J. Math. Soc. Japan
Vol. 50, No. 2, 1998

Brownian potentials and Besov spaces*

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(Received Mar. 1, 1996)

1. Introduction.

This paper presents techniques enabling one to connect potentials of the Brownian motion process with a particular type of interpolation spaces known as Besov spaces. Earlier works of Komatsu [6] and Yoshikawa [11], developed interpolation theory by using semigroup resolvent; it turned out that their approach is very useful, and it parallels recent new results dealing with characterization of some excessive functions of the Brownian motion in terms of Γ-potentials, obtained by Glover, Rao, Šikić, and Song [5].

However, the main result of this paper states that Brownian potentials of finite measures, given over bounded domains, belong to Besov spaces. This result, interesting by itself, provides several important applications.

Namely, consider the Schrödinger equation

$$\left(\frac{1}{2}\triangle + q\right)u = 0, \tag{1}$$

understood in the sense of distributions. In order to study this equation from the probabilistic point of view, the crucial notion of the gauge function was introduced in earlier work by Chung and Rao [3]. This paper shows that under some assumptions, the gauge function belongs to a Besov space; it measures its degree of continuity. An earlier work by Aizenman and Simon [1], (p. 217, Theorem 1.5 and Remark), shows that any solution of equation (1) is a continuous function, but it may not be Hölder continuous of any order. (For probabilistic approach consider Pop-Stojanović and Rao [8]). However, this paper shows more than its continuity, namely, that it belongs locally to a Besov space.

Finally, it has been shown that every bounded solution u of the Schrödinger equation $((\triangle/2)+q)u=-f$ has the form $u=h+w$, where h is a bounded harmonic function, and w belongs to Besov space $B_{pp}^{2\theta}(\Omega)$, with $p<n/(n-2+2\theta)$.

2. Besov Spaces and Brownian Motion Process.

Let $\Omega \subseteq R^n$, $n \geq 3$, be a bounded C^∞-domain, and let $D'(\Omega)$ denote the set of all distributions on Ω. Furthermore, throughout this paper real numbers 2θ and p satisfy

* The publication is based on work sponsored by the Croatian–American Joint Fund in cooperation with the Ministry of Science and Technology (Croatia), and the National Science Foundation (USA) under project No. JF159.

$0 < 2\theta < 1$ and $1 \le p < +\infty$. Following Triebel [10], the Besov space $B^{2\theta}_{pp}(\Omega)$ is defined as

$$B^{2\theta}_{pp}(\Omega) = \{f; f \in D'(\Omega);\ \text{there exists } g \in B^{2\theta}_{pp}(R^n),\ g|_\Omega = f\}, \tag{2}$$

where

$$B^{2\theta}_{pp}(R^n) = \left\{f;\ \|f\|_p + \left\{\int_{R^n \times R^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+2\theta p}}\, dx\, dy\right\}^{1/p} < +\infty\right\}. \tag{3}$$

REMARK 1. Observe that assumptions made at the outset of the paper are needed to obtain the main result of the paper. Several definitions and results are valid for unbounded domain Ω and for $n = 1, 2$, as well. Finally, requirement for C^∞-domain is needed only in the above definition. (See Triebel [10] for more details). □

Let $(X_t;\ t \ge 0)$ be a Brownian motion process killed upon exit from Ω and τ the first exit time of the process from Ω. Let (T_t) denote the corresponding semigroup of operators, i.e., $(T_t f)(x) = E^x[f(X_t)]$, and G the potential operator of (X_t), i.e.,

$$(Gf)(x) = E^x\left[\int_0^{+\infty} f(X_t)dt\right] = E^x\left[\int_0^\tau f(X_t)dt\right]. \tag{4}$$

The Brownian semigroup (T_t) is a C_0-semigroup of contractions on $L^p(\Omega)$, whose infinitesimal generator is $(1/2)\triangle$; here, the Laplacian $\triangle$ is taken with zero boundary conditions. Since, $\mathscr{D}((1/2)\triangle) = \mathscr{D}(\triangle) = \mathscr{D}(-\triangle)$, where $\mathscr{D}$ denotes the domain of an operator with respect to L^p space, the interpolation space

$$(\mathscr{D}(\triangle), L^p(\Omega))_{\theta,p} \tag{5}$$

will be considered using approaches of Komatsu and Yoshikawa, and that of Besov spaces. Consider the second approach first.

