} bs \\ s \end{pmatrix} \in \boldsymbol{C}^{2k+N} : s \in \boldsymbol{C}^N \right\}.$$

In terms of the moving frame f_y, the sections are

$$(4.3)\qquad f_y \cdot s = \begin{pmatrix} x(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1} bs \\ -(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1} bs \\ s \end{pmatrix} + \operatorname{im} J.$$

Sections of $\mathscr{V}|_{\{X=\lambda Y\}}$ are similarly represented by

$$(4.4)\qquad g \cdot s = \begin{pmatrix} \left(\alpha_2 + \dfrac{\lambda - 1}{\lambda + 1}\boldsymbol{I}\right)^{-1} bs \\ 0 \\ s \end{pmatrix} + \operatorname{im} J$$

as s varies in $\boldsymbol{C}^N$.

The evaluation map $H^0(\{X = \lambda Y\}, \mathscr{V}) \to \mathscr{V}_p$ is given by $s \mapsto (4.4) + \operatorname{im} J_p$. To compute

$$H^0(\{\bar{z}X - zY + W = 0\}, \mathscr{V}) \to \mathscr{V}_p \to H^0(\{X + \lambda Y\}, \mathscr{V})$$

we take the expression (4.2) and add an element of $\operatorname{im} J_p$ to obtain a representative of the form (4.4).

At the point of intersection $p = [\lambda, 1, z - \lambda\bar{z}]$

$$J_p = (1+\lambda) \begin{pmatrix} \alpha_1 + \dfrac{z - \lambda\bar{z}}{1+\lambda} \\ \alpha_2 + \dfrac{\lambda - 1}{1+\lambda} \\ a \end{pmatrix}.$$

When $\lambda \in \boldsymbol{C}^*$, we translate (4.2) into the form of (4.4) by adding

$$\frac{2}{1-\lambda} J_p \left(\left(\alpha_2 + \frac{\lambda - 1}{1+\lambda}\right)^{-1} (\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1} bs \right).$$

In the language of moving frames

$$(gE_\lambda)s = f_y s = \begin{pmatrix} (\alpha_1 + x\alpha_2 + iy\boldsymbol{I})\left(\alpha_2 + \dfrac{\lambda-1}{1+\lambda}\right)^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}bs \\ 0 \\ \left(\boldsymbol{I} + a\left(\alpha_2 + \dfrac{\lambda-1}{1+\lambda}\right)^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}b\right)s \end{pmatrix} + \operatorname{im} J$$

$$= g\left(\boldsymbol{I} + a\left(\alpha_2 + \frac{\lambda-1}{1+\lambda}\right)^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}b\right)s.$$

We conclude that

$$E_\lambda = \boldsymbol{I} + a\left(\alpha_2 + \frac{\lambda-1}{1+\lambda}\right)^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}b; \tag{4.5}$$

when $\lambda = 1$, $S = E_1$ is the sought expression of Theorem A.

Since the parametrisations of $H^0(\{X = \lambda Y\}, \mathscr{V})$ agree at [0, 0, 1], as do the parametrisations of $H^0(\{X + Y = 0\}, \mathscr{V})$ and $H^0(\{\bar{z}X - zY + W = 0\}, \mathscr{V})$ at $[-1, 1, z + \bar{z}]$, this agrees with the full computation of the diagram (2.8).

REMARK 4.6. We will show in a future paper [**An3**] that this construction of unitons can be generalised to give a construction of certain solutions of Ward's chiral model, and in this more general context it will be easy to see that the generalised equations are satisfied. In this case, however, it is easy to verify that the solutions are unitary and to use a symbolic calculator to evaluate the equations on specific solutions or families of solutions.

