J. Math. Soc. Japan
Vol. 50, No. 2, 1998

Face algebras I—A generalization of quantum group theory

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(Received Feb. 29, 1996)

§ 0. Introduction

In the last decade, it was recognized that monoidal categories or categories with tensor product play essential roles in many branches of mathematics and mathematical physics such as Jones' index theory, low-dimensional topology and conformal field theory. For example, Ocneanu's classification theory of II_1-subfactors deeply depends on the structure of the monoidal categories of bimodules over von Neumann algebras (cf. [**O**]). On the other hand, it is known that a certain class of monoidal additive categories gives rise to 2-dimensional topological quantum field theories (see Turaev [**T**]).

These developments naturally stimulate to construct non-trivial examples of monoidal categories. Many examples are constructed using representation theory of bialgebras (quantum groups). However, there still exist monoidal categories which have no representation-theoretic interpretation; for example, II_1-subfactors of type D–E have no representation-theoretic counterparts.

In this paper, we begin to study a new algebraic structure named face algebra, which is a generalization of bialgebra. Although, its definition is much more complicated than that of bialgebra, the category of its (co-)modules still has the structure of monoidal abelian category. By considering additional structures on it (such as an antipode, a universal R-matrix, a ribbon structure and a $*$-structure), we obtain monoidal categories with rich additional structures.

In this paper, we concentrate our attention on elementary properties of face algebras and their (co-)module categories. Non-trivial examples and applications will be given elsewhere (cf. [**H1–6**]).

In Sections 1, 2 and 3, We show several basic formulas for face algebras, their antipodes and their universal R-matrices. In Sections 4 and 5, we show that the categories of modules and comodules of face algebras naturally become monoidal categories. When face algebras have antipodes or universal R-matrices, we also discuss the rigidity or the braiding structure of these categories.

Throughout this paper, we work on a fixed ground field $\boldsymbol{K}$. For an algebra A, we denote its product by $m = m_A$. For a coalgebra C, we denote its coproduct and its counit by $\Delta = \Delta_C$ and $\varepsilon = \varepsilon_C$ respectively. We also use the "sigma" notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ $(a \in C)$ (cf. [**S**]).

§ 1. Commutative separable algebras

In this section, we discuss some aspects of commutative separable algebras, which play basic roles throughout this paper. Let R be a commutative algebra over a field $\boldsymbol{K}$. Let e be an element of $R \otimes R$ and let $\{e_i\}$ and $\{e_i'\}$ be elements of R such that $e = \sum_i e_i \otimes e_i'$ and that $\{e_i\}$ are linearly independent. We say that (R, e) is a *separable algebra* if the following two relations are satisfied:

$$\sum_i \lambda e_i \otimes e_i' = \sum e_i \otimes \lambda e_i' \quad (\lambda \in R), \quad \sum_i e_i e_i' = 1. \tag{1.1}$$

We call e a *separating idempotent* of R. This terminology is justified since

$$e^2 = \sum_{ij} e_i \otimes e_i' e_j e_j' = e.$$

It is known that R has a separating idempotent if and only if R is a finite direct product of separable field extensions of $\boldsymbol{K}$ (see e.g. [**P**]). Moreover e is unique if it exists. In fact, we have

$$f = \sum_{ik} f_k \otimes f_k' e_i e_i' = \sum_{ik} f_k e_i \otimes f_k' e_i' = e$$

for another separating idempotent $f = \sum_k f_k \otimes f_k'$. The following proposition shows that (R, e) may be viewed as a "self dual quantum group."

PROPOSITION 1.1. *Let (R, e) be a commutative separable algebra.*
(1) *Then R becomes a cocommutative coalgebra with coproduct Δ_R defined by $\Delta_R(\lambda) := \sum_i \lambda e_i \otimes e_i'$ $(\lambda \in R)$. Moreover, we have Δ_R $(\lambda\mu) = \Delta_R(\lambda)\Delta_R(\mu)$ for each $\lambda, \mu \in R$.*
(2) *Let $\mathring{R} := \mathrm{Hom}_{\boldsymbol{K}}(R, \boldsymbol{K})$ be the dual algebra of (R, Δ_R) equipped with the dual coalgebra structure of R. Then, the map $\mathring{R} \to R$; $\varphi \mapsto \mathring{\varphi} := \sum_i \langle \varphi, e_i \rangle e_i'$ is an algebra and a coalgebra isomorphism. Its inverse $R \to \mathring{R}$; $\lambda \mapsto \mathring{\lambda}$ satisfies $\mathring{\lambda} = \sum_i \langle \mathring{e}_i, \lambda \rangle \mathring{e}_i'$.*
(3) *The element $\mathring{e} := \sum_i \mathring{e}_i \otimes \mathring{e}_i'$ is a separating idempotent of $\mathring{R}$. The coproduct of $\mathring{R}$ satisfies $\Delta_{\mathring{R}}(\varphi) = \sum_i \varphi \mathring{e}_i \otimes \mathring{e}_i'$ $(\varphi \in \mathring{R})$.*
(4) *For each $\varphi, \psi \in \mathring{R}$ and $\lambda, \mu \in R$, we have*

$$\varepsilon_R(\mathring{\varphi}\lambda) = \langle \varphi, \lambda \rangle = \varepsilon_{\mathring{R}}(\varphi\mathring{\lambda}), \tag{1.2}$$

$$\langle \varphi, \mathring{\psi} \rangle = \langle \mathring{\varphi}, \psi \rangle, \quad \langle \lambda, \mathring{\mu} \rangle = \langle \mathring{\lambda}, \mu \rangle, \tag{1.3}$$

$$\langle \varphi\psi, \lambda \rangle = \langle \varphi, \mathring{\psi}\lambda \rangle. \tag{1.4}$$

(5) *The sets $\{e_i\}$ and $\{e_i'\}$ are linear bases of R, whose dual bases are respectively $\{\mathring{e}_i'\}$ and $\{\mathring{e}_i\}$.*

PROOF. The coassociativity of Δ_R easily follows from the first relation of (1.1), while the relation $\Delta_R(\lambda)\Delta_R(\mu) = \Delta_R(\lambda\mu)$ follows from $e^2 = e$. Because of the uniqueness of e, we have

$$\sum_i e_i \otimes e_i' = \sum_i e_i' \otimes e_i, \tag{1.5}$$

which proves the cocommutativity of Δ_R. To show the existence of the counit, we show that the map $\varphi \mapsto \mathring{\varphi}$ ($\varphi \in \mathring{R}$) is bijective. Let ψ be an element of the dual coalgebra $\mathring{R}$ such that $\mathring{\psi} = 0$. Since

$$\begin{aligned}
&\left\langle \sum_{(\psi)} (\psi_{(1)})^\circ \otimes \psi_{(2)}, \varphi \otimes \lambda \right\rangle \\
&= \sum_i \sum_{(\psi)} \langle \psi_{(1)}, e_i \rangle \langle \psi_{(2)}, \lambda \rangle \langle e'_i, \varphi \rangle \\
&= \sum_i \langle \psi, e_i \lambda \rangle \langle e'_i, \varphi \rangle \\
&= \sum_i \langle \psi, e_i \rangle \langle e'_i \lambda, \varphi \rangle \\
&= \langle \mathring{\psi} \lambda, \varphi \rangle = 0
\end{aligned}$$

for each $\varphi \in \mathring{R}$ and $\lambda \in R$, we have $\sum_{(\psi)} (\psi_{(1)})^\circ \otimes \psi_{(2)} = 0$. Hence,

$$\begin{aligned}
\langle \psi, \lambda \rangle &= \sum_i \langle \psi, e_i e'_i \lambda \rangle \\
&= \sum_i \sum_{(\psi)} \langle \psi_{(1)}, e_i \rangle \langle \psi_{(2)}, e'_i \lambda \rangle \\
&= \sum_{(\psi)} \langle \psi_{(2)}, (\psi_{(1)})^\circ \lambda \rangle \\
&= 0
\end{aligned}$$

for each $\lambda \in R$. This implies the injectivity of the map $\varphi \mapsto \mathring{\varphi}$. Since $\dim_K R = \dim_K \mathring{R} < \infty$, $\varphi \mapsto \mathring{\varphi}$ is a linear isomorphism. Hence there exists an element ε_R of $\mathring{R}$ such that $1_R = \mathring{\varepsilon}_R = \sum_i \varepsilon_R(e_i) e'_i$. Using (1.1) and (1.5), we conclude that $(R, \Delta_R, \varepsilon_R)$ is a cocommutative coalgebra with counit ε_R. The verification of the fact that $\varphi \mapsto \mathring{\varphi}$ is an algebra map is straightforward. For example, we obtain

$$\begin{aligned}
(\varphi\psi)^\circ &= \sum_{i,j} \langle \varphi \otimes \psi, e_i e_j \otimes e'_j \rangle e'_i \\
&= \sum_{i,j} \langle \varphi \otimes \psi, e_i \otimes e_j \rangle e'_i e'_j \\
&= \mathring{\varphi} \mathring{\psi} \quad (\varphi, \psi \in \mathring{R}).
\end{aligned}$$

Next we show part (4). We calculate

$$\varepsilon_R(\mathring{\varphi}\lambda) = \sum_i \langle \varphi, e_i \rangle \varepsilon_R(e'_i \lambda) = \sum_i \langle \varphi, e_i \lambda \rangle \varepsilon_R(e'_i) = \langle \varphi, \lambda \rangle. \tag{1.6}$$

The first relation of (1.3) follows from (1.5), and the second relation follows from the first relation. The relation (1.4) follows from (1.6) and $(\varphi\psi)^\circ = \mathring{\varphi}\mathring{\psi}$. The second equality of (1.2) follows from (1.4). Part (3) easily follows from the first assertion of (2). The second assertion of (2) follows from (1.3). Since the map $\mathring{R} \to R$; $\varphi \mapsto \mathring{\varphi} = \sum_i \langle \varphi, e_i \rangle e'_i = \sum_i \langle \varphi, e'_i \rangle e_i$ is surjective, both $\{e_i\}$ and $\{e'_i\}$ span R. By the assumption on $\{e_i\}$, this proves the first assertion of (5). The other assertion of (5) follows from $e'_j = (\mathring{e}'_j)^\circ = \sum_i \langle \mathring{e}'_j, e_i \rangle e'_i$. □

EXAMPLE. (1) Let $R = \boldsymbol{Q}(\sqrt{n})$ be a quadratic field extension of the rational number field $\boldsymbol{K} = \boldsymbol{Q}$. Then, we have $e = (1 \otimes 1 + \sqrt{n} \otimes (1/\sqrt{n}))/2$ and $\varepsilon(q + r\sqrt{n}) = 2q$ $(q, r \in \boldsymbol{Q})$. Since $\varepsilon(1) \neq 1$, R is not a bialgebra.

