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Asymptotic behaviour of solutions to non-isothermal phase separation model with constraint in one-dimensional space

By Akio Ito and Nobuyuki Kenmochi

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1. Introduction.

Let us consider a one-dimensional model for non-isothermal phase separation, which is given by the following system, denoted by (PSC):

$$[\rho(u) + \lambda(w)]_t - u_{xx} = f(t, x) \quad \text{in } Q := (0, +\infty) \times J, \tag{1.1}$$

$$w_t - \{-\kappa w_{xx} + \xi + g(w) - \lambda'(w)u\}_{xx} = 0 \quad \text{in } Q, \tag{1.2}$$

$$\xi \in \partial I_{[\sigma_*, \sigma^*]}(w) \quad \text{in } Q, \tag{1.3}$$

$$-u_x(t, -L) + n_0 u(t, -L) = h_-(t), \quad u_x(t, L) + n_0 u(t, L) = h_+(t) \quad \text{for } t > 0, \tag{1.4}$$

$$w_x(t, -L) = w_x(t, L) = 0 \quad \text{for } t > 0, \tag{1.5-1}$$

$$[-\kappa w_{xx}(t, \cdot) + \xi(t, \cdot) + g(w(t, \cdot)) - \lambda'(w(t, \cdot))u(t, \cdot)]_x|_{x = \pm L} = 0 \quad \text{for } t > 0, \tag{1.5-2}$$

$$u(0, x) = u_0(x), \quad w(0, x) = w_0(x) \quad \text{for } x \in J. \tag{1.6}$$

Here $J := (-L, L)$ with a positive number L; $k > 0$ and $n_0 > 0$ are constants; $\rho(u)$ is an increasing function of u, and $\lambda(w)$, $\lambda'(w) = (d/dw)\lambda(w)$, $g(w)$ are smooth functions of w; $\partial I_{[\sigma_*, \sigma^*]}$ is the subdifferential of the indicator function $I_{[\sigma_*, \sigma^*]}$ of the interval $[\sigma_*, \sigma^*] \subset R$; $f(t, x)$, $h_\pm(t)$, $u_0(x)$ and $w_0(x)$ are given data.

The above system arises in the phase separation of a binary mixture with components A and B. In this context, $\theta := \rho(u)$ represents the absolute temperature and $w := w_A$ the order parameter which is the local concentration of the component A; note that $\sigma_* = 0 \le w_A(t, x) \le 1 = \sigma^*$, and $w_A(t, x) = 1$ (resp. $w_A(t, x) = 0$) means that the phase (the physical situation of the system) at (t, x) is of pure A (resp. pure B), while $0 < w_A(t, x) < 1$ means that the phase at (t, x) is of mixture. Along the same approach as [**1**, **13**], the system (1.1)–(1.3) can be derived from a free energy functional of Landau-Ginzburg type

$$F_\Omega(\theta; w) := \int_J \left\{ \frac{\kappa\theta}{2} |w_x|^2 + \tau(\theta) + \theta(I_{[0,1]}(w) + \hat{g}(w)) + \lambda(w) \right\} dx \quad \text{for } w \in H^1(J),$$

where $\hat{g}$ is a primitive of g and $\tau(\theta)$ is a smooth function of θ satisfying $\theta = \tau(\theta) - \theta\tau'(\theta)$ $(= \rho(u))$.

An existence-uniqueness result for (PSC) was established in Kenmochi & Niezgódka [**11**]. When ρ is bi-Lipschitz on R, the large time behaviour of the solution $\{u, w\}$ was discussed in Kenmochi & Niezgódka [**10**]; in fact, under some assumptions on the convergences of the data $f(t) \to 0$ and $h_{\pm}(t) \to h_\infty$, it was proved there that $u(t)$ converges to u_∞ $(= h_\infty/n_0 = \text{const.})$ as $t \to +\infty$ and any ω-limit point w_∞ of $w(t)$ is a solution of the stationary problem, denoted by $\mathrm{P}(\sigma_*, \sigma^*; u_\infty, m_0)$:

$$-\kappa w_{\infty xx} + \xi_\infty + g(w_\infty) - \lambda'(w_\infty)u_\infty = v \quad \text{in } J, \tag{1.7}$$

$$\xi_\infty \in \partial I_{[\sigma_*,\sigma^*]}(w_\infty) \quad \text{in } J, \tag{1.8}$$

$$w_{\infty x}(-L) = w_{\infty x}(L) = 0, \tag{1.9}$$

$$\int_J w_\infty(x)\,dx = m_0, \tag{1.10}$$

$$v = \frac{1}{2L}\int_J \{\xi_\infty + g(w_\infty) - \lambda'(w_\infty)u_\infty\}\,dx, \tag{1.11}$$

where $m_0 := \int_J w_0\,dx$.

The purpose of the present paper is to investigate the structure of the solution set of $\mathrm{P}(\sigma_*, \sigma^*; u_\infty, m_0)$ and further some common properties of the ω-limit points of w.

In Shen & Zheng [**14**], the problem without constraint (1.3) was independently studied. They proved existence, uniqueness and asymptotic convergence of the solution. As far as the asymptotic behaviour of the solution as $t \to +\infty$ is concerned, our situation is much more complicated than theirs. Indeed, in our case, the order parameter $w(t, x)$ does not asymptotically converge and may oscillate as $t \to +\infty$, although it is very slow in time; this might come from constraint (1.3).

In particular, when the temperature $\theta = \rho(u)$ is supposed to be constant (hence u to be constant), system (1.2)–(1.3) is called "Cahn-Hilliard model with constraint", which was treated so far in [**2**, **12**]. This model was introduced as the quench limit of temperature $\theta \downarrow 0$ and studied in the case of $g(w) - \lambda'(w)u \equiv -cw$ with a positive constant c by Blowey & Elliott [**2**] and a more general case by Kenmochi, Niezgódka & Pawlow [**12**]. In [**2**], the expression of any solution to the corresponding stationary problem was obtained. In this paper, assuming $m_0 = 0$, we shall show that this type of expression of solutions still holds in our non-isothermal setting, even though nonlinear term $g(w) - \lambda'(w)u$ is of general N-shape in w. Furthermore, the structure of the solution set of $\mathrm{P}(\sigma_*, \sigma^*, u_\infty, 0)$ and the ω-limit set of w will be more precisely studied. This paper gives not only some generalizations but also improvements of results [**2**] to the non-isothermal case.

We refer to [**8**, **15**, **16**, **17**] for related works to the Cahn-Hilliard equation without constraints, and to [**3**, **4**, **7**] for the phase field model with constraint.

NOTATIONS. For simplicity we use the following notations:

$H^1(J)$: the usual Sobolev space with norm $|\cdot|_{H^1(J)}$ given by

$$|z|_{H^1(J)} = \{|z_x|^2_{L_2(J)} + n_0(|z(-L)|^2 + |z(L)|^2)\}^{1/2};$$

$H^1(J)^*$: the dual space of $H^1(J)$;

$(\cdot,\cdot)$: the standard inner product in $L^2(J)$;

$\langle\cdot,\cdot\rangle$: the duality pairing between $H^1(J)^*$ and $H^1(J)$;

F: the duality mapping from $H^1(J)$ onto $H^1(J)^*$;

$$a(v,z) := \int_J v_x(x)z_x(x)\,dx \quad \text{for } v,z \in H^1(J);$$

$$k_1 \vee k_2 = \max\{k_1,k_2\}, \quad k_1 \wedge k_2 = \min\{k_1,k_2\}.$$

We denote by $L^2(J)_0$ the Hilbert space

$$\left\{z \in L^2(J); \int_J z\,dx = 0\right\}$$

with the inner product $(\cdot,\cdot)_0$ induced from $(\cdot,\cdot)$, and by $H^1(J)_0$ the space

$$\left\{z \in H^1(J); \int_J z\,dx = 0\right\}$$

with the norm $|\cdot|_{H^1(J)_0}$ given by

$$|z|_{H^1(J)_0} = |z_x|_{L^2(J)}.$$

In this case, the identification of $L^2(J)_0$ with its dual yields that

$$H^1(J)_0 \subset L^2(J)_0 \subset H^1(J)^*_0$$

with compact and densely defined injections, where $H^1(J)^*_0$ is the dual space of $H^1(J)_0$. We define a mapping $\pi_0 : L^2(J) \to L^2(J)_0$ by

$$\pi_0[z](x) := z(x) - \frac{1}{2L}\int_J z(y)\,dy \quad \text{for } x \in J;$$

π_0 is the projection from $L^2(J)$ onto $L^2(J)_0$ and from $H^1(J)$ onto $H^1(J)_0$.

Also, we denote by $\langle\cdot,\cdot\rangle_0$ the duality pairing between $H^1(J)^*_0$ and $H^1(J)_0$, and by F_0 the duality mapping from $H^1(J)_0$ onto $H^1(J)^*_0$; by definition,

$$\langle F_0 v, z\rangle_0 = \int_J v_x z_x\,dx \quad \text{for all } v,z \in H^1(J)_0.$$

2. Global estimate of solutions.

Problem (PSC) is discussed under the following assumptions:

(A1) ρ is a maximal monotone graph in $R \times R$ whose domain $D(\rho)$ and range $R(\rho)$ are open in R, and is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto

$R(\rho)$, and furthermore there are constants $A_0 > 0$ and α with $1 \le \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \ge \frac{A_0|r_1 - r_2|}{|r_1 r_2|^{\alpha} + 1} \quad \text{for all } r_1, r_2 \in D(\rho).$$

(A2) $\kappa > 0$, $n_0 > 0$, σ_* and σ^*, with $\sigma_* < \sigma^*$, are constants.

(A3) $\lambda : R \to R$ is of C^3-function, and $g : R \to R$ is of C^2-function.

(A4) $f \in W^{1,2}_{\text{loc}}(R_+; L^2(J)) \cap L^2(R_+; L^2(J))$ such that

$$\sup_{t \ge 0} |f|_{W^{1,2}(t,t+1;L^2(J))} < +\infty,$$

and $h_+, h_- \in W^{1,2}_{\text{loc}}(R_+)$ such that

$$\sup_{t \ge 0} \{|h_+|_{W^{1,2}(t,t+1)} + |h_-|_{W^{1,2}(t,t+1)}\} < +\infty,$$

and for some constant h_∞

$$h_\pm - h_\infty \in L^2(R_+).$$

(A5) $(h_\pm(t)/n_0) \in \overline{D(\rho)}$ for all $t \ge 0$ and there are positive constants A_1 and A_1' such that

$$\rho(r)(n_0 r - h_\pm(t)) \ge -A_1|r| - A_1' \quad \text{for all } r \in D(\rho) \text{ and all } t \ge 0.$$

(A6) $u_0 \in H^1(J)$ with $\rho(u_0) \in L^2(J)$, and $w_0 \in H^2(J)$ such that

$$w_{0x}(-L) = w_{0x}(L) = 0, \quad \sigma_* \le w_0 \le \sigma^* \quad \text{on } \bar{J},$$

$$2L\sigma_* < m_0 := \int_J w_0(x)\,dx < 2L\sigma^*,$$

and there is $\xi_0 \in L^2(J)$ satisfying

$$\xi_0 \in \partial I_{[\sigma_*,\sigma^*]}(w_0) \quad \text{a.e. in } J, \quad -\kappa w_{0xx} + \xi_0 \in H^1(J).$$

Next, we give a weak variational formulation for (PSC).

