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Nonlinear small data scattering for the wave equation in $\boldsymbol{R}^{4+1}$

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1. Introduction.

In this paper we discuss nonlinear small data scattering problem for the following wave equation

$$(1.1) \qquad \square u = F(u), \quad t \in \boldsymbol{R}, x \in \boldsymbol{R}^n.$$

Here $\square = \partial_t^2 - \Delta = \partial_t^2 - \sum_{j=1}^n \partial_j^2$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ and $F(u) = \lambda |u|^{p-1} u$ with some $\lambda \in \boldsymbol{R} \backslash \{0\}$, $p > 1$. Although our proof also works for complex-valued solutions, we consider only real-valued solutions for simplicity throughout this paper.

Let us first review the previous works on small data scattering for (1.1). Although we shall deal with 4-space dimensional case only, the large number of papers has been devoted to the study of small data scattering for (1.1) in various spaces of functions and in general space dimension $n \geq 2$. Denote by $\dot{W}^{1,p}(\boldsymbol{R}^n)$ $(1 < p < \infty)$ the completion of test functions with respect to the seminorm $\|\nabla v\|_{L^p}$. $\dot{W}^{-1,p'}(\boldsymbol{R}^n)$ $(1/p + 1/p' = 1)$ means the dual space of $\dot{W}^{1,p}(\boldsymbol{R}^n)$. We simply denote $\dot{W}^{1,2}(\boldsymbol{R}^n)$ by $\dot{H}^1(\boldsymbol{R}^n)$. Note that $\dot{H}^1(\boldsymbol{R}^n)$ is identical to $\{v = v(x) | v \in L^{2n/(n-2)}(\boldsymbol{R}^n), \nabla v \in L^2(\boldsymbol{R}^n)\}$ when $n \geq 3$. Set $E(\boldsymbol{R}^n) = \dot{H}^1(\boldsymbol{R}^n) \times L^2(\boldsymbol{R}^n)$. For more information on the definitions of spaces, norms and operators, see Section 2. There are two fundamental problems in the nonlinear scattering theory. One of them is to prove the existence of the wave operators. Pecher [15] established the space-time mixed norm estimates of free solutions of finite energy and proved the existence of the wave operators for (1.1) as mappings from a neighborhood of 0 in $E(\boldsymbol{R}^n)$ into $E(\boldsymbol{R}^n)$, assuming $p = 1 + 4/(n-2)$ and $n = 3, 4, 5$. Ginibre and Velo [2] have eliminated the restriction of n in the result of Pecher and proved the same result for all $n \geq 3$ by making better use of the space-time integrability and estimating fractional derivatives of the nonlinear term in the Besov spaces (see Proposition 3.3 in [2]). $E(\boldsymbol{R}^n)$ is called the energy space and it is the largest space of data for which we may construct the wave operators for (1.1) in the usual sense. Hence a class of data in the results of Pecher, Ginibre and Velo is the largest, but the allowed value of p is $1 + 4/(n-2)$ only. For a smaller class of data we can discuss the scattering theory for a more general perturbation operator. In fact, Strauss [19] proved that the wave operators can be defined for (1.1) as mappings from a neighborhood of 0 in $(\dot{H}^1 \cap \dot{W}^{1,1+1/p}) \times (L^2 \cap L^{1+1/p})$ into $(\dot{H}^1 \cap L^{p+1}) \times (L^2 \cap \dot{W}^{-1,p+1})$, assuming $p_1(n) < p \leq 1 + 4/(n-1)$, $n \geq 2$. Here $p_1(n) = (n + 2 + \sqrt{n^2 + 8n})/2(n-1)$. Later, Mochizuki and Motai [13] reduced the lower bound $p_1(n)$ to a smaller value $p_2(n)$ $(n \geq 2)$ by working in a different space. The lower bound for p in [13], however, does not seem optimal because in the case of space dimension $n=2$ or 3 Pecher [16], Kubota and Mochizuki [10] and Tsutaya [20] have proved the existence of the wave operators as mappings from a neighborhood of 0 in weighted $C^3(\boldsymbol{R}^n)\times C^2(\boldsymbol{R}^n)$ into weighted $C^2(\boldsymbol{R}^n)\times C^1(\boldsymbol{R}^n)$ for (1.1) with $\rho>\rho_0(n)=(n+1+\sqrt{n^2+10n-7})/2(n-1)$. Here there holds $\rho_0(n)<\rho_2(n)$ for all $n\geq 2$. The crucial point of the proofs in [10], [16], [20] is to establish a pointwise decay estimate for the nonlinear term of the corresponding integral equation (see, e.g., Lemma 4.1 in [20]). Although such a method as in [10], [16], [20] does not seem applicable directly to higher dimensional case, Kubo and Kubota [9] have recently constructed the wave operators in higher and odd space dimensions (i.e. $n=5,7,\cdots$) in a similar, but more complicated way for small and spherically symmetric data in a weighted Banach space, assuming $\rho_0(5)<\rho$ if $n=5$ and $\rho_0(n)<\rho<1+4/(n-3)$ if $n=7,9,\cdots$. Note that in any space dimension $\rho_0(n)$ will be the smallest value for us to construct the wave operators for small and smooth data in view of the blow up theorems (see [3], [6], [17], [18]). Thus it has been expected to construct the wave operators for (1.1) with $\rho>\rho_0(n)$ for small data in a suitable Banach space when $n\geq 4$. Moreover, since we are discussing small data scattering problem, it is also desirable that the wave operators should be constructed for (1.1) with $\rho>\rho_0(n)$ as mappings from a neighborhood of 0 in a suitable Banach space into the Banach space itself. The first aim of this paper is to construct the wave operators for (1.1) with $\rho>\rho_0(4)=2$ as mappings from a neighborhood of 0 in a Hilbert space Σ (defined just below) into Σ in the case of four space dimensions. More precisely, let us introduce

