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Ginzburg-Landau equation with magnetic effect: non-simply-connected domains

Dedicated to Professor Kôji KUBOTA on the occasion of his 60th birthday

By Shuichi JIMBO* and Jian ZHAI

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§ 1. Introduction and preliminaries.

We deal with the Ginzburg-Landau (GL) functional with its variational equation (GL equation) and study the existence of many kinds of local minimizers (stable solutions) in non-trivially geometrical situations. We consider the following GL functional:

$$\mathscr{H}_\lambda(\Phi, A) = \int_\Omega \left(\frac{1}{2}|(\nabla - iA)\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2\right) dx + \int_{R^3} \frac{1}{2}|\mathrm{rot}\, A|^2\, dx. \tag{1.1}$$

for the variable (Φ, A), where Φ is a $\boldsymbol{C}$-valued function in Ω and A is an $\boldsymbol{R}^3$-valued function in $\boldsymbol{R}^3$ and $\lambda > 0$ is a parameter. This type of functional appears in the theory of the (low-temperature) superconductivity (cf. [**18**]). Note that the first and second terms correspond to the energy of the electrons in the material Ω and that of the magnetic field, respectively. It should be emphasized that the magnetic field occurs in the whole space $\boldsymbol{R}^3$. The theory suggests that a physically realizable state corresponds to a local minimizer (of such an energy functional) and hence, in our case, it becomes a solution (Φ, A) to the following variational equation (1.2) (GL equation):

$$\begin{cases} (\nabla - iA)^2\Phi + \lambda(1 - |\Phi|^2)\Phi = 0 & \text{in } \Omega, \\ \dfrac{\partial \Phi}{\partial \nu} - i\langle A \cdot \nu\rangle\Phi = 0 & \text{on } \partial\Omega, \\ \mathrm{rot}\,\mathrm{rot}\, A + (i(\bar{\Phi}\nabla\Phi - \Phi\nabla\bar{\Phi})/2 + |\Phi|^2 A)\Lambda_\Omega = 0 & \text{in } \boldsymbol{R}^3. \end{cases} \tag{1.2}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product of vectors in $\boldsymbol{R}^3$, ν is the unit outward normal vector on $\partial\Omega$ and Λ_Ω is the characteristic function of Ω, i.e. $\Lambda_\Omega(x) = 1$ in Ω and $\Lambda_\Omega(x) = 0$ in $\boldsymbol{R}^3\backslash\Omega$. In [**15**], it was proved for the case of a ring-shaped (rotationally symmetric) domain Ω, that many kinds of stable steady state solutions coexist for large $\lambda > 0$. The purpose of this paper is to extend this result to a general non-trivial domain Ω (cf. Fig. 1). Here the general non-trivial domain means the general domain that is not simply-connected. In [**16**] the GL functional and its variational equation (cf. (1.3), (1.4)) simplified by neglecting the magnetic effect, were studied and several kinds of

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stable solutions were constructed in a certain situation. Those are obtained by putting $A \equiv 0$ in (1.1) and (1.2).

$$\mathscr{H}_\lambda^0(\Phi) = \int_\Omega \left(\frac{1}{2}|\nabla\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2\right) dx, \tag{1.3}$$

$$\Delta\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \text{ in } \Omega, \quad \partial\Phi/\partial\nu = 0 \text{ on } \partial\Omega (\Phi : \boldsymbol{C}\text{-valued}). \tag{1.4}$$

To make clear the problem discussed in this paper and our approach, we briefly review the ideas and results in [**16**]. Denote the set of the continuous maps from $\overline{\Omega}$ into $S^1 = \{z \in \boldsymbol{C} \,|\, |z| = 1\}$ by $\mathscr{M}$, i.e. $\mathscr{M} = C^0(\overline{\Omega}; S^1)$. In view of the functional (1.4), the absolute value of a local minimizer Φ_λ may approach 1 as λ grows up. It suggests that Φ_λ approaches a certain map in $\mathscr{M}$. Actually it was proved in [**16**] that for any given homotopy class in $\mathscr{M}$, there exists a stable solution Φ_λ of (1.4) for large $\lambda > 0$ which uniformly approaches the harmonic map ($\in \mathscr{M}$) belonging to that homotopy class as $\lambda \to \infty$.

In certain physical situations, (1.4) is regarded as an approximate model equation of (1.2) and so it is natural problem to compare the solutions of (1.2) with those of (1.4). Indeed given Φ we see that $\mathscr{H}_\lambda(\Phi, A)$ is convex in the variable A and it admits only global minimizers (essentially unique). This implies that when we seek for a local minimizer, Φ is more important variable than A because A is determined almost uniquely by Φ. In other words, $\mathscr{H}_\lambda(\Phi, A)$ can be controlled only by the variable Φ. This suggests that the situation of local minimizers for $\mathscr{H}_\lambda(\Phi, A)$ may be similar to that of $\mathscr{H}_\lambda^0(\Phi)$ for large $\lambda > 0$. We will construct a solution (Φ, A) of (1.2) such that Φ behaves like an element with an arbitrarily prescribed homotopy type in $\mathscr{M}$. For the construction of solutions, we first deal with the limit problem $\lambda = \infty$ of (1.1) and (1.2) and next consider $0 \ll \lambda < \infty$ as a perturbation. We consider the stability problem by investigating asymptotic behaviors of certain linearized equations and eigenvalue problems.

We remark that a similar nice work on the existence of local minimizers of (1.1) with their topological characterization for non-simply connected superconductors is done in-

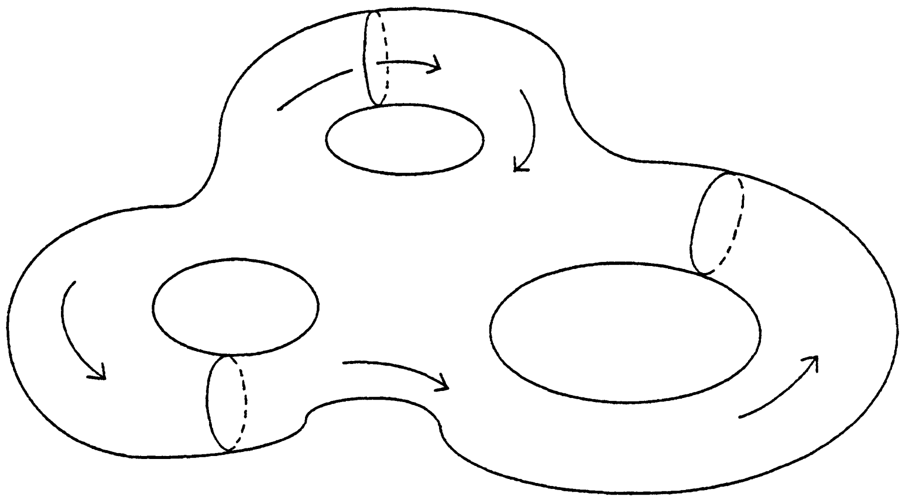

Fig. 1: $\Omega \subset \boldsymbol{R}^3$ (Doughnut with 3 holes)

dependently by Rubinstein and Sternberg [**22**], recently. They are dealing with the case that Ω has the same topological type of a solid torus (i.e. $\pi_1(\Omega) = \boldsymbol{Z}$), while their method is also applicable to general cases such as ours. The stability inequality such as (2.3) in our main theorem (Th. 4) is not given there. Since many years ago there have been many important works on the solutions of Ginzburg-Landau equations with or without magnetic effects in different situations. See [**3**], [**4**], [**5**], [**6**], [**9**], [**10**], [**11**], [**13**], [**17**], [**21**], [**23**], [**24**] and the references therein.

We formulate the problem more precisely. We consider the functional (1.1) for (Φ, A) satisfying

$$A \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3), \quad \nabla A \in L^2(\boldsymbol{R}^3; \boldsymbol{R}^{3\times 3}), \quad \Phi \in H^1(\Omega; \boldsymbol{C}). \tag{1.5}$$

The conditions concerning A come from the following consideration. From (1.1) $\operatorname{rot} A$ should belong to L^2. Taking account of the gauge transformation (which will be mentioned later), we can assume without loss of generality that $\operatorname{div} A = 0$ and it leads to $\nabla A \in L^2(\boldsymbol{R}^3; \boldsymbol{R}^{3\times 3})$ (cf. Lemma 7 in §3). The last line of (1.2) can be regarded as a part of the time stationary Maxwell equation,

$$\begin{cases} \operatorname{rot} \operatorname{rot} A = J \\ J = -(i(\bar{\Phi}\nabla\Phi - \Phi\nabla\bar{\Phi})/2 + |\Phi|^2 A)\Lambda_\Omega, \end{cases} \tag{1.6}$$

where J naturally corresponds to the electric current. If J is a given $\boldsymbol{R}^3$-valued function satisfying $\operatorname{div} J = 0$ in $\boldsymbol{R}^3$, the function defined by

$$A(x) = \frac{1}{4\pi} \int_{\boldsymbol{R}^3} \frac{J(y)}{|x-y|} \, dy \tag{1.7}$$

satisfies $\operatorname{rot} \operatorname{rot} A = J$. In our situation if there exists a C^1 solution (Φ, A) to (1.2), J has a compact support and satisfies $\operatorname{div} J = 0$ in $\boldsymbol{R}^3$ (in the distribution sense), hence A is expressed as in (1.7) and it implies $A(x) \to 0$ as $|x| \to \infty$. Therefore, using the Sobolev's inequality (cf. Lemma 7 in §3 or [**12**; Gilbarg-Trudinger]) with $\nabla A \in L^2$, we get $A \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)$.