DEFINITION 2.1. For $0 < T < +\infty$ a couple $\{u, w\}$ of functions $u : [0,T] \to H^1(J)$ and $w : [0,T] \to H^1(J)$ is called a (weak) solution of (PSC) on $[0,T]$, if the following conditions (w1)–(w4) are fulfilled:

(w1) $u \in L^\infty(0,T; H^1(J))$,

$\rho(u)$ is weakly continuous from $[0,T]$ into $L^2(J)$ with

$$\rho(u)'\left(= \frac{d}{dt}\rho(u)\right) \in L^1(0,T; H^1(J)^*),$$

$w \in L^\infty(0,T; H^1(J)) \cap L^2(0,T; H^2(J))$, $w' \in L^2(0,T; H^1(J)^*)$, $\lambda(w)' \in L^1(0,T; H^1(J)^*)$.

(w2) $\rho(u)(0) = \rho(u_0)$ and $w(0) = w_0$.

(w3) For a.e. $t \in [0, T]$ and all $z \in H^1(J)$,

$$\begin{aligned}&\frac{d}{dt}(\rho(u(t)) + \lambda(w(t)), z) + a(u(t), z)\\&\qquad + (n_0 u(t, -L) - h_-(t))z(-L) + (n_0 u(t, L) - h_+(t))z(L)\\&= (f(t), z).\end{aligned} \tag{2.1}$$

(w4) For a.e. $t \in [0, T]$,

$$w_x(t, -L) = w_x(t, L) = 0,$$

and there is a function $\xi \in L^2(0, T; L^2(J))$ such that

$$\xi \in \partial I_{[\sigma_*, \sigma^*]}(w) \quad \text{a.e. in } (0, T) \times J \tag{2.2}$$

and

$$\frac{d}{dt}(w(t), \eta) + \kappa(w_{xx}(t), \eta_{xx}) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t), \eta_{xx}) = 0 \tag{2.3}$$

for all $\eta \in H^2(J)$ with $\eta_x(-L) = \eta_x(L) = 0$ and a.e. $t \in [0, T]$.

As is easily seen from the above definition, for any solution $\{u, w\}$ of (PSC) on $[0, T]$ it holds that

$$\int_J w(t, x)\, dx = \int_J w_0(x)\, dx \quad \text{for all } t \in [0, T],$$

and

$$(\rho(u) + \lambda(w))' \in L^\infty(0, T; H^1(J)^*).$$

Also, the inequalities "$2L\sigma_* \le \int_J w_0\, dx \le 2L\sigma^*$" are necessary in order for (PSC) to have a solution; if $\int_J w_0\, dx = 2L\sigma_*$ (resp. $2L\sigma^*$), then we see that $w \equiv \sigma_*$ (resp. σ^*).

We say that a couple $\{u, w\}$ of functions $u : R_+ \to H^1(J)$ and $w : R_+ \to H^1(J)$ is a solution of (PSC) on R_+, if it is a solution of (PSC) on $[0, T]$ for every finite $T > 0$.

We now recall an existence-uniqueness result.

Theorem 2.1 (cf. [**11**; Theorem 2.4]). *Assume that* (A1)–(A6) *hold. Then* (PSC) *has one and only one solution $\{u, w\}$ on R_+, and it satisfies that for every finite $T > 0$*

$$\begin{cases} u \in L^2(0, T; H^2(J)), & u' \in L^2(0, T; L^2(J)), \\ w \in L^\infty(0, T; H^2(J)), & w' \in L^\infty(0, T; H^1(J)^*) \cap L^2(0, T; H^1(J)), \\ \xi \in L^\infty(0, T; L^2(J)), & \end{cases} \tag{2.4}$$

where ξ is the function as in (w4) *of Definition* 2.1.

From Definition 2.1 and regularity (2.4) in Theorem 2.1 we see that

$$\rho(u)' \in L^\infty(0, T; H^1(J)^*), \quad \lambda(w)' \in L^2(0, T; L^2(J)) \quad \text{for every } T > 0, \tag{2.5}$$

so that for a.e. $t \in R_+$

$$\rho(u)'(t) + \lambda(w)'(t) + Fu(t) = \tilde{f}(t) \text{ in } H^1(J)^* \tag{2.6}$$

$$\begin{gathered} F_0^{-1}w'(t) + \kappa F_0[\pi_0(w(t))] + \pi_0[\xi(t) + g(w(t)) - \lambda'(w(t))u(t)] = 0 \\ \text{in } H^1(J)_0^* \text{ (actually in } L^2(J)_0), \end{gathered} \tag{2.7}$$

$$\xi(t) \in \partial I_{[\sigma_*,\sigma^*]}(w(t)) \text{ a.e. in } \bar{J}, \tag{2.8}$$

subject to the initial condition (1.6), where $\tilde{f}(t) \in H^1(J)^*$ defined by

$$\langle \tilde{f}(t), z\rangle = (f(t), z) + h_+(t)z(L) + h_-(t)z(-L) \text{ for } z \in H^1(J).$$

As to global estimates for solutions we have:

THEOREM 2.2. *Assume that* (A1)–(A6) *hold and* $u_\infty := h_\infty/n_0 \in D(\rho)$. *Let* $\{u, w\}$ *be the solution of* (PSC) *on* R_+. *Then*:

$$u - u_\infty \in L^2(R_+; H^1(J)), \quad u \in L^\infty(R_+; H^1(J)), \tag{2.9}$$

$$\sup_{t\ge 0} |u'|_{L^2(t,t+1;L^2(J))} < +\infty, \tag{2.10}$$

$$w \in L^\infty(R_+; H^2(J)), \quad w' \in L^\infty(R_+; H^1(J)^*) \cap L^2(R_+; H^1(J)^*) \tag{2.11}$$

and

$$\sup_{t\ge 0} |w'|_{L^2(t,t+1;H^1(J))} < +\infty. \tag{2.12}$$

The global estimates (2.9)–(2.12) can be inferred in the same way as employed in the proof of [**11**; Theorem 2.4]. We give briefly their proofs under the same conditions and notations as in Theorem 2.2.

LEMMA 2.1. $u - u_\infty \in L^2(R_+; H^1(J))$, $w \in L^\infty(R_+; H^1(J))$ *and* $w' \in L^2(R_+; H^1(J)^*)$.

PROOF. Let ρ^* be the primitive of ρ^{-1} such that $\rho^*(\rho(u_\infty)) = 0$. First, multiplying (2.6) by $u(t) - u_\infty$, we have

$$\begin{aligned} &\frac{d}{dt}\left\{\int_J \rho^*(\rho(u(t)))\,dx - (\rho(u(t)) + \lambda(w(t)), u_\infty)\right\} \\ &\quad + (\lambda(w)'(t), u(t)) + \delta_1|u(t) - u_\infty|^2_{H^1(J)} \\ &\le C_{\delta_1}\{|f(t)|^2_{L^2(J)} + |h_+(t) - h_\infty|^2 + |h_-(t) - h_\infty|^2\} \quad \text{for a.e. } t \ge 0, \end{aligned} \tag{2.13}$$

where δ_1 is any small positive number and C_{δ_1} is a positive constant dependent only on δ_1 (and the given data). Next, multiplying (2.7) by $w'(t)$, we have

$$|w'(t)|^2_{H^1(J)_0^*} + \frac{d}{dt}\left\{\frac{\kappa}{2}|w_x(t)|^2_{L^2(J)} + \int_J \hat{g}(w(t))\,dx\right\} = (\lambda(w)'(t), u(t)) \quad \text{for a.e. } t \ge 0, \tag{2.14}$$

where $\hat{g}$ is a primitive of g such that $\hat{g} \geq 0$ on $[\sigma_*, \sigma^*]$. Adding (2.13) and (2.14) yields

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_J \rho^*(\rho(u(t)))\,dx - (\rho(u(t)) + \lambda(w(t)), u_\infty) + \frac{\kappa}{2}|w_x(t)|^2_{L^2(J)} + \int_J \hat{g}(w(t))\,dx\right\} \\
&\quad + |w'(t)|^2_{H^1(J)_0^*} + \delta_1|(u(t) - u_\infty|^2_{H^1(J)} \\
&\leq C_{\delta_1}\{|f(t)|^2_{L^2(J)} + |h_+(t) - h_\infty|^2 + |\, h_-(t) - h_\infty|^2\} \quad \text{for a.e. } t \geq 0.
\end{aligned} \tag{2.15}$$

Here, note that

$$\rho^*(\rho(r)) - \rho(r)u_\infty \geq -\rho(u_\infty)u_\infty \quad \text{for all } r \in D(\rho).$$

Then, from (2.15) we get the required global estimates. ◇

LEMMA 2.2. *There are positive constants K_1 and K_2 such that*

$$\begin{aligned}
&\sup_{s \leq t \leq T}(t-s)|u(t)|^2_{H^1(J)} + \sup_{s \leq t \leq T}(t-s)|w'(t)|^2_{H^1(J)_0^*} + \int_s^T (t-s)|w'_x|^2_{L^2(J)}\,dt \\
&\quad + \int_s^T (t-s)(\rho(u)', u')\,dt \leq K_1 \int_s^T (t-s)\int_J |w'|^2|u|\,dx\,dt + K_2
\end{aligned} \tag{2.16}$$

for all s, T with $0 \leq s \leq T(\leq s+1)$.

PROOF. First, multiply (2.6) by $u'(t)$. Then, for a.e. $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}|u_x(t)|^2_{L^2(J)} + \frac{n_0}{2}(|u(t,-L)|^2 + |u(t,L)|^2) - h_-(t)u(t,-L)\right. \\
&\quad \left. - h_+(t)u(t,L) - (f(t), u(t))\right\} + (\rho(u)'(t), u'(t)) + (\lambda(w)'(t), u'(t)) \\
&\leq |h'_-(t)||u(t,-L)| + |h'_+(t)||u(t,L)| + |f'(t)|_{L^2(J)}|u(t)|_{L^2(J)}.
\end{aligned} \tag{2.17}$$

Next, assuming that w and ξ are smooth in time t, differentiate (2.7) in t and multiply the resultant by $w'(t)$. Then, for a.e. $t \geq 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w'(t)|^2_{H^1(J)_0^*} + \kappa|w'_x(t)|^2_{L^2(J)} + (\xi'(t), w'(t)) \\
&\quad \leq M(g')|w'(t)|^2_{L^2(J)} + (\lambda(w)'(t), u'(t)) + M(\lambda'')\int_J |w'(t)|^2\,|\,u(t)|\,dx,
\end{aligned} \tag{2.18}$$

where $M(g') = \max_{\sigma_* \leq r \leq \sigma^*}|g'(r)|$ and $M(\lambda'') = \max_{\sigma_* \leq r \leq \sigma^*}|\lambda''(r)|$. By the monotonicity of $\partial I_{[\sigma_*, \sigma^*]}$ we have formally

$$(\xi'(t), w'(t)) \geq 0 \quad \text{for a.e. } t \geq 0, \tag{2.19}$$

and besides note the interpolation inequality

$$|z|^2_{L^2(J)_0} \leq c_0|z_x|_{L^2(J)}|z|_{H^1(J)_0^*} \quad \text{for all } z \in H^1(J)_0, \tag{2.20}$$

where c_0 is a positive constant. Add (2.17) and (2.18) multiplied by $(t-s)$ with