(2) Let R be a finite direct product $\prod_{i \in \mathscr{V}} \boldsymbol{K}_i$, where $\boldsymbol{K}_i$ $(i \in \mathscr{V})$ is a copy of $\boldsymbol{K}$. Then we may take e_i and e'_i as the primitive idempotent corresponding to $\boldsymbol{K}_i$, that is, $e_i = e'_i = (\delta_{ij} 1_{\boldsymbol{K}_i})_{j \in \mathscr{V}}$. The counit is given by $\varepsilon_R(e_i) = 1$.

NOTATION. Since R is cocommutative, we have $\sum_i e_i \otimes e'_i = \sum_i e'_i \otimes e_i$. Hence there exists no essential difference between the bases $\{e_i\}$ and $\{e'_i\}$. In the following, we denote e by $\sum_i e_i \otimes e_i$ instead of $\sum_i e_i \otimes e'_i$, in order to simplify the notation. The letters φ and ψ stand for elements of $\mathring{R}$ and the letters λ and μ stand for elements of R.

§ 2. Face algebras and their antipodes

2.1. Face algebras

For a commutative separable algebra R, we set $\mathfrak{E}(R) := \mathring{R} \otimes R$ and regard it as an algebra via Proposition 1.1 (2). We also define a coalgebra structure on $\mathfrak{E}(R)$ via the identification $\mathfrak{E}(R) \simeq \mathrm{End}_{\boldsymbol{K}}(R)^*$. By Proposition 1.1 (5) and (4), the coproduct and the counit of $\mathfrak{E}(R)$ are given by

$$\Delta(\varphi \otimes \lambda) = \sum_i (\varphi \otimes e_i) \otimes (\mathring{e}_i \otimes \lambda), \tag{2.1}$$

$$\varepsilon(\varphi \otimes \lambda) = \langle \varphi, \lambda \rangle = \varepsilon_R(\mathring{\varphi}\lambda) = \varepsilon_{\mathring{R}}(\varphi\mathring{\lambda}) \quad (\varphi \in \mathring{R}, \quad \lambda \in R), \tag{2.2}$$

where ε_R and $\varepsilon_{\mathring{R}}$ denote the unit of (R, Δ_R) and $(\mathring{R}, \Delta_{\mathring{R}})$ respectively. Using (2.1) and $e^2 = e$, we see that $\Delta(ab) = \Delta(a)\Delta(b)$ for each $a, b \in \mathfrak{E}(R)$.

Let $\mathfrak{H}$ be a $\boldsymbol{K}$-algebra equipped with a coalgebra structure $(\mathfrak{H}, \Delta, \varepsilon)$ such that $\Delta(ab) = \Delta(a)\Delta(b)$ for each $a, b \in \mathfrak{H}$. Let (R, e) be a commutative separable algebra over $\boldsymbol{K}$ and let $\Omega = \Omega_{\mathfrak{H}} : \mathfrak{E}(R) \to \mathfrak{H}$ be both an algebra and a coalgebra map. For simplicity, we denote $\Omega(\varphi \otimes \lambda) \in \mathfrak{H}$ by $\varphi\lambda$ for each $\varphi \in \mathring{R}$ and $\lambda \in R$. We say that $\mathfrak{H} = (\mathfrak{H}, \Omega)$ is an *R-face algebra* if

$$\sum_i \varepsilon(ae_i)\varepsilon(\mathring{e}_i b) = \varepsilon(ab) \tag{2.3}$$

for each $a, b \in \mathfrak{H}$. A linear map $f : \mathfrak{H} \to \mathfrak{H}'$ of R-face algebras is called a *map of R-face algebras* if f is both an algebra and a coalgebra map such that $f \circ \Omega_{\mathfrak{H}} = \Omega_{\mathfrak{H}'}$, and a subspace $\mathfrak{J} \subset \mathfrak{H}$ is called a *biideal* if it is both an ideal and a coideal. The quotient $\mathfrak{H}/\mathfrak{J}$ is an R-face algebra precisely when $\mathfrak{J}$ is a biideal of $\mathfrak{H}$.

EXAMPLE (1) For a commutative separable algebra R, $\mathfrak{C}(R)$ becomes an R-face algebra via $\Omega = \mathrm{id}$. Moreover, R itself becomes an R-face algebra via $\Omega : \mathfrak{C}(R) \to R$; $\varphi \otimes \lambda \mapsto \mathring{\varphi}\lambda$ $(\varphi \in \mathring{R}, \lambda \in R)$.

(2) Each bialgebra $\mathfrak{H}$ naturally becomes a $\boldsymbol{K}$-face algebra. Moreover, an R-face algebra is a bialgebra if and only if $R \simeq \boldsymbol{K}$ (see Corollary 4.6).

(3) Let V be an R-bimodule such that $\dim_{\boldsymbol{K}}(V) < \infty$. Then $\mathfrak{H}(V) := \bigoplus_{r \geq 0} \mathrm{End}_{\boldsymbol{K}}(V^{(r)})^*$ naturally becomes an R-face algebra, where $V^{(0)} = R$, $V^{(1)} = V$ and $V^{(r+1)} = V^{(r)} \otimes_R V$. Moreover, every finitely generated R-face algebra is isomorphic to $\mathfrak{H}(V)/\mathfrak{I}$ for some V and a biideal $\mathfrak{I} \subset \mathfrak{H}(V)$. See **[H6]** for a proof of this result.

(4) Let $R = \prod_{i \in \mathscr{V}} \boldsymbol{K}_i$, $\boldsymbol{K}_i \simeq \boldsymbol{K}$ and e_i be as in Example (2) of Section 1. In this case, we call an R-face algebra a *$\mathscr{V}$-face algebra*. An algebra $\mathfrak{H}$ is a $\mathscr{V}$-face algebra if and only if it is a coalgebra equipped with elements $\{e^i_j \mid i, j \in \mathscr{V}\}$ which satisfy the following relations:

$$e^i_j e^m_n = \delta_{im}\delta_{jn} e^i_j \quad (i, j, m, n \in \mathscr{V}), \quad \sum_{i,j} e^i_j = 1,$$

$$\Delta(e^i_j) = \sum_k e^i_k \otimes e^k_j \quad (i, j \in \mathscr{V}), \quad \varepsilon(e^i_j) = \delta_{ij} \quad (i, j \in \mathscr{V}),$$

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \varepsilon(ab) = \sum_{i,j,k} \varepsilon(ae^i_k)(e^k_j b) \quad (a, b \in \mathfrak{H}).$$

In fact, $e^i_j := \Omega(\mathring{e}_i \otimes e_j)$ satisfies these relations.

We refer the reader to **[H1, 2, 4, 5]** for more non-trivial examples of face algebras.

For an R-face algebra $\mathfrak{H}$, let $\mathfrak{H}^{\mathrm{op}}$ (resp. $\mathfrak{H}^{\mathrm{cop}}$) be a copy of $\mathfrak{H}$ equipped with the same coproduct (resp. product) as $\mathfrak{H}$ and the opposite product $m^{\mathrm{op}} : a \otimes b \mapsto ba$ (resp. the opposite coproduct $\Delta^{\mathrm{cop}} : a \mapsto \sum_{(a)} a_{(2)} \otimes a_{(1)}$) of $\mathfrak{H}$. Then $\mathfrak{H}^{\mathrm{op}}$ (resp. $\mathfrak{H}^{\mathrm{cop}}$) becomes an R-face algebra via $\Omega_{\mathfrak{H}^{\mathrm{op}}}(\varphi \otimes \lambda) = \Omega_{\mathfrak{H}}(\varphi \otimes \lambda)$ (resp. $\Omega_{\mathfrak{H}^{\mathrm{cop}}}(\varphi \otimes \lambda) = \Omega_{\mathfrak{H}}(\mathring{\lambda} \otimes \mathring{\varphi})$).

LEMMA 2.1. *Let $\mathfrak{H}$ be an R-face algebra. For each $a \in \mathfrak{H}$, $\varphi, \psi \in \mathring{R}$ and $\lambda, \mu \in R$, we have the following identities*:

$$\Delta(\varphi\lambda) = \sum_i \varphi e_i \otimes \mathring{e}_i \lambda, \tag{2.4}$$

$$\sum_i \varepsilon(e_i) e_i = 1, \tag{2.5}$$

$$\varepsilon(a\lambda) = \varepsilon(a\mathring{\lambda}), \quad \varepsilon(\lambda a) = \varepsilon(\mathring{\lambda} a), \tag{2.6}$$

$$\sum_{(a)} a_{(1)} \varepsilon(\lambda a_{(2)} \mu) = \lambda a \mu, \tag{2.7}$$

$$\sum_{(a)} \varepsilon(\lambda a_{(1)} \mu) a_{(2)} = \mathring{\lambda} a \mathring{\mu}, \tag{2.8}$$

$$\sum_{(a)} \lambda a_{(1)}\mu \otimes a_{(2)} = \sum_{(a)} a_{(1)} \otimes \mathring{\lambda} a_{(2)} \mathring{\mu}, \tag{2.9}$$

$$\Delta(\varphi\lambda a\psi\mu) = \sum_{(a)} \varphi a_{(1)}\psi \otimes \lambda a_{(2)}\mu, \tag{2.10}$$

$$\Delta(a) = \sum_i \sum_{(a)} e_i a_{(1)} \otimes \mathring{e}_i a_{(2)} = \sum_i \sum_{(a)} a_{(1)} e_i \otimes a_{(2)} \mathring{e}_i. \tag{2.11}$$

PROOF. Equations (2.4) and (2.5) are immediate consequences of (2.1), (2.2). Setting $b = \mathring{\lambda}$ in (2.3), and then using (1.1) and (2.5), we obtain $\varepsilon(a\mathring{\lambda}) = \varepsilon(a\lambda)$. The other formula in (2.6) also follows from (2.3) by setting $a = \lambda$. Equation (2.11) follows from $\Delta(a) = \Delta(1)\Delta(a) = \Delta(a)\Delta(1)$ and equation (2.9) follows from (2.11) and (1.1). Equation (2.10) follows from (2.4) and (2.11). Equation (2.8) follows from (2.9) and equation (2.7) follows from (2.6) and (2.9). □

2.2. Antipodes

Let $x^{\pm}$, $e^{\pm}$ be four elements of an algebra A. We say that x^- is an (e^+, e^-)-*generalized inverse* of x^+ if the following four relations are satisfied:

$$x^-x^+ = e^+, \quad x^+x^- = e^-,$$
$$x^+x^-x^+ = x^+, \quad x^-x^+x^- = x^-. \tag{2.12}$$

LEMMA 2.2. *Let x^- be an (e^+, e^-)-generalized inverse of x^+ in an algebra A. Then*:
(1) *x^+ is an (e^-, e^+)-generalized inverse of x^-.*
(2) *We have the following formulas*:

$$e^-x^+ = x^+ = x^+e^+, \quad e^+x^- = x^- = x^-e^-, \quad (e^{\pm})^2 = e^{\pm}. \tag{2.13}$$

(3) *Let a and b be elements of A which commute both e^+ and e^-, then $x^+a = bx^+$ if and only if $ax^- = x^-b$.*
(4) *The (e^+, e^-)-generalized inverse of x^+ is unique.*