The formulation for the stability of the solution is completely similar to that in [**15**]. Note that the functional (1.1) as well as the equation (1.2) is invariant under the following (gauge) transformation:

$$\begin{cases} (\Phi, A) \mapsto (\Phi', A') \\ \Phi' = e^{i\rho}\Phi, \quad A' = A + \nabla\rho \quad (\rho : \boldsymbol{R}\text{-valued function in } \boldsymbol{R}^3). \end{cases} \tag{1.8}$$

This transformation creates a family of solutions (denoted by $C(\Phi, A)$) from one solution (Φ, A) and $C(\Phi, A)$ itself corresponds to one physical state. This observation leads us to study the variation of $\mathscr{H}_\lambda$ in the direction transversal to $C(\Phi, A)$ to see the stability of (Φ, A). We note that the tangent space of $C(\Phi, A)$ at this point is described as

$$T(\Phi, A) = \{(i\xi\Phi, \nabla\xi) \mid \xi : \boldsymbol{R}\text{-valued function on } \boldsymbol{R}^3\}.$$

We take a space $N(\Phi, A)$ which is transversal to $T(\Phi, A)$ with $T(\Phi, A) \cap N(\Phi, A) =$

$\{(0,0)\}$ and consider the second variation

$$\mathscr{L}_\lambda(\Phi, A, \Psi, B) = \frac{d^2}{d\varepsilon^2}\,\mathscr{H}_\lambda(\Phi + \varepsilon\Psi, A + \varepsilon B)_{|\varepsilon=0}$$

for $(\Psi, B) \in N(\Phi, A)$. We precisely define several spaces appearing in the above arguments and give some important properties. Let $\Phi \in C^1(\overline{\Omega}; \boldsymbol{C})$, $A \in C^1(\boldsymbol{R}^3; \boldsymbol{R}^3)$ and $\nabla A \in L^2(\boldsymbol{R}^3; \boldsymbol{R}^{3\times 3})$. We note that the solution (Φ, A) which we will construct in §4 satisfies these conditions. For convenience, we sometimes deal with Φ-component in terms of real functions by taking its real and imaginary parts. We put $\Phi = u + vi$ and $\Psi = \phi + \psi i$. Hereafter we also denote $\mathscr{H}_\lambda(\Phi, A)$, $\mathscr{L}_\lambda(\Phi, A, \Psi, B)$, $T(\Phi, A)$ and $N(\Phi, A)$ by $\mathscr{H}_\lambda(u, v, A)$, $\mathscr{L}_\lambda(u, v, A, \phi, \psi, B)$, $T(u, v, A)$ and $N(u, v, A)$, respectively. Put

$$Z = \{B \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3) \mid \nabla B \in L^2(\boldsymbol{R}^3; \boldsymbol{R}^{3\times 3})\}.$$

The tangent space $T(\Phi, A)$ is defined as follows,

$$T(\Phi, A) = T(u, v, A) = \{(-v\xi, u\xi, \nabla\xi) \mid \xi \in L^2_{loc}(\boldsymbol{R}^3), \nabla\xi \in Z\}. \tag{1.9}$$

To define a subspace $N(\Phi, A) = N(u, v, A)$ which is transversal to $T(\Phi, A)$, we use the Helmholtz decomposition (cf. [**19**]). It is known that $L^6(\Omega; \boldsymbol{R}^3)$ and $L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)$ have the decompositions:

$$L^6(\Omega; \boldsymbol{R}^3) = X_1 \oplus X_2, \quad L^6(\boldsymbol{R}^3; \boldsymbol{R}^3) = Y_1 \oplus Y_2,$$

where

$$\begin{aligned}
X_1 &= \{\nabla\xi \mid \xi \in L^6(\Omega), \nabla\xi \in L^6(\Omega; \boldsymbol{R}^3)\},\\
X_2 &= \{B \in L^6(\Omega; \boldsymbol{R}^3) \mid \operatorname{div} B = 0 \text{ in } \Omega, \langle B \cdot \nu\rangle = 0 \text{ on } \partial\Omega\},\\
Y_1 &= \{\nabla\xi \mid \xi \in L^6_{loc}(\boldsymbol{R}^3), \nabla\xi \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)\},\\
Y_2 &= \{B \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3) \mid \operatorname{div} B = 0 \text{ in } \boldsymbol{R}^3\}.
\end{aligned}$$

Let us define

$$\overline{N}(\Phi, A) = \overline{N}(u, v, A) = \left\{(\phi, \psi, B) \in H^1(\Omega)^2 \times Z \mid \int_\Omega (v\phi - u\psi)\, dx = 0, B_{|\Omega} \in X_2\right\},$$

$$N(\Phi, A) = N(u, v, A) = \left\{(\phi, \psi, B) \in H^1(\Omega)^2 \times Z \mid \int_\Omega (v\phi - u\psi)\, dx = 0, B \in Y_2\right\}.$$

For these spaces, we have the properties in the following propositions. Their proofs can be carried out quite similarly as in §2 in [**15**] and so we omit them.

PROPOSITION 1.

$$H^1(\Omega) \times H^1(\Omega) \times Z = T(u, v, A) + \overline{N}(u, v, A). \tag{1.10}$$

PROPOSITION 2.

$$H^1(\Omega) \times H^1(\Omega) \times Z = T(u, v, A) \oplus N(u, v, A). \tag{1.11}$$

FORMULA OF THE SECOND VARIATION OF $\mathscr{H}_\lambda$

$$(1.12)\quad \mathscr{L}_\lambda(\Phi, A, \Psi, B) = \mathscr{L}_\lambda(u, v, A, \phi, \psi, B) = \frac{d^2}{d\varepsilon^2}\mathscr{H}_\lambda(u+\varepsilon\phi, v+\varepsilon\psi, A+\varepsilon B)_{|\varepsilon=0}$$

$$= \int_\Omega \{|\nabla\phi + \psi A|^2 + |\nabla\psi - \phi A|^2 - \lambda(1-u^2-v^2)(\phi^2+\psi^2)$$

$$+ 2\lambda(u\phi + v\psi)^2\}\,dx + \int_{\boldsymbol{R}^3} |\mathrm{rot}\,B|^2\,dx$$

$$+ \int_\Omega (u^2+v^2)B^2\,dx + 4\int_\Omega \langle A\cdot B\rangle(u\phi + v\psi)\,dx$$

$$- 2\int_\Omega \{\phi\langle\nabla v\cdot B\rangle - \psi\langle\nabla u\cdot B\rangle + u\langle\nabla\psi\cdot B\rangle - v\langle\nabla\phi\cdot B\rangle\}\,dx.$$

This is calculated directly from (1.1). The following property is also proved by a direct calculation.

PROPOSITION 3. *Let* $(\Phi, A) \in H^1(\Omega; \boldsymbol{C}) \times Z$ *be a* C^1*-solution of* (1.2). *Then*

$$(1.13)\qquad \mathscr{L}_\lambda(\Phi, A, \Psi, B) = \mathscr{L}_\lambda(\Phi, A, \Psi', B')$$

provided that (Ψ, B), $(\Psi', B') \in H^1(\Omega; \boldsymbol{C}) \times Z$ *and* $(\Psi - \Phi', B - B') \in T(\Phi, A)$.

§ 2. Main results.

In this section we present the main results. Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with C^3 boundary. We impose the following topological condition on the domain Ω.

(A) There exists a continuous map of $\overline{\Omega}$ into $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$ which is not homotopic to a constant valued map.

Under this assumption, there are infinitely many homotopy classes in the continuous mappings of $\overline{\Omega}$ into S^1. Because if θ_0 is a map satisfying (A), then all the maps $\theta_0(x)^2, \theta_0(x)^3, \theta_0(x)^4, \dots$ belong to distinct homotopy classes of $\mathscr{M}$. Here S^1 is regarded as a group. We also remark that in our situation that Ω is a domain in $\boldsymbol{R}^3$, it is known that the above condition (A) is equivalent to that Ω is not simply-connected (see Appendix in [**16**]).

We construct non-trivial solutions to (1.2) for this Ω and prove their stability.

We seek for a solution (Φ, A) in the form $\Phi(x) = W(x)e^{i\theta(x)}$ where $W = W(x) > 0$ and θ is a continuous map of Ω into S^1 with an arbitrarily prescribed homotopy type.

The following theorem is the main result of this paper.

THEOREM 4. *Assume* (A). *For any* $\theta_0 \in \mathscr{M} = C^0(\overline{\Omega}; S^1)$, *there exists a* $\lambda_0 > 0$ *such that* (1.2) *has a solution* $(\Phi_\lambda, A_\lambda)$ *for any* $\lambda \geqq \lambda_0$ *such that* $\Phi_\lambda \in C^2(\overline{\Omega})$, $A \in Z \cap C^1(\boldsymbol{R}^3; \boldsymbol{R}^3)$ *and*

$$(2.1)\qquad \Phi_\lambda(x) = W_\lambda(x)e^{i\theta_\lambda(x)},$$

$$(2.2)\qquad \lim_{\lambda\to\infty}\sup_{x\in\Omega}|W_\lambda(x) - 1| = 0,$$

and the map $\theta_\lambda : \overline{\Omega} \longrightarrow S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$ is homotopic to θ_0. Moreover it is stable in the sense that there exists a constant $c > 0$ such that

$$\mathscr{L}_\lambda(\Phi_\lambda, A_\lambda, \Psi, B) \geqq c(\|\Psi\|^2_{H^1(\Omega;\boldsymbol{C})} + \|B\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\nabla B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times3})}) \tag{2.3}$$

for $(\Psi, B) \in N(\Phi_\lambda, A_\lambda)$ and $\lambda \geqq \lambda_0$.

We will prove this theorem in the following sections. In § 3 we deal with the equation for the limit case $\lambda = \infty$ and prove the existence of solutions by the variational method. In § 4 we deal with (1.2) for large $\lambda > 0$ and construct solutions as a "perturbation" from the limit case $\lambda = \infty$. In § 5 we prove the stability by the spectral analysis on a certain linearized problem.

§ 3. Existence of solutions for $\lambda = \infty$.