$0 \le s \le t \le T(\le s+1)$ and integrate the resultant over $[s, T']$ with any $s \le T' \le T$. Then, by (2.19), (2.20) and the global estimates obtained in Lemma 2.1, we have

$$\begin{aligned}
&(T'-s)\Big\{\frac{1}{2}|u_x(T')|^2_{L^2(J)} + \frac{n_0}{2}(|u(T',-L)|^2 + |u(T',L)|^2) \\
&\quad - h_-(T')u(T',-L) - h_+(T')u(T',L) - (f(T'),u(T')) + \frac{1}{2}|w'(T')|^2_{H^1(J)_0^*}\Big\} \\
&\quad + \int_s^{T'}(t-s)(\rho(u)',u')\,dt + \frac{\kappa}{4}\int_s^{T'}(t-s)|w'_x|^2_{L^2(J)}\,dt \\
&\le \int_s^{T'}(t-s)\int_J |w'|^2|u|\,dx\,dt + K_3\int_s^{T'}\{|u|^2_{H^1(J)} + |w'|^2_{H^1(J)_0^*} \\
&\quad + |h_-|^2 + |h'_-|^2 + |h_+|^2 + |h'_+|^2 + |f|^2_{L^2(J)} + |f'|^2_{L^2(J)}\}\,dt,
\end{aligned} \tag{2.21}$$

where K_3 is a positive constant independent of s, T' and T. It is easy to derive an inequality of the form (2.16) for suitable constants K_1 and K_2 from (2.21) with the help of condition (A4). When w and ξ are not smooth enough, inequalities (2.18) and (2.19) do not hold in general. But, by showing them for approximate solutions of $\{u, w\}$ we can rigorously obtain (2.16); see [**11**; sections 5, 8] in details. ◇

LEMMA 2.3.

$$\begin{aligned}
&\int_s^T (t-s)\int_J |w'|^2|u|\,dx\,dt \\
&\quad \le \delta_2\int_s^T (t-s)|w'_x|^2_{L^2(J)}\,dt + C_{\delta_2}\Big\{\sup_{s\le t\le T}(t-s)|w'(t)|^2_{H^1(J)_0^*}\Big\}\int_s^T |u|^2_{H_1(J)}\,dt
\end{aligned} \tag{2.22}$$

for all $0 \le s \le T(\le s+1)$,

where δ_2 *is an arbitrary positive number and* C_{δ_2} *is a positive constant dependent only on* δ_2.

PROOF. With a positive constant c'_0 satisfying

$$|z|_{L^\infty(J)} \le c'_0|z|_{H^1(J)} \quad \text{for all } z \in H^1(J),$$

we observe that

$$\begin{aligned}
&\int_s^T\int_J (t-s)|w'|^2|u|\,dx\,dt \\
&\le c_0c'_0\int_s^T (t-s)|u|_{H^1(J)}|w'|_{H^1(J)_0^*}|w'_x|_{L^2(J)}\,dt \quad \text{(cf. (2.20))} \\
&\le c_0c'_0\Big\{\sup_{s\le t\le T}(t-s)^{1/2}|w'(t)|_{H^1(J)_0^*}\Big\}\int_s^T (t-s)^{1/2}|u|_{H^1(J)}|w'_x|_{L^2(J)}\,dt
\end{aligned}$$

$$\leq c_0 c_0' \left\{ \sup_{s \leq t \leq T} (t-s)^{1/2} |w'(t)|_{H^1(J)_0^*} \right\} \left\{ \int_s^T |u|^2_{H^1(J)} \, dt \right\}^{1/2} \left\{ \int_s^T (t-s) |w'_x|^2_{L^2(J)} \, dt \right\}^{1/2}$$

$$\leq \delta_2 \int_s^T (t-s) |w'_x|^2_{L^2(J)} \, dt + \frac{(c_0 c_0')^2}{4\delta_2} \left\{ \sup_{s \leq t \leq T} (t-s) |w'(t)|^2_{H^1(J)_0^*} \right\} \int_s^T |u|^2_{H^1(J)} \, dt$$

for any positive δ_2. Hence (2.22) holds for $C_{\delta_2} := \dfrac{(c_0 c_0')^2}{4\delta_2}$. ◇

Proof of Theorem 2.2. Choose a small positive number δ_2 so that

$$K_1 \delta_2 \leq \frac{1}{2},$$

and then for this δ_2 choose a small positive number ε_0 so that

$$K_1 C_{\delta_2} \int_s^{s+\varepsilon_0} |u|^2_{H^1(J)} \, dt \leq \frac{1}{2} \quad \text{for all } s \in R_+,$$

which is possible by Lemma 2.1. For such δ_2 and ε_0 it follows from Lemmas 2.2 and 2.3 that

$$\sup_{s \leq t \leq T} (t-s) |u(t)|^2_{H^1(J)} + \frac{1}{2} \sup_{s \leq t \leq T} (t-s) |w'(t)|^2_{H^1(J)_0^*}$$
$$+ \frac{1}{2} \int_s^T (t-s) |w'_x|^2_{L^2(J)} \, dt + \int_s^T (t-s) (\rho(u)', u') \, dt \leq K_2$$

for all s, T with $0 \leq s \leq T \leq s + \varepsilon_0$. Therefore, the arbitrariness of s and T together with Theorem 2.1 implies that $u \in L^\infty(R_+; H^1(J))$, $w' \in L^\infty(R_+; H^1(J)_0^*)$,

$$\sup_{t \geq 0} |w'_x|_{L^2(t,t+1;L^2(J))} < +\infty$$

and

$$\sup_{t \geq 0} |u'|_{L^2(t,t+1;L^2(J))} < +\infty \quad \text{(cf. condition (A1))}.$$

Thus the global estimates (2.9)–(2.12), except $w \in L^\infty(R_+; H^2(J))$, have been obtained. The estimate $w \in L^\infty(R_+; H^2(J))$ follows immediately from the facts that

$$-\kappa w_{xx}(t) + \pi_0(\xi(t)) = l(t) := -F_0^{-1} w'(t) - \pi_0[g(w(t)) - \lambda'(w(t)) u(t)]$$

and $l \in L^\infty(R_+; L^2(J))$. ◇

To study the large time behaviour of the order parameter w, let us consider the following stationary problem, denoted by $\mathrm{P}(\sigma_*, \sigma^*; u_\infty, m_0)$:

$$-\kappa v_{xx} + \gamma + g(v) - \lambda'(v) u_\infty = \nu \quad \text{a.e. in } J, \tag{2.23-1}$$

$$\gamma \in L^2(J), \ \gamma \in \partial I_{[\sigma_*, \sigma^*]}(v) \quad \text{a.e. in } J, \tag{2.23-2}$$

$$v_x(-L) = v_x(L) = 0, \tag{2.23-3}$$

$$v = \frac{1}{2L}\int_J \{\gamma + g(v) - \lambda'(v)u_\infty\}\,dx, \tag{2.23-4}$$

$$\int_J v\,dx = m_0. \tag{2.23-5}$$

We say that v is a solution of $P(\sigma_*, \sigma^*; u_\infty, m_0)$, if $v \in H^2(J)$ and (2.23-1)–(2.23-5) are satisfied.

As an easy consequence of Theorem 2.2 we have the following theorem.

Theorem 2.3. *Under the same assumptions as in Theorem 2.2, the following statements hold*:

(a) $u(t) \to u_\infty \left(= \frac{h_\infty}{n_0}\right)$ *weakly in* $H^1(J)$ *as* $t \to +\infty$.

(b) *The* ω*-limit set* $\omega(u_0, w_0) := \{z \in H^1(J); w(t_n) \to z$ *in* $H^1(J)$ *(as* $n \to \infty$*) for some* $t_n \uparrow +\infty\}$ *is non-empty, compact and connected in* $H^1(J)$. *Also* $\omega(u_0, w_0)$ *is bounded in* $H^2(J)$.

(c) $\lim_{t\to+\infty}\left\{\frac{\kappa}{2}|w_x(t)|^2_{L^2(J)} + \int_J (\hat{g}(w(t)) - \lambda(w(t))u_\infty)\,dx\right\}$ *exists, where* $\hat{g}$ *is any primitive of* g.

(d) *Any* ω*-limit point* $v \in \omega(u_0, w_0)$ *is a solution of* $P(\sigma_*, \sigma^*; u_\infty, m_0)$.

Proof. (a) is an immediate consequence of (2.9) and (2.10). Hence $u(t,\cdot) \to u_\infty$ uniformly on $\bar{J}$ as $t \to +\infty$. This means that the closure of $\{u(t,x); x \in \bar{J}, t \geq M\}$ for sufficiently large $M > 0$ is contained in $D(\rho)$, so that

$$\sup_{t\geq M} |\rho(u)'|_{L^2(t,t+1;L^2(J))} < +\infty \quad \text{and} \quad \rho(u) \in L^\infty([M,+\infty) \times \bar{J}).$$

With the help of these estimates, (b) and (d) can be proved in a way similar to that of [**10**; Theorem 3.1]. Moreover, (c) follows from (2.15). ◇

3. Stationary problem.

In this section, assumptions (A1), (A2) and (A3) are always fulfilled, and u_∞ is a given constant. Under some restrictions on u_∞, g and λ we consider the stationary problem with $m_0 = 0$, namely $P(\sigma_*, \sigma^*; u_\infty, 0)$ which is very restricted, but still interesting from the physical point of view.

We denote by S^* the solution set of $P := P(\sigma_*, \sigma^*; u_\infty, 0)$. The structure of S^* is to be investigated in the following decomposition:

$$S^* = S_c + S_0 + S_1,$$

where

$$S_c := \{v; v \text{ is a constant solution of P}\},$$

$$S_0 := \{v \in H^2(J);\quad v \text{ is a non-constant solution of P such that } \sigma_* < v < \sigma^* \text{ on } \bar{J}\}$$

and

$$S_1 := \{v \in H^2(J);\quad v \text{ is a non-constant solution of P such that } v(x) = \sigma_* \text{ or } \sigma^* \text{ for some } x \in \bar{J}\}.$$

In the study of P we suppose that the function

$$q(v) := g(v) - \lambda'(v)u_\infty, \quad v \in R,$$

satisfies the following properties (q1), (q2) and (q3):

(q1) (oddness) $q(v) = -q(-v)$ for all $v \in R$.

(q2) (N-shape condition) There are points ζ_{0-}, ζ_M, ζ_m and ζ_{0+} such that

$$\zeta_{0-} < \zeta_M < 0 < \zeta_m < \zeta_{0+},$$

$$q(\zeta_{0-}) = q(\zeta_{0+}) = 0,$$

$$q'\left(= \frac{d}{dv}q\right) > 0 \quad \text{on } (-\infty, \zeta_M) \cup (\zeta_m, +\infty),$$

$$q' < 0 \quad \text{on } (\zeta_M, \zeta_m),$$

and

$$q'(\zeta_M) = q'(\zeta_m) = 0.$$

(q3) (convexity-concavity) $q'' \geq 0$ on $(0, +\infty)$, $q''(0) = 0$, $q'' \leq 0$ on $(-\infty, 0)$ and q'' is not identically zero on any neighbourhood of 0.