REMARK 2. Notation used in (5) for the interpolation space, is that of Yoshikawa [11]. The same space is obtained, but differently denoted, in Bergh and Löfström [2] by using the J-method from Banach spaces $L^p(\Omega)$, with its usual norms, and $\mathscr{D}(\triangle)$, endowed with the norm $\|f\|_p + \|\triangle f\|_p$. Notice that in the notation of Bergh and Löfström, the spaces appear in the reverse order of that used in this paper, as well as in Yoshikawa's. Again, the same spaces, although under different notation, appear in Komatsu [6]. □

It is well-known (see, for example, Theorem 6.7.4, p. 160 in Bergh and Löfström [2]) that

$$B^{2\theta}_{pp}(R^n) = (\mathscr{D}(\triangle), L^p(R^n))_{\theta,p}, \tag{6}$$

where the Laplacian $\triangle$ is considered over the entire R^n. The set of C^∞ functions with compact supports in Ω is the core for the Laplacian on Ω with zero boundary conditions. Therefore, every $f \in \mathscr{D}(\triangle)$ over Ω is also in $\mathscr{D}(\triangle)$ over R^n, and corresponding norms coincide. Of course, every $f \in L^p(\Omega)$ is also in $L^p(R^n)$, and their two norms coincide. Since only these spaces and their norms are involved in the J-method, one concludes that

$$\text{for every } f \in (\mathscr{D}(\triangle), L^p(\Omega))_{\theta,p} \text{ there exists} \tag{7}$$
$$g \in (\mathscr{D}(\triangle), L^p(R^n))_{\theta,p} \text{ such that } g|_\Omega = f.$$

Now, consider the other approach. It has been proved in Komatsu [**6**], (see also Proposition 2.1. in Yoshikawa [**11**]), that the interpolation space $(\mathscr{D}(\triangle), L^p(\Omega))_{\theta,p}$ is the set of all $f \in L^p(\Omega)$ such that

$$\int_0^{+\infty} \|\lambda^\theta A R_\lambda f\|_p^p \frac{d\lambda}{\lambda} < +\infty, \tag{8}$$

where $A = -(\triangle/2)$, and (R_λ) is the resolvent of the semigroup (T_t). It is obvious, that the same statement could be obtained by replacing the Laplacian $\triangle$ in (8) instead of A. Using Fubini theorem, Jensen inequality and the fact that $\lambda e^{-\lambda t}$ is the density of a probability measure on $[0, +\infty)$, (8) yields to

$$\begin{aligned}
\int_0^{+\infty} \lambda^{\theta p-1} \|A R_\lambda f\|_p^p \, d\lambda &= \int_0^{+\infty} \lambda^{\theta p-1} \|\lambda R_\lambda f - f\|_p^p \, d\lambda \\
&= \int_0^{+\infty} \lambda^{\theta p-1} \left\| \int_0^{+\infty} \lambda e^{-\lambda t}(T_t f - f)\, dt \right\|_p^p d\lambda \\
&\le \int_0^{+\infty} \lambda^{\theta p-1} \left[\int_0^{+\infty} \lambda e^{-\lambda t} \|T_t f - f\|_p \, dt \right]^p d\lambda \\
&\le \int_0^{+\infty} \lambda^{\theta p-1} \int_0^{+\infty} \lambda e^{-\lambda t} \|T_t f - f\|_p^p \, dt \, d\lambda \\
&= \int_0^{+\infty} \|T_t f - f\|_p^p \int_0^{+\infty} \lambda^{\theta p} e^{-\lambda t} \, d\lambda \, dt \\
&= \Gamma(\theta p + 1) \int_0^{+\infty} \left\| \frac{T_t f - f}{t^\theta} \right\|_p^p \frac{dt}{t}.
\end{aligned}$$

Combining this with (2), (6), (7), and (8), one gets the following

THEOREM 1. *Let f be in $L^p(\Omega)$.*

a) *If* $\int_0^{+\infty} \|\lambda^\theta \triangle R_\lambda f\|_p^p \frac{d\lambda}{\lambda} < +\infty$, then $f \in B_{pp}^{2\theta}(\Omega)$.

b) *If*

$$\int_0^{+\infty} \left\| \frac{E^\cdot[f(X_t)] - E^\cdot[f(X_0)]}{t^\theta} \right\|_p^p \frac{dt}{t} < +\infty, \; \textit{then } f \in B_{pp}^{2\theta}(\Omega).$$

In both cases one obtains that

$$\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+2\theta p}} \, dx \, dy \tag{9}$$

is finite.

