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Normal intermediate subfactors

By Tamotsu TERUYA

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1. Introduction.

The index theory for type II_1 factors was initiated by V. Jones [**11**] and the classification of subfactors has been studied by many people ([**4**], [**8**], [**9**], [**10**], [**12**], [**13**], [**15**], [**16**], [**17, 18**], [**24**], etc.). A. Ocneanu [**22**] introduced the concept of a paragroup to classify subfactors. By using the so-called standard invariants equivalent to the paragroups, S. Popa [**27**], [**26**] classified subfactors under more general conditions. The paragroups or the standard invariants for an inclusion of type II_1 factors with finite Jones index is a group like object which generalizes finite groups. So the theory of finite groups may be considered as part of the index theory for an inclusion of type II_1 factors with finite Jones index. It is well known that if $\alpha : G \to Aut(N)$ is an outer action of a finite group G on a type II_1 factor N and K is an intermediate subfactor for $N \subset N \rtimes_\alpha G$, then there is a subgroup H of G such that $K = N \rtimes_\alpha H$ (see for instance [**21**]). On the other hand, Y. Watatani [**33**] showed that there exist only finitely many intermediate subfactors for an irreducible inclusion with finite index. So it is natural to consider intermediate subfactors as "quantized subgroups" in the index theory for an inclusion of type II_1 factors. The notion of normality for subgroups plays an important role in the theory of finite groups. In this paper we introduce the notion of normality for intermediate subfactors of irreducible inclusions.

D. Bisch [**1**] and A. Ocneanu [**23**] gave a nice characterization of intermediate subfactors of a given irreducible inclusion $N \subset M$ in terms of Jones projections and Ocneanu's Fourier transform $\mathscr{F} : N' \cap M_1 \to M' \cap M_2$. We define normal intermediate subfactors as follows:

DEFINITION. Let $N \subset M$ be an irreducible inclusion of type II_1 factors with finite index and K an intermediate subfactor of the inclusion $N \subset M$. Then K is a *normal intermediate subfactor* of the inclusion $N \subset M$ if $e_K \in \mathscr{Z}(N' \cap M_1)$ and $\mathscr{F}(e_K) \in \mathscr{Z}(M' \cap M_2)$, where e_K is the Jones projection for the inclusion $K \subset M$.

Every finite dimensional Hopf C^*-algebra (Kac algebra) gives rise to an irreducible inclusion of AFD II_1 factors, which is characterized by depth 2 (see for example [**23**], [**30**], [**31**], [**36**]). Let M be the crossed product algebra $N \rtimes \mathbf{H}$ of N by an outer action

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of a finite dimensional Hopf C^*-algebra $\mathbf{H}$. Unfortunately, there is no one-to-one correspondence between the intermediate subfactors of $N \subset M$ and the subHopf C^*-algebras of $\mathbf{H}$ in general. But we get the next result:

THEOREM. *Let $N \subset M$ be an irreducible, depth 2 inclusion of type* II_1 *factors with finite index, i.e., M is described as the crossed product algebra $N \rtimes \mathbf{H}$ of N by an outer action of a finite dimensional Hopf C^*-algebra $\mathbf{H}$. Let K be an intermediate subfactor of $N \subset M$ and e_K is the Jones projection for $K \subset M$. Then K is described as the crossed product algebra $N \rtimes \mathbf{K}$ of N by an outer action of a subHopf C^* algebra $\mathbf{K}$ of $\mathbf{H}$ if and only if e_K is an element of the center of the relative commutant algebra $N' \cap M_1$ where M_1 is the basic extension for $N \subset M$.*

Let $N \subset M$ be an irreducible inclusion of type II_1 factors with finite index and M_1 the basic extension for $N \subset M$. Let K be an intermediate subfactor of $N \subset M$ and K_1 the basic extension for $K \subset M$. Then K_1 is an intermediate subfactor of $M \subset M_1$. For the Jones projections e_K and e_{K_1} for the inclusions $K \subset M$ and $K_1 \subset M_1$, respectively, since $\mathscr{F}(e_K) = \lambda e_{K_1}$ for some scalar λ, we get the next theorem:

THEOREM. *If the depth of a given irreducible inclusion $N \subset M$ is 2, then an intermediate subfactor K of $N \subset M$ is normal in $N \subset M$ if and only if the depths of $N \subset K$ and $K \subset M$ are both 2.*

The author [**32**] showed that if M is the crossed product $N \rtimes G$ of type II_1 factor N by a finite group G and K is given by $N \rtimes H$, then H is a normal subgroup of G if and only if $(K \subset M) \simeq (K \subset K \rtimes F)$ for some finite group F, i.e., the depth of $K \subset M$ is 2. Hence we see by the previous theorem that H is a normal subgroup of G if and only if K is a normal intermediate subfactor of $N \subset M$. Therefore our notion of normality for intermediate subfactors is an extension of that in the theory of finite groups.

We show that if the depth of $N \subset M$ is 2, then the set of all normal intermediate subfactors of $N \subset M$ is a sublattice of intermediate subfactor lattice for $N \subset M$ and we have the Jordan-Dedekind chain condition holding in normal intermediate subfactor lattices.

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2. Preliminaries.

2.1. Intermediate subfactors.

We recall here some results for intermediate subfactors. Let $N \subset M$ be a pair of type II_1 factors. We denote by $\mathscr{L}(N \subset M)$ the set of all intermediate von Neumann subalgebras of $N \subset M$. The set $\mathscr{L}(N \subset M)$ forms a lattice under the two operations $\vee$ and $\wedge$ defined by

$$K_1 \vee K_2 = (K_1 \cup K_2)'' \text{ and } K_1 \wedge K_2 = K_1 \cap K_2.$$

If the relative commutant algebra $N' \cap M$ is trivial, then $\mathscr{L}(N \subset M)$ is exactly the lattice of intermediate subfactors for $N \subset M$. In fact for any $K \in \mathscr{L}(N \subset M)$, $\mathscr{Z}(K) = K' \cap K \subset N' \cap M = \boldsymbol{C}$. If M is the crossed product $N \rtimes_\alpha G$ for an outer action α of a finite group G, then it is well known that the intermediate subfactor lattice $\mathscr{L}(N \subset M)$ is isomorphic to the subgroup lattice $\mathscr{L}(G)$ (see [**20**], [**21**]). In [**33**] Y. Watatani proved the next theorem.

THEOREM (Watatani). *Let $N \subset M$ be a pair of type* II_1 *factors. If $[M : N] < \infty$ and $N' \cap M = \boldsymbol{C}$, then $\mathscr{L}(N \subset M)$ is a finite lattice.*

This theorem was also shown by S. Popa implicitly [**25**].

From now on we assume that $[M : N] < \infty$ and $N' \cap M = \boldsymbol{C}$. Let $N \subset M \subset M_1 \subset M_2$ be the Jones tower of $N \subset M$ obtained by iterating the basic extension. Let $e_N \in M_1$ and $e_M \in M_2$ be the Jones projections for $N \subset M$ and $M \subset M_1$, respectively. We denote by $\mathscr{F}$, Ocneanu's Fourier transform from $N' \cap M_1$ onto $M' \cap M_2$ i.e.,

$$\mathscr{F}(x) = [M : N]^{-3/2} E_{M'}^{N'}(x e_M e_N), \quad x \in N' \cap M_1,$$

where $E_{M'}^{N'}$ is the conditional expectation from N' onto M'. For $K \in \mathscr{L}(N \subset M)$, if e_K is the Jones projection for $K \subset M$, then e_K is an element of $N' \cap M_1$. In fact $K_1 = \langle M, e_K \rangle = J_M K' J_M \subset J_M N' J_M = M_1$ and hence $e_K \in K' \cap K_1 \subset N' \cap M_1$.

D. Bisch [**1**] and A. Ocneanu [**23**] gave the next characterization of intermediate subfactors in terms of Jones projections in $N' \cap M_1$.

THEOREM (Bisch-Ocneanu). *Let p be a projection in $N' \cap M_1$. There exists an intermediate subfactor $K \in \mathscr{L}(N \subset M)$ such that $p = e_K$ if and only if*

(1) $p \geq e_N$,

(2) *$\mathscr{F}(p) = \lambda q$ for some $\lambda \in \boldsymbol{C}$ and some projection $q \in M' \cap M_2$.*

In this case, q is the Jones projection e_{K_1} for $K_1 \subset M_1$.

For the convenience, we prove the next lemmas (see for example [**1**], [**29**]).

LEMMA 2.1. *With the above notations, we have*

$$e_K = [K : N][M : N] E_{M_1}^{M_2}(e_M e_N e_{K_1}),$$

where $E_{M_1}^{M_2}$ is the trace preserving conditional expectation from M_2 onto M_1.