We consider the limit problem of (1.1), (i.e. $\lambda = \infty$). By putting $\Phi(x) = w(x)e^{i\theta(x)}$ where w is a positive function and θ is an $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$-valued function, the functional (1.1) is rewritten as

$$\mathscr{H}_\lambda(\Phi, A) = \int_\Omega \frac{1}{2}|w(\nabla\theta - A)|^2\,dx + \int_{\boldsymbol{R}^3} \frac{1}{2}|\mathrm{rot}\,A|^2\,dx + \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{\lambda}{4}(1 - w^2)^2\right)dx. \tag{3.1}$$

In this paper we are seeking for a local minimizer $(\Phi, A) = (we^{i\theta}, A)$ of (1.1) with $\Phi \neq 0$ in Ω and so if λ is very large, w might approach 1. The order of the convergence will be $w - 1 = O(1/\lambda)$ (this turns to be true in §4). This consideration suggests us the following functional as the limit case $\lambda = \infty$ of (1.1).

$$\mathscr{H}_\infty(\theta, A) = \frac{1}{2}\int_\Omega |\nabla\theta - A|^2\,dx + \frac{1}{2}\int_{\boldsymbol{R}^3} |\mathrm{rot}\,A|^2\,dx, \tag{3.2}$$

where θ is an S^1-valued function in Ω and A is an $\boldsymbol{R}^3$-valued function in $\boldsymbol{R}^3$. Remark that $\nabla\theta$ can be naturally regarded as an $\boldsymbol{R}^3$-valued function. This choice of the functional will turn out to be nice afterwards. The Euler-Lagrange equation of (3.2) is

$$\begin{cases} \mathrm{div}(\nabla\theta - A) = 0 & \text{in } \Omega, \\ \langle \nabla\theta - A \cdot \nu\rangle = 0 & \text{on } \partial\Omega, \\ \mathrm{rot}\,\mathrm{rot}\,A + (A - \nabla\theta)\Lambda_\Omega = 0 & \text{in } \boldsymbol{R}^3. \end{cases} \tag{3.3}$$

We will construct a solution (θ, A) such that θ is homotopic to θ_0 of any given homotopy type in $\mathscr{M} = C^0(\overline{\Omega}, S^1)$. For a technical reason, we use $\boldsymbol{R}$-valued functions in place of θ. We can assume without loss of generality that θ_0 is C^3 because we can mollify θ_0 without changing its homotopy class. Let $\hat{\Omega}$ be the universal covering space of Ω which is endowed with the canonical metric, i.e. the covering map $\imath_1 : \hat{\Omega} \longrightarrow \Omega$ is locally isometric. On the other hand $\boldsymbol{R}$ is the universal covering of $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$. Let $\imath_2 : \boldsymbol{R} \longrightarrow S^1$ be the covering map. Fix a point $p \in \Omega$ and let $q = \theta_0(p) \in S^1$. Take $\hat{p} \in \hat{\Omega}$ and $\hat{q} \in \boldsymbol{R}$ such that $\imath_1(\hat{p}) = p$, $\imath_2(\hat{q}) = q$. For any $\theta \in C^0(\Omega, S^1)$ such that

$\theta(p) = q$, there exists a unique continuous map:

$$\hat{\theta} : \hat{\Omega} \longrightarrow \boldsymbol{R} = \hat{S}^1,$$

such that

$$\theta(\iota_1(z)) = \iota_2(\hat{\theta}(z)) \quad (\forall\, z \in \overline{\Omega}), \quad \hat{\theta}(\hat{p}) = \hat{q}.$$

$$\begin{array}{ccc} \hat{\Omega} & \xrightarrow{\ \hat{\theta}\ } & \boldsymbol{R} = \hat{S}^1 \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ \Omega & \xrightarrow[\ \theta\ ]{} & S^1 \end{array}$$

Proposition 5 ([**16**; §3]). *For any $\theta \in C^0(\Omega, S^1)$ such that $\theta \sim \theta_0$ and $\theta(p) = q$, the function $\eta(z) = \hat{\theta}(z) - \hat{\theta}_0(z)$ in $\hat{\Omega}$ can be identified with an $\boldsymbol{R}$-valued function in Ω. That is, there exists a unique $\boldsymbol{R}$-valued function η' in Ω such that $\eta(z) = \eta'(\iota_1(z))$. On the other hand, for any $\boldsymbol{R}$-valued continuous function η' on Ω with $\eta'(p) = 0$, there exists a unique $\theta \in C^0(\Omega, S^1)$ such that $\theta(p) = q$ and $\eta'(\iota_1(z)) = \hat{\theta}(z) - \hat{\theta}_0(z)$ in $\hat{\Omega}$.*

By the identification in Proposition 5, we denote η' also by η. Translating the unknown function θ into η, we can rewrite the functional (3.2) and the equation (3.3), respectively,

$$\mathscr{H}'_\infty(\eta, A) = \frac{1}{2}\int_\Omega |\nabla\eta + X_0 - A|^2\,dx + \frac{1}{2}\int_{\boldsymbol{R}^3} |\mathrm{rot}\,A|^2\,dx, \tag{3.4}$$

$$\begin{cases} \mathrm{div}(\nabla\eta + X_0 - A) = 0 & \text{in } \Omega, \\ \langle \nabla\eta + X_0 - A \cdot \nu \rangle = 0 & \text{on } \partial\Omega, \\ \mathrm{rot}\,\mathrm{rot}\,A + (A - \nabla\eta - X_0)\Lambda_\Omega = 0 & \text{in } \boldsymbol{R}^3, \end{cases} \tag{3.5}$$

where $X_0 = \nabla\theta_0$ is a C^2 class $\boldsymbol{R}^3$-valued function. The important point is that η is an $\boldsymbol{R}$-valued function in Ω. Clearly (3.5) is the Euler-Lagrange equation of (3.4). On the other hand, for a solution of (η, A) of (3.5) with $\eta(p) = 0$ (this can be easily satisfied by adding an adequate constant to η) we can get a solution (θ, A) of (3.3) by the aid of Proposition 5.

We consider the minimizing problem for (3.4) in the space:

$$D = \{(\eta, A) \in H^1(\Omega) \times L^6(\boldsymbol{R}^3; \boldsymbol{R}^3) \mid \mathrm{rot}\,A \in L^2(\boldsymbol{R}^3; \boldsymbol{R}^3)\}.$$

Proposition 6. *There exists a minimizer $(\eta_\infty, A_\infty) \in D$ of (3.4) such that*

$$\mathrm{div}\,A_\infty = 0 \ \textit{in}\ \boldsymbol{R}^3, \quad \int_\Omega \eta_\infty\,dx = 0. \tag{3.6}$$

Moreover (η_∞, A_∞) belongs to $C^{2+\gamma}(\overline{\Omega}) \times C^{1+\gamma}(\boldsymbol{R}^3; \boldsymbol{R}^3)$ for any $\gamma \in [0, 1)$ and it is a solution of (3.5).

Hereafter in this section we will prove Proposition 6. Before that we recall some technical tools, which will be used later in this paper.

LEMMA 7. *There exists a constant* $C > 0$ *such that*

$$\|\varphi\|_{L^6(\boldsymbol{R}^3)} \leqq C\|\nabla\varphi\|_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)} \quad (\forall\, \varphi \in L^6(\boldsymbol{R}^3)),$$

$$\|\nabla B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times 3})} = \|\mathrm{div}\, B\|^2_{L^2(\boldsymbol{R}^3)} + \|\mathrm{rot}\, B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)}$$

if $B \in L^2_{loc}(\boldsymbol{R}^3;\boldsymbol{R}^3)$, $\nabla B \in L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times 3})$.

(Proof of Lemma 7) The first one is the Sobolev's inequality (cf. [**12**]). The second identity can be proved by the Fourier transform. □

(Proof of Proposition 6) Let $\{(\eta_n, A_n)\}_{n=1}^{\infty}$ be a minimizing sequence in D. Using the Helmholtz decomposition in L^6 (cf. [**19**]), each A_n is decomposed as follows:

$$(3.7)\qquad \begin{cases} A_n = \nabla\xi_n + B_n \in Y_1 \oplus Y_2 & \text{in } L^6(\boldsymbol{R}^3;\boldsymbol{R}^3), \\ \mathrm{div}\, B_n = 0 & \text{in } \boldsymbol{R}^3, \\ \displaystyle\int_\Omega (\eta_n - \xi_n)\, dx = 0. & \end{cases}$$

From the boundedness of

$$\mathscr{H}'_\infty(\eta_n, A_n) = \frac{1}{2}\int_\Omega |\nabla(\eta_n - \xi_n) + X_0 - B_n|^2\, dx + \frac{1}{2}\int_{\boldsymbol{R}^3} |\mathrm{rot}\, B_n|^2\, dx \quad (n \geqq 1),$$

the quantity

$$(3.8)\qquad \int_{\boldsymbol{R}^3} |\nabla B_n|^2\, dx = \int_{\boldsymbol{R}^3} |\mathrm{rot}\, B_n|^2\, dx + \int_{\boldsymbol{R}^3} |\mathrm{div}\, B_n|^2\, dx \quad (\text{cf. Lemma7}),$$

is bounded. Therefore $\{B_n\}$ is bounded in $L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)$ from the first inequality of Lemma 7. At the same time, we have the boundedness of $\int_\Omega |\nabla(\eta_n - \xi_n)|^2\, dx (n \geqq 1)$. Using the Poincaré inequality and the last line of (3.7), we obtain the boundedness of $\{\eta_n - \xi_n\}$ in $H^1(\Omega)$. Thus we obtain a weakly convergent subsequence of $\{\eta_n - \xi_n\}$ and $\{B_n\}$ (which we denote by the same notation) and their limits, $(\eta_\infty, A_\infty) \in H^1(\Omega) \times L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)$, such that

$$(3.9)\qquad \begin{cases} \eta_n - \xi_n \to \eta_\infty & \text{weakly in } H^1(\Omega), \\ B_n \to A_\infty & \text{weakly in } L^6(\boldsymbol{R}^3;\boldsymbol{R}^3), \\ \nabla B_n \to \nabla A_\infty & \text{weakly in } L^2(\boldsymbol{R}^2;\boldsymbol{R}^{3\times 3}), \end{cases}$$

as $n \to \infty$. From the lower semi-continuity of the norm under the weak convergence, we get

$$\liminf_{n\to\infty} \mathscr{H}'_\infty(\eta_n, A_n) \geqq \mathscr{H}'_\infty(\eta_\infty, A_\infty).$$

$(\eta_\infty, A_\infty) \in H^1(\Omega) \times (H^1_{\mathrm{loc}}(\boldsymbol{R}^3;\boldsymbol{R}^3) \cap L^6(\boldsymbol{R}^3;\boldsymbol{R}^3))$ is the minimizer of (3.3) and satisfies $\mathrm{div}\, A_\infty = 0$ in $\boldsymbol{R}^3$. Moreover from the regularity argument of weak solutions in the theory of the elliptic boundary value problem, $\eta_\infty \in H^2(\Omega)$ (cf. Chap.3 in [**20**]). Using $\operatorname{rot}\operatorname{rot} A_\infty = -\Delta A_\infty$ and applying the Schauder estimate (cf. [**12**]) to (3.5), we obtain the desired regularity:

$$(\eta_\infty, A_\infty) \in C^{2+\gamma}(\overline{\Omega}) \times C^{1+\gamma}(\boldsymbol{R}^3; \boldsymbol{R}^3) \text{ for any } \gamma \in [0,1). \qquad \square$$

By adding an adequate constant we get a solution of (3.5) with $\eta_\infty(p) = 0$ and consequently, we obtain a solution $(\theta_\infty, A_\infty)$ of (3.3) such that $\theta_\infty \in C^{2+\gamma}(\overline{\Omega}; S^1)$, $\theta_\infty(p) = q$ and $\hat{\theta}_\infty = \hat{\theta}_0 + \eta_\infty$ through Proposition 5.