PROOF. Straightforward. □

For an R-face algebra $\mathfrak{H}$, we define linear operators $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{E}_-$ and $\mathscr{E}'_-$ on $\mathfrak{H}$ as follows:

$$\mathscr{E}(a) = \sum_i \varepsilon(ae_i)e_i, \quad \mathscr{E}'(a) = \sum_i \varepsilon(e_i a)\mathring{e}_i,$$
$$\mathscr{E}_-(a) = \sum_i \varepsilon(ae_i)\mathring{e}_i, \quad \mathscr{E}'_-(a) = \sum_i \varepsilon(e_i a)e_i \quad (a \in \mathfrak{H}). \tag{2.14}$$

We say that a linear operator $S \in \mathrm{End}(\mathfrak{H})$ (resp. $S_- \in \mathrm{End}(\mathfrak{H})$) is an *antipode* (resp. a *pode*) of $\mathfrak{H}$ if S is an $(\mathscr{E}, \mathscr{E}')$-generalized inverse (resp. S_- is an $(\mathscr{E}_-, \mathscr{E}'_-)$-generalized inverse) of $\mathrm{id}_{\mathfrak{H}}$ with respect to the convolution product $*$ (resp. opposite convolution product $*_{\mathrm{op}}$) given by $(f * g)(a) = \sum_{(a)} f(a_{(1)})g(a_{(2)})$ (resp. $(f *_{\mathrm{op}} g)(a) = \sum_{(a)} f(a_{(2)})g(a_{(1)})$)

$(f, g \in \mathrm{End}(\mathfrak{H}), a \in \mathfrak{H})$. We say that $\mathfrak{H}$ is an *R-Hopf face algebra* if it has an antipode. By definition, both the antipode and the pode are unique if they exist. It is easy to verify that $S \in \mathrm{End}(\mathfrak{H})$ is an antipode if and only if the following three relations are satisfied:

$$\sum_{(a)} S(a_{(1)})a_{(2)} = \mathscr{E}(a), \quad \sum_{(a)} a_{(1)}S(a_{(2)}) = \mathscr{E}'(a),$$

$$\sum_{(a)} S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a) \quad (a \in \mathfrak{H}).$$

LEMMA 2.3. *For $\varphi \in \mathring{R}$, $\lambda \in R$ and $a \in \mathfrak{H}$, we have*:

$$\sum_{(a)} S(a_{(1)})\lambda a_{(2)} = \mathscr{E}(\lambda a), \quad \sum_{(a)} a_{(1)}\varphi S(a_{(2)}) = \mathscr{E}'(a\varphi), \tag{2.15}$$

$$\sum_{(a)} S_{-}(a_{(2)})\varphi a_{(1)} = \mathscr{E}_{-}(\varphi a), \quad \sum_{(a)} a_{(2)}\lambda S_{-}(a_{(1)}) = \mathscr{E}'_{-}(a\lambda). \tag{2.16}$$

PROOF. Using (2.7) twice, we obtain

$$\begin{aligned}\sum_{(a)} S(a_{(1)})\lambda a_{(2)} &= \sum_{(a)} S(a_{(1)})a_{(2)}\varepsilon(\lambda a_{(3)}) \\ &= \sum_{(a)} \mathscr{E}(a_{(1)})\varepsilon(\lambda a_{(2)}) \\ &= \mathscr{E}(\lambda a).\end{aligned}$$

The others are derived in a similar manner. □

THEOREM 2.4. *For an R-face algebra $\mathfrak{H}$, its antipode S (resp. its pode S_{-}) is an antialgebra, anticoalgebra map. Moreover we have*:

$$S(\varphi\lambda) = \mathring{\lambda}\mathring{\varphi} = S_{-}(\varphi\lambda) \quad (\varphi \in \mathring{R}, \lambda \in R). \tag{2.17}$$

PROOF (cf. [S, Proposition 4.0.1]). An operator S_{-} is a pode of $\mathfrak{H}$ if and only if it is an antipode of $\mathfrak{H}^{\mathrm{op}}$. Hence it suffices to prove the assertions for S. To start with, we show

$$S(\varphi\lambda a\psi\mu) = \mathring{\mu}\mathring{\psi}S(a)\mathring{\lambda}\mathring{\varphi} \quad (\lambda, \mu \in R, \varphi, \psi \in \mathring{R}, a \in \mathfrak{H}). \tag{2.18}$$

We set $f(a) = \mathring{\mu}S(a)\mathring{\lambda}$ $(a \in \mathfrak{H})$. It follows from the lemma above that

$$(\mathrm{id} * f)(a) = \sum_{(a)} a_{(1)}\mathring{\mu}S(a_{(2)})\mathring{\lambda} = \mathscr{E}'(a\mathring{\mu})\mathring{\lambda} = \mathscr{E}'(\lambda a\mu),$$

where the last equality follows from (2.6) and (1.1). Using (2.7) twice and the fourth equality of (2.13), we compute

$$\sum_{(1)} S(a_{(1)})\mathscr{E}'(\lambda a_{(2)}\mu) = \sum_{(a)} S(a_{(1)})\mathscr{E}'(a_{(2)})\varepsilon(\lambda a_{(3)}\mu)$$
$$= \sum_{(a)} S(a_{(1)})\varepsilon(\lambda a_{(2)}\mu)$$
$$= S(\lambda a\mu).$$

Combining these two formulas, we obtain

$$S(\lambda a\mu) = (S * \mathrm{id} * f)(a) = \sum_{(a)} \mathscr{E}(a_{(1)})\mathring{\mu}S(a_{(2)})\mathring{\lambda} = \mathring{\mu}S(a)\mathring{\lambda},$$

where the last equality follows from the third equality of (2.13). Similarly we obtain $S(\varphi a\psi) = \mathring{\psi}S(a)\mathring{\varphi}$ which completes the proof of (2.18). Next, we show that S is an anticoalgebra map. We define maps $g, h : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ by $g(a) = \Delta(S(a))$ and $h(a) = \sum_{(a)} S(a_{(2)}) \otimes S(a_{(1)})$. Using (2.15) and (2.7), we compute

$$(\Delta * h)(a) = \sum_i \sum_{(a)} a_{(1)}\varepsilon(e_i a_{(2)})S(a_{(3)}) \otimes \mathring{e}_i$$
$$= \sum_i \sum_{(a)} e_i a_{(1)}S(a_{(2)}) \otimes \mathring{e}_i$$
$$= \Delta(\mathscr{E}'(a)).$$

Hence, by (2.13), we obtain

$$(g * \Delta * h)(a) = \Delta((S * \mathscr{E}')(a)) = \Delta(S(a)).$$

On the other hand, using (2.15), (2.8) and (2.11), we calculate

$$(g * \Delta * h)(a) = \sum_{(a)} \Delta(S(a_{(1)})a_{(2)})h(a_{(3)})$$
$$= \sum_i \Delta(e_i)h(a\mathring{e}_i)$$
$$= \sum_j \sum_{(a)} S(a_{(2)}\mathring{e}_j) \otimes S(a_{(1)}e_j)$$
$$= h(a),$$

where the third equality follows from (2.4), (2.10) and (2.18). Thus we get $\Delta(S(a)) = \sum_{(a)} S(a_{(2)}) \otimes S(a_{(1)})$. Using (2.3), we compute

$$\varepsilon\left(\sum_{(a)} S(a_{(1)})a_{(2)}\right) = \sum_i \sum_{(a)} \varepsilon(S(a_{(1)})e_i)\varepsilon(\mathring{e}_i a_{(2)})$$
$$= \sum_i \sum_{(a)} \varepsilon(S(e_i a_{(1)}))\varepsilon(\mathring{e}_i a_{(2)})$$
$$= \varepsilon(S(a)),$$

where the third equality follows from (2.6) and (2.18), while the last equality follows from (2.11). On the other hand, by (2.5), we have $\varepsilon(\mathscr{E}(a)) = \varepsilon(a)$. Therefore $\varepsilon(S(a)) = \varepsilon(a)$ follows from (2.15). Lastly, we show that S is an antialgebra map. Using (2.18), (2.15) and (2.5), we obtain

$$S(1) = \sum_i S(e_i)\mathring{e}_i = \mathscr{E}(1) = 1.$$

Let $p, q : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$ be maps defined by $p(a \otimes b) = S(ab)$ and $q(a \otimes b) = S(b)S(a)$. Using (2.15), (2.6) and (2.3), we compute

$$\begin{aligned}(m * q)(a \otimes b) &= \sum_i \sum_{(a)} a_{(1)}\mathring{e}_i \varepsilon(\mathring{e}_i b) S(a_{(2)}) \\ &= \sum_i \sum \mathscr{E}'(a\mathring{e}_i)\varepsilon(\mathring{e}_i b) \\ &= \mathscr{E}'(ab).\end{aligned}$$

Hence by (2.13), we obtain $(p * m * q)(a \otimes b) = S(ab)$. On the other hand, since $(p * m)(a \otimes b) = \mathscr{E}(ab)$, we have

$$\begin{aligned}(p * m * q)(a \otimes b) &= \sum_{(a)(b)} \varepsilon(a_{(1)}b_{(1)})S(b_{(2)})S(a_{(2)}) \\ &= \sum_i \sum_{(a)(b)} \varepsilon(a_{(1)}e_i)\varepsilon(\mathring{e}_i b_{(1)})S(b_{(2)})S(a_{(2)}) \\ &= S(b)S(a)\end{aligned}$$

where the first equality follows from (2.18) and (2.11), while the last equality follows from (2.6), (2.8) and (2.18). Thus we get $S(ab) = S(b)S(a)$. The relation (2.17) follows from (2.18) and $S(1) = 1$. □

PROPOSITION 2.5. *Let $\mathfrak{H}$ be an R-Hopf face algebra. Then its antipode S is bijective if and only if $\mathfrak{H}$ has a pode S_-. Moreover, we have $S_- = S^{-1}$.*

PROOF. Suppose $\mathfrak{H}$ has a pode S_-. Using the theorem above, we obtain

$$(S * (S \circ S_-))(a) = \sum_{(a)} S(S_-(a_{(2)})a_{(1)}) = \mathscr{E}(a).$$

Computing similarly, we see that $S \circ S_-$ is the $(\mathscr{E}', \mathscr{E})$-generalized inverse of S. By Lemma 2.2 (1) and (4), this implies that $S \circ S_- = \mathrm{id}_{\mathfrak{H}}$. Similarly we obtain $S_- \circ S = \mathrm{id}_{\mathfrak{H}}$. The proof of the if-part is straightforward. □

PROPOSITION 2.6. *Let $f : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a map of R-face algebras. If both $\mathfrak{H}_1$ and $\mathfrak{H}_2$ have antipodes S (resp. the pode S_-), then $f \circ S = S \circ f$ (resp. $f \circ S_- = S_- \circ f$).*

PROOF (cf. [S, Lemma 4.0.4]). A straightforward computation shows that both $f \circ S$ and $S \circ f$ are $(\mathscr{E}, \mathscr{E}')$-generalized inverse of f in the algebra $\mathrm{Hom}_K(\mathfrak{H}_1, \mathfrak{H}_2)$ with respect to the convolution product, where $\mathscr{E}, \mathscr{E}' \in \mathrm{Hom}_{\boldsymbol{K}}(\mathfrak{H}_1, \mathfrak{H}_2)$ are formally the same as in (2.14). Therefore, the assertion follows from the uniqueness of the $(\mathscr{E}, \mathscr{E}')$-generalized inverse. □

§ 3. Braiding structure

3.1. Invertible Skew pairings

Instead of discussing the braiding structure directly, we begin with considering slightly more general setting which is called invertible skew pairing. In the bialgebra case, invertible skew pairing is first introduced by [**DT**], while its dual version is first considered by Reshetikhin and Semenov-Tian-Shansky [**RS**]. In a forthcoming paper, we will use results of this subsection to construct a functor from a certain category of quantum semigroups to a category of quantum groups.

