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Bounded topological orbit equivalence and C^*-algebras

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§1. Introduction.

The interplay between measurable dynamics and the theory of von Neumann algebras has a long and successful history (see [**4**, **7**, **12**]). A high point in this development (through Murray-von Neumann, Dye, Krieger, Connes, Haagerup ...) is the equivalence of two classification problems: the classification of nonsingular ergodic transformations of Lebesgue space up to orbit equivalence, and the classification of approximately finite dimensional von Neumann factors. Over the years, inevitably people have contemplated the possibility of an analogous theory relating C^*-algebras and topological dynamics.

Perhaps the first dramatic progress in this direction came with the work of Giordano, Putnam and Skau [**4**] on minimal homeomorphisms of the Cantor set. For these systems, GPS wrote a dictionary relating properties of the topological dynamics, the associated transformation group (crossed product) C^*-algebra, and the K-theory of that algebra. In particular, they classified these systems up to orbit equivalence, and they classified the associated algebras up to isomorphism and restricted isomorphism (i.e., isomorphism respecting the subalgebra of continuous functions). Some of this work has been extended to more general zero dimensional systems [**3**].

In this paper we have two main results. A purely topological result is the Bounded Orbit Equivalence Theorem (Theorem 2.3): if S and T are topologically free homeomorphisms of a compact Hausdorff space with the same orbits, and $Tx = S^{n(x)}x$ where the cocycle n is bounded, then the complement of a closed nowhere dense set of periodic points is the union of disjoint open invariant sets A and B, with $T|_A \cong S|_A$ and $T|_B \cong S^{-1}|_B$. The cocycle is cohomologous to 1 on A and to -1 on B. Stronger conclusions in the case of a continuous cocycle are easy corollaries (Theorems 3.1, 3.2). These results can be viewed as topological analogues of Belinskaya's Theorem (Remark 3.3). What gives this topological situation a somewhat different flavor is that one cannot neglect periodic points. For example, our most technical argument (Proposition 2.2) is a triviality in the absence of periodic points (or in the measurable category).

Our other main result, Theorem 3.6, is a complete characterization of restricted

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isomorphism for the crossed product C^*-algebras of topologically free homeomorphisms S, T of compact Hausdorff spaces. The restricted isomorphism exists if and only if there exist decompositions of the domains such that $S|_{X_1} \cong T|_{Y_1}$ and $S|_{X_2} \cong T^{-1}|_{Y_2}$. In the connected or transitive case, this amounts to flip conjugacy (i.e., X_1 or X_2 must be empty).

The paper is organized into three sections. Section 2 contains the work on bounded orbit equivalence. Section 3 harvests the consequences for continuous cocycles and C^*-algebras, and provides some context. Section 4 provides examples indicating the necessity of some hypotheses and some limits on generalization.

§2. Bounded orbit equivalence.

Let X be an arbitrary compact Hausdorff space. Let S and T be homeomorphisms of X having the same orbits. Then there exists an integer valued function n such that $Tx = S^{n(x)}x$, for all x in X. The "jump function" n generates the function $f : \boldsymbol{Z} \times X \to \boldsymbol{Z}$ defined by

$$\begin{aligned} f(k,x) &= n(x) + n(Tx) + \cdots + n(T^{k-1}x) && \text{for } k > 0 \\ f(k,x) &= -[n(T^{-1}x) + n(T^{-2}x) + \cdots + n(T^k x)] && \text{for } k < 0, \\ f(0,x) &= 0. \end{aligned}$$

We then have, for every k and x,

$$T^k x = S^{f(k,x)}x. \tag{2-1}$$

The function f satisfies the cocycle equation

$$f(k+\ell, x) = f(k, T^\ell x) + f(\ell, x) \tag{2-2}$$

and f is referred to as the cocycle for T generated by n. Abusing notation, we sometimes refer to the function n as a cocycle. Note that $n(x)$ is uniquely determined by S and T if the point x is aperiodic, and that a point is periodic for S iff it is periodic for T. Also, the function $x \mapsto f(p,x)$ is invariant on an orbit of cardinality p, since it is the sum of the values of n over the points in the orbit.

DEFINITION 2.1. A homeomorphism $S : X \to X$ is *topologically free* if the set of its aperiodic points are dense.

The following proposition is a basic starting point in our discussions.

PROPOSITION 2.2 (Bijection of coordinates). *In the situation described above, suppose that the homeomorphism S is topologically free, the cocycle n is bounded on X, and n is continuous at every point in the orbit of a point x_0.*

Then the map $k \to f(k, x_0)$ is a bijection of the group $\boldsymbol{Z}$.