$$\Sigma=\Big\{(f_1,f_2)\in L^2(\boldsymbol{R}^4)\times L^2(\boldsymbol{R}^4)\,|$$

$$\|(f_1,f_2)\|_\Sigma\equiv\sum_{|\alpha|\leq 2}\|\langle\cdot\rangle^{|\alpha|}\partial_x^\alpha f_1\|_{L^2}+\sum_{|\alpha|=3}\|\langle\cdot\rangle^2\partial_x^\alpha f_1\|_{L^2}$$

$$+\sum_{|\alpha|\leq 1}\|\langle\cdot\rangle^{|\alpha|+1}\partial_x^\alpha f_2\|_{L^2}+\sum_{|\alpha|=2}\|\langle\cdot\rangle^2\partial_x^\alpha f_2\|_{L^2}<\infty\Big\}.$$

Here and after $\partial_x^\alpha=\partial_1^{\alpha_1}\cdots\partial_4^{\alpha_4}$ for a multi-index $\alpha=(\alpha_1,\cdots,\alpha_4)$ with $|\alpha|=\alpha_1+\cdots+\alpha_4$ and $\langle x\rangle=\sqrt{1+|x|^2}$. It should be noticed that the persistence holds in the space Σ for free solutions in the sense that $(u,\partial_t u)\in C(\boldsymbol{R};\Sigma)$ for $u(t)=(\cos\omega t)f_1+(\omega^{-1}\sin\omega t)f_2$ if $(f_1,f_2)\in\Sigma$. We shall prove that the wave operators $W_\pm$ are defined for (1.1) with $\rho>2$ as one–one and continuous mappings from a neighborhood of 0 in Σ into Σ (Theorems 1 and 3). The inequality (3.7) plays an important role to show Range $(W_\pm)\subset\Sigma$.

Another important problem is to show the asymptotic completeness of the wave operators. Pecher (for $n=3,4,5$), Ginibre and Velo (for all $n\geq 3$) have proved the asymptotic completeness of the wave operators for small data in $E(\boldsymbol{R}^n)$, provided that $\rho=1+4/(n-2)$ [2], [15]. On the other hand, the asymptotic completeness of the wave operators can not be proved in [9], [10], [13], [16], [19], [20], notwithstanding that their attentions are confined to the case of small data. Thus, it has been completely an open problem in any space dimension whether or not we can prove the asymptotic completeness for small data in a suitable space under the assumption of $\rho > \rho_0(n)$. In this paper we shall make a first step toward answering this question by showing the asymptotic completeness of the wave operators for small data in Σ. Namely, in the case of four space dimensions we shall prove that the ordinary Cauchy problem for (1.1), with data $(u(0), \partial_t u(0)) = (f, g)$ at $t = 0$ in Σ, has a unique global solution $u = u(t, x)$ provided that $\rho > 2$ and $\|(f, g)\|_\Sigma$ is small. Note that this is a refinement of a recent result of Zhou [21] on a class of Cauchy data. Moreover, with the help of (3.7), (3.10), (3.17)–(3.18) it will be shown that the solution u has the asymptotic states in Σ. That is, there is a unique pair of functions (f^+, g^+), $(f^-, g^-) \in \Sigma$ satisfying

$$\|u(t,\cdot) - u^{\pm}(t,\cdot)\|_e \to 0 \quad \text{as} \quad t \to \pm\infty,$$

(the double sign in the same order) for $u^{\pm}(t) = (\cos \omega t) f^{\pm} + (\omega^{-1} \sin \omega t) g^{\pm}$ (Theorem 2). Hence, combining Theorems 1 and 3 with Theorem 2, we conclude that the scattering operator S can be defined for (1.1) with $\rho > 2$ as a one–one and continuous mapping from a neighborhood of 0 in Σ into Σ (Theorem 4).