We can prove that a solution to (3.5) is essentially unique. Precisely we have the following result.

PROPOSITION 8. *The solution* (η_∞, A_∞) *of* (3.5) *is unique under the condition* (3.6).

(Proof of Proposition 8) Suppose that (η, A), (η', A') are two solutions of (3.5) which satisfy (3.6) and let $\tilde{\eta} = \eta - \eta'$, $\tilde{A} = A - A'$. Clearly, $(\tilde{\eta}, \tilde{A})$ satisfies

$$\begin{cases} \operatorname{rot}\operatorname{rot}\tilde{A} + (\tilde{A} - \nabla\tilde{\eta})\Lambda_\Omega = 0 & \text{in } \boldsymbol{R}^3, \\ \operatorname{div}(\tilde{A} - \nabla\tilde{\eta}) = 0 & \text{in } \Omega, \\ \langle \tilde{A} - \nabla\tilde{\eta}, \nu \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\tilde{\eta} \in C^2(\overline{\Omega})$ and $\partial\Omega$ is C^3 we can extend $\tilde{\eta}$ to $\tilde{\eta}_* \in C^2(\boldsymbol{R}^3)$ with a compact support. Thus we get

$$\operatorname{rot}\operatorname{rot}(\tilde{A} - \nabla\tilde{\eta}_*) + (\tilde{A} - \nabla\tilde{\eta}_*)\Lambda_\Omega = 0.$$

Multiplying the equation by $\tilde{A} - \nabla\tilde{\eta}_*$ and integrating it over $\boldsymbol{R}^3$, we get

$$\int_{\boldsymbol{R}^3} |\operatorname{rot}(\tilde{A} - \nabla\tilde{\eta}_*)|^2\,dx + \int_\Omega |\tilde{A} - \nabla\tilde{\eta}_*|^2\,dx = 0.$$

Then, we have

(3.10) $$\operatorname{rot}(\tilde{A} - \nabla\tilde{\eta}_*) \equiv 0 \text{ in } \boldsymbol{R}^3, \quad \tilde{A} - \nabla\tilde{\eta}_* \equiv 0 \text{ in } \Omega.$$

On the other hand $\tilde{A} - \nabla\tilde{\eta}_* \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)$, there exists ξ such that $\nabla\xi \in L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)$ and

(3.11) $$\operatorname{div}(\tilde{A} - \nabla\tilde{\eta}_* - \nabla\xi) = 0 \text{ in } \boldsymbol{R}^3.$$

Noting $\operatorname{rot}(\tilde{A} - \nabla\tilde{\eta}_* - \nabla\xi) = 0$ and (3.11) with Lemma 7, we get

$$\tilde{A} - \nabla(\tilde{\eta}_* + \xi) = 0.$$

From $\operatorname{div}\tilde{A} = 0$, we have $\tilde{A} \equiv 0$ in $\boldsymbol{R}^3$ and $\tilde{\eta}_* \equiv 0$ in Ω. This completes the proof of Proposition 8. $\square$

REMARK. If we replace the condition $\int_\Omega \eta_\infty\,dx = 0$ by $\eta_\infty(p) = 0$ in (3.6), the conclusion in Proposition 8 is still true.

§ 4. Existence of solutions for $0 \ll \lambda < \infty$

In this section we construct a solution $(\Phi_\lambda, A_\lambda)$ to (1.2) for large $\lambda > 0$. We seek for a solution such that Φ-component has the form: $\Phi(x) = w(x)e^{i\theta(x)}$ where

$$w : \Omega \to (0, \infty), \quad \theta : \Omega \to S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}.$$

The functional (1.1) is rewritten in the variable (w, θ, A) (cf. (3.1))

$$\mathscr{H}_\lambda(\Phi, A) = \int_\Omega \frac{1}{2}|w(\nabla\theta - A)|^2\,dx + \int_{\boldsymbol{R}^3} \frac{1}{2}|\mathrm{rot}\,A|^2\,dx + \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{\lambda}{4}(1 - w^2)^2\right) dx. \tag{4.1}$$

We consider a local minimizer (w, θ, A) such that θ is homotopic to θ_0 and $\theta(p) = q$. Note that $\theta_0 \in \mathscr{M}$ is a given map, which can be assumed to be C^3 without loss of generality. As in § 3 we use the change of the unknown variable θ to η by Proposition 5 (Recall $\eta(\imath_1(z)) = \hat{\theta}(z) - \hat{\theta}_0(z)$ in $\hat{\Omega}$). The functional $\mathscr{H}_\lambda$ is transformed into the following one:

$$\mathscr{H}'_\lambda(w, \eta, A) = \int_\Omega \frac{1}{2}|w(\nabla\eta + X_0 - A)|^2\,dx + \int_{\boldsymbol{R}^3} \frac{1}{2}|\mathrm{rot}\,A|^2\,dx + \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{\lambda}{4}(1 - w^2)^2\right) dx, \tag{4.2}$$

where $X_0 = \nabla\theta_0$. The Euler-Lagrange equation of (4.2) is the following system of equations (4.3) $\sim$ (4.5):

$$\begin{cases} \Delta w + (\lambda(1 - w^2) - |\nabla\eta + X_0 - A|^2)w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

$$\begin{cases} \mathrm{div}(w^2(\nabla\eta + X_0 - A)) = 0 & \text{in } \Omega, \\ \langle \nabla\eta + X_0 - A \cdot \nu \rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

$$\mathrm{rot}\ \mathrm{rot}\,A + (A - \nabla\eta - X_0)w^2 A_\Omega = 0 \quad \text{in } \boldsymbol{R}^3. \tag{4.5}$$

We recall the solution (η_∞, A_∞) of the equation (cf. § 3):

$$\begin{cases} \mathrm{div}(\nabla\eta_\infty + X_0 - A_\infty) = 0 & \text{in } \Omega, \\ \langle \nabla\eta_\infty + X_0 - A_\infty \cdot \nu \rangle = 0 & \text{on } \partial\Omega, \\ \eta_\infty(p) = 0, \end{cases} \tag{4.6}$$

$$\mathrm{rot}\ \mathrm{rot}\,A_\infty + (A_\infty - \nabla\eta_\infty - X_0)A_\Omega = 0, \quad \mathrm{div}\,A_\infty = 0 \text{ in } \boldsymbol{R}^3. \tag{4.7}$$

We construct a solution (w, η, A) to the above system (4.3)–(4.5) as a perturbation from (4.6)–(4.7) by a fixed point theorem. More precisely, for a certain given (η, A) we find a unique solution $w = w(\eta, A) > 0$ of (4.3). This map will be proved to be continuous and compact. For this $w(\eta, A)$ we consider (4.4)–(4.5) and get a solution $(\tilde{\eta}(\eta, A), \tilde{A}(\eta, A))$ and prove that this correspondence is continuous. We denote the composition of these two maps by T_λ, to which we apply the Schauder fixed point theorem and get a solution of the "full system" (4.3)–(4.5). From now we carry out this program. Let us define precisely the set E and $F(\delta)$ for the definition of the map T_λ.

$$F(\delta) = \{w \in C^{1+\alpha}(\overline{\Omega}) \mid \|w - 1\|_{C^{1+\alpha}(\overline{\Omega})} \leqq \delta\},$$

$$E = \{(\eta, A) \in C^{2+\alpha}(\overline{\Omega}) \times (C^{1+\alpha}(\overline{\Omega}; \boldsymbol{R}^3) \cap L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)) \mid (\eta, A) \text{ satisfies } (4.8)\},$$

$$\begin{cases} \eta(p) = 0, \|\eta - \eta_\infty\|_{C^{1+\alpha}(\overline{\Omega})} \leqq 1, \quad \operatorname{div} A = 0 & \text{in } \boldsymbol{R}^3, \\ \|A - A_\infty\|_{C^{1+\alpha}(\overline{\Omega}; \boldsymbol{R}^3)} \leqq 1, \quad \|A - A_\infty\|_{L^6(\boldsymbol{R}^3; \boldsymbol{R}^3)} \leqq 1. \end{cases} \tag{4.8}$$

Note that if $0 < \delta < 1$, any element of $F(\delta)$ is positive. Given (η, A), we can get a solution $w(\lambda, \eta, A)$ to (4.3) with some detailed asymptotic properties for large $\lambda > 0$. The existence of $w(\lambda, \eta, A)$ is proved by a standard upper-lower solution method and its asymptotic property is obtained by the Hölder space estimate of the resolvent of the 2nd order elliptic operator due to S. Campanato. An almost same arguments in this procedure is found in the proof of Prop.4.2 in [**16**]. Hence we describe the result in our situation without proofs.

PROPOSITION 9. *Let $0 < \delta < 1$. There exist $\lambda_0 = \lambda_0(\delta) > 0$ and $c_0 > 0$ such that* (4.3) *has a solution $w(\lambda, \eta, A) \in F(\delta)$ for any $(\eta, A) \in E$, which is unique among positive functions* (*also in $F(\delta)$*) *and satisfies*

$$\begin{cases} 1 - \dfrac{c_0}{\lambda} < w(\lambda, \eta, A; x) \leqq 1 \quad \text{for} \quad x \in \Omega (\lambda \geqq \lambda_0), \\ \|w(\lambda, \eta, A) - 1\|_{C^\alpha(\overline{\Omega})} \leqq \dfrac{c_0}{\lambda} (\|\nabla \eta\|^2_{C^\alpha(\overline{\Omega})} + \|X_0\|^2_{C^\alpha(\overline{\Omega})} + \|A\|^2_{C^\alpha(\overline{\Omega})}), \\ \lim\limits_{\lambda \to \infty} \sup\limits_{(\eta, A) \in E} \|w(\lambda, \eta, A) - 1\|_{C^{2+\alpha}(\overline{\Omega})} = 0. \end{cases} \tag{4.9}$$

From (4.9), there exists $\lambda_1 = \lambda_1(\delta) > 0$ such that $w(\lambda, \eta, A) \in F(\delta)$ for $\lambda \geqq \lambda_1$ and $(\eta, A) \in E$ and moreover the image of this map $E \longrightarrow F(\delta)$ is compact. By a standard argument we can show that the map is also continuous.

Next we consider a solution (η, A) of (4.4)–(4.5) for $w \in F(\delta)$. It can be constructed by a variational method which is completely similar as that in §3. So we only state the result.