Let $\mathfrak{H}^*$ denote the dual algebra of the underlying coalgebra of an R-face algebra $\mathfrak{H}$ (cf. [**S**]). We regard $\mathfrak{H}^*$ as an $\mathfrak{E}(R)$-bimodule via the following formula (cf. [**H3**]):

$$\langle \varphi\lambda X \psi\mu, a\rangle = \langle X, \varphi \mathring{\psi} a \mathring{\lambda} \mu\rangle \quad (X \in \mathfrak{H}^*, \quad a \in \mathfrak{H}, \quad \varphi, \psi \in \mathring{R}, \quad \lambda, \mu \in R). \tag{3.1}$$

By (2.6), we have $(\varphi\lambda)1_{\mathfrak{H}^*} = 1_{\mathfrak{H}^*}(\varphi\lambda) =: \varphi\lambda$ for each $\varphi \in \mathring{R}$ and $\lambda \in R$.

Let $\mathfrak{R}$ be another R-face algebra and let $\tau^+ : \mathfrak{H} \otimes \mathfrak{R} \to \boldsymbol{K}$ be a bilinear pairing with (e^+, e^-)-generalized inverse τ^- in the algebra $(\mathfrak{H} \otimes \mathfrak{R})^*$, where $e^\pm \in (\mathfrak{H} \otimes \mathfrak{R})^*$ are defined by

$$e^+ = \sum_i e_i \otimes \mathring{e}_i, \quad e^- = \sum_i \mathring{e}_i \otimes e_i. \tag{3.2}$$

We say that τ^+ is an *invertible skew pairing* on $(\mathfrak{H}, \mathfrak{R})$ if

$$\tau^+(ab, x) = \sum_{(x)} \tau^+(a, x_{(1)})\tau^+(b, x_{(2)}), \tag{3.3}$$

$$\tau^+(a, xy) = \sum_{(a)} \tau^+(a_{(1)}, y)\tau^+(a_{(2)}, x), \tag{3.4}$$

$$\tau^+(a, 1) = \varepsilon(a), \quad \tau^+(1, x) = \varepsilon(x) \tag{3.5}$$

for each $a, b \in \mathfrak{H}$ and $x, y \in \mathfrak{R}$. We call τ^- the *generalized inverse* of τ^+.

PROPOSITION 3.1. *Let τ^+ be an invertible skew pairing on $(\mathfrak{H}, \mathfrak{R})$. For each $a, b \in \mathfrak{H}$, $x, y \in \mathfrak{R}$, $\varphi, \psi \in \mathring{R}$ and $\lambda, \mu \in R$, we have the following formulas*:

$$\tau^+(\varphi\lambda, x) = \varepsilon(\lambda x \varphi), \quad \tau^+(a, \varphi\lambda) = \varepsilon(\varphi a \lambda), \tag{3.6}$$

$$\tau^-(\varphi\lambda, x) = \varepsilon(\varphi x \lambda) \quad \tau^-(a, \varphi\lambda) = \varepsilon(\lambda a \varphi), \tag{3.7}$$

$$\tau^+(\varphi\lambda a \psi\mu, x) = \tau^+(a, \mathring{\lambda}\mu x \varphi\mathring{\psi}), \tag{3.8}$$

$$\tau^-(\varphi\lambda a \psi\mu, x) = \tau^-(a, \psi\mathring{\varphi} x \mathring{\mu}\lambda), \tag{3.9}$$

$$(\varphi \otimes 1)\tau^+(\lambda \otimes 1) = (1 \otimes \mathring{\varphi})\tau^+(1 \otimes \mathring{\lambda}),$$

$$(1 \otimes \varphi)\tau^-(1 \otimes \lambda) = (\mathring{\varphi} \otimes 1)\tau^-(\mathring{\lambda} \otimes 1), \tag{3.10}$$

$$(\lambda \otimes \varphi)\tau^+ = \tau^+(\varphi \otimes \lambda), \quad (\varphi \otimes \lambda)\tau^- = \tau^-(\lambda \otimes \varphi), \tag{3.11}$$

$$\tau^-(ab, x) = \sum_{(x)} \tau^-(b, x_{(1)})\tau^-(a, x_{(2)}), \tag{3.12}$$

$$\tau^-(a, xy) = \sum_{(a)} \tau^-(a_{(1)}, x)\tau^-(a_{(2)}, y). \tag{3.13}$$

Proof. The formulas (3.10) immediately follow from $\tau^\pm = e^\mp \tau^\pm e^\pm$. By (3.1), (3.10) is equivalent to:

$$\tau^+(\varphi a \mu, x) = \tau^+(a, \mu x \varphi), \quad \tau^-(\lambda a \psi, x) = \tau^-(a, \psi x \lambda).$$

Hence (3.6) follows from (3.5). Using (3.3), (3.6), (2.7) and (2.8), we obtain

$$\tau^+(\lambda a \psi, x) = \sum_{(x)} \tau^+(\lambda, x_{(1)})\tau^+(a, x_{(2)})\tau^+(\psi, x_{(3)})$$
$$= \tau^+(a, \mathring{\lambda} x \mathring{\psi}),$$

or equivalently, $(\lambda \otimes \varphi)\tau^+ = \tau^+(\varphi \otimes \lambda)$. Therefore (3.11) follows from Lemma 2.2 (3). The formulas (3.8) and (3.9) follow from (3.10) and (3.11). In order to show (3.12), we set $f := (m \otimes \mathrm{id})^* : (\mathfrak{H} \otimes \mathfrak{R})^* \to (\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{R})^*$. Then (3.3) and (3.12) are equivalent to $f(\tau^+) = \tau_{13}^+\tau_{23}^+$ and $f(\tau^-) = \tau_{23}^-\tau_{13}^-$ respectively, where $\tau_{13}^\pm, \tau_{23}^\pm \in (\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{R})^*$ are defined by $\tau_{13}^\pm(a, b, x) = \varepsilon(b)\tau^\pm(a, x)$ and $\tau_{23}^\pm(a, b, x) = \varepsilon(a)\tau^\pm(b, x)$. Since $f(XY) = f(X)f(Y)$ for $X, Y \in (\mathfrak{H} \otimes \mathfrak{R})^*$, $f(\tau^-)$ is a $(f(e^+), f(e^-))$-generalized inverse of $f(\tau^+)$. On the other hand, using (3.10) and (3.11), we see that $\tau_{23}^-\tau_{13}^-$ is also a $(f(e^+), f(e^-))$-generalized inverse of $f(\tau^+) = \tau_{13}^+\tau_{23}^+$. Hence (3.12) follows from the uniqueness of the generalized inverse. The proof of (3.13) is similar. □

Corollary 3.2. *Let τ^+ be an invertible skew pairing on $(\mathfrak{H}, \mathfrak{R})$ with generalized inverse τ^-. Then $\sigma^+ : x \otimes a \mapsto \tau^-(a, x)$ gives an invertible skew pairing on $(\mathfrak{R}, \mathfrak{H})$ with generalized inverse $\sigma^- : x \otimes a \mapsto \tau^+(a, x)$.*

Lemma 3.3. *Let $\mathfrak{H}$ and $\mathfrak{R}$ be R-face algebras and τ^+ a bilinear pairing on $(\mathfrak{H}, \mathfrak{R})$, which satisfies* (3.3), (3.4) *together with the relations* (3.14) (*resp.* (3.15)) *below. If $\mathfrak{H}$ has the antipode S (resp. $\mathfrak{R}$ has the pode S_-), then τ^+ is an invertible skew pairing with generalized inverse τ^- given by $\tau^-(a, x) = \tau^+(S(a), x)$ (resp. $\tau^-(a, x) = \tau^+(a, S_-(x))$).*

$$\tau^+(\varphi\lambda, x) = \varepsilon(\lambda x \varphi), \quad \tau^+(a, 1) = \varepsilon(a), \tag{3.14}$$

$$\tau^+(1, x) = \varepsilon(x), \quad \tau^+(a, \varphi\lambda) = \varepsilon(\varphi a \lambda). \tag{3.15}$$

PROOF. Using (3.3), (2.15) and (3.14), we see that $\tau^-(a,x) := \tau^+(S(a),x)$ satisfies

$$(\tau^-\tau^+)(a,x) = \sum_{(a)} \tau^+(S(a_{(1)})a_{(2)},x) = e^+(a,x)$$

and $\tau^+\tau^- = e^-$. On the other hand, the relations $\tau^+e^+ = e^+$ and $e^+\tau^- = \tau^-$ easily follow from

$$\begin{aligned}\tau^+(\varphi\lambda a\psi\mu, x) &= \sum_{(x)} \tau^+(\varphi\lambda, x_{(1)})\tau^+(a, x_{(2)})\tau^+(\psi\mu, x_{(3)}) \\ &= \tau^+(a, \mathring{\lambda}\mu x\varphi\mathring{\psi}).\end{aligned}$$

□

3.2. CQT face algebras

In this subsection, we will discuss the braiding structure on face algebras. Instead of Drinfeld's quasitriangular (QT) Hopf algebras, we give a generalization of coquasitriangular (CQT) bialgebras, because it is more suitable for our examples, as we will show elsewhere. CQT bialgebra is the dual notion of QT bialgebra, and it has been studied by Larson and Towber, Majid, Schauenburg, the author and many other people.

Let $\mathfrak{H}$ be an R-face algebra and let $\mathscr{R}^+$ be a bilinear form on $\mathfrak{H}$ with (e^+, e^-)-generalized inverse $\mathscr{R}^-$, where e^+ and e^- are as in (3.2). We say that $(\mathfrak{H}, \mathscr{R}^+)$ is *coquasitriangular* (CQT) if the following relations hold:

$$\mathscr{R}^+ m^*(X)\mathscr{R}^- = (m^{\mathrm{op}})^*(X) \quad (X \in \mathfrak{H}^*), \tag{3.16}$$

$$(m \otimes \mathrm{id})^*(\mathscr{R}^+) = \mathscr{R}^+_{13}\mathscr{R}^+_{23}, \quad (\mathrm{id} \otimes m)^*(\mathscr{R}^+) = \mathscr{R}^+_{13}\mathscr{R}^+_{12}. \tag{3.17}$$

Here m^{op} denotes the opposite product of $\mathfrak{H}$, and Z_{ij} ($Z \in (\mathfrak{H} \otimes \mathfrak{H})^*$, $\{i,j,k\} = \{1,2,3\}$) denotes an element of $(\mathfrak{H} \otimes \mathfrak{H} \otimes \mathfrak{H})^*$ defined by $\langle Z_{ij}, a_1 \otimes a_2 \otimes a_3\rangle = \langle Z, a_i \otimes a_j\rangle\varepsilon(a_k)$.