PROOF. The claim is immediate if x_0 is aperiodic, so suppose that x_0 is periodic. We assert first that the map $k \mapsto f(k, x_0)$ is injective. By the continuity of n on the orbit of x_0, for each k the function $x \mapsto f(k,x)$ is continuous at x_0. Therefore, given integers k and ℓ, there exists a neighborhood U of x_0 on which both functions $x \mapsto f(k,x)$ and $x \mapsto f(\ell,x)$ are constant. By topological freeness, U contains an aperiodic point y. If $k \neq \ell$, then $f(k,y) \neq f(\ell,y)$, and therefore $f(k,x_0) \neq f(\ell,x_0)$. This proves injectivity.

Next, let p be the period of x_0 (i.e., the cardinality of the orbit of x_0). It follows from (2-1) that $f(p,x_0)$ is an integral multiple of p. Let M be the nonzero integer such that $f(p,x_0) = Mp$. For $0 \leq k < p$, the numbers $f(k,x_0)$ are distinct modulo p, and for all integers j it follows from (2-2) that

$$f(k+jp,x_0) = f(k,T^{jp}x_0) + f(jp,x_0) = f(k,x_0) + jMp.$$

Therefore to prove the bijectivity, it remains to show that $|M| = 1$. For an argument by contradiction, we assume $M > 1$ (the case $M < -1$ may be treated in a similar way).

Let $N = \max|n(x)|$, and choose an integer R so that

$$MRp \geq p + (p+2)N. \tag{2-3}$$

By the continuity of $n(x)$ on the orbit of x_0 and by topological freeness we can find an aperiodic point y near x_0 such that for both $z = y$ and $z = S^p(y)$,

$$f(i,z) = f(i,x_0), \qquad \text{for} \quad 0 \leq i \leq Rp, \tag{2-4}$$

and moreover,

$$f(p,S^jz) = f(p,S^jx_0), \qquad \text{for} \quad 0 \leq j \leq p + (p+2)N. \tag{2-5}$$

Define the integer J by $p = f(J,y)$, hence

$$T^J y = S^{f(J,y)}y = S^p y.$$

There are two cases for J.

<u>The case $J > 0$.</u> In this case, since $f(i,x_0) \neq p$ for all i, it follows from (2-4) that $J > Rp$. Now

$$f(Rp,y) = f(Rp,x_0) = MRp \geq p + (p+2)N,$$
$$f(J,y) = p$$

and for all i,

$$|f(i+1,y) - f(i,y)| = |n(T^i y)| \leq N. \tag{2-6}$$

Therefore there is a largest integer H such that

$$Rp < H < J \quad \text{and} \quad f(H,y) \geq p + (p+1)N. \tag{2-7}$$

Then since

$$\begin{aligned}(p+1)N &= |p - (p + (p+1)N)| \\ &\leq |p - f(H,y)| = |f(J-H, T^H y)| \\ &\leq (J-H)N,\end{aligned}$$

we must have $J - H > p$ and $p + H < J$. Because $H + 1 < J$ and $H + 1$ cannot satisfy the conditions (2-7), we have from (2-6) that

$$f(H, y) \leq p + (p+2)N.$$

Hence writing $f(H, y) = kp + \ell$ $(0 \leq \ell \leq p-1)$, we see by (2-5) that

$$\begin{aligned} f(p, S^{f(H,y)}y) &= f(p, S^{f(H,y)}x_0) = f(p, S^{\ell}x_0) \\ &= f(p, x_0) = Mp. \end{aligned}$$

Therefore,

$$\begin{aligned} f(H+p, y) &= f(H, y) + f(p, T^H y) \\ &= f(H, y) + f(p, S^{f(H,y)}(y)) \\ &= f(H, y) + Mp > f(H, y). \end{aligned}$$

Since $H + p < J$, this contradicts the choice of H as the largest integer satisfying the conditions (2-7).

The case $J < 0$. Note here that

$$y = S^{-p}(S^p y) = T^{-J}(S^p y),$$

that is, $f(-J, S^p y) = -p$. Therefore, we may essentially repeat the argument of the first case, with the triplet $(S^p y, y, -J)$ in place of $(y, S^p y, J)$, and with the relation $-p = f(-J, S^p y)$ in place of $p = f(J, y)$. This completes the proof. □

With this proposition we can clarify the structure of topological orbit equivalence for bounded cocycles and then for continuous cocycles.