PROPOSITION 10. *Let $\delta > 0$. For any $w \in F(\delta)$, there exists a solution $(\eta(w), A(w)) \in H^1(\Omega) \times (H^1_{loc} \cap L^6(\boldsymbol{R}^3; \boldsymbol{R}^3))$ to* (4.4)–(4.5) *with $(\eta(w), A(w)) \in C^{2+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega}; \boldsymbol{R}^3)$. Moreover the solution is unique if the conditions $\operatorname{div} A(w) = 0$ in $\boldsymbol{R}^3$ and $\eta(w; p) = 0$ are imposed.*

We prove that the image of the map defined in $F(\delta)$ is included in E if $\lambda > 0$ is large. Let (η, A) denote the unique solution of (4.4)–(4.5) with $\eta(p) = 0$ and $\operatorname{div} A = 0$ in $\boldsymbol{R}^3$ for $w \in F(\delta)$. Using (4.4)–(4.6) and (4.5)–(4.7), we consider the equations of $\eta - \eta_\infty$ and $A - A_\infty$ respectively.

$$(4.10)\qquad \begin{cases} \operatorname{div}(w^2(\nabla(\eta-\eta_\infty)-(A-A_\infty))) \\ \quad +\operatorname{div}((w^2-1)(\nabla\eta_\infty+X_0-A_\infty))=0 & \text{in } \Omega, \\ \langle\nabla(\eta-\eta_\infty)-(A-A_\infty)\cdot\nu\rangle=0 & \text{on } \partial\Omega, \end{cases}$$

$$(4.11)\qquad \begin{cases} \operatorname{rot}\operatorname{rot}(A-A_\infty)+(A-A_\infty-\nabla(\eta-\eta_\infty))w^2\Lambda_\Omega \\ \quad +(w^2-1)(A_\infty-\nabla\eta_\infty-X_0)\Lambda_\Omega=0 & \text{in } \boldsymbol{R}^3. \end{cases}$$

From these equations we get the following integral identities,

$$(4.12)\qquad \begin{cases} \int_{\boldsymbol{R}^3}|\operatorname{rot}(A-A_\infty)|^2\,dx+\int_\Omega(A-A_\infty-\nabla(\eta-\eta_\infty))(A-A_\infty)w^2\,dx \\ =\int_\Omega(1-w^2)(A_\infty-\nabla\eta_\infty-X_0)(A-A_\infty)\,dx, \end{cases}$$

$$(4.13)\qquad \begin{cases} \int_\Omega(|\nabla(\eta-\eta_\infty)|^2-(A-A_\infty)\nabla(\eta-\eta_\infty))w^2\,dx \\ =-\int_\Omega(1-w^2)(A_\infty-\nabla\eta_\infty-X_0)\nabla(\eta-\eta_\infty)\,dx. \end{cases}$$

Adding (4.12) to (4.13), we get

$$(4.14)\qquad \begin{cases} \int_{\boldsymbol{R}^3}|\operatorname{rot}(A-A_\infty)|^2\,dx+\int_\Omega|A-A_\infty-\nabla(\eta-\eta_\infty)|^2w^2\,dx \\ =\int_\Omega(1-w^2)(A_\infty-\nabla\eta_\infty-X_0)(A-A_\infty-\nabla(\eta-\eta_\infty))\,dx \\ \leqq\left(\int_\Omega|A-A_\infty-\nabla(\eta-\eta_\infty)|^2\,dx+\int_\Omega|A_\infty-\nabla\eta_\infty-X_0|^2\,dx\right) \\ \quad\times\frac{1}{2}\|1-w^2\|^2_{L^\infty(\Omega)}. \end{cases}$$

Using Lemma 7, we see that there exists $\delta_0>0$ such that

$$(4.15)\qquad \|A-A_\infty\|^2_{L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)}+\|\nabla(A-A_\infty)\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times3})}+\|\nabla(\eta-\eta_\infty)\|^2_{L^2(\Omega;\boldsymbol{R}^3)}\leqq\gamma_1(\delta)$$

for and $w\in F(\delta)$ and $\delta\in(0,\delta_0)$. Here $\gamma_1(\delta)>0$ is a function with $\lim_{\delta\to0}\gamma_1(\delta)=0$.

From (4.11) and $A,A_\infty\in L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)$ and $\operatorname{div}A\equiv0$, $\operatorname{div}A_\infty\equiv0$, $A-A_\infty$ has the following expression by the aid of the fundamental solution of the Laplacian

$$(4.16)\qquad \begin{cases} A(x)-A_\infty(x)=\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{J(y)}{|x-y|}\,dy, \\ J(x)=-(A-A_\infty-\nabla(\eta-\eta_\infty))w^2\Lambda_\Omega \\ \qquad +(1-w^2)(A_\infty-\nabla\eta_\infty-X_0)\Lambda_\Omega. \end{cases}$$

Applying the estimates of the singular integral operator (cf. [**2**], [**7**]), we have the following estimate,

$$\|A - A_\infty\|_{H^2(|x| \leqq M; \boldsymbol{R}^3)} \leqq c_M \|J\|_{L^2(\Omega; \boldsymbol{R}^3)}, \tag{4.17}$$

where $c_M > 0$ is a constant depending on $M > 0$ such that $\bar{\Omega} \subset \{|x| < M\}$. We used that J has a compact support. On the other hand J is estimated through (4.16) by A, η, w as follows,

$$\begin{aligned} \|J\|_{L^2(\Omega; \boldsymbol{R}^3)} \leqq \|A - A_\infty\|_{L^2(\Omega; \boldsymbol{R}^3)} + \|\nabla(\eta - \eta_\infty)\|_{L^2(\Omega; \boldsymbol{R}^3)} \\ + \|w^2 - 1\|_{L^\infty(\Omega)} \|A_\infty - \nabla\eta_\infty - X_0\|_{L^2(\Omega; \boldsymbol{R}^3)} \end{aligned} \tag{4.18}$$

Using (4.15), we get an estimate

$$\|A - A_\infty\|_{H^2(|x| \leqq M; \boldsymbol{R}^3)} \leqq \gamma_2(\delta)$$

where $\gamma_2(\delta) > 0$ is a function with $\lim_{\delta \to 0} \gamma_2(\delta) = 0$.

Next we consider the estimates of the solution $\eta - \eta_\infty$ of the elliptic boundary value problem (4.10) which can be rewritten as follows

$$\begin{cases} \operatorname{div}(w^2(\nabla(\eta - \eta_\infty))) & \\ = \operatorname{div}(w^2(A - A_\infty)) - \operatorname{div}((w^2 - 1)(\nabla\eta_\infty + X_0 - A_\infty)) & \text{in } \Omega, \\ \dfrac{\partial}{\partial\nu}(\eta - \eta_\infty) = \langle A - A_\infty \cdot \nu \rangle & \text{on } \partial\Omega, \end{cases} \tag{4.10)'$$

Taking into consideration the additional condition $\eta(p) = 0$, we have the estimate

$$\begin{aligned} \|\eta - \eta_\infty\|_{H^2(\Omega)} &\leqq c(\|\operatorname{div}((w^2 - 1)(\nabla\eta_\infty + X_0 - A_\infty)) + \operatorname{div}(w^2(A - A_\infty))\|_{L^2(\Omega)} \\ &+ \|\langle A - A_\infty \cdot \nu\rangle\|_{H^{1/2}(\partial\Omega)}) \leqq c'(\gamma_3(\delta) + \|A - A_\infty\|_{H^1(\Omega; \boldsymbol{R}^3)}) \end{aligned}$$

Here c, c' are positive constants and $\gamma_3(\delta) \to 0$ for $\delta \to 0$. Thus we obtained that $\|A - A_\infty\|_{H^2(|x| \leqq M; \boldsymbol{R}^3)}$ and $\|\eta - \eta_\infty\|_{H^2(\Omega)}$ are small if $\delta > 0$ is taken small. Applying a similar regularity argument to (4.10)′ and (4.11), repeatedly, we get the following estimates:

$$\|A - A_\infty\|_{C^{1+\alpha}(\bar{\Omega}; \boldsymbol{R}^3)} \leqq \gamma_4(\delta), \quad \|\eta - \eta_\infty\|_{C^{2+\alpha}(\bar{\Omega})} \leqq \gamma_5(\delta), \quad (0 < \alpha < 1),$$

where $\gamma_4(\delta)$, $\gamma_5(\delta)$ satisfy the same conditions as $\gamma_1(\delta)$, $\gamma_2(\delta)$, $\gamma_3(\delta)$.

Summing up these arguments, we see that if $\delta > 0$ is small and $\lambda > 0$ is taken large, then T_λ is well-defined and $T_\lambda(E) \subset E$ holds. The continuity of the map T_λ can be proved in a similar way below. The continuity and compactness of $E \ni (\eta, A) \mapsto w(\lambda, \eta, A) \in F(\delta)$ has been already argued and so we deal with the part $w \mapsto (\eta(w), A(w)) \in E$. Let (η_j, A_j) be the solution of (4.3)–(4.4) for $w_j \in E$ $(j = 1, 2)$. By a similar calculation as in (4.10) $\sim$ (4.14), we obtain

$$\begin{cases} \displaystyle\int_{\boldsymbol{R}^3} |\operatorname{rot}(A_1 - A_2)|^2\,dx + \int_\Omega (A_1 - A_2 - \nabla(\eta_1 - \eta_2))(A_1 - A_2)w_1^2\,dx \\ \displaystyle + \int_\Omega (A_2 - \nabla\eta_2 - X_0)(A_1 - A_2)(w_1^2 - w_2^2)\,dx = 0, \end{cases} \tag{4.19}$$

$$
(4.20)\qquad \begin{cases} \int_\Omega (|\nabla(\eta_1-\eta_2)|^2-(A_1-A_2)\nabla(\eta_1-\eta_2))w_1^2\,dx \\ \quad + \int_\Omega (\nabla\eta_2+X_0-A_2)\nabla(\eta_1-\eta_2)(w_1^2-w_2^2)\,dx=0, \end{cases}
$$

and we get

$$
(4.21)\qquad \begin{cases} \int_{\boldsymbol{R}^3} |\mathrm{rot}(A_1-A_2)|^2\,dx+\int_\Omega |A_1-A_2-\nabla(\eta_1-\eta_2)|^2 w_1^2\,dx \\ \quad + \int_\Omega (A_2-\nabla\eta_2-X_0)(A_1-A_2-\nabla(\eta_1-\eta_2))(w_1^2-w_2^2)\,dx=0. \end{cases}
$$

From this identity and Lemma 7, if w_2 is close to w_1 in $F(\delta)$, then $\|\nabla(\eta_2-\eta_1)\|_{L^2(\Omega;\boldsymbol{R}^3)}$, $\|\nabla(A_2-A_1)\|_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times 3})}$ and $\|A_2-A_1\|_{L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)}$ are small. Applying the Schauder estimate to the system of equations concerning $\eta_2-\eta_1$, A_2-A_1 (which are obtained similarly as (4.10)–(4.11)), we can prove that $\|(\eta_2-\eta_1)\|_{C^{2+\alpha}(\bar\Omega;\boldsymbol{R}^3)}$, $\|A_2-A_1\|_{C^{1+\alpha}(\bar\Omega;\boldsymbol{R}^3)}$ are small.