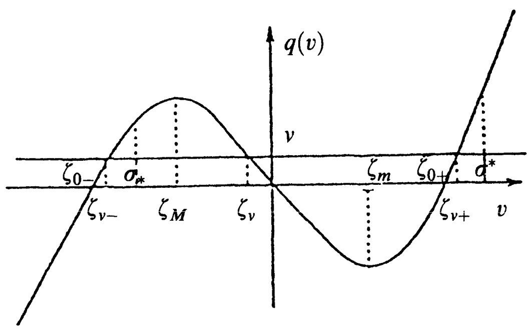

Figure 1

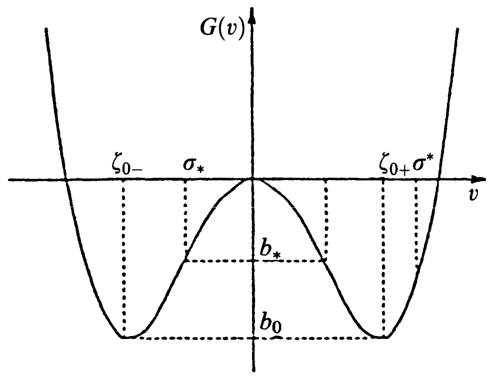

Figure 2

EXAMPLE 3.1. As physically relevant examples of $q(v)$ we consider the following two cases: (i) $q(v) := v(v^2 + u_\infty)$ which is given by $g(v) = v^3$, $\lambda(v) = -(1/2)v^2$ with a negative constant u_∞; (ii) $q(v) := v(v^2 - 1)$ which is given by $g(v) = v(v^2 - 1)$ and $\lambda(v) = v$ with $u_\infty = 0$. Clearly, both cases satisfy the above conditions. The case when $q(v) := -v$ is excluded, since $q'' \equiv 0$. This case was treated in [**2**] for the Cahn-Hilliard model.

We further suppose that

$$\sigma_* < 0 < \sigma^*. \tag{3.1}$$

We define $G : R \to R$ with the primitive $\hat{g}$ of g with $\hat{g}(0) = 0$ by

$$G(v) := \hat{g}(v) - \lambda(v)u_\infty + \lambda(0)u_\infty \left(= \int_0^v q(s)\, ds \right)$$

and set

$$b_0 := G(\zeta_{0-})(= G(\zeta_{0+})), \quad b_* := \max\{G(\zeta_{0-} \vee \sigma_*), G(\zeta_{0+} \wedge \sigma^*)\}.$$

Clearly, by (q1), (q2) and (3.1),

$$b_0 \le b_* < 0.$$

For each $v \in R$ with $q(\zeta_m) < v < q(\zeta_M)$, the (algebraic) equation $q(\eta) = v$ has exactly three roots $\eta = \zeta_{v-}, \zeta_v, \zeta_{v+}$ with $\zeta_{v-} < \zeta_v < \zeta_{v+}$ and $\zeta_{v-} < 0 < \zeta_{v+}$. Also, for $v = q(\zeta_m)$ (resp. $v = q(\zeta_M)$), the equation $q(\eta) = v$ has exactly two roots $\eta = \zeta_{v-}, \zeta_m$ with $\zeta_{v-} < 0 < \zeta_m$ (resp. $\eta = \zeta_M, \zeta_{v+}$ with $\zeta_M < 0 < \zeta_{v+}$). With these points, we define

$$b_{0v} := \max\{G(\zeta_{v-}) - v\zeta_{v-}, G(\zeta_{v+}) - v\zeta_{v+}\},$$

$$b_{*v} := \max\{G(\zeta_{v-} \vee \sigma_*) - v(\zeta_{v-} \vee \sigma^*), G(\zeta_{v+} \wedge \sigma^*) - v(\zeta_{v+} \wedge \sigma^*)\}.$$

for v with $q(\zeta_m) \le v \le q(\zeta_M)$.

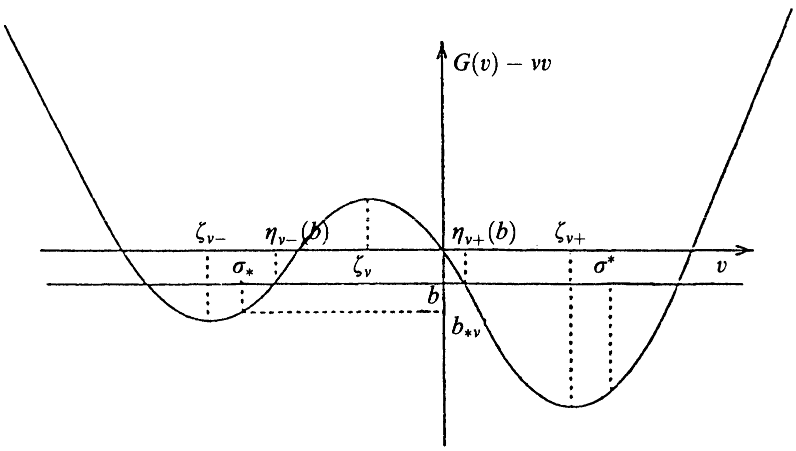

Figure 3

Besides, given v with $q(\zeta_m) < v < q(\zeta_M)$ and $b \in R$, we see that

(1) if $b_{0v} \le b < G(\zeta_v) - v\zeta_v$, then $G(\eta) - v\eta = b$ has exactly two roots $\eta = \eta_{v-}(b)$, $\eta_{v+}(b)$ in $[\zeta_{v-}, \zeta_{v+}]$ with $\eta_{v-}(b) < \eta_{v+}(b)$;

(2) if $b = G(\zeta_v) - v\zeta_v$, then $G(\eta) - v\eta = b$ has exactly one root $\eta = \zeta_v$ in $[\zeta_{v-}, \zeta_{v+}]$.

By definition any solution v $(\in H^2(-L, L))$ of P satisfies

$$-\kappa v_{xx} + \gamma + q(v) = \nu \quad \text{a.e. in } J, \tag{3.2-1}$$

$$\gamma \in L^2(J), \gamma \in \partial I_{[\sigma_*,\sigma^*]}(v) \quad \text{a.e. in } J, \tag{3.2-2}$$

$$v_x(-L) = v_x(L) = 0, \tag{3.2-3}$$

$$\nu = \frac{1}{2L} \int_J \{\gamma + q(v)\}\, dx, \tag{3.2-4}$$

$$\int_J v\, dx = 0. \tag{3.2-5}$$

Lemma 3.1. *Let v be any solution of* P *and*

$$\nu := \frac{1}{2L} \int_J \{\gamma + q(v)\}\, dx \quad \text{and} \quad b := G(v(-L)) - \nu v(-L)$$

Then:

(i) *$G(v(x)) - \nu v(x) \geq b$ for all $x \in \bar{J}$. $v_x(x) = 0$ if and only if $G(v(x)) - \nu v(x) = b$; hence $G(v(L)) - \nu v(L) = b$.*

(ii) *If v is constant on $\bar{J}$, then $v \equiv 0$ on $\bar{J}$. In this case, $\nu = b = 0$.*

(iii) *If v is non-constant on $\bar{J}$, then*

$$q(\zeta_m) < \nu < q(\zeta_M), \tag{3.3}$$

$$b_{*\nu} \leq b < G(\zeta_\nu) - \nu\zeta_\nu, \tag{3.4}$$

and

$$\eta_{\nu-}(b) \leq v \leq \eta_{\nu+}(b) \quad \text{on } \bar{J}. \tag{3.5}$$

Proof. Multiplying (3.2-1) by v_x and integrating it over $[-L, x]$, we have

$$-\frac{\kappa}{2}|v_x(x)|^2 + G(v(x)) - \nu v(x) = b \quad \text{for all } x \in \bar{J}. \tag{3.6}$$

Hence (i) holds.

Assume v is constant on $\bar{J}$. Then, by (3.2-5), $v \equiv 0$ must hold. Since $\sigma_* < 0 < \sigma^*$, $\partial I_{[\sigma_*,\sigma^*]}(0) = \{0\}$ and $q(0) = 0$, it follows that $\gamma \equiv 0$ on $\bar{J}$, $\nu = 0$ and $b = 0$. Thus (ii) holds.

In the rest of the proof we assume that v is non-constant on $\bar{J}$.

Now, assume $\nu \geq q(\zeta_M)$. In this case, there is exactly one point η_0 ($> \zeta_{0+}$) such that

$$G(r) - \nu r \text{ is strictly decreasing for } r \in (-\infty, \eta_0],$$

$$G(r) - \nu r \geq G(\eta_0) - \nu\eta_0 \quad \text{for all } r \in R,$$

and

$$G(r) - \nu r \text{ is strictly increasing for } r \in [\eta_0, +\infty).$$

Hence, it follows from this together with (i) that either (α) or (β) below hold:

(α) $v(-L) \le \eta_0$ and v is decreasing on $\bar{J}$.

(β) $v(-L) \ge \eta_0$ and v is increasing on $\bar{J}$.

In both cases (α) and (β) we have $v_x \ne 0$ on $(-L, L]$ which contradicts the boundary condition $v_x(L) = 0$. Therefore we obtain $\nu < q(\zeta_M)$. Similarly $\nu > q(\zeta_m)$. Thus (3.3) holds.

Next we show (3.4) by contradiction. Assuming that

$$b_{*\nu} > b \ge \min_{r \in R} \{G(r) - \nu r\},$$

we consider for instance the case where

$$\zeta_{\nu-} \vee \sigma_* = \sigma_*, \quad \zeta_{\nu+} \wedge \sigma^* = \zeta_{\nu+}, \quad b_{*\nu} = G(\sigma_*) - \nu\sigma_*. \tag{3.7}$$

In this case, the equation $G(\eta) - \nu\eta = b$ has at most two roots $\eta = \eta_1, \eta_2$ in the interval $[\sigma_*, \sigma^*]$ with $\zeta_\nu < \eta_1 \le \zeta_{\nu+} \le \eta_2$. This implies that $v(-L) = \eta_1$ or η_2. On account of (i) there are a point $x_1 \in \bar{J}$ and a constant $\delta > 0$ such that

$$v \le \eta_1 - \delta \text{ (hence } v_x < 0) \text{ on } [x_1, L]$$

or

$$v \ge \eta_2 + \delta \text{ (hence } v_x > 0) \text{ on } [x_1, L].$$

In both cases we have $v_x(L) \ne 0$ which is a contradiction. Therefore $b \ge b_{*\nu}$ must hold true. Assuming that $b \ge G(\zeta_\nu) - \nu\zeta_\nu$, we have a similar contradiction; in particular, if $b = G(\zeta_\nu) - \nu\zeta_\nu$ and $v(-L) = \zeta_\nu$, then (i) implies $v \equiv \zeta_\nu$, so that this case is excluded. Thus (3.4) holds under (3.7), and it holds true in any other possible cases of $\zeta_{\nu-} \vee \sigma_*$, $\zeta_{\nu+} \wedge \sigma^*$ and $b_{*\nu}$.