5. Interpretation of uniton number

We now prove part 3 of Theorem B. Recall from [**Uhl**] that $S \in \operatorname{Harm}(\boldsymbol{S}^2, \mathrm{U}(N))$ has *uniton number n* if S admits an extended solution of the form

$$\tilde{E}_\lambda = \lambda^{-n}T_0 + \lambda^{1-n}T_1 + \cdots + T_n$$

and this is the shortest possible such solution. We will assume, without loss of generality, that $\tilde{E}_\lambda$ is a fixed shortest-length extended solution in Uhlenbeck normal form, i.e.

$$\operatorname{span}\{\operatorname{im} T_0(z) : z \in \boldsymbol{C}\} = \boldsymbol{C}^N.$$

LEMMA 5.1. *Let $\mathscr{V}$ be the uniton bundle represented by the data $(\alpha_1', \gamma, a', b')$, and let $P_0^{(l)}$ be the lth formal neighbourhood of the line. The following numbers are the same.*

1. *the largest l such that there exists a map*

$$\mathcal{O}_{P_0^{(l)}}(1) \xrightarrow{s} \mathscr{V}_{P_0^{(l)}}, \quad s|_{P_0} \neq 0;$$

2. *the largest l such that there exists a map*

$$\mathcal{V}_{P_0^{(l)}} \xrightarrow{s} \mathcal{O}_{P_0^{(l)}}(-1), \quad s|_{P_0} \neq 0;$$

3. *the smallest l such that $\gamma^{l+1} = 0$.*

We will call l the *length* of the jump at P_0. See [**Ti**] for another definition of length.

PROOF. Observe that if p_1 and p_2 are two points on $P_0 \subset \boldsymbol{P}^2$, then

$$\ker K_{p_2} \circ J_{p_1} = \{s \in H^0(P_0, \mathcal{V}) : s(p_1) = 0\} \cong H^0(P_0, \mathcal{V}(-1)),$$

and that this works just as well for $P_0^{(l)}$ in which case all objects are defined over $\boldsymbol{C}[\lambda]/(\lambda^{l+1})$ instead of over $\boldsymbol{C}$. The equivalence of 1. and 3. can now be reduced to the calculation of

$$\begin{aligned} \ker(\lambda + \gamma)|_{P_0^{(l)}} &= \{\lambda^i \gamma^j \cdot \lambda^{l+1}(\lambda+\gamma)^{-1} : i,j \geq 0\} \\ &= \{\lambda^i\gamma^j \cdot (\gamma^l - \lambda\gamma^{l-1} + \cdots + (-\lambda)^{l-1}\gamma + (-\lambda)^l) : i,j \geq 0\}. \end{aligned} \tag{5.2}$$

Since the dual bundle is described by the transposed monad, 2. and 3. are also equivalent. See [**Ti**] for another definition of length. □

Recall that the triviality properties of a uniton bundle $\mathcal{V}$ define two sorts of framings of $\mathcal{V}$: g over $G_\infty \cup T\boldsymbol{C}^*$ and f_y over G_y where y is in a neighbourhood of the real sections in $\boldsymbol{C}^3$. The monodromy diagram (2.8) can be interpreted as relating the two frames via the standard extended solution:

$$f_y = gE_\lambda(y). \tag{5.3}$$

Among extended solutions, E_λ is determined by the property $E_\lambda(\infty) = \boldsymbol{I}$. Uniqueness of extended solutions implies that

$$E_\lambda(y) = \tilde{E}_\lambda(\infty)^{-1}\tilde{E}_\lambda(y).$$

The reality condition on E_λ implies that,

$$E_\lambda^{-1} = (E_{\bar{\lambda}^{-1}})^* = T_n^* + \cdots + \lambda^{n-1}T_1^* + \lambda^n T_0^*.$$

Substituting these expressions into (5.3), for example, tells us that

$$f_y \cdot \tilde{E}_\lambda(y)^{-1} = g \cdot \tilde{E}_\lambda(\infty)^{-1}$$

are holomorphic sections on a neighbourhood of P_0 in $\widetilde{T\boldsymbol{P}}^1$ and have full rank away from P_0.

5.1. UNITON NUMBER $\leq$ LENGTH $+1$. Since $\tilde{E}_\lambda$ is in Uhlenbeck normal form, we can find z_0, z_1 such that

$$T_n(z_0)^* T_0(z_1) \neq 0.$$

The reality condition on $\tilde{E}_\lambda$ implies that the N sections

$$\lambda^n f_y \cdot \tilde{E}_\lambda(y)^{-1}\tilde{E}_\lambda(z_0) = \lambda^n g \cdot \tilde{E}_\lambda(\infty)^{-1}\tilde{E}_\lambda(z_0)$$

are zero on $P_0^{(n-1)} \cap G_{(z_0,0,-\bar{z}_0)}$, but are not all zero on $P_0 \cap G_{(z_1,0,-\bar{z}_1)}$. This shows that $l \geq n-1$ (by the first characterization of length).