THEOREM 2.3 (Bounded Orbit Equivalence). *Let S and T be topologically free homeomorphisms of a compact Hausdorff space X with the same orbits. Suppose S and T are related by a bounded cocycle, that is, there is a function $n : X \to \mathbf{Z}$ such that $Tx = S^{n(x)}x$ and $\max_x |n(x)| = N < \infty$. Let P be the union of all orbits which intersect the set of points at which n is discontinuous. Then*

(1) *P is a closed nowhere dense set of periodic points of period at most $2N$.*

(2) *There exists a decomposition of the complement of P into disjoint invariant open sets A and B such that there is a continuous integer-valued function a on $A \cup B$ satisfying*

$$\begin{aligned} n(x) &= 1 + a(x) - a(Tx), \qquad x \in A, \\ &= -1 + a(x) - a(Tx), \quad x \in B, \end{aligned}$$

and the homeomorphism $g(x) = S^{a(x)}(x)$ defines topological conjugacies of restrictions,

$$T|_A \cong S|_A \quad \textit{and} \quad T|_B \cong S^{-1}|_B.$$

REMARK. The above function $(a - a \circ T)$ is called a coboundary, the function a is called a transfer function, and we say that the cocycle n is cohomologous to $(1_A - 1_B)$ via the continuous transfer function a.

PROOF. Let $A_i = \{x : n(x) = i\}$, then X is the union of those A_i for which $|i| \leq N$. The function n is continuous in the interior of each A_i and discontinuous elsewhere. For $-N \leq i < j \leq N$, let D_{ij} denote the intersection of the boundaries of A_i and A_j. Then n is discontinuous at x if and only if x is in some D_{ij}, in which case

$$S^i x = Tx = S^j x, \quad \text{and} \quad S^{i-j}x = x \text{ with } |i-j| \leq 2N.$$

Thus each D_{ij} is a closed set of periodic points of period at most $2N$, and P is a union of finitely many sets of the form $S^k D_{ij}$. Because P is a closed set of periodic points and S is topologically free, P has empty interior. This proves the assertion (1). The proof of (2) needs a series of lemmas.

Below, we will use interval notation for sets of consecutive integers; for example, we let $[m,n]$ denote $\{k \in \boldsymbol{Z} : m \leq k \leq n\}$.

LEMMA 2.4. *Given a positive integer M and a point x_0 in $X \backslash P$, there exist a neighborhood U of x_0 and an integer $\overline{M} > 0$ such that for every y in U,*

$$[-M, M] \subseteq \{f(k,y) : k \in [-\overline{M}, \overline{M}]\}.$$

PROOF. By Proposition 2.2, the set $\{f(k,x) : k \in \boldsymbol{Z}\}$ exhausts the group $\boldsymbol{Z}$ at every point of $X \backslash P$. Take a neighborhood U of x_0 whose closure is still contained in $X \backslash P$. Then for each point y in $\overline{U}$ we can find a positive integer $\overline{M}_y$ such that

$$[-M, M] \subseteq \{f(k,y) : k \in [-\overline{M}_y, \overline{M}_y]\}.$$

Since each function $x \mapsto f(k,x)$ is continuous on $X \backslash P$, we may assume that the above inclusion holds on a neighborhood of y. Covering the compact set $\overline{U}$ with finitely many such neighborhoods, we may choose a positive integer $\overline{M}$ such that

$$[-M, M] \subseteq \{f(k,y) : k \in [-\overline{M}, \overline{M}]\}$$

for every y in U. This completes the proof of the lemma. □

Given a positive integer m, we define

$$A_m = \{x \in X \backslash P : \forall n \geq m, \ f(n,x) > 0 \text{ and } f(-n,x) < 0\},$$

$$B_m = \{x \in X \backslash P : \forall n \geq m, \ f(n,x) < 0 \text{ and } f(-n,x) > 0\}.$$

By definition A_m and B_ℓ are disjoint for any m and ℓ.

LEMMA 2.5. *The sets $A = \bigcup_{m>0} A_m$ and $B = \bigcup_{m>0} B_m$ are disjoint invariant open sets whose union is $X \backslash P$.*