PROOF. Since $e_M \leq e_{K_1}$, we have $e_M e_N e_{K_1} = e_M e_{K_1} e_N e_{K_1} = e_M E_{K_1}^{M_1}(e_N)$. Since $E_{K_1}^{M_1}(e_N) e_K = E_{K_1}^{M_1}(e_N e_K) = E_{K_1}^{M_1}(e_N)$, we have by [**24**]

$$\begin{aligned} E_{K_1}^{M_1}(e_N) &= [M:K]E_M^{K_1}(E_{K_1}^{M_1}(e_N)e_K)e_K \\ &= [M:K]E_M^{M_1}(e_N)e_K = \frac{[M:K]}{[M:N]}e_K \\ &= \frac{1}{[K:N]}e_K. \end{aligned}$$

Therefore we have $e_Me_Ne_{K_1} = (1/[K:N])e_Me_K$. And hence we have

$$E_{M_1}^{M_2}(e_Me_Ne_{K_1}) = \frac{1}{[K:N]}E_{M_1}^{M_2}(e_M)e_K = \frac{1}{[K:N][M:N]}e_K.$$

We get the result. □

Lemma 2.2. *Let K be an intermediate subfactor for $N \subset M$. Let $K \subset M \subset K_1 \subset K_2$ and $N \subset M \subset M_1 \subset M_2$ be the Jones towers for $K \subset M$ and $N \subset M$, respectively. If e_{K_1} is the Jones projection for $K_1 \subset M_1$, then there exists a *-isomorphism φ of K_2 onto $e_{K_1}M_2e_{K_1}$ such that $\varphi(x) = xe_{K_1}$ for $x \in K_1$ and $\varphi(e_M^{K_1}) = e_M$, where $e_M^{K_1}$ and e_M are the Jones projections for $M \subset K_1$ and $M \subset M_1$, respectively.*

Proof. Since $e_{K_1} \in K_1' \subset M'$ on $L^2(M_1)$, it is obvious that $(M \subset K_1) \simeq (Me_{K_1} \subset K_1e_{K_1})$. Therefore it is enough to show that $e_{K_1}M_2e_{K_1}$ is the basic extension for $Me_{K_1} \subset K_1e_{K_1}$ with the Jones projection e_M. By the fact that $e_M = e_{K_1}e_Me_{K_1}$, e_M is an element of $e_{K_1}M_2e_{K_1}$. Let $\tilde{K}_2$ be the basic extension for $K_1 \subset M_1$. Since $e_{K_1}\tilde{K}_2e_{K_1} = K_1e_{K_1}$, we get by the proof of Lemma 2.1,

$$\begin{aligned} E_{K_1e_{K_1}}^{e_{K_1}M_2e_{K_1}}(e_M) &= E_{e_{K_1}\tilde{K}_2e_{K_1}}^{e_{K_1}M_2e_{K_1}}(e_M) = e_{K_1}E_{\tilde{K}_2}^{M_2}(e_M)e_{K_1} \\ &= E_{\tilde{K}_2}^{M_2}(e_M) = \frac{1}{[M:K]}e_{K_1}. \end{aligned}$$

We can see that $Me_{K_1} = (K_1 \cap \{e_M\}')e_{K_1} = K_1e_{K_1} \cap \{e_M\}'$. Therefore $e_{K_1}M_2e_{K_1}$ is the basic extension for $Me_{K_1} \subset K_1e_{K_1}$ by **[24]**. □

2.2. Finite dimensional Hopf C^*-algebras.

In this subsection we recall some facts about finite dimensional Hopf C^*-algebras.

Let $\mathbf{H}$ be a finite dimensional Hopf C^*-algebra with a comultiplication $\Delta_{\mathbf{H}}$ and an anti-pode $S_{\mathbf{H}}$. Let $\mathbf{K}$ be a subHopf C^*-algebra of $\mathbf{H}$, i.e., $\mathbf{K}$ is a *-subalgebra of $\mathbf{H}$, $S_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K}$ and $\Delta_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K} \otimes \mathbf{K}$.

Lemma 2.3. *Define the subset $\mathbf{K}^{\perp}$ of $\mathbf{H}^*$ by*

$$\mathbf{K}^{\perp} = \{f \in \mathbf{H}^* \mid (f,k) = 0, \forall k \in \mathbf{K}\},$$

where $(\ ,\ ) : \mathbf{H}^* \times \mathbf{H} \to \boldsymbol{C}$ is the dual pairing defined by $(f,h) = f(h)$, $f \in \mathbf{H}^*, h \in \mathbf{H}$. Then $\mathbf{K}^{\perp}$ is an ideal of $\mathbf{H}^*$.

PROOF. Let g be an element of $\mathbf{K}^{\perp}$ and f an element of $\mathbf{H}^*$. Then the element gf of $\mathbf{H}^*$ is determined by the equation

$$(gf, h) = (g \otimes f, \triangle_{\mathbf{H}}(h)), \quad \forall h \in \mathbf{H}.$$

By virtue of $\triangle_{\mathbf{H}}(\mathbf{K}) \subset \mathbf{K} \otimes \mathbf{K}$, we get

$$(gf, k) = (g \otimes f, \triangle_{\mathbf{H}}(k)) = 0, \quad \forall k \in \mathbf{K}.$$

Therefore gf is an element of $\mathbf{K}^{\perp}$. Similarly, $fg \in \mathbf{K}^{\perp}$. □

By the above lemma, there exists a central projection $p \in \mathbf{H}^*$ such that $\mathbf{K}^{\perp} = p\mathbf{H}^*$. We put $e_{\mathbf{K}} = 1 - p$.

PROPOSITION 2.4. *With the above notation, the reduced algebra* $e_{\mathbf{K}}\mathbf{H}^*$ *is the dual Hopf* C^**-algebra of* $\mathbf{K}$.

PROOF. Suppose that $k \in \mathbf{K}$ and $(y, k) = 0, \forall y \in e_{\mathbf{K}}\mathbf{H}^*$. Then

$$(f, k) = (e_{\mathbf{K}}f, k) + (pf, k) = (e_{\mathbf{K}}f, k) = 0, \quad \forall f \in \mathbf{H}^*.$$

Therefore $k = 0$. Conversely, suppose that $y \in e_{\mathbf{K}}\mathbf{H}^*$ and $(y, k) = 0, \forall k \in \mathbf{K}$. Then $y \in \mathbf{K}^{\perp} \cap e_{\mathbf{K}}\mathbf{H}^* = \{0\}$. Hence the form $(\ ,\ )|_{e_{\mathbf{K}}\mathbf{H}^* \times \mathbf{K}}$ establishes a duality between $\mathbf{K}$ and $e_{\mathbf{K}}\mathbf{H}^*$. So we can identify $e_{\mathbf{K}}\mathbf{H}^*$ with $\mathbf{K}^*$. Then for $y \in \mathbf{K}^*$ and $k_1, k_2 \in \mathbf{K}$, we have

$$(y, k_1 k_2) = (\triangle_{\mathbf{H}^*}(y), k_1 \otimes k_2) = (\triangle_{\mathbf{H}^*}(y)(e_{\mathbf{K}} \otimes e_{\mathbf{K}}), k_1 \otimes k_2).$$

Hence $\triangle_{\mathbf{K}^*}(y) = \triangle_{\mathbf{H}^*}(y)(e_{\mathbf{K}} \otimes e_{\mathbf{K}})$. Similarly, we have $S_{\mathbf{K}^*} = S_{\mathbf{H}^*|\mathbf{K}^*}$ by the fact that

$$\overline{(y^*, k^*)} = (S_{\mathbf{H}^*}(y), k), \quad \forall y \in \mathbf{K}^*, \quad \forall k \in \mathbf{K}.$$

Therefore $e_{\mathbf{K}}\mathbf{H}^*$ is again a Hopf C^*-algebra with the dual algebra $\mathbf{K}$. □

THEOREM 2.5. *Let* $\mathbf{H}$ *be a finite dimensional Hopf* C^**-algebra. The number of subHopf* C^**-algebras of* $\mathbf{H}$ *is finite.*

PROOF. By the above proposition, the map $\mathbf{K} \mapsto e_{\mathbf{K}}$ from the set of subHopf C^*-algebras of $\mathbf{H}$ to central projections of $\mathbf{H}^*$ is injective. Since the number of central projections of $\mathbf{H}^*$ is finite, so is that of subHopf C^* algebras of $\mathbf{H}$. □

REMARK. Since every finite dimensional Hopf C^*-algebra (Kac algebra) admits an "outer" action on the AFD II_1 factor [**36**], the above theorem immediately follows from [**33**, Theorem 2.2].

DEFINITION. Let $\mathbf{H}$ be a Hopf algebra.

(1) The left adjoint action of $\mathbf{H}$ on itself is given by

$$(ad_l h)(k) = \sum_{(h)} h_1 k(S_{\mathbf{H}}(h_2)), \quad \text{for all } h, k \in \mathbf{H}.$$

(2) The right adjoint action of $\mathbf{H}$ on itself is given by

$$(ad_r h)(k) = \sum_{(h)} (S_{\mathbf{H}}(h_1))kh_2, \quad \text{for all } h, k \in \mathbf{H}.$$

(3) A subHopf algebra $\mathbf{K}$ of $\mathbf{H}$ is called *normal* if both $(ad_l\mathbf{H})(\mathbf{K}) \subset \mathbf{K}$ and $(ad_r\mathbf{H})(\mathbf{K}) \subset \mathbf{K}$ hold. (See [**19**, pp. 33].)

The next proposition will be useful later.

PROPOSITION 2.6. *Let* $\mathbf{H}$ *be a finite dimensional Hopf algebra with a counit* $\varepsilon_{\mathbf{H}}$ *and* $\mathbf{K}$ *a subHopf algebra of* $\mathbf{H}$. *Then* $\mathbf{K}$ *is normal if and only if* $\mathbf{HK}^+ = \mathbf{K}^+\mathbf{H}$, *where* $\mathbf{K}^+ = \mathbf{K} \cap \ker \varepsilon_{\mathbf{H}}$.

See for a proof [**19**, pp. 35].

2.3. Bimodules.

In this subsection we recall some facts about the bimodule calculus associated with an inclusion of type II_1 factors (see for example [**23**], [**34**]).