Consequently we have proved that the map $T_\lambda : E\to E$ is well-defined for large $\lambda>0$ and it is compact and continuous. Therefore from the Schauder fixed point theorem, we have a fixed point in E and we get a solution $(W_\lambda,\eta_\lambda,A_\lambda)$ to (4.3) $\sim$ (4.5) for large $\lambda>0$. We have established the existence of a solution (Φ_λ,A_λ) in Theorem 4.

The obtained solution $(\Phi_\lambda,A_\lambda)=(W_\lambda e^{i\theta_\lambda},A_\lambda)$ satisfies the following properties, which are verified through the above construction. The last one in (4.22) is proved by the equation (4.3).

PROPOSITION 11. *There exists a constant $c_1>0$ (independent of λ) such that*

$$
(4.22)\qquad \begin{cases} \lambda\|W_\lambda-1\|_{C^\alpha(\bar\Omega)}+\|\nabla\theta_\lambda\|_{C^{1+\alpha}(\bar\Omega)}\leqq c_1 \\ \lim_{\lambda\to\infty}\|W_\lambda-1\|_{C^{2+\alpha}(\bar\Omega)}=0, \\ \lim_{\lambda\to\infty}\sup_{x\in\Omega}|\lambda(1-W_\lambda^2)-|\nabla\theta_\lambda|^2|=0, \end{cases}
$$

for any $\alpha\in(0,1)$.

§5. Stability of (Φ_λ,A_λ)

In this section we prove the stability of (Φ_λ,A_λ), which we constructed in §3 and §4. The procedure below is almost similar as was done in **[15]** (while the estimate is modified). We estimate the second variation of the functional $\mathscr{H}_\lambda$ at $(\Phi_\lambda,A_\lambda)=(u_\lambda,v_\lambda,A_\lambda)$. We express Φ_λ in terms of real valued functions, i.e., we put $u_\lambda(x)=W_\lambda(x)\cos\theta_\lambda$, $v_\lambda(x)=W_\lambda(x)\sin\theta_\lambda$. Let us consider $\mathscr{L}_\lambda(u_\lambda,v_\lambda,A_\lambda,\phi,\psi,B)$ on $\bar N(\Phi_\lambda,A_\lambda)=\bar N(u_\lambda,v_\lambda,A_\lambda)$. Hereafter we denote $\mathscr{L}_\lambda(u_\lambda,v_\lambda,A_\lambda,\phi,\psi,B)$ by $\mathscr{L}_\lambda(\phi,\psi,B)$ for simplicity. From (1.12), we have

$$
(5.1)\qquad \mathscr{L}_\lambda(\phi,\psi,B)=I_1(\phi,\psi)+I_2(B)+I_3(\phi,\psi,B)
$$

where

$$I_1(\phi,\psi) = \int_\Omega (|\nabla\phi + \psi A_\lambda|^2 + |\nabla\psi - \phi A_\lambda|^2 - \lambda(1 - u_\lambda^2 - v_\lambda^2)(\phi^2 + \psi^2) + 2\lambda(u_\lambda\phi + v_\lambda\psi)^2)\,dx$$

$$I_2(B) = \int_{R^3} |\mathrm{rot}\, B|^2\,dx + \int_\Omega (u_\lambda^2 + v_\lambda^2)B^2\,dx,$$

$$I_3(\phi,\psi,B) = 4\int_\Omega \langle A_\lambda \cdot B\rangle(u_\lambda\phi + v_\lambda\psi)\,dx$$
$$- 2\int_\Omega \{\phi\langle\nabla v_\lambda \cdot B\rangle - \psi\langle\nabla u_\lambda \cdot B\rangle + u_\lambda\langle\nabla\psi\cdot B\rangle - v_\lambda\langle\nabla\phi\cdot B\rangle\}\,dx.$$

We change the variables ϕ, ψ into $\hat{\phi}$, $\hat{\psi}$ by

$$\begin{pmatrix}\hat{\phi}(x)\\ \hat{\psi}(x)\end{pmatrix} = R(-\theta_\lambda(x))\begin{pmatrix}\phi(x)\\ \psi(x)\end{pmatrix} \text{ where } R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{5.2}$$

and we express I_1 and I_3 in terms of $\hat{\phi}$, $\hat{\psi}$, B. I_1 and I_3 are rewritten as follows,

$$I_1(\phi,\psi) = \hat{I}_1(\hat{\phi},\hat{\psi}) = \int_\Omega (|\nabla\hat{\phi} + (A_\lambda - \nabla\theta_\lambda)\hat{\psi}|^2 + |\nabla\hat{\psi} - (A_\lambda - \nabla\theta_\lambda)\hat{\phi}|^2$$
$$- \lambda(1 - W_\lambda^2)(\hat{\phi}^2 + \hat{\psi}^2) + 2\lambda W_\lambda^2\hat{\phi}^2)\,dx, \tag{5.3}$$

$$I_3(\phi,\psi,B) = \hat{I}_3(\hat{\phi},\hat{\psi},B) \tag{5.4}$$
$$= 4\int_\Omega \langle A_\lambda\cdot B\rangle W_\lambda\hat{\phi}\,dx - 4\int_\Omega (W_\lambda\langle\nabla\theta_\lambda\cdot B\rangle\hat{\phi} - \langle\nabla W_\lambda\cdot B\rangle\hat{\psi})\,dx$$

We used that $\operatorname{div} B = 0$ in Ω and $\langle B\cdot\nu\rangle = 0$ on $\partial\Omega$ since $(\phi,\psi,B)\in\bar{N}(u_\lambda,v_\lambda,A_\lambda)$.

To investigate the coerciveness of I_1, we consider the eigenvalue problem of an elliptic operator,

$$\begin{cases}\Delta\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \begin{pmatrix}\operatorname{div}((A_\lambda - \nabla\theta_\lambda)\psi) + \langle(A_\lambda - \nabla\theta_\lambda)\cdot\nabla\psi\rangle\\ -\operatorname{div}((A_\lambda - \nabla\theta_\lambda)\phi) - \langle(A_\lambda - \nabla\theta_\lambda)\cdot\nabla\phi\rangle\end{pmatrix} & \\ +(\lambda(1 - W_\lambda^2) - |A_\lambda - \nabla\theta_\lambda|^2)\begin{pmatrix}\phi\\ \psi\end{pmatrix} - 2\lambda W_\lambda^2\begin{pmatrix}\phi\\ 0\end{pmatrix} + \mu\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} & \text{in } \Omega,\\ \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega.\end{cases} \tag{5.5}$$

Let

$$\{\mu_l(\lambda)\}_{l=1}^\infty \text{ and } \left\{\begin{pmatrix}\phi_{l,\lambda}\\ \psi_{l,\lambda}\end{pmatrix}\right\}_{l=1}^\infty \subset L^2(\Omega)\times L^2(\Omega)$$

be the eigenvalues arranged in increasing order (with counting multiplicity) and a complete system of the corresponding orthonormal eigenfunctions of (5.5). We can apply quite a similar argument as in **[15]** or **[16]** for such type of the eigenvalue problem as

(5.5) and we get their asymptotic behaviors of the eigenvalues and eigenfunctions. So we give the results without proofs.

LEMMA 12.

$$\lim_{\lambda\to\infty} \mu_l(\lambda) = \mu_l \quad (l \geqq 1), \tag{5.6}$$

$$\lim_{\lambda\to\infty} (\|\nabla\phi_{l,\lambda}\|^2_{L^2(\Omega)} + \lambda\|\phi_{l,\lambda}\|^2_{L^2(\Omega)}) = 0 \quad (l \geqq 1), \tag{5.7}$$

where $\{\mu_l\}_{l=1}^{\infty}$ *is the set of the eigenvalues arranged in increasing order* (*with counting multiplicity*) *of the following eigenvalue problem*:

$$\begin{cases} \Delta\psi + \mu\psi = 0 & \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

We recall an auxiliary property concerning a complete orthonormal basis of the product of two Hilbert spaces, which we use in the proof of Lemma 15.

LEMMA 13 ([**15**]). *Let* H_1 *and* H_2 *be two real Hilbert spaces with inner products* $(\cdot,\cdot)_{H_1}$ *and* $(\cdot,\cdot)_{H_2}$, *respectively and let* H *be the product Hilbert space* $H_1 \times H_2$ *with the inner product*:

$$\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\phi'\\ \psi'\end{pmatrix}\right)_H \equiv (\phi,\phi')_{H_1} + (\psi,\psi')_{H_2} \text{ for } \begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\phi'\\ \psi'\end{pmatrix} \in H.$$

If there exists an orthonormal basis $\left\{\begin{pmatrix}\phi_n\\ \psi_n\end{pmatrix}\right\}_{n=1}^{\infty} \subset H$, *then*

$$\begin{cases} \sum_{n=1}^{\infty}(\phi,\phi_n)_{H_1}(\psi,\psi_n)_{H_2} = 0 & \text{for any } \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in H, \\ \|\phi\|^2_{H_1} = \sum_{n=1}^{\infty}(\phi,\phi_n)^2_{H_1}, \quad \|\psi\|^2_{H_2} = \sum_{n=1}^{\infty}(\psi,\psi_n)^2_{H_2}. \end{cases} \tag{5.9}$$

The following result directly follows from the above lemmas.