PROOF. By the previous lemma, to each point x in $X \backslash P$ we may associate an open neighborhood U of x and a positive integer K (depending on U) such that for every y in U,

$$[-N, N] \subseteq \{f(k,y) : k \in (-K, K)\}.$$

Suppose $y \in U$. By Proposition 2.2, the map $k \mapsto f(k,y)$ is a bijection of $\boldsymbol{Z}$. In particular, if $k \geq K$, then $|f(k,y)| > N$, and since

$$|f(k \pm 1, y) - f(k,y)| \leq N$$

it follows that $f(k \pm 1, y)$ and $f(k,y)$ have the same sign. Thus the sign is constant in each of the sets $Z_1 = \{f(k,y) : k \geq K\}$ and $Z_2 = \{f(k,y) : k \leq -K\}$. Because

$$\boldsymbol{Z} = Z_1 \cup Z_2 \cup \{f(k,y) : k \in (-K,K)\}$$

it follows that the sign in Z_1 must be the opposite of the sign in Z_2. Thus either $y \in A_K$ or $y \in B_K$, and we have a decomposition

$$U = (U \cap A_K) \bigcup (U \cap B_K).$$

For each $m > 0$, A_m and B_m are closed sets of $X \backslash P$, hence the sets $U \cap A_K$ and $U \cap B_K$ are relatively open in U. Therefore they are open in X. Taking the union over such neighborhoods U of points in $X \backslash P$, we see that $X \backslash P$ is the union of open sets of the form $(U \cap A_K)$ or $(U \cap B_K)$. It follows that A and B are disjoint open sets with union $X \backslash P$.

If $x \in X \backslash P$, then $x \in A$ if and only if $\lim_{k \to +\infty} f(k,x) = +\infty$. For every x, we have $|f(k+1,x) - f(k,Tx)| = |n(x)| \leq N$, and it follows that the sets A and B are T-invariant. This proves the lemma. □

Given an integer M and x in A, define

$$\begin{aligned} c_M(x) &= \#\big((-M,\infty) \cap \{f(i,x) : i \leq 0\}\big), \\ a_M(x) &= c_M(x) - M, \\ a(x) &= \lim_{M \to \infty} a_M(x). \end{aligned}$$

The functions c_M and a_M are continuous on A. Because any point in A has a neighborhood U on which a_M is constant for all sufficiently large M, it follows that a also is continuous on A.

LEMMA 2.6. *For x in A,*

$$n(x) = 1 + a(x) - a(Tx).$$

PROOF. Suppose $M + n(x) > 0$. Then $f(0,Tx) = 0 \in (-M - n(x), \infty)$, and

$$\begin{aligned} c_M(x) &= \#\big((-M,\infty) \cap \{f(i,x) : i \leq 0\}\big) \\ &= \#\big((-M,\infty) \cap \{f(i-1,Tx) + n(x) : i \leq 0\}\big) \\ &= \#\big((-M - n(x),\infty) \cap \{f(i-1,Tx) : i \leq 0\}\big) \\ &= \#\big((-M - n(x),\infty) \cap \{f(i,Tx) : i \leq 0\}\big) - 1 \\ &= c_D(Tx) - 1, \end{aligned}$$

where D denotes $M+n(x)$. For all large M, $c_D(x)=c_M(x)+n(x)$, and therefore for all large M,

$$n(x)=c_D(x)-c_M(x)=c_D(x)-c_D(Tx)+1$$
$$=c_M(x)-c_M(Tx)+1=a(x)-a(Tx)+1.$$

This proves the lemma. □

Proof of the theorem. The argument of Lemma 2.6 was applied to the triple (S,T,n) to define the function a on A. For x in B, we have $Tx=(S^{-1})^{-n(x)}x$, and we apply the argument of Lemma 2.6 to the triple $(S^{-1},T,-n)$ to produce a continuous function b on B such that for x in B,

$$-n(x)=1+b(x)-b(Tx).$$

Defining $a=-b$ on B, we get

$$n(x)=-1+a(x)-a(Tx),\quad x\in B.$$

This proves the cocycle relations claimed for n and a.

Now for x in A, $(Sg)x=S^{\{1+a(x)\}}x$ and $(gT)x=S^{\{n(x)+a(Tx)\}}x$. Thus by Lemma 2.6, $Sg=gT$ on A. Because S and T have the same orbits and g sends each orbit into itself, the restriction $g|_A$ will be a bijection if and only if it is a bijection on each orbit O. Automatically, $g:O\to O$ is surjective (since for all k, $S^kg=gT^k$). If g collapses distinct points x and T^ix in O, then the cardinality of the image of O is at most $|i|$, and by the surjectivity O cannot be infinite. On the other hand, any surjection from a finite set O to itself must be injective. This shows $g|_A$ is a bijection. Moreover, because $g=S^a$ is a local homeomorphism, it then follows that $g|_A$ is a homeomorphism and thus gives a conjugacy $T|_A\cong S|_A$. The argument that $g|_B$ induces a conjugacy, $T|_B\cong S^{-1}|_B$ is essentially the same. This finishes the proof. □

A homeomorphism $X\to X$ is topologically transitive if every nonempty invariant open set is dense. Two homeomorphisms are flip conjugate if they are conjugate or one is conjugate to the inverse of the other.