In order to prove our theorems we need to extend a recent result of Zhou. In [21] he discussed the ordinary Cauchy problem with data given at $t = 0$ and proved remarkably that in four space dimensions the equation (1.1) has a unique global solution in a suitable space if $\rho > 2$ and sufficiently smooth, small data decay fast at spatial infinity. However, the decay condition he assumed on data is too strong to develop the scattering theory according to the standard formulation, such as that in [19]. Thus, one of our main tasks is to weaken the decay condition significantly and solve the ordinary Cauchy problem for small data in Σ which is a larger class than that in Zhou [21]. We need to investigate the precise commutation relations between the fundamental solution and the operators L_j, $\Omega_{k\ell}$, L_0 and then make effective use of the inequalities (3.16), (3.18) and the Sobolev estimate $\|v\|_{L^4} \le C\|\nabla v\|_{L^2}$ to improve a class of data in [21]. See Propositions 3.2, 4.3 below. Another difference to be stressed between the result of Zhou and ourselves is in a class of the asymptotic states. Although he makes no reference to the asymptotic behavior of solutions, it follows immediately from his proof that the solutions to (1.1) constructed in [21] have the asymptotic states in $H^3(\boldsymbol{R}^4) \times H^2(\boldsymbol{R}^4)$. But this is not enough to prove the asymptotic completeness. We shall prove by using (3.7), (3.10), (3.17)–(3.18) that all the solutions to (1.1), with small data in Σ at $t = 0$, have the asymptotic states in Σ. See Proposition 4.7 and Remark 4.2.

To solve the equation (1.1) by giving data at $t = \pm\infty$ we shall follow the same line as in Hidano and Tsutaya [4]. We have only to carry out simple, but careful limiting procedures. Thus, this paper will be mostly devoted to the proof of Theorem 2, which is an extension of the result of Zhou.

Finally, we refer to a recent work of Lindblad and Sogge [12]. Making use of the Strichartz-type inequalities, they have developed the scattering theory for (1.1) with small data in the homogeneous Sobolev space $\dot{H}^{\gamma}(\boldsymbol{R}^n) \times \dot{H}^{\gamma-1}(\boldsymbol{R}^n)$ with $\gamma = n/2 - 2/(\rho - 1)$. This value of γ is closely related to the homogeneity of the equation (1.1) and naturally arises from a simple scaling argument. Their interest is in the global existence and asymptotic behavior of the solutions to (1.1) with low regular data and they have proved the existence of the wave operators and the asymptotic completeness for small data in $\dot{H}^{\gamma}(\boldsymbol{R}^n) \times \dot{H}^{\gamma-1}(\boldsymbol{R}^n)$ with γ given above when $1 + 4/(n-1) \le \rho$ if $n = 2$ or 3 and $1 + 4/(n-1) \le \rho \le 1 + 4/(n-3)$ if $n \ge 4$ (see Theorem 2.2 in [12]). Of course, $\rho_0(n) < 1 + 4/(n-1)$ for all $n \ge 2$, and in our theorems ρ will be only assumed to be strictly larger than $\rho_0(4)$. While the continuity of the scattering operator can not be shown in [12] without the case of $\rho = (n+3)/(n-1)$ (see [12] on page 423), we shall prove the continuity of the scattering operator in the Σ-topology for any $\rho > \rho_0(4)$.

2. Notations and theorems.

Following Klainerman [7], [8], we introduce several partial differential operators as follows: $\partial_0 = \partial_t$, $L_j = t\partial_j + x_j\partial_t (j = 1, \cdots, n)$, $\Omega_{jk} = x_j\partial_k - x_k\partial_j (1 \le j < k \le n)$, $L_0 = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n$. The operators $\partial_0, \cdots, \partial_n$, $L_1, \cdots, L_n$, $\Omega_{12}, \cdots, \Omega_{n-1n}$, and L_0 are denoted by $\Gamma_0, \cdots, \Gamma_\mu$ in this order, where $\mu = (n^2 + 3n + 2)/2$. In order to make use of the different behavior of the solutions in the neighborhood of the characteristic cone and away from it, we introduce the norm for $1 \le p,\ q \le \infty$

$$(2.1) \qquad \|v(\cdot)\|_{L^{p,q}} = \|v(r\zeta)r^{(n-1)/p}\|_{L^p(\boldsymbol{R}_+; L^q(S^{n-1}))}$$

$$= \left(\int_0^\infty \left(\int_{S^{n-1}} |v(r\zeta)|^q d\zeta\right)^{p/q} r^{n-1} dr\right)^{1/p}$$