Let A, B, C be type II_1 factors and let $\alpha = {}_AH_B$, $\beta = {}_AK_B$, $\gamma = {}_BL_C$ be A-B, A-B and B-C Hilbert bimodules, respectively. We write $\alpha\gamma$ for the A-C Hilbert bimodule ${}_AH_B \otimes_{BB} L_C$. We denote by $\langle \alpha, \beta \rangle$ the dimension of the space of A-B intertwiners from ${}_AH_B$ to ${}_AK_B$. The conjugate Hilbert space H^* of ${}_AH_B$ is naturally a B-A bimodule with B-A actions defined by

$$b \cdot \xi^* \cdot a = (a^*\xi b^*)^* \quad \text{for } a \in A \text{ and } b \in B,$$

where $\xi^* = \langle \cdot, \xi \rangle_H \in H^*$ for $\xi \in {}_AH_B$. We denote by $\bar{\alpha}$ the conjugate B-A Hilbert bimodule associated with α.

PROPOSITION 2.7 (Frobenius reciprocity). *Let* A, B, C *be type* II_1 *factors, and* $\alpha = {}_AH_B$, $\beta = {}_BK_C$ *and* $\gamma = {}_AL_C$ *be Hilbert bimodules. Then*

$$\langle \alpha\beta, \gamma \rangle = \langle \alpha, \gamma\bar{\beta} \rangle = \langle \beta, \bar{\alpha}\gamma \rangle.$$

See for a proof [**23**], [**34**].

EXAMPLE 2.8. Let M be a type II_1 factor with the normalized trace τ_M. As usual we let $L^2(M)$ be the Hilbert space obtained by completing M in the norm $\|x\|_2 = \sqrt{\tau_M(x^*x)}$, $x \in M$. Let $\eta : M \to L^2(M)$ be the canonical implementation. Let $J : L^2(M) \to L^2(M)$ be the modular conjugation defined by $J\eta(x) = \eta(x^*)$, $x \in M$. For $\theta \in \mathrm{Aut}(M)$, we define ${}_M(L^2(M)_\theta)_M$, the M-M Hilbert bimodule, by

(1) ${}_M(L^2(M)_\theta)_M = L^2(M)$ as a Hilbert space,

(2) $x \cdot \xi \cdot y = xJ\theta(y)^*J\xi$, $x, y \in M$, $\xi \in L^2(M)$.

Then for $\theta, \theta_1, \theta_2 \in \mathrm{Aut}(M)$ we have

$$\overline{{}_M(L^2(M)_\theta)_M} \simeq {}_M(L^2(M)_{\theta^{-1}})_M$$

$${}_M(L^2(M)_{\theta_1})_M \bigotimes_M {}_M(L^2(M)_{\theta_2})_M \simeq {}_M(L^2(M)_{\theta_1\theta_2})_M.$$

A bimodule $\alpha = {}_AH_B$ is called irreducible if $\langle\alpha,\alpha\rangle = 1$, i.e., $\mathrm{End}_{A\text{-}B}({}_AH_B) \simeq \boldsymbol{C}$. If $\langle\alpha,\alpha\rangle < \infty$, $\alpha = {}_AH_B$, then we can get an A-B irreducible bimodule by cutting ${}_AH_B$ by a minimal projection in $\mathrm{End}_{A\text{-}B}({}_AH_B)$.

Example 2.9. Let $N \subset M$ be an inclusion of type II_1 factors. We define the N-M bimodule ${}_NL^2(M)_M$ by actions $x \cdot \xi \cdot y = xJy^*J\xi$, $\xi \in L^2(M)$, $x \in N$, $y \in M$. Then we can see that $\mathrm{End}({}_NL^2(M)_M) \simeq N' \cap M$. In particular, if $N' \cap M = \boldsymbol{C}$, then ${}_NL^2(M)_M$ is an irreducible N-M bimodule.

The next lemma is well known.

Lemma 2.10. *Let $N \subset M$ be a pair of type II_1 factors with finite index and M_1 the basic extension for the inclusion $N \subset M$. For $\theta \in \mathrm{Aut}(N)$, ${}_N(L^2(M)_\theta)_N \simeq {}_NL^2(M)_N$ if and only if there exists a unitary $u \in M_1$ such that $uxu^* = \theta(x)$, for all $x \in N$, where ${}_N(L^2(M)_\theta)_N$ $(= L^2(M)$ as a Hilbert space$)$ is defined as in Example 2.8.*

Example 2.11. Let $\gamma : G \to \mathrm{Aut}(N)$ be an outer action of a finite group G on a type II_1 factor N. Let $M = N \rtimes_\gamma G$ be the crossed product and ρ the N-M bimodule ${}_NL^2(M)_M$ defined as in Example 2.9. If $\{\lambda_g \mid g \in G\}$ is a unitary implementation for the crossed product, then each element $x \in M$ is written in the form $x = \sum_{g\in G} x_g\lambda_g$, $x_g \in N$. This implies that the irreducible decomposition of $\rho\bar{\rho} = {}_NL^2(M)_N$ is

$$\bigoplus_{g\in G} {}_N(\overline{N\lambda_g}^{\|\cdot\|_2})_N \simeq \bigoplus_{g\in G} {}_N(L^2(N)_{\gamma_g})_N,$$

where ${}_N(L^2(N)_{\gamma_g})_N$ is the N-N bimodule as in Example 2.8.

3. Definition of normal intermediate subfactors.

In this section, we shall introduce the notion of normality for intermediate subfactors and study its properties.

Let $N \subset M$ be a pair of type II_1 factors with $[M : N] < \infty$. Let $N \subset M \subset M_1 \subset M_2$ be the Jones tower of $N \subset M$, obtained by iterating the basic extensions. We denote by $\mathscr{F}$, Ocneanu's Fourier transform from $N' \cap M_1$ onto $M' \cap M_2$, i.e.,

$$\mathscr{F}(x) = [M : N]^{-3/2} E_{M'}^{N'}(xe_Me_N), \quad x \in N' \cap M_1,$$

where $E_{M'}^{N'}$ is the conditional expectation from N' onto M'.

Definition 3.1. Let K be an intermediate subfactor of $N \subset M$ and e_K the Jones projection for the inclusion $K \subset M$. Then we call that K is *normal* in $N \subset M$ if e_K and $\mathscr{F}(e_K)$ are elements of the centers of $N' \cap M_1$ and $M' \cap M_2$, respectively.

Lemma 3.2. *Let K be an intermediate subfactor for an irreducible inclusion $N \subset M$ of type II_1 factors with finite index. Let K_1 and M_1 be the basic extensions for $K \subset M$ and $N \subset M$, respectively. Then K is normal in $N \subset M$ if and only if K_1 is normal in $M \subset M_1$.*

PROOF. Since $\mathscr{F}(e_K) = \lambda e_{K_1}$ for some $\lambda \in \boldsymbol{C}$, it is obvious by the definition. $\square$

PROPOSITION 3.3. *Let* $\alpha : G \to \mathrm{Aut}(P)$ *be an outer action of a finite group* G *on a type* II_1 *factor* P *and* H *a subgroup of* G. *Let* M *be the fixed point algebra* $P^{(H,\alpha)}$ *and* N *the fixed point algebra* $P^{(G,\alpha)}$. *For* $K \in \mathscr{L}(N \subset M)$, *there is a subgroup* A *of* G *such that* $H \subset A \subset G$ *and* $K = P^{(A,\alpha)}$. *Then* K *is a normal intermediate subfactor of* $N \subset M$ *if and only if* $AgH = HgA$ *for* $\forall g \in G$. *In particular,* K *is normal in* $N \subset P$ *if and only if* A *is a normal subgroup of* G.

PROOF. Let $\{u_g | g \in G\}$ be unitary operators on $L^2(P)$ defined by $u_g\eta(x) = \eta(\alpha_g(x))$, $x \in P$, where $L^2(P)$ and η are defined as in Example 2.8. Let P_1 be the basic extension for $N \subset P$. Then $N' \cap P_1 = \{\sum_{g \in G} x_g u_g | x_g \in \boldsymbol{C}\} \simeq \boldsymbol{C}G$. Let e_M^P be the Jones projection for $M \subset P$. Then $e_M^P = (1/{}^{\#}H)\sum_{h \in H} u_h$. Let M_1 be the basic extension for $N \subset M$. Then by Lemma 2.2,

$$N' \cap M_1 \simeq e_M^P(N' \cap P_1)e_M^P = \left\{\sum_{g \in G}\sum_{h,k \in H} x_g u_{hgk} | x_g \in \boldsymbol{C}\right\}.$$

Therefore

$$e_K^M \in \mathscr{Z}(N' \cap M_1) \Leftrightarrow e_M^P e_K^P e_M^P \left(= e_K^P = \frac{1}{{}^{\#}A}\sum_{a \in A} u_a\right) \in \mathscr{Z}(e_M^P(N' \cap P_1)e_M^P)$$

$$\Leftrightarrow \sum_{a \in A}\sum_{h,k \in H} u_{ahgk} = \sum_{a \in A}\sum_{h,k \in H} u_{hgka} \text{ for } \forall g \in G$$

$$\Leftrightarrow AgH = HgA \text{ for } \forall g \in G.$$

Since $M' \cap M_2$ is a commutative algebra, we get the the result. $\square$

We obviously have the dual version of this proposition by Lemma 3.2.

In [33] Y. Watatani introduced the notion of *quasi-normal intermediate subfactors* to study the modular identity for intermediate subfactor lattices.