Corollary 2.7. *Suppose S and T are topologically transitive homeomorphisms satisfying the assumptions of Theorem* (2.3). *Then one of the sets A and B in Theorem* (2.3) *is empty, and the restrictions of S and T to $X\backslash P$ are flip conjugate.*

§3. Continuous cocycles and C^*-algebras.

Specializing Theorem 2.3 to the case of a continuous cocycle gives us the following.

Theorem 3.1. *Suppose S and T are topologically free homeomorphisms of a compact Hausdorff space X with the same orbits, and $n:X\to\mathbf{Z}$ is a continuous function such that $Tx=S^{n(x)}x$. Then there exists a decomposition of X into disjoint invariant open sets A and B such that $T|_A$ is conjugate to $S|_A$, and $T|_B$ is conjugate to $S^{-1}|_B$.*

In both cases the conjugacy g may be given the form $gx = S^{a(x)}x$, where the transfer function a is continuous.

The following result, proved in [2] in the metrizable case, is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Keep the assumptions of Theorem* 3.1 *and suppose also that S or T is topologically transitive. Then one of the sets A and B is empty, and T is flip conjugate to S by a homeomorphism $gx = S^{a(x)}x$, where the transfer function a is continuous.*

Remark 3.3. Theorem 3.2 is a topological analogue of Belinskaya's theorem [1] in ergodic theory (see [6] for an elegant exposition). Belinskaya proved that if S and T are ergodic automorphisms of a Lebesgue probability space with the same orbits by an integrable cocycle n, then T is flip conjugate (in the measurable category) to S by a measure-preserving transformation $gx = S^{a(x)}x$, where the transfer function a is measurable. (The function a need not be integrable, even if n is bounded.) Here the assumption of ergodicity parallels the assumption of transitivity in Theorem 3.2. If the assumption of ergodicity is relaxed to aperiodicity, then the conclusion parallels Theorem 3.1: the Lebesgue space is the disjoint union of invariant measurable sets A and B such that $T|_A \cong S|_A$, and $T|_B \cong S^{-1}|_B$, with n cohomologous to 1 on A and to -1 on B. □

Remark 3.4. Let S and T denote homeomorphisms of a compact Hausdorff space X with the same orbits, with $Tx = S^{n(x)}x$. If X is connected and n is continuous, then clearly $S = T$ or $S = T^{-1}$. However, each of the following periodicity-connectivity conditions (which make no reference to the cocycle n) will also guarantee the rigidity conclusion, that $S = T$ or $S = T^{-1}$.

(1) The complement of the periodic points is path-connected and dense [8].

(2) S has no periodic points and X is connected [4].

These results follow from a theorem of Sierpinski ([13] or [9, p. 173]): if a connected compact Hausdorff space X is the union of a countable family of disjoint closed sets, then all but one of these sets is empty. (Kupka gave a category proof for the special case $X = [0, 1]$, and Giordano-Putnam-Skau appealed directly to Sierpinski's result.) In Kupka's situation [8], for example, it follows that the cocycle n must be constant on any arc of aperiodic points, and thus on a dense set; thus for some integer q, the equality $T = S^q$ holds on a dense set, hence everywhere, and the existence of aperiodic points forces $|q| = 1$. □

We've stated our results for homeomorphisms S and T with the same orbits. The results are easily reformulated into the situation where S is a homeomorphism of a space X, T is a homeomorphism of a space Y, and there exists a homeomorphism h sending the orbits of S to those of T (for the reformulation, replace S in the statements with hSh^{-1}).

For a homeomorphism S, we let $\mathscr{A}(S)$ denote its transformation group C^*-algebra, i.e., the crossed product algebra generated by the action of S on $C(S)$, the continuous functions into $\boldsymbol{C}$. We write $\mathscr{A}(S) \cong \mathscr{A}(T)$ to indicate isomorphism of the C^*-algebras, and we write $(\mathscr{A}(S), C(S)) \cong (\mathscr{A}(T), C(T))$ if there is a restricted isomorphism of $\mathscr{A}(S)$ and $\mathscr{A}(T)$, i.e. an isomorphism of C^*-algebras $\mathscr{A}(S) \to \mathscr{A}(T)$ which sends the subalgebra $C(S)$ onto $C(T)$. In the interplay between topological dynamics and C^*-theory, one naturally wants to understand these relations.