REMARK 3. To get the main result, one needs only the part a) of Theorem 1. However, the part b) is interesting, since it gives a probabilistic sufficient condition for a function to belong in the Besov space. Observe that one can consider R^n instead of Ω (even for any $n \in N$), and use (2) and Theorem 4.3 in Komatsu [**6**]. Hence, one gets the following (probabilistic) characterization of the Besov space: $f \in B_{pp}^{2\theta}(R^n)$ if and only if $f \in L^p(R^n)$ and

$$\int_0^{+\infty} \left\| \frac{E^{\bullet}[f(B_t)] - E^{\bullet}[f(B_0)]}{t^\theta} \right\|_p^p \frac{dt}{t} < +\infty, \tag{10}$$

where (B_t) is a standard Brownian motion process on R^n, unlike (X_t) which is Brownian motion process killed upon exit from Ω. It follows also that corresponding norms are equivalent as well. □

Now, the main result of the paper. Following Glover, Rao, Šikić, and Song [**5**], for $0 < \alpha < 1$ and $f \geq 0$, one defines the Γ-potential:

$$V^\alpha f(x) = \frac{1}{\Gamma(\alpha)} E^x \left[\int_0^{+\infty} t^{\alpha-1} f(X_t)\, dt \right]. \tag{11}$$

It follows that

$$Gf(x) = V^\alpha V^{1-\alpha} f(x), \tag{12}$$

and the same holds true for a positive measure μ instead of function f.

THEOREM 2. *If $s = G\mu$, where μ is a finite positive measure on Ω, then, for every $p < n/(n-2+2\theta)$, $s \in B_{pp}^{2\theta}(\Omega)$, i.e.,*

$$\int_{\Omega\times\Omega} \frac{|s(x) - s(y)|^p}{|x-y|^{n+2\theta p}}\, dx\, dy \tag{13}$$

is finite.

PROOF. Since $n \geq 3$, the Green potential G is dominated by function $C|x-y|^{-(n-2)}$, where C is a constant depending on n and Ω only. Therefore, on a bounded domain and with respect to the finite measure μ, $s = G\mu$ is in $L^p(\Omega)$ for $1 \leq p < n/(n-2)$. Since $p < n/(n-2+2\theta) < n/(n-2)$, one has that $s \in L^p(\Omega)$.

Considering Theorem 1. a), it is sufficient to check that s satisfies equation (8). Function s is excessive. Hence, $-s^\lambda = \lambda R_\lambda s - s = (\triangle/2) R_\lambda s$ satisfies the equation $s = s^\lambda + \lambda R_\lambda s$, for each $\lambda > 0$, and s^λ is a unique λ-excessive function with such a property (see Rao [**9**] for details). Therefore, it is sufficient to show that

$$\int_0^{+\infty} \lambda^{\theta p - 1} \int_\Omega [s^\lambda(x)]^p\, dx\, d\lambda < +\infty. \tag{14}$$

Using Lemma 1, p. 49, in Mazja [**7**], one can estimate the integral in (14).

$$\int_0^{+\infty} \lambda^{p-1} \int_\Omega [\lambda^{\theta-1} s^\lambda(x)]^p \, dx \, d\lambda = \int_\Omega \int_0^{+\infty} \lambda^{p-1} [\lambda^{\theta-1} s^\lambda(x)]^p \, d\lambda \, dx$$
$$\leq \int_\Omega \frac{1}{p} \left(\int_0^{+\infty} \lambda^{\theta-1} s^\lambda(x) \, d\lambda \right)^p dx.$$

Hence, it is sufficient to prove that

$$x \mapsto \int_0^{+\infty} \lambda^{\theta-1} s^\lambda(x) \, d\lambda \tag{15}$$

is in $L^p(\Omega)$.

The potential $s = G\mu$ is also purely excessive, i.e., it allows the Γ-potential representation $s = V^\theta g$, with

$$g = \frac{1}{\Gamma(\theta)\Gamma(1-\theta)} \int_0^{+\infty} \lambda^{\theta-1} s^\lambda \, d\lambda, \tag{16}$$

wherefrom one sees that (15) and (16) only differ for a multiplicative constant (see Glover, Rao, Šikić, Song [**5**] for derivation of (16)). However,

$$s = G\mu = V^\theta V^{1-\theta} \mu, \tag{17}$$

and $g = V^{1-\theta}\mu$, as shown in Glover, Rao, Šikić, Song [**5**]. Therefore, it remains to show that $V^{1-\theta}\mu \in L^p(\Omega)$. Denote by $p_t(x, y)$ the transition probability density for (X_t). Then, one gets the following estimate:

$$(V^{1-\theta}\mu)(x) = \frac{1}{\Gamma(1-\theta)} \int_0^{+\infty} \int_\Omega t^{(1-\theta)-1} p_t(x, y) \mu(dy) \, dt$$
$$\leq M \int_\Omega \frac{1}{|x-y|^{n-2(1-\theta)}} \, \mu(dy),$$

where M is a constant depending on Ω, n and θ, only. Since μ is a finite measure, function

$$x \mapsto \int_\Omega \frac{1}{|x-y|^{n-2(1-\theta)}} \, \mu(dy) \tag{18}$$

belongs to $L^p(\Omega)$, for $p < n/(n-2+2\theta)$, which terminates the proof. □