LEMMA 3.4. *For a* CQT *R-face algebra $(\mathfrak{H}, \mathscr{R}^+)$, $\mathscr{R}^+$ is an invertible skew pairing on $(\mathfrak{H}, \mathfrak{H})$ with generalized inverse $\mathscr{R}^-$.*

PROOF. It suffices to verify (3.5). We calculate

$$\begin{aligned}\mathscr{R}^+(1,a) &= \sum_{(a)} \mathscr{R}^+(1, a_{(1)})e^-(1, a_{(2)}) \\ &= \sum_{(a)} \mathscr{R}^+(1, a_{(1)})\mathscr{R}^+(e_i, a_{(2)})\mathscr{R}^-(\mathring{e}_i, a_{(3)}) \\ &= \sum_{(a)} \mathscr{R}^+(e_i, a_{(1)})\mathscr{R}^-(\mathring{e}_i, a_{(2)}) \\ &= \varepsilon(a),\end{aligned}$$

where the first equality follows from (2.5) and the definition of e^-, the second and the last equalities follow from $\mathscr{R}^+\mathscr{R}^- = e^-$, and the third equality follows from (3.3). The proof of $\mathscr{R}^+(a,1) = \varepsilon(a)$ is similar. □

Proposition 3.5. *Let* $(\mathfrak{H}, \mathscr{R}^+)$ *be a* CQT *R-face algebra. For each* $a \in \mathfrak{H}$, $\varphi, \psi \in \mathring{R}$ *and* $\lambda, \mu \in R$, *we have the following formulas*:

$$\mathscr{R}^+(a, \varphi\lambda) = \varepsilon(\varphi a \lambda), \quad \mathscr{R}^+(\varphi\lambda, a) = \varepsilon(\lambda a \varphi), \tag{3.18}$$

$$\mathscr{R}^-(a, \varphi\lambda) = \varepsilon(\lambda a \varphi), \quad \mathscr{R}^-(\varphi\lambda, a) = \varepsilon(\varphi a \lambda), \tag{3.19}$$

$$\mathscr{R}^+(\varphi\lambda a \psi\mu, b) = \mathscr{R}^+(a, \mathring{\lambda}\mu b \varphi \mathring{\psi}), \quad \mathscr{R}^-(\varphi\lambda a \psi\mu, b) = \mathscr{R}^-(a, \psi \mathring{\varphi} b \mathring{\mu} \lambda), \tag{3.20}$$

$$(m \otimes \mathrm{id})^*(\mathscr{R}^-) = \mathscr{R}^-_{23}\mathscr{R}^-_{13}, \quad (\mathrm{id} \otimes m)^*(\mathscr{R}^-) = \mathscr{R}^-_{12}\mathscr{R}^-_{13}, \tag{3.21}$$

$$\mathscr{R}^\pm_{12}\mathscr{R}^\pm_{13}\mathscr{R}^\pm_{23} = \mathscr{R}^\pm_{23}\mathscr{R}^\pm_{13}\mathscr{R}^\pm_{12}, \tag{3.22}$$

$$\mathscr{R}^\mp_{23}\mathscr{R}^\pm_{12}\mathscr{R}^\pm_{13} = \mathscr{R}^\pm_{13}\mathscr{R}^\pm_{12}\mathscr{R}^\mp_{23}, \quad \mathscr{R}^\mp_{13}\mathscr{R}^\mp_{23}\mathscr{R}^\pm_{12} = \mathscr{R}^\pm_{12}\mathscr{R}^\mp_{23}\mathscr{R}^\mp_{13}. \tag{3.23}$$

Proof. The relations (3.18–21) immediately follow from Proposition 3.1 and the lemma above. The proof of (3.22) is quite similar to the original one (cf. [**D1**]). Using (3.11) and (3.10), we obtain

$$\begin{aligned}\mathscr{R}^-_{23}(\mathscr{R}^+_{12}\mathscr{R}^+_{13}\mathscr{R}^+_{23})\mathscr{R}^-_{23} &= \sum_i \mathscr{R}^-_{23}\mathscr{R}^+_{12}(e_i \otimes \mathring{e}_i \otimes 1)\mathscr{R}^+_{13} \\ &= \mathscr{R}^-_{23}\mathscr{R}^+_{12}\mathscr{R}^+_{13}.\end{aligned}$$

Similarly we get $\mathscr{R}^-_{23}(\mathscr{R}^+_{23}\mathscr{R}^+_{13}\mathscr{R}^+_{12})\mathscr{R}^-_{23} = \mathscr{R}^+_{13}\mathscr{R}^+_{12}\mathscr{R}^-_{23}$. This proves one of the relations (3.23). Other formulas of (3.23) are similarly proved. □

By virtue of Lemma 3.3 and Corollary 3.2, we have the following proposition.

Proposition 3.6. *For a* CQT *R-Hopf face algebra* $(\mathfrak{H}, \mathscr{R}^+)$, *we have the following*:

$$(S \otimes \mathrm{id})^*(\mathscr{R}^+) = \mathscr{R}^-, \quad (\mathrm{id} \otimes S)^*(\mathscr{R}^-) = \mathscr{R}^+, \tag{3.24}$$

$$(S \otimes S)^*(\mathscr{R}^\pm) = \mathscr{R}^\pm. \tag{3.25}$$

§ 4. Comodules

In this section, we study right comodule theory of an R-face algebra $\mathfrak{H}$. We put emphasis on category-theoretic aspects of the comodules, rather than representation-theoretic one's. Unless otherwise stated, $\mathfrak{H}$-comodule means right $\mathfrak{H}$-comodule. For a comodule L, we denote its coaction $L \to L \otimes \mathfrak{H}$ by $\rho = \rho_L$. We also use the "sigma" notation $\rho(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)} (u \in L)$ (cf. [**S**]).

4.1. The truncated tensor product

Let $\mathfrak{H}$ be an R-face algebra and let $\mathscr{C}om_{\mathfrak{H}}$ be the category of all right $\mathfrak{H}$-comodules. In this subsection, we construct a binary operation $\overline{\otimes} : \mathscr{C}om_{\mathfrak{H}} \times \mathscr{C}om_{\mathfrak{H}} \to \mathscr{C}om_{\mathfrak{H}}$ and show that $(\mathscr{C}om_{\mathfrak{H}}, \overline{\otimes})$ is a monoidal category. We refer the reader to the book [**M**] of Mac Lane for elementary aspects of monoidal category theory.

Let L be an $\mathfrak{H}$-comodule. Using (2.7), we see that L becomes an R-bimodule via

$$\lambda u\mu = \sum_{(u)} u_{(0)}\varepsilon(\lambda u_{(1)}\mu) \quad (u \in L, \quad \lambda, \mu \in R). \tag{4.1}$$

Using (2.8) and (2.7), we obtain:

$$\sum_{(u)} \lambda u_{(0)}\mu \otimes u_{(1)} = \sum_{(u)} u_{(0)} \otimes \mathring{\lambda} u_{(1)}\mathring{\mu}, \tag{4.2}$$

$$\rho(\lambda u\mu) = \sum_{(u)} u_{(0)} \otimes \lambda u_{(1)}\mu \tag{4.3}$$

for each $u \in L$ and $\lambda, \mu \in R$. For an $\mathfrak{H}$-comodule map $f : L \to L'$, we have

$$\lambda f(u)\mu = f(\lambda u\mu) \quad (u \in L, \quad \lambda, \mu \in R). \tag{4.4}$$

Let M be another $\mathfrak{H}$-comodule. We define a linear endomorphism $^- : L \otimes M \to L \otimes M$ by $\overline{u \otimes v} := \sum_i ue_i \otimes e_i v$ and denote its image by $L \mathbin{\overline{\otimes}} M$.

PROPOSITION 4.1. *For right $\mathfrak{H}$-comodules L and M, $L \mathbin{\overline{\otimes}} M$ becomes a right $\mathfrak{H}$-comodule via*

$$\begin{aligned} \rho(\overline{u \otimes v}) &= \sum_{(u)(v)} u_{(0)} \otimes v_{(0)} \otimes u_{(1)}v_{(1)} \\ &= \sum_{(u)(v)} \overline{u_{(0)} \otimes v_{(0)}} \otimes u_{(1)}v_{(1)} \quad (u \in L, v \in M). \end{aligned} \tag{4.5}$$

PROOF. For $u \in L$ and $v \in M$, let $\tilde{\rho}(u \otimes v)$ denote the right-hand side of the first equality of (4.5). Using (4.2), we obtain

$$\begin{aligned} \tilde{\rho}(u \otimes v) &= \sum_{(u)(v)} \sum_i u_{(0)} \otimes v_{(0)} \otimes u_{(1)}\mathring{e}_i \mathring{e}_i v_{(1)} \\ &= \sum_{(u)(v)} \overline{u_{(0)} \otimes v_{(0)}} \otimes u_{(1)}v_{(1)} \\ &\in (L \mathbin{\overline{\otimes}} M) \otimes \mathfrak{H}. \end{aligned}$$

Hence $\tilde{\rho}$ defines a linear map from $L \otimes M$ into $(L \mathbin{\overline{\otimes}} M) \otimes \mathfrak{H}$. On the other hand, by (4.3), we have $\tilde{\rho}(\overline{u \otimes v}) = \tilde{\rho}(u \otimes v)$. This proves that $\rho := \tilde{\rho}|_{L \overline{\otimes} M}$ is a well-defined map into $(L \mathbin{\overline{\otimes}} M) \otimes \mathfrak{H}$. The relation $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho$ follows from $(\tilde{\rho} \otimes \mathrm{id}) \circ \tilde{\rho} = (\mathrm{id} \otimes \Delta) \circ \tilde{\rho}$. Finally, using (2.3), (2.6) and (4.1), we obtain

$$(\mathrm{id} \otimes \varepsilon) \circ \rho(\overline{u \otimes v}) = \overline{u \otimes v}. \tag{4.6}$$