Giordano, Putnam and Skau [**4**] solved these problems completely within the class of minimal homeomorphisms of the Cantor set. For S and T in that class, they proved there is a restricted isomorphism of $\mathscr{A}(S)$ and $\mathscr{A}(T)$ if and only if S and T are flip conjugate. Their characterization for $\mathscr{A}(S) \cong \mathscr{A}(T)$ showed that isomorphism in general does not imply restricted isomorphism. (In fact, the Strong Orbit Realization Theorem of Ormes [**10**] shows that among minimal homeomorphisms of the Cantor set, for any S the set of T such that $\mathscr{A}(S) \cong \mathscr{A}(T)$ exhibits a vast range of dynamics.)

One ingredient in the Giordano-Putnam-Skau [**4**] classification of minimal homeomorphisms of the Cantor set up to restricted isomorphism of their crossed product algebras was the metrizable case of Theorem 3.2. Appealing to Theorem 3.1 and the general result [**15**, Theorem 2], we will give a general dynamical characterization for restricted isomorphism. (A characterization of isomorphism does not seem close.) For completeness we first include the proof of a necessary but easy lemma.

LEMMA 3.5. *Suppose S and T are topologically free homeomorphisms of a compact Hausdorff space with the same orbits, and there is a continuous function n such that $T = S^n$.*

Then there is a continuous function m such that $S = T^m$.

PROOF. We continue the notation of Section 2. By Proposition 2.2, for any point x there exists a unique integer $k(x)$ such that $f(k(x), x) = 1$. Define $m(x) = k(x)$, then

$$T^{m(x)}x = S^{f(k(x),x)}x = Sx.$$

Because n is continuous on X, for every integer j the function $f_j : x \mapsto f(j, x)$ is continuous on X. For each j, the set $\{x : m(x) = j\}$ is equal to the closed open set $\{x : f_j(x) = 1\}$. It follows that m is continuous. $\square$

Below, $X = X_1 \cup X_2$ is a decomposition of X if the sets X_1 and X_2 are closed and disjoint (one may be empty).

THEOREM 3.6. *Suppose S and T are topologically free homeomorphisms of compact Hausdorff spaces X and Y. Then the following are equivalent.*

(1) $(\mathscr{A}(S), C(S)) \cong (\mathscr{A}(T), C(T))$.

(2) *There exist decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $S|X_1$ is conjugate to $T|Y_1$ and $S|X_2$ is conjugate to $T^{-1}|Y_2$.*

If S and T are topologically transitive or the spaces are connected, then these conditions hold if and only if S and T are flip conjugate.

PROOF. According to Theorem 2 of [**15**], the restricted isomorphism holds if and only if there is a homeomorphism $h : X \to Y$ and continuous functions $m : X \to \boldsymbol{Z}$ and $n : Y \to \boldsymbol{Z}$ such that

$$T = (hSh^{-1})^n \qquad \text{and} \qquad S = (h^{-1}Th)^m.$$

Now the theorem follows immediately from Lemma 3.5 and Theorem 3.1. □

REMARK 3.7. Theorem 3.6 was proved in [**15**] for transitive S and metrizable X via [**15**, Theorem 2] and [**2**].

REMARK 3.8. Proposition 2.2, Theorem 2.3 and Theorem 3.1 all fail without the assumption of topological freeness. For example, let S be the rational rotation of the circle through angle $2\pi/5$ (so, $S^5 = \mathrm{Id}$), and let $T = S^2$, with the constant cocycle $n \equiv 2$. Here, it is easy to check S and T are not flip conjugate, and even their C^*-algebras $\mathscr{A}(S), \mathscr{A}(T)$ are not isomorphic [**5**, **17**].

§4. Examples.

Throughout this section, S and T are topologically free homeomorphisms of a compact metric space with the same orbits. By Lemma 3.5, if $T = S^n$ with n continuous, then $S = T^m$ with m continuous. The following example shows that this statement is false if "continuous" is replaced by "bounded".