Finally we show (3.5). By (3.3) and (3.4), the equation $G(\eta) - \nu\eta = b$ has at most four roots $\eta = \tilde{\eta}_{\nu\pm}(b)$, $\eta_{\nu\pm}(b)$ in $[\sigma_*, \sigma^*]$ such that

$$\tilde{\eta}_{\nu-}(b) \le \zeta_{\nu-} \le \eta_{\nu-}(b) < \zeta_\nu < \eta_{\nu+}(b) \le \zeta_{\nu+} \le \tilde{\eta}_{\nu+}(b).$$

We have by (i) the following three possibilities:

$$v \le \tilde{\eta}_{\nu-}(b) \quad \text{on } \bar{J}, \tag{3.8}$$

$$v \ge \tilde{\eta}_{\nu+}(b) \quad \text{on } \bar{J}, \tag{3.9}$$

and

$$\eta_{\nu-}(b) \le v \le \eta_{\nu+}(b) \quad \text{on } \bar{J}. \tag{3.10}$$

In a way similar to those in the proofs of (3.3) and (3.4), we can show that (3.8) and (3.9) are impossible. Accordingly (3.10) must hold true. ◇

Lemma 3.2. *Let* $q(\zeta_m) < \nu < q(\zeta_M)$ *and* $b_{0\nu} < b < G(\zeta_\nu) - \nu\zeta_\nu$. *Define* $I_0(\nu, b)$ *and* $I_1(\nu, b)$ *by*

$$I_0(\nu,b) = \left(\frac{\kappa}{2}\right)^{1/2} \int_{\eta_{\nu-}(b)}^{\eta_{\nu+}(b)} \frac{1}{\{G(v) - \nu v - b\}^{1/2}} dv \tag{3.11}$$

and

$$I_1(\nu,b) = \left(\frac{\kappa}{2}\right)^{1/2} \int_{\eta_{\nu-}(b)}^{\eta_{\nu+}(b)} \frac{v}{\{G(v) - \nu v - b\}^{1/2}} dv. \tag{3.12}$$

Then:

(1) *For any fixed* ν, $I_0(\nu,b)$ *is continuous and strictly decreasing in* b, *and*

$$\lim_{b\uparrow G(\zeta_\nu)-\nu\zeta_\nu} I_0(\nu,b) = (\kappa)^{1/2} \frac{\pi}{|q'(\zeta_\nu)|^{1/2}}, \tag{3.13}$$

$$\lim_{b\downarrow b_{0\nu}} I_0(\nu,b) = +\infty. \tag{3.14}$$

(2) *For any fixed* ν, $I_1(\nu,b)$ is continuous in b, *and for all* b

$$\begin{cases} I_1(\nu,b) > 0 & \text{if } \nu < 0, \\ I_1(0,b) = 0, & \\ I_1(\nu,b) < 0 & \text{if } \nu > 0. \end{cases} \tag{3.15}$$

Proof. (1) is due to [**6**; section 5]. We prove here (2). For the continuity (and some further regularity) of I_1 we refer to [**5**; section 4]. We show (3.15) below. By the oddness of q, it is clear that $I_1(0,b) = 0$ for all b.

Assume $\nu > 0$. We employ a new variable y, $0 \le y \le 1$, of integration in (3.12), which is given by

$$y^2 = \frac{G(v) - \nu v - G(\zeta_\nu) + \nu\zeta_\nu}{b - G(\zeta_\nu) + \nu\zeta_\nu}, \quad 0 \le y \le 1; \tag{3.16}$$

for each y, (3.16) has two roots $v = v_1(\nu,b,y)$ and $v = v_2(\nu,b,y)$ in $[\eta_{\nu-}(b), \eta_{\nu+}(b)]$ such that

$$v_1(\nu,b,y) \in [\eta_{\nu-}(b), \zeta_\nu], \quad v_2(\nu,b,y) \in [\zeta_\nu, \zeta_{\nu+}(b)].$$

With this new variable y, $I_1(\nu,b)$ can be written in the form

$$I_1(\nu,b) = \{2\kappa(G(\zeta_\nu) - \nu\zeta_\nu - b)\}^{1/2} \int_0^1 \frac{y}{\{1-y^2\}^{1/2}} \left[\frac{v_1}{q(v_1) - \nu} - \frac{v_2}{q(v_2) - \nu}\right] dy,$$

where $v_i(y) := v_i(\nu,b,y)$, $i = 1,2$. To see $I_1(\nu,b) < 0$, it is enough to verify that

$$Q(y) := \frac{v_1(y)}{q(v_1) - \nu} - \frac{v_2(y)}{q(v_2) - \nu} \le 0 \quad \text{for } 0 < y < 1$$

and $Q \not\equiv 0$. Since $q(v_1) - v > 0$, $v_1 < 0$ and $q(v_2) - v < 0$, it follows that

$$Q(y) < 0 \quad \text{for } y \text{ with } v_2(y) \le 0$$

and hence $Q \not\equiv 0$. For any y with $v_2(y) > 0$ we see from the relation $v_1(y) < -v_2(y)$ and condition (q3) that

$$\frac{q(v_1(y))}{v_1(y)} \ge \frac{q(v_2(y))}{v_2(y)}, \text{ i.e. } v_1(y)q(v_2(y)) \ge v_2(y)q(v_1(y)),$$

so that

$$Q(y) = \frac{1}{(q(v_1) - v)(q(v_2) - v)}\{v_1q(v_2) - v_2q(v_1) - v(v_1 - v_2)\} \le 0.$$

Thus $I_1(v,b) < 0$, if $v > 0$. Similarly $I_1(v,b) > 0$, if $v < 0$. Accordingly (2) holds.

◇

4. Expression of non-constant solutions.

We keep assumptions (A1)–(A3), (3.1), (q1)–(q3) and $m_0 = 0$ in this section, too.

To give a general expression for non-constant solutions of $\mathrm{P} := \mathrm{P}(\sigma_*, \sigma^*; u_\infty, 0)$, for given v and b with

$$q(\zeta_m) < v < q(\zeta_M), \quad b_{0v} \le b < G(\zeta_v) - v\zeta_v,$$

we consider the auxiliary problem

$$-\kappa V_{xx} + q(V) = v \quad \text{in } R, \tag{4.1-1}$$

$$V(0) = \zeta_v, \quad V_x(0) = \left\{\frac{2}{\kappa}(G(\zeta_v) - v\zeta_v - b)\right\}^{1/2}. \tag{4.1-2}$$

By the well-known theory on ODEs, problem (4.1) has a unique solution $V \in C^4(R)$. It is easy to see that there exists a compact interval $[\tau, \tau_1]$ with $\tau := \tau(v,b) < 0 < \tau_1 := \tau_1(v,b)$ such that

$$\begin{cases} V(\tau) = \eta_{v-}(b), \quad V(\tau_1) = \eta_{v+}(b), \quad V(2\tau_1 - \tau) = \eta_{v-}(b), \\ V_x(\tau) = V_x(\tau_1) = V_x(2\tau_1 - \tau) = 0, \\ V_x > 0 \quad \text{on } (\tau, \tau_1), \quad V_x < 0 \quad \text{on } (\tau_1, 2\tau_1 - \tau), \end{cases} \tag{4.2}$$

and V is periodic with period $2(\tau_1 - \tau)$ on R and V is symmetric with respect to $x = \tau$ and $x = \tau_1$, i.e.

$$V(\tau - x) = V(\tau + x) \text{ and } V(\tau_1 - x) = V(\tau_1 + x) \quad \text{for all } x \in R.$$

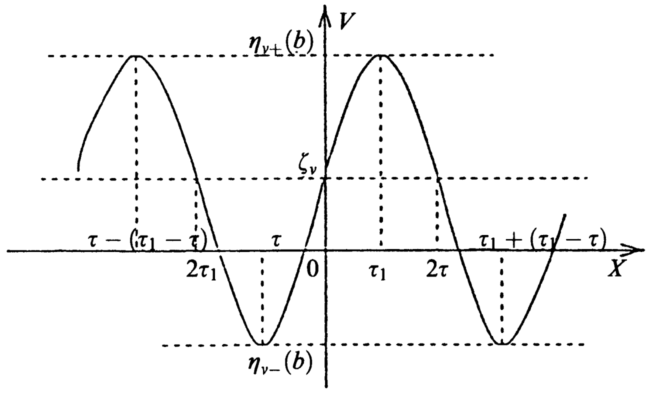

Figure 4

Moreover, we prove:

LEMMA 4.1. (cf. [**5**, **6**]) *Let V be the solution of* (4.1) *under the same assumptions on ν and b as above. Also, let $\tau = \tau(\nu, b)$ and $\tau_1 = \tau_1(\nu, b)$ be as above. Then,*

$$\tau_1 - \tau = I_0(\nu, b) \tag{4.3}$$

and

$$\int_\tau^{\tau_1} V\,dx = I_1(\nu, b). \tag{4.4}$$

In the sequel we denote by $V^{\nu,b}$ the solution V of problem (4.1), and by $\overline{V}^{\nu,b}$ the function $V(x + 2\tau_1)$; we put $\bar{\tau}(\nu, b) = -\tau_1(\nu, b)$ and $\bar{\tau}_1(\nu, b) = -\tau(\nu, b)$. The function $V^{\nu,b}$ and $\overline{V}^{\nu,b}$ are called the principal parts of the solution to our problem P.

LEMMA 4.2. *Let v be any non-constant solution of* P, *and ν, γ be the corresponding number and function in system* (3.2). *Also, put $b = G(v(-L)) - \nu v(-L)$. Let (x_L, x_R) be any connected component of the set $\{x \in \bar{J}; v_x(x) \neq 0\}$. Then, on (x_L, x_R), v coincides with a translation of $V^{\nu,b}$ or $\overline{V}^{\nu,b}$ in the space-variable x; more precisely, there is a number x_0 in (x_L, x_R) such that*

$$\begin{cases} x_L = \tau(\nu, b) + x_0, \quad x_R = \tau_1(\nu, b) + x_0, \\ v(x) = V^{\nu,b}(x - x_0) \quad \text{for all } x \in [x_L, x_R], \end{cases} \tag{4.7}$$

or

$$\begin{cases} x_L = \bar{\tau}(\nu, b) + x_0, \quad x_R = \bar{\tau}_1(\nu, b) + x_0, \\ v(x) = \overline{V}^{\nu,b}(x - x_0) \quad \text{for all } x \in [x_L, x_R]. \end{cases} \tag{4.8}$$

Moreover,

$$x_R - x_L = I_0(\nu, b), \quad \int_{x_L}^{x_R} v\,dx = I_1(\nu, b). \tag{4.9}$$

PROOF. Note that $v(x_L) = \eta_{\nu-}(b)$ or $\eta_{\nu+}(b)$, and $v_x(x_L) = v_x(x_R) = 0$. Assuming $v(x_L) = \eta_{\nu-}(b)$, we have $v_x > 0$ on (x_L, x_R), which shows that $\sigma_* < v < \sigma^*$ on (x_L, x_R), hence $\gamma = 0$ on (x_L, x_R). Therefore v satisfies

$$\begin{cases} -\kappa v_{xx} + q(v) = \nu & \text{in } (x_L, x_R), \\ v(x_L) = \eta_{\nu-}(b), \quad v_x(x_L) = 0. \end{cases}$$

From the uniqueness result for Cauchy problems of ODEs and (4.2) it follows that (4.7) holds for $x_0 = x_L - \tau(\nu, b)$. Also, (4.9) is a direct consequence of Lemma 4.1. In the case of $v(x_L) = \eta_{\nu+}(b)$ we have (4.8) with (4.9). ◇