5.2. UNITON NUMBER $\geq$ LENGTH $+1$. Assume, on the other hand, that $l > n-1$ and let s be a section

$$\mathscr{V} \xrightarrow{s} \mathscr{O}_{P_0^{(l)}}(-1); \quad s|_{P_0} \neq 0.$$

We can find a $y_0 \in \boldsymbol{R}^3$ and a $v \in \mathscr{V}_{P_0 \cap G_{y_0}}$ such that $s(v) \neq 0$. Let $v = f_{y_0} \cdot v'$. It follows that the section

$$\lambda^n f_y \cdot E_\lambda(y)^{-1} E_\lambda(y_0) v'$$

on $P_0^{(n)} \cap G_{y_0}$ is not mapped to zero by s, but since its image is a section of $\mathscr{O}_{P_0^{(n)}}(-1) \cong \mathscr{O}_{P_0}(-1)^{\oplus(n+1)}$, it must be zero, contradicting the assumption $l > n-1$. We conclude that $n = l+1$, which is part 3 of Theorem B.

6. Real triviality

In [**An1**] we formulated the definition of a uniton bundle and found it necessary to include the assumption that the bundle was trivial on real sections. Using the monad representation of the bundle, however, we are able to show that this assumption is unnecessary.

Since the group of real translations in the space of sections acts transitively, and the induced action on $\widetilde{\boldsymbol{TP}}^1$ pulls uniton bundles back to uniton bundles, it is enough to prove

LEMMA 6.1. *Any bundle which satisfies the reality property of a uniton bundle, and which is trivial on nonpolar fibres, and the section at infinity is also trivial on the zero section.*

PROOF. The assumed triviality properties are enough to identify such bundles with bundles over $\boldsymbol{P}^2$, and to consider monad representatives. In the language of monads, we must show that any set of monad data α_1', γ nilpotent, a', b' satisfying the monad equation and nondegeneracy conditions (see Theorem B) also satisfies (4.1)

$$0 \neq \det \alpha_1 = \det \begin{pmatrix} -\alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix},$$

where ϕ_1 and ϕ_2 are given by (3.9).

If M is a complex matrix, $(u, v) = u^* M^* M v$ defines a nonnegative (possibly degenerate) symmetric sesquilinear form. The sum of such forms is again such a form, so we see from (3.9) that $i\phi_1$ and $-i\phi_2$ define such forms.

Choose a basis for $\boldsymbol{C}^k$ which is orthogonal with respect to $i\phi_1$ by first choosing a basis for $\ker \alpha_1'$ and extending it to a basis of the whole space. In other words, in the chosen basis the first columns of α_1' will be zero and the remaining columns have full rank, and ϕ_1 will be diagonal.

Let θ be a parametrix for α_1', by which we mean an invertible matrix such that

$$\theta\alpha_1' = \begin{pmatrix} 0 & \\ & I \end{pmatrix}$$

in the chosen basis.

Fix the hermitian metric in which the chosen basis is unitary. Taking adjoints with respect to this metric,

$$\alpha_1'^*\theta^* = \begin{pmatrix} 0 & \\ & I \end{pmatrix}$$

tells us that $\theta^* : \ker \alpha_1' \hookrightarrow \ker \alpha_1'^*$.

Use θ to rewrite the determinant as

$$\frac{1}{|\det \theta|^2} \det \begin{pmatrix} -I & \\ & \theta \end{pmatrix} \begin{pmatrix} -\alpha_1'^* & \phi_1 \\ \phi_2 & \alpha_1' \end{pmatrix} \begin{pmatrix} \theta^* & \\ & I \end{pmatrix} \tag{6.2}$$

$$= \frac{1}{|\det \theta|^2} \det \begin{pmatrix} \begin{pmatrix} 0 & \\ & I \end{pmatrix} & -\phi_1 \\ \theta\phi_2\theta^* & \begin{pmatrix} 0 & \\ & I \end{pmatrix} \end{pmatrix}; \tag{6.3}$$