EXAMPLE 4.1. Let X be a one-point compactification of the group $\boldsymbol{Z}$ and let $S : X \to X$ be the map $x \mapsto x + 1$, fixing the point ∞. Partition the integers into disjoint integer intervals A_k, $k \in \boldsymbol{Z}$, such that A_k contains $3(|k| + 1)$ points. We denote the integers in A_k in order as $a_0^k, a_1^k, \dots, a_{3|k|+2}^k$, and we place A_{k+1} to the right of A_k, i.e. $a_{(3|k|+2)}^k + 1 = a_0^{k+1}$. We will define a cocycle $n : X \to \boldsymbol{Z}$ such that $T = S^n$ is a homeomorphism with the same two orbits as S, and T moves through the sets A_k as ordered by k, but within a set A_k the map T will act basically as a 3-steps up and down shift. Explicitly, set $n(\infty) = 1$, and for each k define n on three "transition points" by the rule

$$n(a_j^k) = 1, \qquad \text{if } j = 1, 3|k|, \text{ or } 3|k| + 2$$

and otherwise define n by the rules

$$\begin{aligned} n(a_j^k) &= -3 \qquad \text{if } j \equiv 1\,(mod\,3) \\ &= +3 \qquad \text{if } j \not\equiv 1\,(mod\,3). \end{aligned}$$

(For example, if $A_3 = [0, 11]$, then T moves through the points of $[0, 12]$ in the order $0, 3, 6, 9, 10, 7, 4, 1, 2, 5, 8, 11, 12$.) It follows that $|n| \le 3$. The cocycle m such that $S = T^m$ is uniquely determined on the infinite orbit $\boldsymbol{Z}$, and

$$a_1^k = S(a_0^k) = T^{2k+1}(a_0^k),$$

so that $m(a_0^k) = 2k + 1$, and m is unbounded. □

REMARK 4.2. The proof of Proposition 2.2 can be considerably simplified when the cocycle n is continuous. Thus it is natural to try to reduce the proof of the Bounded Orbit Equivalence Theorem 2.3 to the continuous result, Theorem 3.1. When *both* cocycles n and m are bounded, this can be carried out in the following way. The restrictions of S and T to $X\backslash P$ extend to homeomorphisms $\tilde{S}$ and $\tilde{T}$ of the Čech compactification $\beta(X\backslash P)$ of the locally compact space $X\backslash P$. A point x in $\beta(X\backslash P)$ is a limit of a net x_α of points in $X\backslash P$, and

$$\tilde{S}x = \lim_\alpha Sx_\alpha = \lim_\alpha T^{k(\alpha)}x_\alpha.$$

Because there are only finitely many possible values above for $k(\alpha)$, we may pass to a subnet on which $k(\alpha)$ is some constant k, and deduce $\tilde{S}x = \tilde{T}^k x$. This with the symmetric argument shows that $\tilde{S}$ and $\tilde{T}$ have the same orbits. Since n extends to a continuous function $\tilde{n}$ such that $\tilde{T} = \tilde{S}^{\tilde{n}}$, we may apply Theorem 3.1 (which is not limited to the metrizable case) to $\tilde{S}$ and $\tilde{T}$, and then obtain the desired results on $X\backslash P$ by restriction.

But if only one of the cocycles m and n is bounded, then the homeomorphisms $\tilde{S}$ and $\tilde{T}$ will not have the same orbits, and this argument breaks down. □

In the next example, S and T are homeomorphisms of a nice connected compact metric space with the same orbits via a bounded cocycle, but S and T are not flip conjugate. In addition, $\mathscr{A}(S)$ and $\mathscr{A}(T)$ are not isomorphic. The authors are indebted to T. Natsume and G. Stuck for discussions of this example.

EXAMPLE 4.3. Let $\boldsymbol{T}^2$ denote the two dimensional torus $\boldsymbol{R}^2/\boldsymbol{Z}^2$. Denote points of $\boldsymbol{T}^2$ as pairs (s,t) with $0 \le s,t < 1$. Let S be the homeomorphism of $\boldsymbol{T}^2$ defined by $S:(s,t) \mapsto (s, s+t)$. Let T be the homeomorphism defined by

$$\begin{aligned} T : x &\mapsto Sx \quad && \text{if } x = (s,t) \text{ with } 0 \le s \le 1/2, \\ &\mapsto S^{-1}x && \text{if } x = (s,t) \text{ with } 1/2 \le s < 1. \end{aligned}$$

Each circle $C_s = \{(s,t) \in \boldsymbol{T}^2 : 0 \le s < 1\}$ is invariant under S and T, which act on C_s as rotations. The homeomorphisms S and T are topologically free, but not topologically transitive. The map T is homotopic to the identity by the homotopy $\{T_r : 0 \le r \le 1\}$ given by

$$\begin{aligned} T_r : (s,t) &\mapsto (s, t+rs), && \text{if } 0 \le s \le 1/2, \\ &\mapsto (s, t + r(1-s)) && \text{if } 1/2 \le s \le 1. \end{aligned}$$

On the other hand, the induced map by S on the fundamental group of $\boldsymbol{T}^2$ is not trivial (it is expressed by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). Thus S is not flip conjugate to T.