According to Lemma 4.2 we have a general expression for a non-constant solution v of P as follows. Let ν be the corresponding number and $b = G(v(-L)) - \nu v(-L)$. Then there is a partition

$$-L = x^0 \le x_L^1 < x_R^1 \le x_L^2 < x_R^2 \le x_L^3 < \cdots \le x_L^l < x_R^l \le x^{l+1} = L \tag{4.10}$$

of the interval $\bar{J}$ such that

$$x_R^i = x_L^i + I_0(\nu, b), \quad i = 1, 2, \ldots, l, \tag{4.11}$$

and one of the following (4.12) and (4.13) holds:

$$\begin{cases} v = \eta_{\nu-}(b) & \text{on } [-L, x_L^1], \\ v = V^{\nu,b}(\tau - x_L^1 + \cdot) & \text{on } (x_L^1, x_R^1), \\ v = \eta_{\nu+}(b) & \text{on } [x_R^1, x_L^2], \\ v = \overline{V}^{\nu,b}(\bar{\tau} - x_L^2 + \cdot) & \text{on } (x_L^2, x_R^2), \\ v = \eta_{\nu-}(b) & \text{on } [x_R^2, x_L^3], \\ \cdots\cdots & \\ v = \begin{Bmatrix} V^{\nu,b}(\tau - x_L^l + \cdot) & \text{if } l \text{ is odd,} \\ \overline{V}^{\nu,b}(\bar{\tau} - x_L^l + \cdot) & \text{if } l \text{ is even,} \end{Bmatrix} & \text{on } (x_L^l, x_R^l), \\ v = \begin{Bmatrix} \eta_{\nu+}(b) & \text{if } l \text{ is odd,} \\ \eta_{\nu-}(b) & \text{if } l \text{ is even,} \end{Bmatrix} & \text{on } [x_R^l, L]; \end{cases} \tag{4.12}$$

$$
\begin{cases}
v = \eta_{v+}(b) & \text{on } [-L, x_L^1], \\
v = \overline{V}^{v,b}(\bar{\tau} - x_L^1 + \cdot) & \text{on } (x_L^1, x_R^1), \\
v = \eta_{v-}(b) & \text{on } [x_R^1, x_L^2], \\
v = V^{v,b}(\tau - x_L^2 + \cdot) & \text{on } (x_L^2, x_R^2), \\
v = \eta_{v+}(b) & \text{on } [x_R^2, x_L^3], \\
\quad \cdots\cdots & \\
v = \left.\begin{cases} \overline{V}^{v,b}(\bar{\tau} - x_L^l + \cdot) & \text{if } l \text{ is odd}, \\ V^{v,b}(\tau - x_L^l + \cdot) & \text{if } l \text{ is even}, \end{cases}\right\} & \text{on } (x_L^l, x_R^l), \\
v = \left.\begin{cases} \eta_{v-}(b) & \text{if } l \text{ is odd}, \\ \eta_{v+}(b) & \text{if } l \text{ is even}, \end{cases}\right\} & \text{on } [x_R^l, L].
\end{cases}
\tag{4.13}
$$

Besides, with the notations

$$J_1 := [-L, x_L^1] \cup [x_R^2, x_L^3] \cup [x_R^4, x_L^5] \cup \cdots \tag{4.14-1}$$

and

$$J_2 := [x_R^1, x_L^2] \cup [x_R^3, x_L^4] \cup [x_R^5, x_L^6] \cup \ldots, \tag{4.14-2}$$

we have

$$lI_0(v, b) + |J_1| + |J_2| = 2L \tag{4.15}$$

and

$$lI_1(v, b) + \eta_{v-}(b)|J_1| + \eta_{v+}(b)|J_2| = 0 \quad \text{in the case of (4.12)}, \tag{4.16-1}$$

$$lI_1(v, b) + \eta_{v+}(b)|J_1| + \eta_{v-}(b)|J_2| = 0 \quad \text{in the case of (4.13)}, \tag{4.16-2}$$

where $|J_i|$, $i = 1, 2$, stands for the linear measure of J_i; (4.16) comes from the constraint (3.2-5).

For simplicity expression (4.12) (resp. (4.13)) with {(4.10), (4.11), (4.14), (4.15), (4.16-1) (resp. (4.16-2))} is called of T(I) (resp. T(II)).

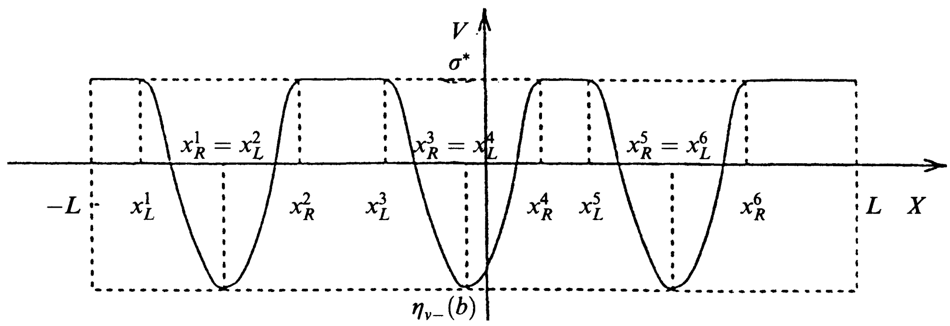

Figure 5

We end this section with the expression of the solution to the problem without constraint:

$$\begin{cases} -\kappa v_{xx} + q(v) = \nu \quad \text{in } J, \\ v_x(-L) = v_x(L) = 0, \\ \nu = \dfrac{1}{2L}\displaystyle\int_J q(v)\,dx, \quad \int_J v\,dx = 0; \end{cases} \tag{4.17}$$

the solution v is considered in $C^4(\bar{J})$. We denote by $\tilde{S}_0$ the set of all non-constant solutions of (4.17). Then we have:

Lemma 4.3. (1) *$v \equiv 0$ is the only constant solution of* (4.17).

(2) *If v is a non-constant solution of* (4.17), *that is $v \in \tilde{S}_0$, then*

$$\nu = 0 \text{ and } b_0 < G(v(-L)) < 0.$$

(3) *$\tilde{S}_0$ is a finite set, i.e. $\tilde{S}_0 := \{\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_p\}$, where $\tilde{V}_i$, $i = 1, 2, \dots, p$, are non-constant solutions of* (4.17) *such that $\tilde{V}_1(-L) < \tilde{V}_2(-L) < \cdots < \tilde{V}_p(-L)$.*

(4) *Let v be any non-constant solution of* (4.17). *Then*

$$v = V^{0,b}(\tau + L + \cdot) \text{ or } \overline{V}^{0,b}(\bar{\tau} + L + \cdot) \quad \text{on } \bar{J}, \tag{4.18}$$

where $b = G(v(-L))$.

Proof. (1) is clear. Let v be any non-constant solution, and ν be the corresponding number. Put $b := G(v(-L)) - \nu v(-L)$. Then, as seen in problem (4.1), the set $\{x \in \bar{J} : v_x(x) = 0\}$ is isolated. Also, by repeating the same argument as Lemmas 4.1 and 4.2, we see that on each connected component (x_L, x_R) of $\{x \in \bar{J}; v_x(x) \neq 0\}$ the solution v coincides with a translation of $V^{\nu,b}$ or $\overline{V}^{\nu,b}$ in space-variable, i.e.

$$v = V^{\nu,b}(\tau - x_L + \cdot) \text{ or } \overline{V}^{\nu,b}(\bar{\tau} - x_L + \cdot), \tag{4.19}$$

and $x_R - x_L = I_0(\nu, b) \le 2L$. Hence the number of such connected components of $\{x \in \bar{J}; v_x(x) \neq 0\}$ is finite. Therefore, for some positive integer $N = N(\nu, b)$, we have

$$NI_0(\nu, b) = 2L, \quad NI_1(\nu, b) = \int_J v\,dx = 0. \tag{4.20}$$

Combining (4.20) with Lemma 3.2 (2), we conclude that

$$\nu = 0 \text{ and } b_0 < b = G(v(-L)) < 0.$$

Moreover, the numbers of $b's$ and $N's$ satisfying $NI_0(0, b) = 2L$ are finite, hence (4.17) has at most a finite number of non-constant solutions, i.e. $\tilde{S}_0$ is a finite set. Thus (1)–(4) have been proved. ◇

Corollary to Lemma 4.3. *In the decomposition $S^* := S_c + S_0 + S_1$ of the solution set S^* for* P, *it holds that*

$$S_c = \{0\}$$

and

$$S_0 = \{\tilde{V}_k \in \tilde{S}_0; G(\tilde{V}_k(-L)) > b_*, 1 \le k \le p\}. \tag{4.21}$$

PROOF. By Lemma 3.1(ii), $S_c = \{0\}$. Let $v \in S_0$ and ν, γ be the corresponding number and function in (3.2). Then, $\gamma = 0$ on J, since $\sigma_* < v < \sigma^*$ on $\bar{J}$. This implies that v is a non-constant solution of (4.17). Hence $v = \tilde{V}_k$ for some $k = 1, 2, \dots, p$, with $G(\tilde{V}_k(-L)) > b_*$, i.e. $S_0 \subset \{\tilde{V}_k \in \tilde{S}_0; G(\tilde{V}_k(-L)) > b_*, 1 \le k \le p\}$. The converse inclusion is easily seen. Hence (4.21) holds. ◇

REMARK 4.1. The results (1)–(3) of Lemma 4.3 are essentially due to Zheng [**17**]; a special case of $q(v) = v^3 - v$ was treated there.

5. Expression of non-constant solutions (continued).

In this section we consider a more precise expression for non-constant solutions of $\mathrm{P} := \mathrm{P}(\sigma_*, \sigma^*; u_\infty, 0)$; conditions (A1)–(A3), (3.1), (q1)–(q3) are still fulfilled throughout this section.

LEMMA 5.1. *Further suppose that*

$$\sigma^* \ge \zeta_{0+}, \quad \sigma_* \le \zeta_{0-}. \tag{5.1}$$

Let v be any non-constant solution of P. *Then $v \in S_0$, and by the way $S_1 = \emptyset$.*

PROOF. Let ν and γ be the corresponding number and function in system (3.2). By (5.1),

$$b_{0\nu} = b_{*\nu} = \begin{cases} G(\zeta_{\nu-}) - \nu\zeta_{\nu-} & \text{if } \nu > 0, \\ G(\zeta_{0-})(= G(\zeta_{0+})) & \text{if } \nu = 0, \\ G(\zeta_{\nu+}) - \nu\zeta_{\nu+} & \text{if } \nu < 0. \end{cases}$$

From Lemmas 3.1 and 3.2 it follows that

$$b_{0\nu} < b := G(v(-L)) - \nu v(-L) < G(\zeta_\nu) - \nu\zeta_\nu$$

and besides

$$\sigma_* \le \eta_{\nu-}(b_{0\nu}) < \eta_{\nu-}(b) \le v \le \eta_{\nu+}(b) < \eta_{\nu+}(b_{0\nu}) \le \sigma^* \quad \text{on } \bar{J}.$$

This shows that $\gamma \equiv 0$ on $\bar{J}$, so that $v \in S_0$. ◇

The complement of (5.1) consists of the following three cases (5.2)–(5.4):

$$0 < \sigma^* < \zeta_{0+}, \quad \sigma^* = -\sigma_*, \tag{5.2}$$

$$0 < \sigma^* < \zeta_{0+}, \quad \sigma^* < -\sigma_*, \tag{5.3}$$

$$\zeta_{0-} < \sigma_* < 0, \quad \sigma_* > -\sigma^*. \tag{5.4}$$

From conditions (q1)–(q3), we see that $(d/d\nu)\zeta_{\nu\pm} > 0$ for $\nu \in (q(\zeta_m), q(\zeta_M))$ and

$G(\zeta_{\nu-} \vee \sigma_*) - \nu(\zeta_{\nu-} \vee \sigma_*)$ (resp. $G(\zeta_{\nu+} \wedge \sigma^*) - \nu(\zeta_{\nu+} \wedge \sigma^*)$) is strictly increasing (resp. decreasing) in $\nu \in (q(\zeta_m), q(\zeta_M))$. Therefore, in the case of (5.3) (resp. (5.4)) there exists exactly one ν^* (resp. ν_*) with $0 < \nu^* < q(\zeta_M)$ (resp. $q(\zeta_m) < \nu_* < 0$) such that

$$G(\sigma^*) - \nu^*\sigma^* = G(\zeta_{\nu^*-} \vee \sigma_*) - \nu^*(\zeta_{\nu^*-} \vee \sigma_*) \tag{5.5}$$

$$(\text{resp. } G(\sigma_*) - \nu_*\sigma_* = G(\zeta_{\nu_*+} \wedge \sigma^*) - \nu_*(\zeta_{\nu_*+} \wedge \sigma^*)).$$

Lemma 5.2. *Let v be any non-constant solution of* P *and ν be the corresponding number in system* (3.2). *Then*:

(1) *If* (5.2) *is satisfied, then* $\nu = 0$.
(2) *If* (5.3) *is satisfied, then* $0 \le \nu \le \nu^*$.
(3) *If* (5.4) *is satisfied, then* $\nu_* \le \nu \le 0$.