This completes the proof of the proposition. □

We call $L \mathbin{\overline{\otimes}} M$ the *truncated tensor product* of L and M. Let $f : L \to L'$ and $g : M \to M'$ be $\mathfrak{H}$-comodule maps. Using (4.4), we see that $f \mathbin{\overline{\otimes}} g := (f \otimes g)|_{L \overline{\otimes} M} : L \mathbin{\overline{\otimes}} M \to L' \mathbin{\overline{\otimes}} M'$ is an $\mathfrak{H}$-comodule map. Thus $\overline{\otimes}$ defines a bifunctor from $\mathcal{C}om_{\mathfrak{H}}$

into itself. Let N be another $\mathfrak{H}$-comodule. Since

$$\lambda(\overline{u \otimes v})\mu = \sum_{(u)(v)} u_{(0)} \otimes v_{(0)} \sum_i \varepsilon(\lambda u_{(1)} e_i)\varepsilon(e_i v_{(1)} \mu)$$
$$= \overline{\lambda u \otimes v\mu} \tag{4.7}$$

for each $u \in L$, $v \in M$ and $\lambda, \mu \in R$, we have

$$\overline{(\overline{u \otimes v}) \otimes w} = \sum_{ij} ue_i \otimes e_i v e_j \otimes e_j w = \overline{u \otimes (\overline{v \otimes w})}$$

for each $u \in L, v \in M$ and $w \in N$. Moreover, we have

$$\rho(\overline{(\overline{u \otimes v}) \otimes w}) = \sum_{(u)(v)(w)} u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \otimes u_{(1)} v_{(1)} w_{(1)}$$
$$= \rho(\overline{u \otimes (\overline{v \otimes w})}).$$

Hence we have $(L \mathbin{\overline{\otimes}} M) \mathbin{\overline{\otimes}} N \simeq L \mathbin{\overline{\otimes}} (M \mathbin{\overline{\otimes}} N)$ as $\mathfrak{H}$-comodules. Next, we define an $\mathfrak{H}$-comodule structure ρ_R on R and linear maps $\gamma_L : L \to R \mathbin{\overline{\otimes}} L$, $\delta_L : L \to L \mathbin{\overline{\otimes}} R$ by $\rho_R(\lambda) = \sum_i e_i \otimes \mathring{e}_i \lambda$ $(\lambda \in R)$ and $\gamma_L(u) = \sum_i e_i \otimes e_i u$, $\delta_L(u) = \sum_i ue_i \otimes e_i$ $(u \in L)$ respectively. Using (4.2) and (4.3), we see that γ_L and δ_L are $\mathfrak{H}$-comodule isomorphisms with inverses $\overline{\lambda \otimes u} \mapsto \lambda u$ and $\overline{u \otimes \lambda} \mapsto u\lambda$ $(u \in L, \lambda \in R)$ respectively. It is easy to verify that $\gamma_R = \delta_R$ and that $\mathrm{id}_L \mathbin{\overline{\otimes}} \gamma_M = \delta_L \mathbin{\overline{\otimes}} \mathrm{id}_M$. We have thus proved the following theorem.

THEOREM 4.2. *Let $\mathfrak{H}$ be an R-face algebra. Then, the category $\mathcal{C}om_{\mathfrak{H}}$ of all right $\mathfrak{H}$-comodules becomes a monoidal category with product $\overline{\otimes}$ and unit object R.*

PROPOSITION 4.3. *Let $\mathfrak{H}$ be an R-face algebra. For a right $\mathfrak{H}$-comodule M, denote by $\mathscr{F}(M)$ the linear space M viewed as an R-bimodule via* (4.1). *Then $\mathscr{F}$ defines a morphism of monoidal categories from $(\mathcal{C}om_{\mathfrak{H}}, \overline{\otimes})$ into the category $(\mathcal{B}imod_R, \otimes_R)$ of all R-bimodules, with functorial isomorphism*

$$\kappa_{LM} : \mathscr{F}(L) \otimes_R \mathscr{F}(M) \xrightarrow{\sim} \mathscr{F}(L \mathbin{\overline{\otimes}} M); \quad u \otimes_R v \mapsto \overline{u \otimes v} \quad (u \in L, v \in M).$$

PROOF. It is easy to verify that κ_{LM} gives a well-defined bijection. Therefore, the proposition easily follows from (4.4) and (4.7). □

COROLLARY 4.4. *Let $\mathfrak{H}$ be an R-face algebra. For each $\mathfrak{H}$-comodule L, both $L\overline{\otimes}-$ and $-\overline{\otimes}L$ give exact endofunctors on $\mathcal{C}om_{\mathfrak{H}}$.*

PROOF. Since R is isomorphic to a finite direct product of fields, $L \otimes_R -$ and $- \otimes_R L$ are exact endofunctors on $\mathcal{B}imod_R$. Accordingly, the assertion follows from the proposition above. □

As an application of the comodule theory, we obtain the following:

PROPOSITION 4.5. *Let R and R′ be commutative separable algebras over a field* $\boldsymbol{K}$. *Let $\mathfrak{H} = (\mathfrak{H}, \Omega)$ be an R-face algebra and let $\Omega' : \mathfrak{C}(R') \to \mathfrak{H}$ be an algebra and a* *coalgebra map. If* $(\mathfrak{H}, \Omega')$ *is an* R'*-face algebra, then there is a* $\boldsymbol{K}$*-algebra isomorphism* $\alpha : R \xrightarrow{\sim} R'$ *such that* $\Omega'(\alpha(\lambda)^\circ \otimes \alpha(\mu)) = \Omega(\mathring{\lambda} \otimes \mu)$ *for each* $\lambda, \mu \in R$.

PROOF. By (4.5) and (4.6), we have

$$\overline{u \otimes v} = \sum_{(u)(v)} u_{(0)} \otimes v_{(0)} \varepsilon(u_{(1)} v_{(1)}) \quad (u \in L, v \in M).$$

This relation shows that the monoidal category $(\mathscr{C}om_{\mathfrak{H}}, \overline{\otimes})$ does not depend on R and Ω. Hence there exist natural isomorphisms $\gamma'_L : L \xrightarrow{\sim} R' \overline{\otimes} L$ and $\delta'_L : L \xrightarrow{\sim} L \overline{\otimes} R'$ such that (R', γ', δ') forms a unit object of $\mathscr{C}om_{\mathfrak{H}}$. Define an isomorphism $\alpha : R \xrightarrow{\sim} R'$ in $\mathscr{C}om_{\mathfrak{H}}$ by $\alpha = \gamma_{R'}^{-1} \circ \delta'_R$. By standard category-theoretic argument, we obtain $(\alpha \overline{\otimes} \mathrm{id}_L) \circ \gamma_L = \gamma'_L$, which is a part of the uniqueness theorem of the unit object. On the other hand, because of the naturality of γ', we have $(\mathrm{id}_{R'} \overline{\otimes} \alpha) \circ \gamma'_R = \gamma'_{R'} \circ \alpha$. Using these two relations we obtain $\alpha \circ \gamma_R^{-1} = (\gamma'_{R'})^{-1} \circ (\alpha \overline{\otimes} \alpha)$, which implies that α is a $\boldsymbol{K}$-algebra isomorphism. In particular, $(\alpha \otimes \alpha)(e)$ is the separating idempotent of R'. Since α is an $\mathfrak{H}$-comodule map, we have

$$\sum_i \alpha(e_i) \otimes \Omega(\mathring{e}_i \otimes \lambda) = (\alpha \otimes \mathrm{id}) \circ \rho(\lambda) = \sum_i \alpha(e_i) \otimes \Omega'(\alpha(e_i)^\circ \otimes \alpha(\lambda))$$

for each $\lambda \in R$. Since $\{\alpha(e_i) | i\}$ is a basis of R', we get $\Omega(\mathring{e}_i \otimes \lambda) = \Omega'(\alpha(e_i)^\circ \otimes \alpha(\lambda))$ for each i. This completes the proof of the proposition. □

COROLLARY 4.6. *An* R*-face algebra becomes a bialgebra over* $\boldsymbol{K}$ *if and only if* $R = \boldsymbol{K}$.

4.2. Dual comodules

In this subsection, we show that every finite-dimensional $\mathfrak{H}$-comodule has a left dual object in the sense of [**D2**], provided that $\mathfrak{H}$ has an antipode. Let C be an arbitrary coalgebra. Let M be a finite-dimensional C-comodule with basis $\{v^k\}$. We define a left C-comodule structure ρ_{M^*} on the dual space $M^* := \mathrm{Hom}_{\boldsymbol{K}}(M, \boldsymbol{K})$ via $\rho_{M^*}(y^m) = \sum_n a^{mn} \otimes y^n$, where $\{y^k\}$ denotes the dual basis of $\{v^k\}$ and a^{mn} denotes the element of C given by $\rho(v^n) = \sum_m v^m \otimes a^{mn}$. Then, the coaction ρ_{M^*} does not depend on the choice of $\{v^k\}$. In fact, it equals to the following composition:

$$M^* \xrightarrow{\sim} M^* \otimes \boldsymbol{K} \xrightarrow{1 \otimes \langle , \rangle^*} M^* \otimes M \otimes M^* \xrightarrow{1 \otimes \rho \otimes 1} M^* \otimes M \otimes C \otimes M^* \xrightarrow{\langle , \rangle \otimes 1 \otimes 1} C \otimes M^*.$$

It is easy to verify that the coaction $\rho_{M^*} : x \mapsto \sum_{(x)} x_{(-1)} \otimes x_{(0)}$ satisfies:

$$\sum_{(x)} x_{(-1)} \langle x_{(0)}, u \rangle = \sum_{(u)} \langle x, u_{(0)} \rangle u_{(1)} \quad (x \in M^*, u \in M), \tag{4.8}$$

$$\sum_k \sum_{(v^k)} v^k_{(0)} \otimes v^k_{(1)} \otimes y^k = \sum_k \sum_{(y^k)} v^k \otimes y^k_{(-1)} \otimes y^k_{(0)}. \tag{4.9}$$

Let $\mathfrak{H}$ be an R-face algebra with antipode S. For a finite-dimensional right $\mathfrak{H}$-comodule M, we define a right $\mathfrak{H}$-comodule $M^\vee$ to be the vector space M^* equipped with coaction $x \mapsto \sum_{(x)} x_{(0)} \otimes S(x_{(-1)}) (x \in M^\vee)$. We call $M^\vee$ the *left dual comodule* of M. We have:

$$\sum_{(x)} \langle x_{(0)}, u \rangle x_{(1)} = \sum_{(u)} \langle x, u_{(0)} \rangle S(u_{(1)}) \quad (u \in M, x \in M^\vee), \tag{4.8$'$}$$

$$\sum_k \sum_{(v^k)} v^k_{(0)} \otimes y^k \otimes S(v^k_{(1)}) = \sum_k \sum_{(y^k)} v^k \otimes y^k_{(0)} \otimes y^k_{(1)}. \tag{4.9$'$}$$

Lemma 4.7. *Let $\mathfrak{H}$ be an R-face algebra with antipode S. Then, for each finite-dimensional $\mathfrak{H}$-comodule $M, M^\vee$ is a left dual object of M in $(\mathcal{C}om_{\mathfrak{H}}, \overline{\otimes})$. That is, there exist $\mathfrak{H}$-comodule maps $\$: M^\vee \overline{\otimes} M \to R$, $\% : R \to M \overline{\otimes} M^\vee$ which satisfy the following equalities:*

$$M \xrightarrow{(\% \overline{\otimes} 1) \circ \gamma} M \overline{\otimes} M^\vee \overline{\otimes} M \xrightarrow{\delta^{-1} \circ (1 \overline{\otimes} \$)} M$$

$$= M \xrightarrow{\mathrm{id}} M, \tag{4.10}$$

$$M^\vee \xrightarrow{(1 \overline{\otimes} \%) \circ \delta} M^\vee \overline{\otimes} M \overline{\otimes} M^\vee \xrightarrow{\gamma^{-1} \circ (\$ \overline{\otimes} 1)} M^\vee$$

$$= M^\vee \xrightarrow{\mathrm{id}} M^\vee. \tag{4.11}$$

Explicitly these maps are given by

$$\$(\overline{x \otimes u}) = \sum_i \langle e_i x, u \rangle e_i = \sum_i \langle x, u e_i \rangle e_i, \tag{4.12}$$

$$\%(\lambda) = \sum_k \lambda v^k \otimes y^k = \sum_k v^k \otimes y^k \lambda, \tag{4.13}$$

where $\{v^k\}$ and $\{y^k\}$ are as above.