To see that the C^*-algebras $\mathscr{A}(S)$ and $\mathscr{A}(T)$ are not isomorphic, we recall for our situation the cyclic six-term exact sequence of Pimsner and Voicelescu [11] (in which R denotes either S or T).

$$\begin{tikzcd}
K_0(C(\boldsymbol{T}^2)) \arrow[r, "Id^*-R^*"] & K_0(C(\boldsymbol{T}^2)) \arrow[r] & K_0(\mathscr{A}(R)) \arrow[d] \\
K_1(\mathscr{A}(R)) \arrow[u] & K_1(C(\boldsymbol{T}^2)) \arrow[l] & K_1(C(\boldsymbol{T}^2)). \arrow[l, "Id^*-R^*"']
\end{tikzcd}$$

It is well known that $K_0(C(\boldsymbol{T}^2)) \cong \boldsymbol{Z}^2 \cong K_1(C(\boldsymbol{T}^2))$ (e.g. [**14**, p. 154] or [**16**, §6.5]), and the induced maps R^* depend only on the homotopy class of R [**16**, §6.4]. Thus for $R = T$, the induced maps $Id^* - R^*$ vanish, and therefore $K_1(\mathscr{A}(T)) \cong \boldsymbol{Z}^4$. Similarly, we can deduce $K_1(\mathscr{A}(S)) \ncong \boldsymbol{Z}^4$ (and therefore $\mathscr{A}(S) \ncong \mathscr{A}(T)$) by checking that the induced map S^* on $K_1(C(\boldsymbol{T}^2))$ is nontrivial.

This last step can be carried out along the lines of [**14**, pp. 161–2] or in an elementary and self-contained way as follows. For U in $GL(\infty, C(\boldsymbol{T}^2))$, the homotopy class of $\det U$ (as a map into the punctured plane, which modulo homotopy we regard as the circle) is an invariant of the $K_1(C(\boldsymbol{T}^2))$ equivalence class of U. The map $\det U$ induces a homomorphism $d_U : \pi_1(\boldsymbol{T}^2) \to \pi_1(S^1)$ (by postcomposition) which depends only on the homotopy class of $\det U$. Because $d_{(S*U)}$ is d_U precomposed by the nontrivial action of S on $\pi_1(\boldsymbol{T}^2)$, we can conclude that the map S^* on $K_1(C(\boldsymbol{T}^2))$ is nontrivial. □

QUESTION 4.4. On a compact manifold (or even a compact connected metric space), are there homeomorphisms with the same orbits whose crossed-product C^*-algebras are isomorphic but not restricted isomorphic? By the work [**4**] of Giordano, Putnam and Skau, such homeomorphisms exist on the Cantor set.

The homeomorphisms in Example 4.3 are not transitive, but it is not the case that transitive homeomorphisms which share orbits by a bounded cocycle must be flip conjugate. To show this we provide the simple example below.

EXAMPLE 4.5. [**2**, Ch. 2, Ex. 4.3] Let x be the doubly infinite sequence on symbols $\{a, 0, 1\}$ defined by $x_0 = a$; $x_n = 0$ if n is even and nonzero; and $x_n = 1$ if n is odd. Let S be the subshift whose domain is the orbit of x (under the shift), and the two point orbit on which it accumulates. Let $T = S^n$, where $n = 1$ on the two point orbit and n is defined on the orbit of x by

$$\begin{aligned} n(S^k x) &= 1 && \text{if } k < 0 \\ &= 2 && \text{if } k = 0 \\ &= -1 && \text{if } k > 0 \text{ and } k \text{ is even} \\ &= 3 && \text{if } k > 0 \text{ and } k \text{ is odd.} \end{aligned}$$

If we identify $S^k x$ with k, then we can summarize this by saying that T runs through the orbit of x in the order $\cdots -4, -3, -2, -1, 0, 2, 1, 4, 3, 6, 5, 8, 7, 10, 9 \ldots.$ Now S and T have the same orbits; S and T are (trivially) transitive and topologically free; and n is bounded. But S and T cannot be flip conjugate because an aperiodic orbit of S^2 has one accumulation point, whereas an aperiodic orbit of T^2 has two accumulation points. □

The idea of Example 4.5 can be developed to produce examples of mixing positive-entropy subshifts S and T with the same orbits by a bounded cocycle, but with S and T not flip conjugate [**2**, Ch. 2, Ex. 4.4].

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