expand this determinant using the fact that ϕ_1 is diagonal in this basis,

$$= \frac{1}{|\det \theta|^2} \sum_{\substack{\ker \alpha_1' \subset V \subset C^k \\ V \text{ is generated} \\ \text{by basis vectors}}} \det(-i\theta\phi_2\theta^*)|_V \det(i\phi_1)|_V, \tag{6.4}$$

where $\phi|_V$ means the restriction of the form to the subspace, i.e. $\pi\phi\iota$ in terms of orthogonal projection π onto V and injection $\iota : V \hookrightarrow C^k$. Since $i\phi_1$ and $-i\theta\phi_2\theta^*$ are nonnegative, their restrictions to V are also nonnegative, as are all the terms in the expansion of the determinant. It remains to show that one of the terms is not zero.

The determinant of a symmetric, sesquilinear form ϕ is nonzero iff the form is nondegenerate iff null $\phi = \{0\}$. We will show in the next lemma that null $i\phi_1 \cap \ker \alpha_1' = \{0\}$. A similar argument shows that null $i\phi_2 \cap \ker \alpha_1'^* = \{0\}$. Putting this together with the fact that θ^* maps $\ker \alpha_1'$ to $\ker \alpha_1'^*$, we have

$$\det(-i\theta\phi_2\theta^*)|_{\ker \alpha_1'} \det(i\phi_1)|_{\ker \alpha_1'} = \det(-i\phi_2)|_{\ker \alpha_1'^*} \det(i\phi_1)|_{\ker \alpha_1'} \neq 0,$$

completing the proof that the determinant is not zero, and that the bundle is not trivial on the zero section. □

LEMMA 6.5. *Let $(u, v) = u^*(i\phi_1)v$ be the form defined by $i\phi_1$, then*

$$\ker \alpha_1' \cap \operatorname{null} \phi_1 = \{0\}.$$

PROOF. The key observation is that

$$(e, e) = 0 \Leftrightarrow a'(I + \gamma)^{-j}\gamma^{j-1}e = 0 \quad \text{for all } j > 0. \tag{6.6}$$

Let $e \in \ker \alpha_1'$. Either $a\gamma^j e = 0$ for all $j \geq 0$, or there exists a $j \geq 0$ such that $a\gamma^j e \neq 0$, and $a\gamma^l e = 0$ for all $l > j$ (because γ is nilpotent).

In the latter case,

$$a'\gamma^n(\boldsymbol{I} + \gamma)^{-(n+1)}e = a'\gamma^n e \neq 0$$

so $(e, e) \neq 0$.

In the former case, applying the monad equation inductively implies $\alpha_1'\gamma^l e = 0$ for all $l \geq 0$. Since γ is nilpotent, there is an l such that $\gamma^{l+1}e = 0$, then (6.6) implies $\gamma^l e \in \ker a'$ and therefore

$$\gamma^l e \in \ker \alpha_1' \cap \ker \gamma \cap \ker a' = \{0\},$$

or $a'\gamma^l e \neq 0$, and $(e, e) \neq 0$ as above. It follows by induction that $e = 0$, or $(e, e) \neq 0$.

It follows that $\phi_1|_{\ker \alpha_1'}$ is nondegenerate, as required. $\square$

7. Time translation

Because

- $\mathscr{V}|_{\underbrace{\text{nonpolar fibres}}}$ is holomorphically trivial,
- $\delta_t : \widetilde{\boldsymbol{TP}}^1 \to \widetilde{\boldsymbol{TP}}^1$ preserves the fibres,
- δ_t fixes G_∞ and its lift $\tilde{\delta}_t$ fixes $\mathscr{V}|_{G_\infty}$, and
- the union of the nonpolar fibres is a Zariski open set,

the lift of time translation to the bundle is unique if it exists. We therefore refer to 'time-invariant' holomorphic bundles and time-invariant jumps.

We will prove in Lemma 7.4 that a local framed jump at P_0 admits a lift of time translation $\eta \mapsto \eta + \lambda t$ iff it admits a lift of the group of fibre-preserving transformations $\eta \mapsto \eta + \lambda(1 + \lambda)t$. Since the latter is equivalent to the linear action

$$X \mapsto X \quad Y \mapsto Y \quad W \mapsto W + tX \tag{7.1}$$

it induces a map of monads. We can calculate the effect of time-translation on the normalized monad data to be $\alpha_1' \mapsto \alpha_1' - t\gamma$.