Proof. First we show (1). Under (5.2), assume that $\nu < 0$. Then $b_{*\nu} = G(\zeta_{\nu+} \wedge \sigma^*) - \nu(\zeta_{\nu+} \wedge \sigma^*) > G(\zeta_{\nu-} \vee \sigma_*) - \nu(\zeta_{\nu-} \vee \sigma_*) = G(\sigma_*) - \nu\sigma_*$. Hence, by Lemmas 3.1 and 3.2,

$$\sigma_* < v \le \zeta_{\nu+} \wedge \sigma^* \le \sigma^* \quad \text{on } \bar{J}.$$

and

$$I_1(\nu, b) > 0 \text{ with } b = G(v(-L)) - \nu v(-L).$$

On account of Lemma 4.2, we derive from the above facts that

$$\int_J v\,dx = NI_1(\nu, b) + (2L - NI_0(\nu, b))\sigma^* > 0,$$

where N is the number of the connected components of $\{x \in \bar{J}; v_x(x) \neq 0\}$. This contradicts the condition $\int_J v\,dx = 0$. Assuming $\nu > 0$, we have a similar contradiction. Consequently $\nu = 0$ must hold.

Now, we show (2). Under (5.3), assume $\nu < 0$. Then we have the same contradiction as in the proof of (1). Next, assume $\nu > \nu^*$. Then, from the definition (5.5) of ν^* it follows that

$$G(\sigma^*) - \nu\sigma^* < G(\zeta_{\nu-} \vee \sigma_*) - \nu(\zeta_{\nu-} \vee \sigma_*) = b_{*\nu}.$$

By Lemmas 3.1 and 3.2, this implies that

$$\sigma_* \le \zeta_{\nu-} \vee \sigma_* \le v < \sigma^* \quad \text{on } \bar{J}$$

and

$$I_1(\nu, b) < 0 \text{ with } b = G(v(-L)) - \nu v(-L).$$

Taking account of these facts and using Lemma 4.2, we have

$$\int_J v\,dx = NI_1(\nu, b) + (2L - NI_0(\nu, b))\sigma_* < 0,$$

where N is the number of the connected components of $\{x \in \bar{J}; v_x(x) \neq 0\}$. This contradicts $\int_{-L}^{L} v\,dx = 0$. Accordingly $0 \le v \le v^*$ holds. The assertion (3) can be similarly proved. $\diamond$

Now, in terms of expressions $\mathrm{T(I)} = \{(4.10)\text{–}(4.15), (4.16\text{-}1)\}$ and $\mathrm{T(II)} = \{(4.10)\text{–}(4.15), (4.16\text{-}2)\}$ we mention one of main results of this paper.

THEOREM 5.1. *Assume that* (A1)–(A3), (3.1) *and* (q1)–(q3) *are fulfilled. Let v be any non-constant solution of P, v and γ be the corresponding number and function in system* (3.2), *$b := G(v(-L)) - vv(-L)$ and l be the number of all connected components of $\{x \in \bar{J}; v_x(x) \neq 0\}$. Then v has an expression of the form T*(I) *or T*(II) *satisfying one of the following* (a), (b), (c) *and* (d).

(a) *If* (5.1) *holds, then $v = 0$, $b_0 < b < 0$ and $|J_1| = |J_2| = 0$.*

(b) *If* (5.2) *holds, then $v = 0$, $b_*(= G(\sigma_*)) \le b < 0$ and $|J_1| = |J_2|$. Moreover,*

$$\text{if } b > b_*, \quad \text{then } |J_1| = |J_2| = 0; \tag{5.6}$$

$$\text{if } |J_1| = |J_2| > 0, \quad \text{then } b = b_* \text{ and } \eta_{0-}(b) = \sigma_*, \quad \eta_{0+}(b) = \sigma^*. \tag{5.7}$$

(c) *If* (5.3) *holds, then $0 \le v \le v^*$ and $b_{*v} \le b < 0$. This case can be divided into the following three possibilities* (c1)–(c3):

(c1) *$v = 0$; in this case, $b_* \le b < 0$ and $|J_1| = |J_2| = 0$.*

(c2) *$0 < v < v^*$; in this case, $b = b_{*v}$, $\eta_{v-}(b) > \sigma_*, \eta_{v+}(b) = \sigma^*$ and*

$$|J_1| = 0, |J_2| > 0 \quad \text{for } T(\mathrm{I}), \tag{5.8-1}$$

$$|J_1| > 0, |J_2| = 0 \quad \text{for } T(\mathrm{II}), \tag{5.8-2}$$

(c3) *$v = v^*$; in this case, $b = b_{*v^*}, |J_1| + |J_2| > 0, \eta_{v^*-}(b) = \zeta_{v^*-} \vee \sigma_*$ and $\eta_{v^*+}(b) = \sigma^*$. Moreover,*

(i) *if $\zeta_{v^*-} > \sigma_*$, then* (5.8) *holds;*

(ii) *if $\zeta_{v^*-} \le \sigma_*$, then*

$$\sigma_*|J_1| + \sigma^*|J_2| > 0 \quad \text{for } T(\mathrm{I}), \tag{5.9-1}$$

$$\sigma^*|J_1| + \sigma_*|J_2| > 0 \quad \text{for } T(\mathrm{II}), \tag{5.9-2}$$

(d) *If* (5.4) *holds, then $v_* \le v \le 0$ and $b_{*v} \le b < 0$. This case can be divided into the following three possibilities* (d1)–(d3):

(d1) *$v = 0$; in this case, $b_* \le b < 0$ and $|J_1| = |J_2| = 0$.*

(d2) *$v_* < v < 0$; in this case, $b = b_{*v}$, $\eta_{v-}(b) = \sigma_*$, $\eta_{v+}(b) < \sigma^*$* and

$$|J_1| > 0, |J_2| = 0 \quad \text{for } T(\mathrm{I}), \tag{5.10-1}$$

$$|J_1| = 0, |J_2| > 0 \quad \text{for } T(\mathrm{II}), \tag{5.10-2}$$

(d3) *$v = v_*$; in this case, $b = b_{*v_*}, |J_1| + |J_2| > 0, \eta_{v_*-}(b) = \sigma_*$ and $\eta_{v_*+}(b) = \zeta_{v_*+} \wedge \sigma^*$. Moreover,*

(i) *if* $\zeta_{\nu_*+} < \sigma^*$, *then* (5.10) *holds;*
(ii) *if* $\zeta_{\nu_*+} \geq \sigma^*$, *then*

$$\sigma_*|J_1| + \sigma^*|J_2| < 0 \quad \textit{for } T(\mathrm{I}),$$

$$\sigma^*|J_1| + \sigma_*|J_2| < 0 \quad \textit{for } T(\mathrm{II}).$$

Proof. (a) is an immediate consequence of Lemmas 4.3 and 5.1.

Next, assume (5.2) holds. Then it follows from Lemmas 3.1 (iii), 3.2 (2) and 5.2 (1) that $\nu = 0$, $b_* \leq b := G(v(-L)) < 0$ and $I_1(0,b) = 0$. Hence, we infer from (4.16) with $\eta_{0-}(b) = -\eta_{0+}(b)$ that $|J_1| = |J_2|$. If $b > b_*$, then $\eta_{0-}(b) > \sigma_*$ and $\eta_{0+}(b) < \sigma^*$ and hence $\sigma_* < v < \sigma^*$ on $\bar{J}$. This shows by the Corollary to Lemma 4.3 that $v \in S_0 \subset \tilde{S}_0$ and $|J_1| = |J_2| = 0$. Therefore (5.6) holds, and (5.7) holds, too. Thus (b) is obtained.

Consider the case when (5.3) is satisfied, and put $b := G(v(-L)) - \nu v(-L)$; note from Lemma 3.1 (iii) that $b_{*\nu} \leq b < G(\zeta_\nu) - \nu\zeta_\nu$. By the definition of ν^* and Lemma 5.2 (2) we observe that

$$b_{*\nu} = G(\sigma^*) - \nu\sigma^* \geq G(\zeta_{\nu-} \vee \sigma_*) - \nu(\zeta_{\nu-} \vee \sigma_*) \tag{5.11}$$

and

$$0 \leq \nu \leq \nu^*.$$

If $b > b_{*\nu}$, then it follows from Lemma 3.1 (iii) and (5.11) that

$$\sigma_* \leq \zeta_{\nu-} \vee \sigma_* < \eta_{\nu-}(b) \leq v \leq \eta_{\nu+}(b) < \sigma^* \quad \text{on } \bar{J},$$

so that $v \in S_0$; hence the Corollary to Lemma 4.3 implies that $\nu = 0$ and $|J_1| = |J_2| = 0$, and by the way $b = G(v(-L)) < 0$. From this argument we see that $b = b_{*\nu}$, if $\nu > 0$.

Now, suppose under (5.3) that $\nu = 0$. Clearly $b_* \leq b < 0$. Also, it follows from Lemma 3.1 (iii) and (5.3) that

$$\sigma_* \leq \zeta_{0-} \vee \sigma_* < \eta_{0-}(b) \leq v \leq \eta_{0+}(b) = \sigma^* \quad \text{on } \bar{J}.$$

This implies in (4.16) that $|J_1| = 0$ or $|J_2| = 0$. Moreover, since $I_1(0,b) = 0$ (cf. Lemma 3.2 (2)), we have consequently that $|J_1| = |J_2| = 0$. Thus (c1) holds.

Next, suppose under (5.3) that $0 < \nu < \nu^*$. In this case, we have $b = b_{*\nu}$, as was already seen. Since $b = b_{*\nu} > G(\zeta_{\nu-} \vee \sigma_*) - \nu(\zeta_{\nu-} \vee \sigma_*)$, it follows from Lemmas 3.1 (iii) and 3.2 (2) that

$$\sigma_* \leq \zeta_{\nu-} \vee \sigma_* < \eta_{\nu-}(b) \leq v \leq \eta_{\nu+}(b) = \sigma^* \quad \text{on } \bar{J}$$

and $I_1(\nu,b) < 0$. From (4.16) with these we derive (c2).

In the case of $\nu = \nu^*$, note from the definition of ν^* that

$$b = b_{*\nu^*} = G(\zeta_{\nu^*-} \vee \sigma_*) - \nu^*(\zeta_{\nu^*-} \vee \sigma_*) = G(\sigma^*) - \nu^*\sigma^*.$$

This implies by Lemma 3.1 (iii) that

$$\zeta_{\nu^*-} \vee \sigma_* = \eta_{\nu^*-}(b) \leq v \leq \eta_{\nu^*+}(b) = \sigma^* \quad \text{on } \bar{J}.$$

It is easy to see (5.8) (resp. (5.9)) from the relation $\zeta_{v^*-} > \sigma_*$ (resp. $\zeta_{v^*-} \le \sigma_*$). Thus (c3) is proved.