3. Applications.

Given a Borel function q belonging to the Kato class and such that the gauge function

$$g(x) = E^x \left[e^{\int_0^\tau q(X_s) \, ds} \right] \tag{19}$$

is bounded. (For a detailed account on the subject consider the book by Chung and Zhao [**4**]). Then, the Schrödinger potential K, defined by

$$Kf(x) = E^x \left[\int_0^\tau e^{\int_0^t q(X_s) \, ds} f(X_t) \, dt \right], \tag{20}$$

satisfies the equation

$$Kf = Gf + G(qKf). \tag{21}$$

By applying Theorem 2 in this situation, one gets several interesting results.

COROLLARY 1. *If* $s = Kf$, *where* $f \geq 0$ *and* $f \in L^1(\Omega)$, *then, for every* $p < n/(n-2+2\theta), s \in B^{2\theta}_{pp}(\Omega)$.

PROOF. Since Kf satisfies (21), it suffices to show that Gf and $G(qKf)$ belong to the Besov space $B^{2\theta}_{pp}(\Omega)$. Since $f \geq 0$ and $f \in L^1(\Omega)$, Theorem 2 implies immediately that $Gf \in B^{2\theta}_{pp}(\Omega)$. To show the rest, one has to show that $qKf \in L^1(\Omega)$, and then to consider its positive and negative parts. Knowing that K is a symmetric operator, one obtains the inequality

$$\int_\Omega |qKf|(x)\,dx \leq \int_\Omega |f(x)|(K|q|)(x)\,dx,$$

wherefrom the conclusion follows, since $f \in L^1(\Omega)$ and $K|q|$ is bounded.

Now, by taking positive and negative parts of f, one concludes that Kf belongs to the Besov space $B^{2\theta}_{pp}(\Omega)$, for every $f \in L^1(\Omega)$. Recall that, since Ω is bounded and q is in the Kato class, q is in $L^1(\Omega)$.

The gauge function g is the solution of the Schrödinger equation

$$\left(\frac{\triangle}{2} + q\right)u = 0 \tag{22}$$

with the boundary condition

$$u|_{\partial\Omega} = 1. \tag{23}$$

Since $g = 1 + K(q)$, one gets the following

COROLLARY 2. *The gauge function* g *belongs to the Besov space* $B^{2\theta}_{pp}(\Omega)$, *for every* $p < n/(n-2+2\theta)$.

Consider even a more general situation. Let $f \in L^1(\Omega)$ and u be a bounded solution of the non-homogenous Schrödinger equation

$$\left(\frac{\triangle}{2} + q\right)u = -f. \tag{24}$$

Let $\{\Omega_n,\ n \in N\}$ be an increasing sequence of relatively compact subdomains of Ω, (so that the closure of Ω_n, $n \in N$ is contained in Ω), and such that $\bigcup_n \Omega_n = \Omega$. Let τ_n, $n = 1, 2, \ldots$, be the exit time from Ω_n, $n = 1, 2, \ldots$. By taking a subsequence if necessary, one gets a harmonic function h (with respect to the Laplacian), as the limit

$$h(x) = \lim_{n \to +\infty} E^x[u(X_{\tau_n}); \tau_n \nearrow \tau]. \tag{25}$$

Then, u satisfies equation

$$u(x) = Kf(x) + v(x), \tag{26}$$

where v is given by

$$v(x) = h(x) + E^x\left[\int_0^\tau e^{\int_0^t q(X_s)\,ds}(qh)(X_t)\,dt\right]. \tag{27}$$

After applying the result of Corollary 1 to (26) and (27), (since f, $qh \in L^1(\Omega)$), one gets the following concluding result of this paper.

COROLLARY 3. *Every bounded solution u of the Schrödinger equation* (24) *satisfies*

$$u = h + w, \tag{28}$$

where h is a bounded harmonic function, and w belongs to $B^{2\theta}_{pp}(\Omega)$, with $p < n/(n-2+2\theta)$. In particular, $u \in B^{2\theta}_{pp}(\Omega')$, for every compact subdomain Ω' of Ω.

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