The existence of a bundle lift is equivalent to the existence of a one parameter subgroup $G(t) \in Gl(k/2)$ which fixes γ, a' and b', and sends α_1' to $\alpha_1' - t\gamma$. Infinitesimally, this says there exists $g \in \mathrm{gl}(k/2)$ such that

$$[g, \alpha_1'] = \gamma, \quad [g, \gamma] = 0, \quad gb' = 0, \quad a'g = 0. \tag{7.2}$$

As we have said, g is unique (up to the group action). When it exists, it is the δ of Theorem B.

The following simple lemma implies that 2-unitons have energy 4 or more. There is evidence that energy is bounded below by the square of the uniton number.

LEMMA 7.3. *No monad with*

$$\gamma \in \left\{ (N_n), \begin{pmatrix} N_2 & \\ & 0 \end{pmatrix}, \begin{pmatrix} N_m & \\ & Z_r \end{pmatrix} : n > 1, m > 2, r > 0 \right\}$$

where

$$N_n \overset{def}{=} \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathrm{gl}(n), \quad \textit{and } Z_r \overset{\mathrm{def}}{=} (0) \in \mathrm{gl}(r)$$

exists

PROOF. Use the relations (7.2) □

We now justify the assumption that invariance under δ_t is equivalent to a linear group action on $\boldsymbol{P}^2$:

LEMMA 7.4. *Let η, λ be affine base and fibre coordinates on $\boldsymbol{P}^1 \times \boldsymbol{C} \to \boldsymbol{C}$. Let $\mathscr{V}$ be a holomorphic vector bundle defined over some neighbourhood of $\{\lambda = 0\}$, with a fixed framing above $\{\eta = \infty\}$ such that $\mathscr{V}|_{\{\lambda \neq 0\}}$ is trivial.*

Let $f(\lambda)$ and $g(\lambda)$ be holomorphic functions on a neighbourhood of $\lambda = 0$, with $g(0) \neq 0$. Then the 1-parameter group of transformations

$$\lambda \mapsto \lambda \quad \eta \mapsto \eta + f(\lambda)t$$

lifts to a family of framed bundle homomorphisms iff

$$\lambda \mapsto \lambda \quad \eta \mapsto \eta + g(\lambda)f(\lambda)t$$

does.

REMARK 7.5. i) It is not true that the induced pull-back actions on the space of framed jumps are equivalent-only that the fixed points are. ii) This lemma suggests filtering the space of framed holomorphic jumps by the subspaces consisting of bundles which admit a lift of $\eta \mapsto \eta + \lambda^n t$.

PROOF. Let $T(\lambda, \eta) = T(\lambda, \eta, 1/\eta)$ be a clutching function for the bundle with respect to the open cover $\{\eta \neq 0\}$, $\{\eta \neq \infty\}$. Since $\lambda = 0$ is the only jumping line, we can assume that $T(\lambda, \eta, 1/\eta)$ is invertible for $\lambda \neq 0$. As described at the beginning of this section, the triviality properties of the bundle and the framing define a unique lift away from $\{\lambda = 0\}$. Assume without loss of generality that this unique lift acts trivially on the trivialization near $\eta = \infty$. It follows that on the other trivialization it acts by

$$L(\lambda, \eta, t) \overset{\mathrm{def}}{=} T(\lambda, \eta + f(\lambda)t) T(\lambda, \eta + t)^{-1}.$$

Of course, this will be singular on the jumping line when the bundle does not admit a lift. The condition that a lift exists is equivalent to the condition that L is continuous near $\lambda = 0$.

Since L defines a group homomorphism (when considered as a function of t), a lift exists globally iff it exists locally iff the linearisation $(\partial L/\partial t)|_{t=0}$ is continuous. Differentiating L and the analogous expression with f replaced by $f \cdot g$, one sees that one is continuous iff the other is. For this calculation it is convenient (if not necessary) to assume that T is in 'jumping-line normal form', see **[Hu]** □

THANKS. I am grateful to Francis Burstall, Arleigh Crawford, Martin Guest, Nigel Hitchin, Paul Norbury, John Rawnsley, and especially Jacques Hurtubise for helpful discussions and friendly suggestions.

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