DEFINITION. Let $N \subset M$ be an inclusion of type II_1 factors with finite index and K an intermediate subfactor of $N \subset M$. Then K is *quasi-normal* (or *doubly commuting*) if for any $L \in \mathscr{L}(N \subset M)$,

$$\begin{array}{ccc} K & \subset & K \vee L \\ \cup & & \cup \\ K \wedge L & \subset & L \end{array}$$

and

$$\begin{array}{ccc} K_1 & \subset & K_1 \vee L_1 \\ \cup & & \cup \\ K_1 \wedge L_1 & \subset & L_1 \end{array}$$

are commuting squares (see for example [**5**]), where K_1 and L_1 are the basic extensions for $K \subset M$ and $L \subset M$, respectively.

Proposition 3.4. *Let $N \subset M$ be an irreducible inclusion of type* II_1 *factors with finite index. If K is a normal intermediate subfactor of $N \subset M$ then K is quasi-normal in $N \subset M$*

Proof. Suppose that the Jones projection e_K for $K \subset M$ is an element of the center of $N' \cap M_1$. Then for any intermediate subfactor L of $N \subset M$, since the Jones projection $e_K^{K \vee L}$ for $K \subset (K \vee L)$ is also a central projection in $K' \cap (K \vee L)_1$, we have

$$\begin{array}{ccc} K & \subset & K \vee L \\ \cup & & \cup \\ K \wedge L & \subset & L \end{array}$$

is a commuting square. Similarly, if $\mathscr{F}(e_K)$ is an element of the center of $M' \cap M_2$, then

$$\begin{array}{ccc} K_1 & \subset & K_1 \vee L_1 \\ \cup & & \cup \\ K_1 \wedge L_1 & \subset & L_1 \end{array}$$

is a commuting square. Therefore if K is normal in $N \subset M$, then K is quasi-normal. ☐

We have a characterization of normal intermediate subfactors in terms of bimodules. Let K be an intermediate subfactor of an irreducible inclusion $N \subset M$ of type II_1 factors with finite index. We note that e_K is in the center of $N' \cap M_1$ if and only if for any $T \in \mathrm{End}({}_N L^2(M)_N)$, $TL^2(K) \subset L^2(K)$.

Proposition 3.5. *Let K be an intermediate subfactor for an irreducible inclusion $N \subset M$ of type* II_1 *factors with finite index. Let α be the N-K bimodule ${}_N L^2(K)_K$ and β the K-M bimodule ${}_K L^2(M)_M$. If ρ is the N-M bimodule $\alpha\beta = {}_N L^2(M)_M$, then K is normal in $N \subset M$ if and only if*

(1) $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$,

(2) $\langle \bar{\beta}\beta, \bar{\rho}\rho \rangle = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle$.

Proof. Since $\mathrm{End}({}_N L^2(K)_N) = N' \cap \langle N, e_N^K \rangle \simeq e_K(N' \cap M_1)e_K$ by Lemma 2.2, if e_K is an element of the center of $N' \cap M_1$, then for any irreducible N-N bimodule σ contained in $\alpha\bar{\alpha}$, the multiplicity of σ in $\alpha\bar{\alpha}$ is equal to the multiplicity of σ in $\rho\bar{\rho}$. Therefore we have $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$. Conversely, suppose that e_K is not an element of the center of $N' \cap M_1$. Then there exist minimal projections $p \sim q$ in $(N' \cap M_1)$ such that $p \in e_K(N' \cap M_1)e_K$ and $q \notin e_K(N' \cap M_1)e_K$. Therefore we have $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle \neq \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$. And hence e_K is an element of the center of $(N' \cap M_1)$ if and only if $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$. Similarly, we can see that e_{K_1} is an element of the center of $(M' \cap M_2)$ if and only if $\langle \bar{\beta}\beta, \bar{\rho}\rho \rangle = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle$. Since $\mathscr{F}(e_K) = \lambda e_{K_1}$ for some $\lambda \in \boldsymbol{C}$, we get the result. ☐

THEOREM 3.6. *Let K be an intermediate subfactor for an irreducible inclusion $N \subset M$ of type* II_1 *factors with finite index. If the depths of $N \subset K$ and $K \subset M$ are both* 2, *then K is normal in $N \subset M$.*

PROOF. Let α be the N-K bimodule ${}_N L^2(K)_K$ and β the K-M bimodule ${}_K L^2(M)_M$. By the assumption, we have

$$\alpha\bar{\alpha}\alpha \simeq \underbrace{\alpha \oplus \alpha \oplus \cdots \oplus \alpha}_{[K:N] \text{ times}}$$

and

$$\bar{\beta}\beta\bar{\beta} \simeq \underbrace{\bar{\beta} \oplus \bar{\beta} \oplus \cdots \oplus \bar{\beta}}_{[M:K] \text{ times}}.$$

And hence $\langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle = \langle \alpha\bar{\alpha}\alpha, \alpha \rangle = [K:N]$ and $\langle \bar{\beta}\beta, \bar{\beta}\beta \rangle = \langle \bar{\beta}\beta\bar{\beta}, \bar{\beta} \rangle = [M:K]$ by Frobenius reciprocity. Since $N \subset M$ is irreducible, if ρ is the N-M bimodule ${}_N L^2(M)_M (= \alpha\beta)$, then $1 = \langle \rho, \rho \rangle = \langle \alpha\beta, \alpha\beta \rangle = \langle \bar{\alpha}\alpha, \beta\bar{\beta} \rangle$. And hence we have

$$\begin{aligned} \langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle &= \langle \alpha\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha} \rangle = \langle \alpha\bar{\alpha}\alpha, \alpha\beta\bar{\beta} \rangle \\ &= [K:N]\langle \alpha, \alpha\beta\bar{\beta} \rangle = [K:N]\langle \bar{\alpha}\alpha, \beta\bar{\beta} \rangle = [K:N], \end{aligned}$$

i.e., $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$. Similarly, we have $\langle \bar{\beta}\beta, \bar{\rho}\rho \rangle = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle$. So we get the result by Proposition 3.5. □

PROPOSITION 3.7. *Let M_0, N_0, K be intermediate subfactors for an irreducible inclusion $N \subset M$ of type* II_1 *factors with finite index such that $N \subset N_0 \subset K \subset M_0 \subset M$. If K is normal in $N \subset M$, then K is also normal in $N_0 \subset M_0$.*

PROOF. Let $\alpha = {}_N L^2(K)_K$, $\alpha_0 = {}_{N_0} L^2(K)_K$, $\beta = {}_K L^2(M)_M$ and $\beta_0 = {}_K L^2(M_0)_{M_0}$. Since

$$\alpha\bar{\alpha} = {}_N L^2(K)_K \otimes_K {}_K L^2(K)_N = {}_N L^2(K)_K \otimes {}_K L^2(K)_K \otimes_K {}_K L^2(K)_N,$$

we have $\langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_K L^2(K)_K \rangle$ by Frobenius reciprocity. Since $\langle \alpha\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha} \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$ by the assumption, we have $\langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_K L^2(K)_K \rangle$, i.e., the irreducible K-K sub-bimodules of $\bar{\alpha}\alpha\bar{\alpha}\alpha$ contained in $\beta\bar{\beta}$ is only ${}_K L^2(K)_K$. Since $\bar{\alpha}_0\alpha_0$ is contained in $\bar{\alpha}\alpha$ and $\beta_0\bar{\beta}_0$ is contained in $\beta\bar{\beta}$, we have $\langle \bar{\alpha}_0\alpha_0\bar{\alpha}_0\alpha_0, \beta_0\bar{\beta}_0 \rangle = \langle \bar{\alpha}_0\alpha_0\bar{\alpha}_0\alpha_0, {}_K L^2(K)_K \rangle$, i.e., $\langle \alpha_0\bar{\alpha}_0, \alpha_0\beta_0\bar{\beta}_0\bar{\alpha}_0 \rangle = \langle \alpha_0\bar{\alpha}_0, \alpha_0\bar{\alpha}_0 \rangle$. By the same argument, we have $\langle \bar{\beta}_0\beta_0, \bar{\beta}_0\bar{\alpha}_0\alpha_0\beta_0 \rangle = \langle \bar{\beta}_0\beta_0, \bar{\beta}_0\beta_0 \rangle$. We have thus proved the proposition. □

4. Normal intermediate subfactors for depth 2 inclusions.

It is well-known that the crossed product of a finite dimensional Hopf C^* algebra (Kac algebra) is characterized by the depth 2 condition. In this section we study normal intermediate subfactors for depth 2 inclusions.

4.1. The action of $K'\cap K_1$ on M.

Let $N\subset M$ be an irreducible, depth 2 inclusion of type II_1 factors with finite index. Let $N\subset M\subset M_1\subset M_2$ be the Jones tower for $N\subset M$. We put $A=N'\cap M_1$ and $B=M'\cap M_2$. Then A and B are a dual pair of Hopf C^*-algebras with pairing

$$(a,b)=[M:N]^2\tau(ae_Me_Nb),\quad \text{for } a\in A \text{ and } b\in B,$$

where e_N and e_M are the Jones projections for $N\subset M$ and $M\subset M_1$, respectively. Define a bilinear map $A\times M\to M$ (denoted by $a\odot x$) by setting

$$a\odot x=[M:N]E_M^{M_1}(axe_N),$$

for $x\in M$ and $a\in A$. This map is a left action of Hopf C^* algebra A and

$$N=M^A=\{x\in M\,|\,a\odot x=\varepsilon(a)x,\ \forall a\in A\},$$

where $\varepsilon:A\to \mathbf{C}$ is the counit determined by $ae_N=\varepsilon(a)e_N$ (see [**31**]).