LEMMA 14. *For any* $(\phi,\psi) \in L^2(\Omega) \times L^2(\Omega)$, *the following equalities hold.*

$$\sum_{l=1}^{\infty}(\phi\cdot\phi_{l,\lambda})_{L^2(\Omega)}(\psi\cdot\psi_{l,\lambda})_{L^2(\Omega)} = 0,$$

$$\|\phi\|^2_{L^2(\Omega)} = \sum_{l=1}^{\infty}(\phi\cdot\phi_{l,\lambda})^2_{L^2(\Omega)}, \quad \|\psi\|^2_{L^2(\Omega)} = \sum_{l=1}^{\infty}(\psi\cdot\psi_{l,\lambda})^2_{L^2(\Omega)}.$$

$\hat{I}_1(\hat{\phi},\hat{\psi})$ is expressed in terms the Fourier coefficients of $\hat{\phi},\hat{\psi}$:

$$\hat{I}_1(\hat{\phi},\hat{\psi}) = \sum_{l=1}^{\infty} \mu_l(\lambda) g_l^2, \tag{5.10}$$

where

$$g_l = (\phi_{l,\lambda}\cdot\hat{\phi})_{L^2(\Omega)} + (\psi_{l,\lambda}\cdot\hat{\psi})_{L^2(\Omega)}. \tag{5.11}$$

We remark that $\phi_{1,\lambda}(x) = 0$, $\psi_{1,\lambda}(x) = e_\lambda W_\lambda(x)$, $\mu_1(\lambda) = \mu_1 = 0$, $\mu_2 > 0$, $e_\lambda \neq 0$ is a certain real number, which satisfies $\lim_{\lambda\to\infty} e_\lambda^2 = 1/|\Omega|$.

We have the following coercive inequality.

LEMMA 15. *For any $c > 0$ and $\eta > 0$, there exists a constant $\lambda_1 = \lambda_1(c,\eta) > 0$ and $c' = c'(c,\eta) > 0$ such that*

$$(5.12) \qquad \hat{I}_1(\hat{\phi},\hat{\psi}) \geqq c\|\hat{\phi}\|^2_{L^2(\Omega)} + (\mu_2(\lambda) - \eta)\|\hat{\psi}\|^2_{L^2(\Omega)} - c'(\hat{\psi}\cdot W_\lambda)^2_{L^2(\Omega)},$$

for any $\hat{\phi},\hat{\psi} \in H^1(\Omega)$ and $\lambda \geqq \lambda_1$. Equivalently,

$$(5.13) \qquad I_1(\phi,\psi) \geqq c\|\phi\cos\theta_\lambda + \psi\sin\theta_\lambda\|^2_{L^2(\Omega)} + (\mu_2(\lambda) - \eta)\|\phi\sin\theta_\lambda - \psi\cos\theta_\lambda\|^2_{L^2(\Omega)} - c'\left(\int_\Omega (\phi v_\lambda - \psi u_\lambda)\,dx\right)^2,$$

for any $\phi,\psi \in H^1(\Omega)$ and $\lambda \geqq \lambda_1$.

(Proof of Lemma 15) In view of $\lim_{m\to\infty}\mu_m = \infty$ and the properties of the eigenvalues of (5.6) in Lemma 12, for given $c > 0$, we can take a natural number N so that $\mu_l(\lambda) \geqq c+1$ for $l > N$ and for any large $\lambda > 0$. Thus we have,

$$\hat{I}_1(\hat{\phi},\hat{\psi}) \geqq \sum_{l=1}^{N}\mu_l(\lambda)g_l^2 + (c+1)\sum_{l=N+1}^{\infty} g_l^2.$$

Substituting (5.11) into the above inequality, we have

$$\begin{aligned}\hat{I}_1(\hat{\phi},\hat{\psi}) \geqq& \sum_{l=1}^{N}\mu_l(\lambda)(\hat{\phi}\cdot\phi_{l,\lambda})^2_{L^2(\Omega)} + (c+1)\sum_{l=N+1}^{\infty}(\hat{\phi}\cdot\phi_{l,\lambda})^2_{L^2(\Omega)}\\ &+ 2\sum_{l=1}^{N}\mu_l(\lambda)(\hat{\phi}\cdot\phi_{l,\lambda})_{L^2(\Omega)}(\hat{\psi}\cdot\psi_{l,\lambda})_{L^2(\Omega)}\\ &+ 2(c+1)\sum_{l=N+1}^{\infty}(\hat{\phi}\cdot\phi_{l,\lambda})_{L^2(\Omega)}(\hat{\psi}\cdot\psi_{l,\lambda})_{L^2(\Omega)} + \sum_{l=1}^{N}\mu_l(\lambda)(\hat{\psi}\cdot\psi_{l,\lambda})^2_{L^2(\Omega)}\\ &+ (c+1)\sum_{l=N+1}^{\infty}(\hat{\psi}\cdot\psi_{l,\lambda})^2_{L^2(\Omega)}.\end{aligned}$$

From Lemma 14,

$$(5.14) \qquad \begin{aligned}I_1(\hat{\phi},\hat{\psi}) \geqq& (c+1)\|\hat{\phi}\|^2_{L^2(\Omega)} + \sum_{l=1}^{N}(\mu_l(\lambda) - c - 1)(\hat{\phi}\cdot\phi_{l,\lambda})^2_{L^2(\Omega)}\\ &+ \sum_{l=1}^{N} 2(\mu_l(\lambda) - c - 1)(\hat{\phi}\cdot\phi_{l,\lambda})_{L^2(\Omega)}(\hat{\psi}\cdot\psi_{l,\lambda})_{L^2(\Omega)} + \mu_2(\lambda)(\|\hat{\psi}\|^2_{L^2(\Omega)} - (\hat{\psi}\cdot\psi_{l,\lambda})^2_{L^2(\Omega)}).\end{aligned}$$

We used $\mu_1(\lambda) = 0$, $\phi_{1,\lambda} = 0$, $\psi_{1,\lambda} = e_\lambda W_\lambda$. Using Lemma 12-(5.7) in the right hand side of the above inequality, we see that the terms including $\phi_{l,\lambda}(1 \leqq l \leqq N)$ can be absorbed in the ones including $\|\hat{\phi}\|^2_{L^2(\Omega)}$ and $\|\hat{\psi}\|^2_{L^2(\Omega)}$ for large $\lambda > 0$. We have,

$$\hat{I}_1(\hat{\phi},\hat{\psi}) \geqq c\|\hat{\phi}\|^2_{L^2(\Omega)} + (\mu_2(\lambda) - \eta)\|\hat{\psi}\|^2_{L^2(\Omega)} - c'(\hat{\psi}\cdot W_\lambda)^2_{L^2(\Omega)}$$

for large $\lambda > 0$. We obtain (5.12). (5.13) follows immediately. □

LEMMA 16. *For any $c > 0$ and $\eta > 0$, there exist $\delta > 0$, $\lambda_1 > 0$ and $c' > 0$ such that*

$$\hat{I}_1(\hat{\phi},\hat{\psi}) \geqq \delta(\|\nabla\hat{\phi}\|^2_{L^2(\Omega)} + \|\nabla\hat{\psi}\|^2_{L^2(\Omega)}) + c\|\hat{\phi}\|^2_{L^2(\Omega)} + (\mu_2(\lambda) - \eta)\|\hat{\psi}\|^2_{L^2(\Omega)} - c'(W_\lambda\cdot\hat{\psi})^2_{L^2(\Omega)}, \tag{5.15}$$

for any $\hat{\phi},\hat{\psi} \in H^1(\Omega)$ and $\lambda > \lambda_1$,

$$I_1(\phi,\psi) \geqq \delta(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}) + c\|\phi\cos\theta_\lambda + \psi\sin\theta_\lambda\|^2_{L^2(\Omega)} + (\mu_2(\lambda) - \eta)\|\phi\sin\theta_\lambda - \psi\cos\theta_\lambda\|^2_{L^2(\Omega)} - c'\left(\int_\Omega(-\phi v_\lambda + \psi u_\lambda)\,dx\right)^2 \tag{5.16}$$

for any $\phi,\psi \in H^1(\Omega)$ and $\lambda > \lambda_1$.

(Proof of Lemma 16) First recall that there exists an $M > 0$ independent of $\lambda > 0$, such that $|\nabla\theta_\lambda(x)| \leqq M$ in Ω and other properties in Proposition 11. $\lim_{\lambda\to\infty} W_\lambda(x) = 1$ uniformly in Ω (cf. Prop.11). Using these facts in (5.3), we see that for any $c > 0$ there exist constants $\lambda_1 > 0$ and $c'' > 0$ such that

$$\hat{I}_1(\hat{\phi},\hat{\psi}) \geqq \frac{1}{2}(\|\nabla\hat{\phi}\|^2_{L^2(\Omega)} + \|\nabla\hat{\psi}\|^2_{L^2(\Omega)}) + (c+1)\|\hat{\phi}\|^2_{L^2(\Omega)} - c''\|\hat{\psi}\|^2_{L^2(\Omega)} \tag{5.17}$$

for $\hat{\phi},\hat{\psi} \in H^1(\Omega)$ and $\lambda > \lambda_1$.

Using Lemma 15 for $c+1$ in place of c with (5.17), we obtain,

$$\begin{aligned}\hat{I}_1(\hat{\phi},\hat{\psi}) &= (1-\varepsilon)\hat{I}_1(\hat{\phi},\hat{\psi}) + \varepsilon\hat{I}_1(\hat{\phi},\hat{\psi})\\ &\geqq \frac{\varepsilon}{2}(\|\nabla\hat{\phi}\|^2_{L^2(\Omega)} + \|\nabla\hat{\psi}\|^2_{L^2(\Omega)}) + (c+1)\|\hat{\phi}\|^2_{L^2(\Omega)}\\ &\quad + ((1-\varepsilon)(\mu_2(\lambda) - \eta) - \varepsilon c'')\|\hat{\psi}\|^2_{L^2(\Omega)} - (1-\varepsilon)c'(\hat{\psi}\cdot W_\lambda)^2_{L^2(\Omega)}.\end{aligned}$$

where $\varepsilon \in (0,1)$. By taking $\varepsilon > 0$ small, we obtain (5.15).