Assuming (5.4) is fulfilled, we can similarly prove (d). $\diamond$

We should note that for each solution v of P all of the possible cases of its expression are completely covered by Theorem 5.1. Also, we note that if $|J_1| = |J_2| = 0$ in the expression T(I) or T(II), then v is a solution of (4.17), i.e. $v \in \tilde{S}_0$.

Remark 5.1. In case when $q(v) \equiv -c_0 v$ for a positive constant c_0, a similar problem was studied and expressions for solutions were obtained by Blowey & Elliott [**2**]. The above theorem gives some generalizations and improvements of their results.

6. Large time behaviour of the order parameter.

For the solution $\{u, w\}$ of (PSC) on R_+, we got in Theorem 2.3 that $u(t) \to u_\infty$ weakly in $H^1(J)$ as $t \to +\infty$. In this section we investigate the large time behaviour of $w(t)$.

Theorem 6.1. *Assume that* (A1)–(A6) *are satisfied as well as* $u_\infty := (h_\infty / n_0) \in D(\rho)$. *Further assume that* (3.1), (q1)–(q3) *and* (5.1) *are satisfied. Let* $\{u, w\}$ *be the solution of* (PSC) *on* R_+. *Then, the* ω*-limit set* $\omega(u_0, w_0)$ *is a singleton consisting of a solution* v *of* (4.17), *i.e.* $\omega(u_0, w_0) = \{v\}$ *with* $v \equiv 0$ *or* $v \in S_0$. *In this case* $w(t) \to v$ *in* $H^1(J)$ *as* $t \to +\infty$.

Proof. By Lemma 5.1, under (5.1) we see that $S^* = \{0\} + S_0$ and it is a finite set. Since $\omega(u_0, w_0) \subset S^*$ and $\omega(u_0, w_0)$ is connected in $H^1(J)$ by Theorem 2.3, the theorem is concluded. $\diamond$

Next, let us consider the complement of (5.1), namely,

$$0 < \sigma^* < \zeta_{0+} \quad \text{or} \quad \zeta_{0-} < \sigma_* < 0, \tag{6.1}$$

which is divided into (5.2), (5.3) and (5.4). In this case, let us further consider a decomposition of the set S_1 in $S^* = \{0\} + S_0 + S_1$, as follows:

$$S_1 := \sum_{l=1}^{l_0} \{S_1^{\mathrm{I}}(l) + S_1^{\mathrm{II}}(l)\}, \tag{6.2}$$

where for each integer $l = 1, 2, \ldots, l_0, S_1^{\mathrm{I}}(l)$ (resp. $S_1^{\mathrm{II}}(l)) := \{v \in S_1;\ v$ is of T(I) (resp. T(II)) and the number of all connected components of $\{x \in \bar{J}; v_x(x) \neq 0\}$ is $l\}$. In fact, we observe from the expressions for solutions to $\mathrm{P} := \mathrm{P}(\sigma_*, \sigma^*; u_\infty, 0)$ obtained in section 5 that $S_1^{\mathrm{I}}(l)$ and $S_1^{\mathrm{II}}(l)$ are compact in $H^1(J)$ for each l, and (6.2) holds for some positive integer l_0 with $l_0 \inf\{I_0(v, b); v_* \le v \le v^*, b_{0v} \le b < 0\} \le 2L$, since $0 < \inf\{I_0(v, b); v_* \le v \le v^*, b_{0v} \le b < 0\} < +\infty$. Of course, S_1^{I} and $S_1^{\mathrm{II}}(l)$ may be empty or a singleton, or contain a continuum.

THEOREM 6.2. *Assume that* (A1)–(A6) *are satisfied and* $u_\infty := (h_\infty/n_0) \in D(\rho)$, *and* (3.1), (q1)–(q3), (6.1) *are satisfied as well. Let* $\{u, w\}$ *be the solution of* (PSC) *on* R_+. *Then one of the following two cases* (6.3)–(6.4) *holds*:

$$\omega(u_0, w_0) \text{ is a singleton of an element in } \{0\} + S_0, \tag{6.3}$$

$$\omega(u_0, w_0) \subset S_1^{\mathrm{I}}(l) \text{ or } S_1^{\mathrm{II}}(l) \quad \text{for some } l \text{ with } 1 \leq l \leq l_0. \tag{6.4}$$

PROOF. Assume (6.3) does not hold. Then, since $\omega(u_0, w_0)$ is compact and connected in $S^* = \{0\} + S_0 + S_1$ (cf. Theorem 2.3), it follows that $\omega(u_0, w_0) \subset S_1$ and more precisely $\omega(u_0, w_0) \subset S_1^{\mathrm{I}}(l)$ or $S_1^{\mathrm{II}}(l)$ for some l. Thus (6.4) holds. ◇

In the case of (6.3), $w(t)$ converges in $H^1(J)$ to a solution of (4.17) as $t \to \infty$, while in the case of (6.4) $w(t)$ does not converge in $H^1(J)$ as $t \to \infty$, in general. As will be seen in an example given below, in the case of (6.4) the large time behaviour of $w(t)$ is very slow with respect to time t and $w(t, \cdot)$ may oscillate within a subinterval of $\bar{J}$.

In the terminology of phase transition (6.4) says that the number of phase transition layers is invariant for large t, though their locations change very slowly in time as $t \to \infty$. In this sense, a pattern of pure phases and mixture is formed as $t \to \infty$.

EXAMPLE 6.1. Consider the case where $J := (-2, 2)$, $\sigma^* = -\sigma_* = (1/2)$, $g(w) = w^3 - w$, $\rho(u) = u$, $\lambda(w) = w$ and $n_0 = 1$. Let z be the solution of

$$\begin{cases} -\kappa z_{xx} + z^3 - z = 0 \quad \text{in } R, \\ z(0) = 0, \\ z_x(0) = \dfrac{1}{2}\left\{\dfrac{1}{\kappa}\left(1 - \dfrac{1}{2^3}\right)\right\}^{1/2}, \end{cases}$$

where κ is chosen so as to satisfy that

$$\left(\frac{\kappa}{2}\right)\int_{-1/2}^{1/2} \frac{dz}{\left\{\dfrac{z^4}{4} - \dfrac{z^2}{2} + \dfrac{7}{64}\right\}^{1/2}} = 1.$$

Clearly, $z_x > 0$ on $(-1/2, 1/2)$ and $z_x(-1/2) = z_x(1/2) = 0$.

Now, define a function v_0 on [−2, 2] by

$$v_0(x) = \begin{cases} -\dfrac{1}{2} & \text{for } -2 \leq x \leq -\dfrac{3}{2}, \\ z(x+1) & \text{for } -\dfrac{3}{2} < x < -\dfrac{1}{2}, \\ \dfrac{1}{2} & \text{for } -\dfrac{1}{2} \leq x \leq \dfrac{1}{2}, \\ z(x) & \text{for } \dfrac{1}{2} < x < \dfrac{3}{2}, \\ -\dfrac{1}{2} & \text{for } \dfrac{3}{2} \leq x \leq 2. \end{cases}$$

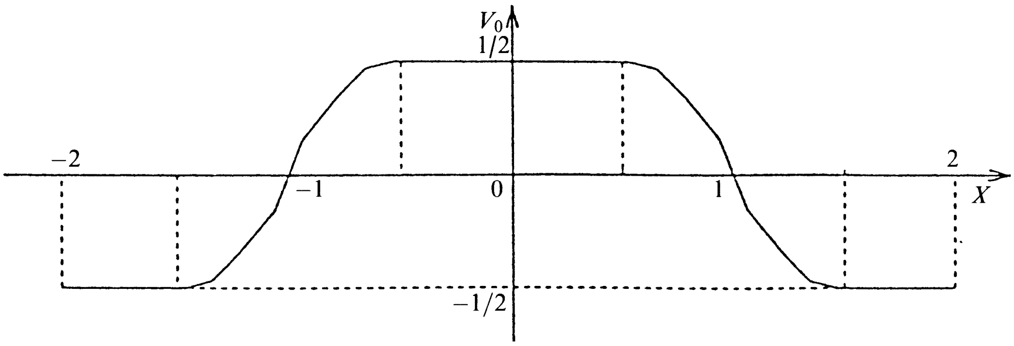

Figure 6

With $y(t) := (1/4)\sin(t+\pi)^{1/3}$ we define

$$w(t,x) := v_0(x + y(t)) \tag{6.5}$$

and

$$u(t,x) := -y'(t)\int_0^x \left[v_0(s+y(t)) + \frac{1}{2}\right] ds \tag{6.6}$$

for $x \in [-2,2]$ and $t \geq 0$. Then we calculate that

$$(u+w)_t - u_{xx} = f(t,x) \quad \text{a.e. in } Q := (0,+\infty)\times(-2,2),$$

$$w_t - \{-\kappa w_{xx} + \xi + w^3 - w - u\}_{xx} = 0 \quad \text{a.e. in } Q,$$

and

$$\xi \in \partial I_{[-1/2,1/2]}(w) \quad \text{a.e. in } Q,$$

where

$$f(t,x) := -\,y''(t)\int_0^x \left[v_0(s+y(t)) + \frac{1}{2}\right] ds - |y'(t)|^2[v_0(x+y(t)) - v_0(y(t))] \\ + 2y'(t)v_0'(x+y(t))$$

and

$$\xi(t,x) = \begin{cases} 0 & \text{for } x \in \left(-\frac{3}{2} - y(t), -\frac{1}{2} - y(t)\right) \cup \left(\frac{1}{2} - y(t), \frac{3}{2} - y(t)\right), \\ \frac{1}{2} - \frac{1}{2^3} & \text{for } x \in \left[-\frac{1}{2} - y(t), \frac{1}{2} - y(t)\right], \\ -\frac{1}{2} + \frac{1}{2^3} & \text{for } x \in \left[-2, -\frac{3}{2} - y(t)\right] \cup \left[\frac{3}{2} - y(t), 2\right]. \end{cases}$$

Moreover, the following initial boundary conditions are fulfilled:

$$\pm u_x(t, \pm 2) + u(t, \pm 2) = h_\pm(t) := -y'(t) \int_0^{\pm 2} \left[v_0(s + y(t)) + \frac{1}{2} \right] ds \quad \text{for } t \geq 0,$$

$$w_x(t, \pm 2) = [-\kappa w_{xx} + \xi + w^3 - w - u]_x(t, \pm 2) = 0 \quad \text{for } t \geq 0,$$

and

$$u_0(x) = -y'(0) \int_0^x \left[v_0(s + y(0)) + \frac{1}{2} \right] ds, \quad w_0(x) = v_0(x + y(0)) \quad \text{on } (-2, 2).$$

The data satisfy all the conditions (A1)–(A6) with $u_\infty = h_\infty = 0$ as well as (3.1), (q1)–(q3) and (5.2). As was checked above, the pair of functions $\{u, w\}$ given by (6.5) and (6.6), is the solution of our problem (PSC). Also, we see that

$$\omega(u_0, w_0) = \left\{ v_0(\cdot + y); |y| \leq \frac{1}{4} \right\},$$

which shows that $\omega(u_0, w_0)$ contains a continuum of solutions to the corresponding stationary problem.

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Akio Ito
Departmant of Mathematics
Graduate School of Science and Technology
Chiba University
Chiba, 263
Japan

Nobuyuki Kenmochi
Department of Mathematics
Faculty of Education
Chiba University
Chiba, 263
Japan