Proposition 4.1. *Let K be an intermediate subfactor of $N\subset M$ and K_1 the basic extension for $K\subset M$. We put $H=K'\cap K_1$. If a is an element of H, then*

$$[M:K]E_M^{K_1}(axe_K)=[M:N]E_M^{M_1}(axe_N),\quad \forall x\in M.$$

This implies $K=M^H=\{x\in M\,|\,a\odot x=\varepsilon(a)x,\ \forall a\in H\}$.

Proof. Since $e_K=([M:N]/[M:K])E_{K_1}^{M_1}(e_N)$ by [**29**], we have

$$\begin{aligned}[M:K]E_M^{K_1}(axe_K)&=[M:K]E_M^{K_1}\left(ax\frac{[M:N]}{[M:K]}E_{K_1}^{M_1}(e_N)\right)\\ &=[M:N]E_M^{K_1}(E_{K_1}^{M_1}(axe_N))\\ &=[M:N]E_M^{M_1}(axe_N)\end{aligned}$$

for $\forall a\in H$ and $\forall x\in M$. □

4.2. Hopf algebra structures on $K'\cap K_1$.

Let $N\subset M$ be an irreducible, depth 2 inclusion of type II_1 factors with finite index and K an intermediate subfactor of $N\subset M$. Then the depth of $K\subset M$ is not 2 in general. In this subsection we shall prove that if the depth of $K\subset M$ is 2, then $H=K'\cap K_1$ is a subHopf C^* algebra of $A=N'\cap M_1$.

By Lemma 2.2, there exists an isomorphism φ of K_2 onto $e_{K_1}M_2e_{K_1}$ such that $\varphi(x)=xe_{K_1}$ for $x\in K_1$ and $\varphi(e_M^{K_1})=e_M$, where $K\subset M\subset K_1\subset K_2$ is the Jones tower for the inclusion $K\subset M$ and $e_M^{K_1}$ is the Jones projection for $M\subset K_1$.

Lemma 4.2. *With the above notation, we have*

$$[M:K]^2\tau(he_M^{K_1}e_Kk)=[M:N]^2\tau(he_Me_N\varphi(k))$$

for $\forall h\in H=K'\cap K_1$ and $\forall k\in M'\cap K_2$.

PROOF. By the fact that $e_{K_1}e_Ne_{K_1}=E_{K_1}^{M_1}(e_N)e_{K_1}=([M:K]/[M:N])e_Ke_{K_1}$, we have $\varphi(e_K)=e_Ke_{K_1}=([M:N]/[M:K])e_{K_1}e_Ne_{K_1}$. Therefore

$$\begin{aligned}[M:K]^2\tau(he_M^{K_1}e_Kk)&=[M:K]^2[K:N]\tau(\varphi(he_M^{K_1}e_Kk))\\&=[M:K]^2[K:N]\frac{[M:N]}{[M:K]}\tau(\varphi(h)e_Me_{K_1}e_Ne_{K_1}\varphi(k))\\&=[M:N]^2\tau(he_Me_N\varphi(k)).\end{aligned}$$ □

LEMMA 4.3. *Let* $N\subset M$ *be an irreducible, depth* 2 *inclusion of type* II_1 *factors with finite index and* K *an intermediate subfactor for* $N\subset M$. *Let* $N\subset M\subset M_1\subset M_2$ *and* $K\subset M\subset K_1\subset K_2$ *be the Jones towers for* $N\subset M$ *and* $K\subset M$, *respectively. If the depth of* $K\subset M$ *is* 2, *then for any* $b\in M'\cap M_2$, *there exist elements* $\{x_i\},\{y_i\}$ *of* $N'\cap M_1$ *such that*

$$b=\sum_i x_ie_My_i$$

and

$$\sum_i E_{K_1}^{M_1}(x_i)e_ME_{K_1}^{M_1}(y_i)\in(K'\cap K_1)e_M(K'\cap K_1),$$

where $E_{K_1}^{M_1}$ *is the trace preserving conditional expectation from* M_1 *onto* K_1.

PROOF. Since the depth of $N\subset M$ is 2,

$$(N'\cap M_1)e_M(N'\cap M_1)=N'\cap M_2.$$

And hence any element $b\in M'\cap M_2$ is written in the form

$$b=\sum_i x_ie_My_i,\quad x_i,y_i\in N'\cap M_1.$$

Since the depth of $K\subset M$ is 2,

$$(K'\cap K_1)e_M^{K_1}(K'\cap K_1)=K'\cap K_2,$$

where $e_M^{K_1}$ is the Jones projection for $M\subset K_1$. By Lemma 2.2, we have

$$(K'\cap K_1)e_M(K'\cap K_1)=e_{K_1}(K'\cap M_2)e_{K_1}.$$

Therefore we have

$$\begin{aligned}e_{K_1}be_{K_1}&=e_{K_1}\left(\sum_i x_ie_My_i\right)e_{K_1}\\&=\sum_i E_{K_1}^{M_1}(x_i)e_ME_{K_1}^{M_1}(y_i)\in(K'\cap K_1)e_M(K'\cap K_1).\end{aligned}$$

We have thus proved the lemma. □

Proposition 4.4. *Suppose that $N \subset M$ is irreducible and the depth of $N \subset M$ is 2. Let K be an intermediate subfactor for $N \subset M$ and K_1 the basic extension for $K \subset M$. If the depth of $K \subset M$ is 2, then $H = K' \cap K_1$ is a subHopf algebra of $A = N' \cap M_1$.*

Proof. Let S_A be an antipode of A, i.e., $S_A : A \to A$ is an anti-algebra morphism determined by

$$(S_A(a), b) = \overline{(a^*, b^*)} \text{ for } \forall a \in A \text{ and } \forall b \in B = M' \cap M_2.$$

Since $Be_NB = N' \cap M_2$ by the assumption, for any $a \in A$ there exist $x_i, y_i \in B$ such that $a = \sum_i x_i e_N y_i$. Then $S_A(a) = \sum_i y_i e_N x_i$ (see for example [**31**]). By the assumption and Lemma 4.2, H and $B_{e_{K_1}} = e_{K_1} B e_{K_1}$ are a dual pair of Hopf algebras with pairing

$$(h, k) = [M : N]^2 \tau(h e_M e_N k) \text{ for } \forall h \in H \text{ and } \forall k \in B_{e_{k_1}}.$$

By the fact that $\varphi(e_K) = ([M : N]/[M : K]) e_{K_1} e_N e_{K_1}$, for $h \in H$ there exist $s_n, t_n \in B_{e_{K_1}}$ such that $he_{K_1} = \varphi(h) = \sum_n s_n e_N t_n$, where φ is defined in Lemma 2.2. Then for $\forall b \in B$, we have

$$\begin{aligned}
(S_A(h), b) &= \overline{(h^*, b^*)} = [M : N]^2 \tau(b e_N e_M h) \\
&= [M : N]^2 \sum_n \tau(b e_N e_M s_n e_N t_n) \\
&= [M : N]^2 \sum_n \tau(b E^{M'}_{M'_1}(e_M s_n) e_N t_n) \\
&= [M : N]^2 \sum_n \tau(e_M s_n) \tau(b e_N t_n) \\
&= [M : N] \sum_n \tau(e_M s_n) \tau(b t_n).
\end{aligned}$$

Since $S_H(h) e_{K_1} = S_{H_{e_{K_1}}}(h e_{K_1}) = \sum_n t_n e_N s_n$ by the fact that $e_{K_1} \in H'$, we have, for $\forall b \in B$,

$$\begin{aligned}
(S_H(h), b) &= [M : N]^2 \tau(S_{H_{e_{K_1}}}(h e_{K_1}) e_M e_N b) \\
&= [M : N]^2 \sum_n \tau(t_n e_N s_n e_M e_N b) \\
&= [M : N] \sum_n \tau(s_n e_M) \tau(t_n b).
\end{aligned}$$

Therefore we have $S_A(h) = S_H(h) \in H$, i.e., $S_A(H) \subset H$.