Next we prove (5.16). By a simple calculation we have

$$\begin{aligned}|\nabla\hat{\phi}|^2 + |\nabla\hat{\psi}|^2 &= |\nabla\phi|^2 + |\nabla\psi|^2 + 2(-\psi\nabla\theta_\lambda\nabla\phi + \phi\nabla\theta_\lambda\nabla\psi) + |\nabla\theta_\lambda|^2(\phi^2+\psi^2)\\ &\geqq \frac{1}{2}(|\nabla\phi|^2 + |\nabla\psi|^2) - |\nabla\theta_\lambda|^2(\phi^2+\psi^2)\\ &= \frac{1}{2}(|\nabla\phi|^2 + |\nabla\psi|^2) - |\nabla\theta_\lambda|^2(\hat{\phi}^2+\hat{\psi}^2),\end{aligned}$$

and using this inequality in (5.17) we get

$$\hat{I}_1(\phi,\psi) \geqq \frac{1}{4}(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}) + (c+1-M^2/2)\|\hat{\phi}\|^2_{L^2(\Omega)} - (c''+M^2/2)\|\hat{\psi}\|^2_{L^2(\Omega)} \tag{5.18}$$

for $\hat{\phi}, \hat{\psi} \in H^1(\Omega)$. We apply a similar argument as (5.15). Using (5.13) (c is replaced by $c+1$) and (5.18), we have

$$
\begin{aligned}
I_1(\phi, \psi) &= (1-\varepsilon) I_1(\phi, \psi) + \varepsilon I_1(\phi, \psi) \\
&\geqq \frac{\varepsilon}{4}(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}) + (c+1-\varepsilon M^2/2)\|\phi\cos\theta_\lambda + \psi\sin\theta_\lambda\|^2_{L^2(\Omega)} \\
&\quad + ((1-\varepsilon)(\mu_2(\lambda) - \eta) - \varepsilon(c'' + M^2/2))\|\hat{\psi}\|^2_{L^2(\Omega)} \\
&\quad - (1-\varepsilon)c'\left(\int_\Omega (-\phi v_\lambda + \psi u_\lambda)\,dx\right)^2.
\end{aligned}
$$

We complete the proof of (5.16) by taking $\varepsilon > 0$ small. □

Proof of theorem 4. We estimate $\mathscr{L}_\lambda(\phi, \psi, B)$ from below. We first consider the case: $(\phi, \psi, B) \in \bar{N}(u_\lambda, v_\lambda, A_\lambda)$. $\mathscr{L}_\lambda$ is expressed as follows,

$$\mathscr{L}_\lambda(\phi, \psi, B) = I_1(\phi, \psi) + I_2(B) + \hat{I}_3(\hat{\phi}, \hat{\psi}, B),$$

where $\hat{I}_3(\hat{\phi}, \hat{\psi}) = I_3(\phi, \psi)$ and $\hat{\phi} = \phi\cos\theta_\lambda + \psi\sin\theta_\lambda$, $\hat{\psi} = \phi\sin\theta_\lambda - \psi\cos\theta_\lambda$.

We estimate $\mathscr{L}_\lambda(\phi, \psi, B)$ by dominating $|\hat{I}_3(\hat{\phi}, \hat{\psi}, B)|$ by I_1 and I_2.

$$
\begin{aligned}
(5.19)\qquad |\hat{I}_3(\hat{\phi}, \hat{\psi}, B)| &\leqq 4\int_\Omega |\langle A_\lambda - \nabla\theta_\lambda \cdot B\rangle W_\lambda \hat{\phi}|\,dx + 4\int_\Omega |\langle \nabla W_\lambda \cdot B\rangle \hat{\psi}|\,dx \\
&\leq 4\sup_{x\in\Omega}|W_\lambda(A_\lambda - \nabla\theta_\lambda)|\int_\Omega (\varepsilon B^2 + \hat{\phi}^2/4\varepsilon)\,dx + 2\sup_{x\in\Omega}|\nabla W_\lambda|\int_\Omega (B^2 + \hat{\phi}^2)\,dx
\end{aligned}
$$

Applying Lemma 16, we obtain

$$
\begin{aligned}
\mathscr{L}_\lambda(\phi, \psi, B) &\geqq \hat{I}_1(\hat{\phi}, \hat{\psi}) + I_2(B) - |\hat{I}_3(\hat{\phi}, \hat{\psi}, B)| \\
&\geqq \delta(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}) \\
&\quad + (c - (1/\varepsilon)\sup_{x\in\Omega}|W_\lambda(A_\lambda - \nabla\theta_\lambda)|)\|\phi\cos\theta_\lambda + \psi\sin\theta_\lambda\|^2_{L^2(\Omega)} \\
&\quad + (\mu_2(\lambda) - \eta - 2\sup_{x\in\Omega}|\nabla W_\lambda|)\|\phi\sin\theta_\lambda - \psi\cos\theta_\lambda\|^2_{L^2(\Omega)} + \int_{\mathbf{R}^3} |\operatorname{rot} B|^2\,dx \\
&\quad + \int_\Omega (W_\lambda^2 - 4\varepsilon\sup_{x\in\Omega}|W_\lambda(A_\lambda - \nabla\theta_\lambda)| - 2\sup_\Omega |W_\lambda|)B^2\,dx
\end{aligned}
$$

We used $(\hat{\psi} \cdot W_\lambda)_{L^2(\Omega)} = 0$ for $(\phi, \psi, B) \in \bar{N}(u_\lambda, v_\lambda, A_\lambda)$. From Prop. 11, there exist $M > 0$ and $\lambda_2 > 0$ such that

$$\sup_{x\in\Omega}|W_\lambda(A_\lambda - \nabla\theta_\lambda)| \leqq M \quad \text{for } \lambda \geqq \lambda_2.$$

Hence we put $\eta = \mu_2/4$, $\varepsilon = (1/16M) + 1$ and $c = (M/\varepsilon) + 1$ in Lemma 16, then we can take a large positive number $\lambda_3 > 0$ such that

$$\mathscr{L}_\lambda(\phi,\psi,B) \geqq \delta(\|\nabla\phi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)}) + \|\phi\cos\theta_\lambda + \psi\sin\theta_\lambda\|^2_{L^2(\Omega)}$$
$$+ (\mu_2/2)\|\phi\sin\theta_\lambda - \psi\cos\theta_\lambda\|^2_{L^2(\Omega)} + \|\mathrm{rot}\, B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)} + (1/2)\int_\Omega B^2\,dx$$
$$\geqq \min(\delta, 1/2, \mu_2/2)(\|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)} + \|B\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\mathrm{rot}\, B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)})$$

for $(\phi,\psi,B) \in \bar{N}(u_\lambda, v_\lambda, A_\lambda)$ and $\lambda > \lambda_3$.

We will obtain a similar inequality on $N(u_\lambda, v_\lambda, A_\lambda)$. Take any $(\phi,\psi,B) \in N(u_\lambda, v_\lambda, A_\lambda)$ and we have the decomposition

$$(\phi,\psi,B) = (-v_\lambda\xi, u_\lambda\xi, \nabla\xi) + (\bar{\phi},\bar{\psi},\bar{B}) \in T(u,v,A) + \bar{N}(u,v,A) \quad (\text{cf. } (1.10)).$$

Note that

$$\mathscr{L}_\lambda(\phi,\psi,B) = \mathscr{L}_\lambda(\bar{\phi},\bar{\psi},\bar{B}). \quad (\text{cf. Prop. } 3)$$

We will prove that there exists a $\delta' > 0$ which is independent of (ϕ,ψ,B) and large $\lambda > 0$ such that

$$(5.20) \qquad \|\bar{\phi}\|^2_{H^1(\Omega)} + \|\bar{\psi}\|^2_{H^1(\Omega)} + \|\bar{B}\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\mathrm{rot}\,\bar{B}\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)}$$
$$\geqq \delta'(\|\phi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)} + \|B\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\mathrm{rot}\, B\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)}).$$

If this is not true, there exists a sequence $(\phi_n,\psi_n,B_n) \in N(u_\lambda, v_\lambda, A_\lambda)$ and $\xi_n (n \geqq 1)$ such that

$$(5.21) \quad (\phi_n,\psi_n,B_n) = (-v_\lambda\xi_n, u_\lambda\xi_n, \nabla\xi_n) + (\bar{\phi}_n,\bar{\psi}_n,\bar{B}_n) \in T(u_\lambda,v_\lambda,A_\lambda) + \bar{N}(u_\lambda,v_\lambda,A_\lambda).$$

with

$$(5.22) \qquad \|\bar{\phi}_n\|^2_{H^1(\Omega)} + \|\bar{\psi}_n\|^2_{H^1(\Omega)} + \|\bar{B}_n\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\mathrm{rot}\,\bar{B}_n\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)} \to 0 \quad (n \to \infty),$$

$$(5.23) \qquad \|\phi_n\|^2_{H^1(\Omega)} + \|\psi_n\|^2_{H^1(\Omega)} + \|B_n\|^2_{L^2(\Omega;\boldsymbol{R}^3)} + \|\mathrm{rot}\, B_n\|^2_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^3)} = 1 \quad (n \geqq 1).$$

(5.21) yields

$$(5.24) \qquad \int_\Omega (u_\lambda^2 + v_\lambda^2)\xi_n(x)\,dx = 0, \quad \Delta\xi_n = 0 \text{ in } \Omega, \quad \frac{\partial\xi_n}{\partial\nu} = \langle B_n \cdot \nu\rangle \text{ on } \partial\Omega,$$

$$(5.25) \qquad \mathrm{rot}\, B_n = \mathrm{rot}\,\bar{B}_n \text{ in } \boldsymbol{R}^3.$$

There exist $\phi, \psi \in H^1(\Omega)$ such that

$$\phi_n \to \phi \quad \psi_n \to \psi \text{ weakly in } H^1(\Omega) \text{ as } n \to \infty.$$

From (5.22) and (5.25) with Lemma 7 and $\mathrm{div}\, B_n = 0$ in $\boldsymbol{R}^3$, $\|\nabla B_n\|_{L^2(\boldsymbol{R}^3;\boldsymbol{R}^{3\times 3})} \to 0$ as $n \to \infty$. Consequently, we have $\lim_{n\to\infty}\|B_n\|_{L^6(\boldsymbol{R}^3;\boldsymbol{R}^3)} = 0$. Using $B_n = \nabla\xi_n + \bar{B}_n$ and $\langle\bar{B}_n \cdot \nu\rangle = 0$ on $\partial\Omega$, we have

$$\int_\Omega B_n^2\,dx = \int_\Omega |\nabla\xi_n|^2\,dx + \int_\Omega \bar{B}_n^2\,dx.$$

This inequality and (5.24) yields $\|\xi_n\|_{H^1(\Omega)} \to 0$ for $n \to \infty$. By considering $H^1(\Omega)$ convergence of $(\phi_n, \psi_n) = (-v_\lambda \xi_n, u_\lambda \xi_n) + (\bar{\phi}_n, \bar{\psi}_n)$ for $n \to \infty$ and we obtain $\phi_n, \psi_n \to 0$ in $H^1(\Omega)$, we see that (5.23) is impossible for large n. This proves the existence of a certain $\delta' > 0$ in (5.20). We have completed the proof of Theorem 4. □

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Shuichi JIMBO,
Department of Mathematics,
Faculty of Science,
Hokkaido University
Sapporo 060 Japan

Jian ZHAI,
Department of Mathematics,
Faculty of Science,
Hokkaido University
Sapporo 060 Japan

Present address
Department of Mathematics,
Zhejiang Unviersity
Hangzhou, P.R.China