Proof. Straightforward calculations based on (4.8)$'$ and (4.9)$'$ yield

$$\langle \lambda x \mu, u \rangle = \langle x, \mu u \lambda \rangle \quad (u \in M, \quad x \in M^\vee, \quad \lambda, \mu \in R), \tag{4.14}$$

$$\sum_k v^k \otimes \lambda y^k \mu = \sum_k \mu v^k \lambda \otimes y^k \quad (\lambda, \mu \in R). \tag{4.15}$$

Using these formulas, we see that (4.12) gives a well-defined map and that the right-hand side of (4.13) belongs to $M \overline{\otimes} M^\vee$. An application of (4.9)$'$ gives

$$\rho \circ \%(\lambda) = \sum_k \sum_{(v^k)} {v^k}_{(0)} \otimes y^k \otimes \lambda {v^k}_{(1)} S({v^k}_{(2)})$$

$$= \sum_k \sum_i e_i v^k \otimes y^k \otimes \mathring{e}_i \lambda$$

$$= (\% \otimes \mathrm{id}_{\mathfrak{H}})(\rho(\lambda)).$$

Similarly, applying (4.8)′, we obtain

$$\begin{aligned}(\$ \otimes \mathrm{id}_{\mathfrak{H}})(\rho(\overline{x \otimes u})) &= \sum_{(u)} \sum_{i} \langle x, u_{(0)} \rangle e_i \otimes S(u_{(1)}) u_{(2)} \mathring{e}_i \\ &= \sum_{ij} \langle x, u e_j \rangle e_i \otimes \mathring{e}_i e_j \\ &= \rho(\$(\overline{x \otimes u})).\end{aligned}$$

The proof of (4.10) and (4.11) is straightforward. □

Let M be a finite-dimensional comodule of an R-face algebra $\mathfrak{H}$ with pode S_-. We define its *right dual comodule* $M^\wedge$ to be the vector space M^* equipped with coaction $x \mapsto \sum_{(x)} x_{(0)} \otimes S_-(x_{(-1)})$. In the same fashion as the lemma above, we see that $M^\wedge$ is a right dual object of M. Thus we have proved the following theorem.

THEOREM 4.8. *For an R-face algebra $\mathfrak{H}$ with bijective antipode, the category of all finite-dimensional right $\mathfrak{H}$-comodules is a rigid monoidal category.*

Let L and M be finite-dimensional comodules of an R-Hopf face algebra $\mathfrak{H}$. For an $\mathfrak{H}$-comodule map $f : L \to M$, its *left dual* $f^\vee : M^\vee \to L^\vee$ is defined to be the following composition:

$$M^\vee \xrightarrow{\delta} M^\vee \overline{\otimes} R \xrightarrow{1 \overline{\otimes} \%} M^\vee \overline{\otimes} L \overline{\otimes} L^\vee \xrightarrow{1 \overline{\otimes} f \overline{\otimes} 1} M^\vee \overline{\otimes} M \overline{\otimes} L^\vee \xrightarrow{\$ \overline{\otimes} 1} R \overline{\otimes} L^\vee \xrightarrow{\gamma^{-1}} L^\vee.$$

By the theory of monoidal category, the correspondence $f \mapsto f^\vee$ defines a contravariant endofunctor of the category of finite-dimensional $\mathfrak{H}$-comodules. We show that $f^\vee$ satisfies

$$\langle f^\vee(y), u \rangle = \langle y, f(u) \rangle \quad (y \in M^\vee, u \in L). \tag{4.16}$$

By a category-theoretic calculation, we see that $\$ \circ (f^\vee \overline{\otimes} \mathrm{id}_L) : M^\vee \otimes L \to R$ agrees with $\$ \circ (\mathrm{id} \overline{\otimes} f)$. Hence the assertion follows from $\langle x, u \rangle = \varepsilon_R \circ \$(\overline{x \otimes u})$ $(x \in L^\vee, u \in L)$.

Let L and M be finite-dimensional comodules of an R-Hopf face algebra $\mathfrak{H}$. By the theory of monoidal categories, we have the following isomorphism:

$$\begin{aligned}\Xi : M^\vee \overline{\otimes} L^\vee &\xrightarrow{(1 \overline{\otimes} \%) \circ \delta} M^\vee \overline{\otimes} L^\vee \overline{\otimes} (L \overline{\otimes} M) \overline{\otimes} (L \overline{\otimes} M)^\vee \\ &\xrightarrow{(1 \overline{\otimes} \gamma^{-1}) \circ (1 \overline{\otimes} \$ \overline{\otimes} 1)} M^\vee \overline{\otimes} M \overline{\otimes} (L \overline{\otimes} M)^\vee \xrightarrow{\gamma^{-1} \circ (\$ \overline{\otimes} 1)} (L \overline{\otimes} M)^\vee.\end{aligned}$$

PROPOSITION 4.9. *Let $\Xi : M^\vee \overline{\otimes} L^\vee \xrightarrow{\sim} (L \overline{\otimes} M)^\vee$ be as above. Then the following diagram is commutative.*

$$\begin{array}{ccc} M^\vee \overline{\otimes} L^\vee & \xrightarrow{\Xi} & (L \overline{\otimes} M)^\vee \\ \cap\downarrow & & \cap\downarrow \\ M^* \otimes L^* & \xrightarrow{\sim} & (L \otimes M)^* \end{array}$$

Here we regard $(L \overline{\otimes} M)^{\vee}$ *as a subspace of* $(L \otimes M)^{*}$ *via the decomposition* $L \otimes M = (L \overline{\otimes} M) \oplus \mathrm{Ker}(^{-}: L \otimes M \to L \otimes M)$.

PROOF. Using $\langle\ ,\ \rangle = \varepsilon_R \circ \$$ and (4.10) for $L \overline{\otimes} M$, we obtain $\langle \Xi(\overline{y \otimes x}), \overline{u \otimes v} \rangle = \sum_i \langle x, ue_i \rangle \langle y, e_i v \rangle$ for each $y \in M^{\vee}$, $x \in L^{\vee}$, $u \in L$ and $v \in M$. This proves the proposition. □

4.3. $\mathrm{Hom}_R(L, M)$

In this subsection, we discuss some additional properties of comodules of face algebras, which we will use in [H5]. Let $\mathfrak{H}$ be an R-face algebra. For an $\mathfrak{H}$-comodule M, we define its subspace $M^{\mathfrak{H}}$ by

$$M^{\mathfrak{H}} = \left\{ u \in M \,|\, \rho(u) = \sum_i ue_i \otimes \mathring{e}_i \right\}. \tag{4.17}$$

LEMMA 4.10. (1) *For each* $u \in M^{\mathfrak{H}}$ *and* $\lambda \in R$, *we have* $\lambda u = u\lambda$. *In particular, we have* $\rho(u) = \sum_i e_i u \otimes \mathring{e}_i$.

(2) *We have the following linear isomorphism*:

$$\mathrm{Hom}_{\mathfrak{H}}(R, M) \xrightarrow{\sim} M^{\mathfrak{H}}; \quad f \mapsto f(1).$$

PROOF. Part (1) follows from (2) and (2) follows from (4.3) and (4.4). □

Now we assume that $\mathfrak{H}$ has an antipode. Let L be a finite-dimensional $\mathfrak{H}$-comodule. We define a linear isomorphism $\tilde{\theta} : M \otimes L^{\vee} \xrightarrow{\sim} \mathrm{Hom}_K(L, M)$ by $\tilde{\theta}(v \otimes x)(u) = \langle x, u \rangle v$ $(v \in M, x \in L^{\vee}, u \in L)$. Using (4.14), we obtain

$$\tilde{\theta}(\overline{v \otimes x})(u\lambda) = \sum_i \langle \lambda e_i x, u \rangle v e_i = \tilde{\theta}(\overline{v \otimes x})(u)\lambda.$$

On the other hand, if $f = \tilde{\theta}(\sum_p v^p \otimes x^p) \in \mathrm{Hom}_R(L, M)$, then

$$f(u) = \sum_i f(ue_i)e_i = \tilde{\theta}\left(\sum_p \overline{v^p \otimes x^p} \right)(u).$$

Thus $\tilde{\theta}$ induces a linear isomorphism $\theta : M \overline{\otimes} L^{\vee} \xrightarrow{\sim} \mathrm{Hom}_R(L, M)$. We define an $\mathfrak{H}$-comodule structure on $\mathrm{Hom}_R(L, M)$ via this isomorphism. By (4.8)′ and (4.14), we have

$$\sum_{(f)} f_{(0)}(u) \otimes f_{(1)} = \sum_{(u)(v)} \langle x, u_{(0)} \rangle v_{(0)} \otimes v_{(1)} S(u_{(1)}), \tag{4.18}$$

$$\lambda f(\mu u) = (\lambda f \mu)(u) \tag{4.19}$$

for $f = \theta(\overline{v \otimes x})$, $u \in L$ and $\lambda, \mu \in R$, where $\lambda f \mu$ is defined by (4.1).

Next, we assume that $\mathfrak{H}$ has a pode S_-. Then there exists a linear isomorphism $\theta_- : {}_R\mathrm{Hom}(L, M) \xrightarrow{\sim} L^{\wedge} \otimes M$ which is compatible with the usual isomorphism $\mathrm{Hom}_K(L, M) \simeq L^* \otimes M$. We define an $\mathfrak{H}$-comodule structure on ${}_R\mathrm{Hom}(L, M)$ via θ_-.