Let Δ_A be the comultiplication of A, i.e., $\Delta_A : A \to A \otimes A$ is determined by

$$(a, b_1 b_2) = (\Delta_A(a), b_1 \otimes b_2) \text{ for } \forall b_1, b_2 \in B.$$

For $h \in H$, we denote $\Delta_A(h)$ by $\sum_{(h)} h_{(1)} \otimes h_{(2)}$. Since $e_M = e_{K_1} e_M$ and $e_{K_1} h = h e_{K_1}$, we have

$$\begin{aligned}
(h, b) &= [M : N]^2 \tau(h e_{K_1} e_M e_N b) \\
&= [M : N]^2 \tau(h e_M e_N b e_{K_1}) \qquad (4.1) \\
&= (h, b e_{K_1}) \text{ for } \forall h \in H \text{ and } \forall b \in B.
\end{aligned}$$

Since e_{K_1} is an element of the center of B by the proof of Theorem 3.6, we have

$$\begin{aligned}(h, b_1 b_2) &= (h, b_1 e_{K_1} b_2 e_{K_1}) = (\Delta_A(h), b_1 e_{K_1} \otimes b_2 e_{K_1}) \\ &= \sum_{(h)} (h_{(1)}, b_1 e_{K_1})(h_{(2)}, b_2 e_{K_1}) \\ &= \sum_{(h)} [M : N]^2 \tau(e_{K_1} h_{(1)} e_M e_N b_1)[M : N]^2 \tau(e_{K_1} h_{(2)} e_M e_N b_2) \\ &= \sum_{(h)} (E_{K_1}^{M_1}(h_{(1)}), b_1)(E_{K_1}^{M_1}(h_{(2)}), b_2), \text{ for } \forall b_1, b_2 \in B.\end{aligned}$$

Since $\sum_{(h)} S_A(h_{(1)}) e_M h_{(2)} \in B$ by [**31**], we have $\Delta_A(H) \subset H \otimes H$ by Lemma 4.3. We have thus proved the proposition. □

Theorem 4.5. *Let $N \subset M$ be an irreducible, depth 2 inclusion of type* II_1 *factors with finite index and K an intermediate subfactor for $N \subset M$. Let $N \subset M \subset M_1 \subset M_2$ and $K \subset M \subset K_1 \subset K_2$ be the Jones towers for $N \subset M$ and $K \subset M$, respectively. Then the depth of $K \subset M$ is 2 if and only if e_{K_1} is an element of the center of $M' \cap M_2$, where e_{K_1} is the Jones projection for $K_1 \subset M_1$.*

Proof. Suppose that the depth of $K \subset M$ is 2. Then by the proof of Theorem 3.6, e_{K_1} is an element of the center of $M' \cap M_2$.

Conversely, suppose that e_{K_1} is an element of the center of $M' \cap M_2$. Then for any $h \in H = K' \cap K_1$, we have

$$\begin{aligned}(S_A(h), b) &= \overline{(h^*, b^*)} = [M : N]^2 \tau(b e_N e_M h) \\ &= [M : N]^2 \tau(e_{K_1} b e_{K_1} e_N e_M h) \\ &= (S_A(h), e_{K_1} b e_{K_1}) \quad \text{for } \forall b \in B = M' \cap M_2\end{aligned}$$

and hence $S_A(H) \subset H$. Similarly, for any $h \in H$, we have

$$\begin{aligned}(\Delta_A(h), x \otimes y) &= (h, xy) = (h, e_{K_1} x e_{K_1} y e_{K_1}) \\ &= (\Delta_A(h), e_{K_1} x e_{K_1} \otimes e_{K_1} y e_{K_1}) \quad \text{for } \forall x, y \in M' \cap M_2,\end{aligned}$$

and hence $\Delta_A(H) \subset H \otimes H$. Therefore H is a subHopf algebra of $N' \cap M_1$. By Proposition 4.1, we have $K = M^H$. So the depth of $K \subset M$ is 2. □

We obviously have the dual version by Lemma 3.2.

Remark. Later we noted by the referee that this theorem follows from the next characterization of depth 2 inclusions by bimodules: Let $N \subset M$ be an irreducible inclusion of type II_1 factors with finite index and ρ the N-M bimodule ${}_N L^2(M)_M$. Then the depth of $N \subset M$ is 2 if and only if for any irreducible bimodule ${}_N X_N \prec \rho\bar{\rho}$, $\dim_N X = \langle X, \rho\bar{\rho} \rangle$.

THEOREM 4.6. *Let $N \subset M$ be an irreducible, depth 2 inclusion of type* II_1 *factors with finite index and K an intermediate subfactor for $N \subset M$. Then K is a normal intermediate subfactor of $N \subset M$ if and only if the depths of $N \subset K$ and $K \subset M$ are both* 2.

PROOF. This immediately follows from Theorem 4.5 and Lemma 3.2. □

THEOREM 4.7. *Let $N \subset M$ be an irreducible, depth* 2 *inclusion of type* II_1 *factors with finite index and K an intermediate subfactor for $N \subset M$. Then K is a normal intermediate subfactor of $N \subset M$ if and only if $K' \cap K_1$ is a normal subHopf algebra of $N' \cap M_1$, where K_1 and M_1 are the basic extensions for $N \subset M$ and $K \subset M$, respectively.*

PROOF. Suppose that K is a normal intermediate subfactor of $N \subset M$. Then $H = K' \cap K_1$ is a subHopf algebra of $A = N' \cap M_1$ by Proposition 4.4. Let ε_H is a counit of H. Then $xe_K = \varepsilon_H(x)e_K$ for $x \in H$. Therefore $H^+ = H \cap \ker \varepsilon_H = H(1 - e_K)$. Since $(1 - e_K)$ is an element of the center of A by the assumption, we have $H^+A = AH^+$. Hence H is a normal subHopf algebra of A by Proposition 2.6. Conversely, we suppose that H is a normal subHopf algebra of A. Then by Proposition 4.4 and Theorem 4.5, e_{K_1} is element of the center of $M' \cap M_2$. Since $H^+ = H(1 - e_K)$, we have $(1 - e_k)A = A(1 - e_K)$ by Proposition 2.6. This implies e_K is an element of the center of $N' \cap M_1$ and hence K is a normal intermediate subfactor of $N \subset M$. □

4.3. Lattices of normal intermediate subfactors.

Let $N \subset M$ be an irreducible, depth 2 inclusion of type II_1 factors with finite index. In this subsection we shall prove that the set of all normal intermediate subfactors of the inclusion $N \subset M$, denoted by $\mathscr{N}(N \subset M)$, is a sublattice of $\mathscr{L}(N \subset M)$. Moreover, $\mathscr{N}(N \subset M)$ is a modular lattice.

LEMMA 4.8. *Let L and K be intermediate subfactors of $N \subset M$ and L_1 and K_1 the basic extensions for $L \subset M$ and $K \subset M$, respectively. Then the basic extension $(L \wedge K)_1$ for $(L \wedge K) \subset M$ is $L_1 \vee K_1$ and the basic extension $(L \vee K)_1$ for $(L \vee K) \subset M$ is $L_1 \wedge K_1$.*

PROOF. By the fact that $(L \cap K)' = (L' \cup K')''$, we have

$$(L \wedge K)_1 = J(L \wedge K)'J = L_1 \vee K_1.$$

Similarly, by the fact that $(L \cup K)' = L' \cap K'$, we have

$$(L \vee K)_1 = J(L \cup K)'J = L_1 \wedge K_1. \qquad \square$$

We note that if we denote by e_A the Jones projection for $A \subset M$, then for $L, K \in \mathscr{L}(N \subset M)$, we have $e_{L \wedge K} = e_L \wedge e_K$. But $e_{L \vee K} \neq e_L \vee e_K$ in general (see [**29**]).

THEOREM 4.9. *Let $N \subset M$ be an irreducible, depth* 2 *inclusion of type* II_1 *factors with finite index. Then the set of all normal intermediate subfactors $\mathscr{N}(N \subset M)$ is a sublattice of $\mathscr{L}(N \subset M)$.*

PROOF. Let L and K be normal intermediate subfactors of $N \subset M$. Since e_L and e_K are elements of the center of $N' \cap M_1$ by the assumption, we have $e_{L \wedge K} = e_L \wedge e_K \in$

$\mathscr{Z}(N' \cap M_1)$ by the above argument. Observe that

$$(L \vee K)' \cap (L \vee K)_1 = (L' \cap L_1) \cap (K' \cap K_1).$$

Since $L' \cap L_1$ and $K' \cap K_1$ are invariant under the left and right adjoint action of $N' \cap M_1$ (see Definition in 2.2), so is $(L \vee K)' \cap (L \vee K)_1$. Therefore we can see that $(L \vee K)' \cap (L \vee K)_1$ is a normal subHopf algebra $N' \cap M_1$ by the definition. Since $L \vee K$ is a normal intermediate subfactor of $N \subset M$ by Theorem 4.7, we have $e_{L \vee K} \in \mathscr{Z}(N' \cap M_1)$. Applying the same argument for the dual inclusion $M \subset M_1$, we conclude that $L \wedge K$ and $L \vee K$ are normal intermediate subfactors of $N \subset M$. □

COROLLARY 4.10. *Let $N \subset M$ be an irreducible, depth 2 inclusion of type* II$_1$ *factors. Then $\mathscr{N}(N \subset M)$ is a modular lattice.*

PROOF. This immediately follows from Proposition 3.4, Theorem 4.9 and [**33**, Theorem 3.9]. □

We recall here Jordan-Dedekind chain condition. In a lattice L, a finite chain $x = x_0 \supseteq x_1 \supseteq \cdots \supseteq x_d = y$ is maximal if $x_i \supsetneqq x_{i+1}$ and $x_i \supseteq a \supseteq x_{i+1}$ implies $x_i = a$ or $x_{i+1} = a$ for $i = 1, 2, \ldots d-1$.

JORDAN-DEDEKIND CHAIN CONDITION: All finite maximal chains between two elements have the same length.

THEOREM 4.11. *Let $N \subset M$ be an irreducible, depth 2 inclusion of type* II$_1$ *factors with finite index. Then the normal intermediate subfactor lattice $\mathscr{N}(N \subset M)$ satisfies the Jordan-Dedekind chain condition.*

PROOF. Since we have the Jordan-Dedekind chain condition holding in modular lattices, this immediately follows from the previous corollary. (see for example [7].) □

EXAMPLE 4.12. We denote by S_n the symmetric group on n letters, $x_1, x_2, \ldots, x_n$ and $\sigma = (1, 2, 3, \ldots, n)$ the element of S_n with order n and $\langle \sigma \rangle$ the cyclic group generated by σ. Let $\gamma : S_n \longrightarrow \mathrm{Aut}(P)$ be an outer action of S_n ($n > 3$) on a type II$_1$ factor P and let $N = P^{\gamma_\sigma} \subset M = P \rtimes_\gamma S_{n-1}$. Then we can see that $S_n = S_{n-1}\langle \sigma \rangle = \langle \sigma \rangle S_{n-1}$ and $S_{n-1} \cap \langle \sigma \rangle = \{e\}$. Therefore the depth of $N \subset M$ is 2 (see [**28, 35**]). We put $K = P \rtimes_\gamma A_{n-1}$, where A_{n-1} is the alternating group consists of the even permutations on $x_1, x_2, \ldots, x_{n-1}$. If n ($\geqq 5$) is odd, then the length of $\mathscr{N}(N \subset M)$ is 3 and if n ($\geqq 4$) is even, then that is 2 (we shall show this fact later in Example 5.4).

5. Some examples.

In this section we shall give some examples of normal intermediate subfactors and non normal ones.

5.1. Group type inclusions.

Let $\gamma : G \longrightarrow \mathrm{Aut}(P)$ be an outer action of a discrete group G on a type II$_1$ factor. Let A and B be finite subgroups of G such that $A \cap B = \{e\}$. Let N be the fixed point algebra $P^{(A,\gamma)}$ and M the crossed product $P \rtimes_\gamma B$. Then $N \subset M$ is an irreducible inclusion by [**2**] and P is normal in $N \subset M$ by Theorem 3.6. In this subsection we consider inclusions of this type.

PROPOSITION 5.1. *With the above notation, let H be a subgroup of B and K the crossed product $P \rtimes H$. Then K is normal in $N \subset M$ if and only if H is a normal subgroup of B and $AH \cap BA = AH \cap HA$.*

PROOF. Let $\alpha = {}_N L^2(P)_P$ and $\beta = {}_P L^2(M)_M$. Let $\beta_1 = {}_P L^2(K)_K$ and $\beta_2 = {}_K L^2(M)_M$. Then we have

$$\bar{\alpha}\alpha = \bigoplus\nolimits_{a\in A} {}_P(L^2(P)_{\gamma_a})_P$$

$$\beta_1\bar{\beta}_1 = \bigoplus\nolimits_{h\in H} {}_P(L^2(P)_{\gamma_h})_P$$

$$\beta\bar{\beta} = \bigoplus\nolimits_{b\in B} {}_P(L^2(P)_{\gamma_b})_P,$$

as in Example 2.11. Since $A \cap B = \{e\}$, we have

$$(ab = a'b', a, a' \in A, b, b' \in B) \iff (a = a' \text{ and } b = b').$$

Therefore if $\rho = {}_N L^2(M)_M (= \alpha\beta)$, then

$$\langle \alpha\beta_1\overline{(\alpha\beta_1)}, \rho\bar{\rho}\rangle = \langle \alpha\beta_1\bar{\beta}_1\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha}\rangle = \langle \bar{\alpha}\alpha\beta_1\bar{\beta}_1, \beta\bar{\beta}\bar{\alpha}\alpha\rangle$$

$$= {}^{\#}(AH \cap BA)$$

and $\langle \alpha\beta_1\overline{(\alpha\beta_1)}, \alpha\beta_1\overline{(\alpha\beta_1)}\rangle = \langle \bar{\alpha}\alpha\beta_1\bar{\beta}_1, \beta_1\bar{\beta}_1\bar{\alpha}\alpha\rangle = {}^{\#}(AH \cap HA)$. Hence $e_K \in \mathscr{Z}(N' \cap M_1)$ if and only if $(AH \cap BA) = (AH \cap HA)$ by Proposition 3.5. Suppose K is normal in $N \subset M$. Then K is also normal in $P \subset M$ by Proposition 3.7. Therefore H is a normal subgroup of B by the dual version of Proposition 3.3. Conversely, if H is a normal subgroup of B, i.e., the depth of $K \subset M$ is 2, then we have

$$\langle \bar{\beta}_2\beta_2, \bar{\rho}\rho\rangle = \langle \bar{\beta}_2\beta_2, \bar{\beta}_2\bar{\beta}_1\bar{\alpha}\alpha\beta_1\beta_2\rangle = \langle \beta_2\bar{\beta}_2\beta_2\bar{\beta}_2, \bar{\beta}_1\bar{\alpha}\alpha\beta_1\rangle$$

$$= [B : H]\langle \beta_2\bar{\beta}_2, \bar{\beta}_1\bar{\alpha}\alpha\beta_1\rangle = [B : H] = \langle \bar{\beta}_2\beta_2, \bar{\beta}_2\beta_2\rangle.$$

This proves the proposition. □

Let G be a finite group with two subgroups A, B satisfying $G = AB$ and $A \cap B = \{e\}$. By the uniqueness of the decomposition of an element in $G = AB = BA$, we can represent ab for $a \in A$, $b \in B$ as $ab = \alpha_a(b)\beta_{b^{-1}}(a^{-1})^{-1} \in BA$. Then the matched pair (A, B, α, β) appears (see for example [**28**]).

PROPOSITION 5.2. *Let (A, B, α, β) be the matched pair defined as above and let*

$$M = P \rtimes_\gamma B \supset N = P^{(A,\gamma)} = \{x \in P | \gamma_a(x) = x, \forall a \in A\},$$

where γ is an outer action of G on II_1 factor P. Then the depth of $N \subset M$ is 2.

See for a proof [**28**, **35**].

THEOREM 5.3. *Let G be a finite group with two subgroups A, B satisfying $G = AB$ and $A \cap B = \{e\}$ and (A, B, α, β) the associated matched pair. Let $\gamma : G \to \mathrm{Aut}(P)$ be an outer action of G on a type* II_1 *factor P and let*

$$M = P \rtimes_\gamma B \supset N = P^{(A,\gamma)} = \{x \in P | \gamma_a(x) = x, \forall a \in A\}.$$

If H is a subgroup of B and $K = P \rtimes_\gamma H \in \mathscr{L}(N \subset M)$, then K is a normal intermediate subfactor for $N \subset M$ if and only if

(1) *H is a normal subgroup of B,*

(2) *$\alpha_a(H) = H$, $\forall a \in A$, i.e., $AH = HA$*

In particular, if G is a semi direct product $B \rtimes A$, then K is normal in $N \subset M$ if and only if H is a normal subgroup of G.

PROOF. Since $BA = AB = G$, we have $(AH \cap BA) = AH$. By Proposition 5.1, we have K is a normal intermediate subfactor in $N \subset M$ if and only if H is a normal subgroup of B and $(AH \cap HA) = AH$, i.e., $AH = HA$ since $^{\#}HA = {}^{\#}AH$. □

EXAMPLE 5.4. Let $N = P^{\gamma_\sigma} \subset M = P \rtimes_\gamma S_{n-1}$ $(n > 3)$ be the irreducible inclusion defined as in Example 4.12. The depth of $N \subset M$ is 2 by Proposition 5.2. We put $K = P \rtimes_\gamma A_{n-1}$. If n is odd, then σ is an even permutation and we can see that $A_n = A_{n-1}\langle\sigma\rangle = \langle\sigma\rangle A_{n-1}$. Therefore K is normal in $N \subset M$ by Theorem 5.3. If n is even, then σ is an odd permutation. Since the product of an even and odd permutation in either order is odd, and the product of two odd permutation is even, $A_{n-1}\langle\sigma\rangle$ is not a subgroup of S_n and hence $A_{n-1}\langle\sigma\rangle \neq \langle\sigma\rangle A_{n-1}$. Therefore K is not normal in $N \subset M$ by Theorem 5.3.

Since S_{n-1} is a maximal subgroup of S_n, we have if $\langle\sigma^k\rangle S_{n-1} = S_{n-1}\langle\sigma^k\rangle$, then $\langle\sigma^k\rangle = \langle\sigma\rangle$ or $k = 0 \pmod{n}$, i.e., there is no normal intermediate subfactor K of $N \subset M$ such that $N \subsetneq K \subsetneq P$ by Proposition 5.2.

REMARK. By Example 5.4, we have completed the proof of Example 4.12.

5.2. Strongly outer actions and intermediate subfactors.

In this subsection we shall study relations between strongly outer actions introduced by Choda and Kosaki [3] and normal intermediate subfactors.

Let $N \subset M$ be a pair of type II_1 factors, and we set

$$\mathrm{Aut}(M, N) = \{\theta \in \mathrm{Aut}(M) | \theta(N) = N\}.$$

Let $N(= M_{-1}) \subset M(= M_0) \subset M_1 \subset M_2 \subset \cdots$ be the Jones tower of the pair $N \subset M$, and $e_k(\in M_k)$ the Jones projection for the pair $M_{k-2} \subset M_{k-1}$. Then each automorphism $\theta \in \mathrm{Aut}(M, N)$ is extended to all M_n subject to the condition $\theta(e_i) = e_i$.

DEFINITION. An automorphism $\theta \in \mathrm{Aut}(M, N)$ is said to be strongly outer if the following condition is satisfied for all $k \geq -1$:

$$a \in M_k \text{ satisfies } ax = \theta(x)a \quad \text{for all } x \in N \Rightarrow a = 0.$$

An action α of a group G into $\mathrm{Aut}(M, N)$ is said to be strongly outer if α_g is strongly outer for all $g \in G$ except for the identity e.

For $\theta \in \mathrm{Aut}(M, N)$, let ${}_N(L^2(M)_\theta)_N$ be the N-N bimodule as in Example 2.8. M. Choda and H. Kosaki [**3**] gave the next characterization of strongly outer automorphisms.