PROPOSITION 4.11. *Let $\mathfrak{H}$ be an R-face algebra with an antipode S (resp. a pode S_-). For an $\mathfrak{H}$-comodule M and a finite-dimensional $\mathfrak{H}$-comodule L, we have the following relation* (4.20) (resp. (4.20′)):

$$\mathrm{Hom}_{\mathfrak{H}}(L, M) = \mathrm{Hom}_R(L, M)^{\mathfrak{H}}, \tag{4.20}$$

$$\mathrm{Hom}_{\mathfrak{H}}(L, M) = {}_R\mathrm{Hom}(L, M)^{\mathfrak{H}}. \tag{4.20′}$$

PROOF. Suppose $f = \theta(\sum_p v^p \otimes x^p) \in \mathrm{Hom}_{\mathfrak{H}}(L, M)$, or equivalently,

$$\sum_p \sum_{(v^p)} \langle x^p, u \rangle v^p{}_{(0)} \otimes v^p{}_{(1)} = \sum_p \sum_{(u)} \langle x^p, u_{(0)} \rangle v^p \otimes u_{(1)}.$$

Then, using (4.18) and (4.19), we obtain $\sum_{(f)} f_{(0)}(u) \otimes f_{(1)} = \sum_i (fe_i)(u) \otimes \mathring{e}_i$. On the other hand, if f belongs to the right-hand side of (4.20), then

$$\begin{aligned}
\rho(f(u)) &= \sum_p \sum_i \sum_{(v^p)} \langle x^p, ue_i \rangle v^p{}_{(0)} \otimes v^p{}_{(1)} e_i \\
&= \sum_p \sum_{(u)(v^p)} \langle x^p, u_{(0)} \rangle v^p{}_{(0)} \otimes v^p{}_{(1)} S(u_{(1)}) u_{(2)} \\
&= \sum_{(f)(u)} f_{(0)}(u_{(0)}) \otimes f_{(1)} u_{(1)} \\
&= \sum_{(u)} \sum_i f(e_i u_{(0)}) \otimes \mathring{e}_i u_{(1)} \\
&= (f \otimes \mathrm{id})(\rho(u)),
\end{aligned}$$

where the first equality follows from $\sum_p v^p \otimes x^p = \sum_p \overline{v^p \otimes x^p}$, (4.14) and (4.3), and the third equality follows from (4.18). This completes the proof of the proposition. □

PROPOSITION 4.12. *Let $\mathfrak{H}$ be an R-face algebra with an antipode (resp. a pode). Let L and M be finite-dimensional right $\mathfrak{H}$-comodules and let N be an arbitrary right $\mathfrak{H}$-comodule. Then the following* (4.12) (resp. (4.21′)) *gives an $\mathfrak{H}$-comodule isomorphism and* (4.22) (resp. (4.22′)) *gives a linear isomorphism*:

$$\mathrm{Hom}_R(L \mathbin{\overline{\otimes}} M, N) \simeq \mathrm{Hom}_R(L, N \mathbin{\overline{\otimes}} M^\vee), \tag{4.21}$$

$${}_R\mathrm{Hom}(L \mathbin{\overline{\otimes}} M, N) \simeq {}_R\mathrm{Hom}(M, L^\wedge \mathbin{\overline{\otimes}} N), \tag{4.21′}$$

$$\mathrm{Hom}_{\mathfrak{H}}(L \mathbin{\overline{\otimes}} M, N) \simeq \mathrm{Hom}_{\mathfrak{H}}(L, N \mathbin{\overline{\otimes}} M^\vee), \tag{4.22}$$

$$\mathrm{Hom}_{\mathfrak{H}}(L \mathbin{\overline{\otimes}} M, N) \simeq \mathrm{Hom}_{\mathfrak{H}}(M, L^\wedge \mathbin{\overline{\otimes}} N). \tag{4.22′}$$

PROOF. The isomorphism (4.21) follows from $(L \mathbin{\overline{\otimes}} M)^\vee \simeq M^\vee \mathbin{\overline{\otimes}} L^\vee$. Since the correspondence $K \mapsto K^{\mathfrak{H}}$ is functorial, (4.22) follows from (4.21) and Proposition 4.11. □

4.4. Braided monoidal categories

Let $(\mathfrak{H}, \mathscr{R}^+)$ be a CQT R-face algebra and let L and M be $\mathfrak{H}$-comodules. We define a linear map $\tilde{\beta}_{LM} : L \otimes M \to M \otimes L$ by

$$\tilde{\beta}_{LM}(u \otimes v) = \sum_{(u)} \sum_{(v)} v_{(0)} \otimes u_{(0)} \mathscr{R}^+(u_{(1)}, v_{(1)}) \quad (u \in L, v \in M).$$

Using (3.20) and (4.3), we obtain $\tilde{\beta}_{LM}(\overline{u \otimes v}) = \tilde{\beta}_{LM}(u \otimes v)$ for each $u \in L$ and $v \in M$. Similarly, using (3.20) and (4.2), we obtain

$$\tilde{\beta}_{LM}(u \otimes v) = \overline{\tilde{\beta}_{LM}(u \otimes v)} \in M \mathbin{\overline{\otimes}} L \quad (u \in L, v \in M).$$

Hence there exists a linear map $\beta_{LM} : L \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} L$ given by

$$\begin{aligned} \beta_{LM}(\overline{u \otimes v}) &= \sum_{(u)(v)} v_{(0)} \otimes u_{(0)} \mathscr{R}^+(u_{(1)}, v_{(1)}) \\ &= \sum_{(u)(v)} \overline{v_{(0)} \otimes u_{(0)}} \mathscr{R}^+(u_{(1)}, v_{(1)}). \end{aligned} \tag{4.23}$$

LEMMA 4.13. *The map $\beta_{LM} : L \mathbin{\overline{\otimes}} M \to M \mathbin{\overline{\otimes}} L$ is an $\mathfrak{H}$-comodule isomorphism with inverse*

$$\begin{aligned} \beta_{LM}^{-1}(\overline{v \otimes u}) &= \sum_{(u)(v)} u_{(0)} \otimes v_{(0)} \mathscr{R}^-(u_{(1)}, v_{(1)}) \\ &= \sum_{(u)(v)} \overline{u_{(0)} \otimes v_{(0)}} \mathscr{R}^-(u_{(1)}, v_{(1)}) \quad (u \in L, v \in M). \end{aligned} \tag{4.24}$$

PROOF. Since $\mathscr{R}^- \mathscr{R}^+ = m^*(1)$, we have $\mathscr{R}^+ m^*(X) = (m^{\text{op}})^*(X) \mathscr{R}^+$ $(X \in \mathfrak{H}^*)$, or equivalently

$$\sum_{(a)(b)} \mathscr{R}^+(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)} = \sum_{(a)(b)} b_{(1)} a_{(1)} \mathscr{R}^+(a_{(2)}, b_{(2)}).$$

It easily follows from this formula that β_{LM} is an $\mathfrak{H}$-comodule map. The proof of the other assertions is straightforward. □

It is easy to see that β_{LM} is natural in L and M, that is, $\beta_{L'M'} \circ f \mathbin{\overline{\otimes}} g = g \mathbin{\overline{\otimes}} f \circ \beta_{LM}$ for each $\mathfrak{H}$-comodule maps $f : L \to L'$ and $g : M \to M'$. Let N be another $\mathfrak{H}$-comodule. Using (3.17), we obtain

$$\beta_{L \overline{\otimes} M, N} = (\beta_{LN} \mathbin{\overline{\otimes}} \mathrm{id}_M) \circ (\mathrm{id}_L \mathbin{\overline{\otimes}} \beta_{MN}), \quad \beta_{L, M \overline{\otimes} N} = (\mathrm{id}_M \mathbin{\overline{\otimes}} \beta_{LN}) \circ (\beta_{LM} \mathbin{\overline{\otimes}} \mathrm{id}_N).$$

Thus we have completed the proof the following theorem.

THEOREM 4.14 (cf. [**D1**], [**JS**]). *For each* CQT *R-face algebra* $(\mathfrak{H}, \mathscr{R}^+)$, *the category* $(\mathscr{C}om_{\mathfrak{H}}, \overline{\otimes})$ *of all right $\mathfrak{H}$-comodules forms a braided monoidal category with braiding given by* (4.23).

§ 5. Module theory

The module theory of an R-face algebra $\mathfrak{H}$ is quite parallel to the comodule theory of $\mathfrak{H}$. In this section, we give analogues of Theorems 4.2 and 4.8 without detailed proof.

Let L and M be left modules of an R-face algebra $\mathfrak{H}$. Let $L \overline{\otimes} M$ be the image of the linear map $^- : L \otimes M \to L \otimes M; u \otimes v \mapsto \sum_i e_i u \otimes \mathring{e}_i v (u \in L, v \in M)$. Since $\Delta(1) = \sum_i e_i \otimes \mathring{e}_i$, $L \overline{\otimes} M$ becomes an $\mathfrak{H}$-module via

$$a(\overline{u \otimes v}) = \sum_{(a)} a_{(1)} u \otimes a_{(2)} v = \sum_{(a)} \overline{a_{(1)} u \otimes a_{(2)} v} \quad (a \in \mathfrak{H}, \quad u \in L, \quad v \in M).$$

On the other hand, we see that $\mathring{R}$ becomes a left $\mathfrak{H}$-module via $a\varphi := \sum_i \varepsilon(e_i a \varphi) \mathring{e}_i$ $(a \in \mathfrak{H}, \varphi \in \mathring{R})$. Moreover, there are $\mathfrak{H}$-module isomorphisms $\gamma_L : L \xrightarrow{\sim} \mathring{R} \overline{\otimes} L$ and $\delta_L : L \xrightarrow{\sim} L \overline{\otimes} \mathring{R}$ given by $\gamma_L(u) = \sum_i \mathring{e}_i \otimes \mathring{e}_i u$ and $\delta_L(u) = \sum_i e_i u \otimes \mathring{e}_i$ respectively. It is easy to verify that the category ${}_{\mathfrak{H}}\mathcal{M}od$ of all left $\mathfrak{H}$-modules becomes a monoidal category with product $\overline{\otimes}$ and unit object $(\mathring{R}, \gamma, \delta)$. Next, suppose $\mathfrak{H}$ has an antipode. For a finite-dimensional $\mathfrak{H}$-module L, we define a left $\mathfrak{H}$-module structure on the dual space $L^\vee := \mathrm{Hom}_{\boldsymbol{K}}(L, \boldsymbol{K})$ by $\langle ax, u \rangle = \langle x, S(a)u \rangle$ $(x \in L^\vee, u \in L, a \in \mathfrak{H})$. Then there exist $\mathfrak{H}$-module maps $\$: M^\vee \overline{\otimes} M \to \mathring{R}$ and $\% : \mathring{R} \to M \overline{\otimes} M^\vee$ given by $\$(\overline{x \otimes u}) = \sum_i \langle \mathring{e}_i x, u \rangle \mathring{e}_i$ $(x \in M^\vee, u \in M)$ and $\%(\varphi) = \sum_k \varphi v^k \otimes y^k (\varphi \in \mathring{R})$ respectively, where $\{v^k\}$ denotes a basis of M and $\{y^k\}$ denotes its dual basis. These maps satisfy the relations (4.10) and (4.11), that is, we have the following theorem.

THEOREM 5.1. *Let $\mathfrak{H}$ be an R-face algebra. Then, the category ${}_{\mathfrak{H}}\mathcal{M}od$ of all left $\mathfrak{H}$-modules becomes a monoidal category with product $\overline{\otimes}$ and unit object $\mathring{R}$. If, in addition, $\mathfrak{H}$ has a bijective antipode, then the category of all finite-dimensional left $\mathfrak{H}$-modules is rigid.*

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