Theorem (Chode-Kosaki). *For $\theta \in \mathrm{Aut}(M, N)$, if ${}_N(L^2(M)_\theta)_N$ does not appear in the irreducible decomposition of $(\rho\bar{\rho})^k$, $k = 1, 2, \dots$, then θ is strongly outer, where ρ is the N-M bimodule ${}_N L^2(M)_M$.*

The next lemma is well-known.

Lemma 5.5. *Let $B \subset A$ be an irreducible pair of type* II$_1$ *factors with finite index. Let $\gamma : G \to \mathrm{Aut}(A, B)$ be an outer action of a finite group G and $\alpha = {}_B L^2(A)_A$. If $\bar{\alpha}\alpha \not\succ {}_A(L^2(A)_{\gamma_g})_A$ for all $g \in G$ except for the identity e, then $B' \cap (A \rtimes_\gamma G) = \boldsymbol{C}$. In particular, if γ is strongly outer, then $B \subset A \rtimes_\gamma G$ is irreducible.*

Proposition 5.6. *Let $B \subset A$ be an irreducible pair of type* II$_1$ *factors with finite index. Let $\gamma : G \to \mathrm{Aut}(A, B)$ be an outer action of a finite group G and $\alpha = {}_B L^2(A)_A$. Then $B \subset A \rtimes_\gamma G$ is irreducible and A is normal in $B \subset A \rtimes_\gamma G$ if and only if $\bar{\alpha}\alpha\bar{\alpha}\alpha \not\succ {}_A(L^2(A)_{\gamma_g})_A$ for all $g \in G$ except for the identity e.*

Proof. Suppose that $\bar{\alpha}\alpha\bar{\alpha}\alpha \not\succ {}_A(L^2(A)_{\gamma_g})_A$ for all $g \in G$ except for the identity e. Since $\bar{\alpha}\alpha \prec \bar{\alpha}\alpha\bar{\alpha}\alpha$, $B \subset A \rtimes_\gamma G$ is irreducible by Lemma 5.5. Let $\beta = {}_A L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G}$ and $\rho = {}_B L^2(A \rtimes_\gamma G)_{A \rtimes_\gamma G} (= \alpha\beta)$. Since $\beta\bar{\beta} \simeq \bigoplus_{g \in G} {}_A(L^2(A)_{\gamma_g})_A$, we have

$$\begin{aligned}\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle &= \langle \alpha\bar{\alpha}, \alpha\beta\bar{\beta}\bar{\alpha} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle \\ &= \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_A L^2(A)_A \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle.\end{aligned}$$

Since $\langle \beta\bar{\beta}, \bar{\alpha}\alpha \rangle = 1$ by Lemma 5.5, we have

$$\begin{aligned}\langle \bar{\beta}\beta, \bar{\rho}\rho \rangle &= \langle \bar{\beta}\beta, \bar{\beta}\bar{\alpha}\alpha\beta \rangle = \langle \beta\bar{\beta}\beta\bar{\beta}, \bar{\alpha}\alpha\beta \rangle \\ &= {}^{\#}G \langle \beta\bar{\beta}, \bar{\alpha}\alpha \rangle = {}^{\#}G = \langle \bar{\beta}\beta, \bar{\beta}\beta \rangle.\end{aligned}$$

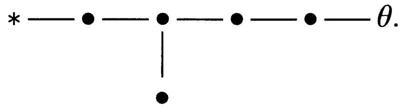
Therefore A is normal in $B \subset A \rtimes_\gamma G$ by Lemma 3.5.

Conversely, suppose that $\bar{\alpha}\alpha\bar{\alpha}\alpha \succ {}_A(L^2(A)_{\gamma_g})_A$ for some $g (\neq e) \in G$. Then we have $\langle \alpha\bar{\alpha}, \rho\bar{\rho} \rangle = \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, \beta\bar{\beta} \rangle \gneqq \langle \bar{\alpha}\alpha\bar{\alpha}\alpha, {}_A L^2(A)_A \rangle = \langle \alpha\bar{\alpha}, \alpha\bar{\alpha} \rangle$. And hence A is not normal in $B \subset A \rtimes_\gamma G$. $\square$

Theorem 5.7. *Let $B \subset A$ be an irreducible pair of type* II$_1$ *factors with finite index. If $\gamma : G \to \mathrm{Aut}(A, B)$ is a strongly outer action of a finite group G, then A is a normal intermediate subfactor for the inclusion $B \subset A \rtimes_\gamma G$.*

Proof. This immediately follows from the previous proposition. $\square$

Example 5.8. Let $B \subset A$ be an irreducible pair of type II$_1$ factors whose principal graph is Dynkin diagram A_{2n-1}. Let θ be a non-trivial automorphism such that the corresponding A-A bimodule ${}_A(L^2(A)_\theta)_A$ appears in the endpoint of the principal graph. As discussed in Izumi [**8**], Kosaki [**14**], Goto [**6**], we can choose θ with $\theta(B) = B$ and $\theta^2 = id$. Let $\alpha = {}_BL^2(A)_A$. Then ${}_A(L^2(A)_\theta)_A \not\prec \bar{\alpha}\alpha\bar{\alpha}\alpha$ for $n = 4, 5, 6, \cdots$. Therefore the intermediate subfactor A is normal in $B \subset A \rtimes_\theta \mathbf{Z}/2\mathbf{Z}$ by Proposition 5.6. If $n = 3$, i.e., the principal graph is A_5, then ${}_A(L^2(A)_\theta)_A \prec \bar{\alpha}\alpha\bar{\alpha}\alpha$. Hence A is not normal in $B \subset A \rtimes_\theta \mathbf{Z}/2\mathbf{Z}$. Similarly let $B \subset A$ be an inclusion of type II_1 factors with the principal graph E_6,



We can also choose $\theta \in \mathrm{Aut}(A, B)$ with $\theta^2 = id$ as above. By Proposition 5.6, A is not normal in $B \subset A \rtimes_\theta \mathbf{Z}/2\mathbf{Z}$.

EXAMPLE 5.9. In the situation of Theorem 5.7, the other intermediate subfactor $B \rtimes_\gamma G$ is not normal in general. In fact let α be an outer action on the AFD II_1 factor R of the symmetric group $S_3 = \mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$ with the obvious generators $a = (1, 2, 3)$, $b = (1, 2)$ for $\mathbf{Z}/3\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}$ respectively. Let us consider the inclusion $B = R \subset A = R \rtimes_\alpha \mathbf{Z}/3\mathbf{Z}$. Consider the period 2 automorphism γ on A defined by

$$\gamma\left(\sum_{g \in \mathbf{Z}/3\mathbf{Z}} x_g \lambda_g\right) = \sum_{g \in \mathbf{Z}/3\mathbf{Z}} \alpha_b(x_g)\lambda_{bgb^{-1}}.$$

The automorphism γ leaves B invariant globally, i.e., $\gamma \in \mathrm{Aut}(A, B)$. Since $\gamma|_B = \alpha_b$, it is an outer automorphism of $B = R$. Notice that γ is also an outer automorphism for A. In fact let us assume that $x = \sum_{g \in \mathbf{Z}/3\mathbf{Z}} x_g\lambda_g \in A$ satisfies $yx = x\gamma(y)$ for each $y \in A$. By just choosing $y \in B$, we see $yx_g = x_g\alpha_{gb}(y)$, $y \in B = R$ for each $g \in \mathbf{Z}/3\mathbf{Z}$. Since α is an outer action and $gb \neq e$, we get $x_g = 0$ for each $g \in \mathbf{Z}/3\mathbf{Z}$, and hence $x = 0$ as desired.

Let β be the dual action (on A) of $\alpha|_{\mathbf{Z}/3\mathbf{Z}}$. Since $B = A^\beta$, we have a irreducible decomposition A-A bimodule ${}_AL^2(A_1)_A \simeq {}_AL^2(A)_A \oplus {}_A(L^2(A)_{\beta_a})_A \oplus {}_A(L^2(A)_{\beta_a^2})_A$. Hence, the two non-trivial automorphisms have period 3. Since $\gamma \in \mathrm{Aut}(A, B)$ has period 2, ${}_A(L^2(A)_\gamma)_A$ can not be in ${}_AL^2(A_1)_A$ and hence it is strongly outer. It is plain to see

$$(B \subset A \subset A \rtimes_\gamma \mathbf{Z}/2\mathbf{Z}) \simeq (R \subset R \rtimes_\alpha \mathbf{Z}/3\mathbf{Z} \subset R \rtimes_\alpha S_3).$$

Through this natural isomorphism, $B \rtimes_\gamma \mathbf{Z}/2\mathbf{Z}$ corresponds to $R \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$. However, this is not a normal intermediate subfactor in $R \rtimes_\alpha S_3$ since $\mathbf{Z}/2\mathbf{Z} = S_2$ is not a normal subgroup in S_3.

The same construction works for

$$S_n = A_n \rtimes \mathbf{Z}/2\mathbf{Z} \supset S_{n-1} = A_{n-1} \rtimes \mathbf{Z}/2\mathbf{Z}.$$

In this case, the above strong outerness is obvious since the alternating group A_n $(n \geq 5)$ is simple and does not admit a non-trivial character and hence ${}_AL^2(A_k)_A$ does not contains a non-trivial one dimensional component (see [**8**]).

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Tamotsu TERUYA

Department of Mathematics,
Hokkaido University,
Sapporo 060
Japan
E-mail address: t-teruya@math.hokudai.ac.jp