with obvious modifications if p or q is infinite. Here $r = |x|$, $\zeta \in S^{n-1}$. It is clear that if $p = q$, then $L^{p,q}$ norm coincides with the usual L^p norm. In [11] Li and Yu first utilized this type of norm for the existence theory of solutions to nonlinear wave equations. Let N be a non-negative integer and Ψ be a characteristic function of a set of $\boldsymbol{R}^{n+1}$. We define the norm

$$(2.2) \qquad \|u(t,\cdot)\|_{\Gamma,N,p,q,\Psi} := \sum_{|\alpha| \le N} \|\Psi(t,\cdot)\Gamma^\alpha u(t,\cdot)\|_{L^{p,q}} \quad (1 \le p, q \le \infty)$$

for any function $u(t,x)$ for which the above right-hand side is finite for every $t \in \boldsymbol{R}$. Here α is a multi-index, $|\alpha| = \alpha_0 + \cdots + \alpha_\mu$ and $\Gamma^\alpha = \Gamma_0^{\alpha_0} \cdots \Gamma_\mu^{\alpha_\mu}$. We also define the norm

$$(2.3) \qquad \|Du(t,\cdot)\|_{\Gamma,N,p,q,\Psi} := \sum_{k=0}^{n} \|\partial_k u(t,\cdot)\|_{\Gamma,N,p,q,\Psi}$$

for a vector $Du = (\partial_t u, \partial_1 u, \cdots, \partial_n u)$. If $\Psi \equiv 1$ in (2.2), (2.3), we omit the sub-index Ψ. If $p = q$, then we omit q. If $N = 0$, then we omit Γ and N. Summing up, we simplify the notations of the norms as follows.

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\Psi} = \begin{cases} \|u(t,\cdot)\|_{\Gamma,N,p,q} & \text{if } \Psi \equiv 1, \\ \|u(t,\cdot)\|_{\Gamma,N,p,\Psi} & \text{if } p = q, \\ \|u(t,\cdot)\|_{p,q,\Psi} & \text{if } N = 0. \end{cases}$$

According to this rule, $\|u(t,\cdot)\|_p$ means the usual L^p-norm. But exceptionally, $\|u(t,\cdot)\|_e$ will mean the energy norm as defined below.

For a non-negative integer s, $W^{s,p}(\boldsymbol{R}^n)$ means the usual Sobolev space on $\boldsymbol{R}^n$ with the norm $\|v\|_{W^{s,p}}$. Especially, we put $H^s(\boldsymbol{R}^n) = W^{s,2}(\boldsymbol{R}^n)$. For any (not necessarily bounded) interval I and any Banach space X $BC(I;X)$ means the set of bounded, continuous functions on I with values in X. We denote by $\mathscr{S}(\boldsymbol{R}^n)$ the space of Schwartz's rapidly decreasing functions. Let $\hat{v}$ or $\mathscr{F}[v]$ mean the Fourier transform of $v \in \mathscr{S}(\boldsymbol{R}^n)$:

$$\hat{v}(\xi) = \mathscr{F}[v](\xi) = \int_{\boldsymbol{R}^n} v(x)\exp[-ix\cdot\xi]dx \quad (i = \sqrt{-1}).$$

We denote by $\mathscr{F}^{-1}[v]$ the inverse Fourier transform of $v \in \mathscr{S}(\boldsymbol{R}^n_\xi)$. The Fourier transform of $v \in \mathscr{S}'(\boldsymbol{R}^n)$ (tempered distribution) is denoted by $\hat{v}$ or $\mathscr{F}[v]$ and the inverse Fourier transform of $v \in \mathscr{S}'(\boldsymbol{R}^n_\xi)$ by $\mathscr{F}^{-1}[v]$, likewise. We denote by $\dot{H}^{-1}(\boldsymbol{R}^n)$ the set of the tempered distributions such that $|\xi|^{-1}\hat{v}(\xi) \in L^2$. We set $\|v\|_{\dot{H}^{-1}} := \|\mathscr{F}^{-1}[|\xi|^{-1}\hat{v}]\|_{L^2}$. Let $\|u(t,\cdot)\|_e$ mean the energy norm $\|u(t,\cdot)\|_e^2 = \{\|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\nabla u(t,\cdot)\|_{L^2}^2\}/2$. For any slowly increasing function $H = H(|\xi|)$ in $\boldsymbol{R}^n_\xi$ we define the operator $H((-\Delta)^{1/2})$ in $\mathscr{S}'(\boldsymbol{R}^n)$ by $H((-\Delta)^{1/2})v = \mathscr{F}^{-1}[H(|\xi|)\hat{v}]$ for $v \in \mathscr{S}'(\boldsymbol{R}^n)$. Put $\omega = (-\Delta)^{1/2}$ for simplicity. Especially, $\omega^{-1}\sin\omega t$ and $\cos\omega t$ are strongly continuous operators with respect to t in $H^s(\boldsymbol{R}^n)$ for any $s \in \boldsymbol{R}$.

Let I be any interval of $\boldsymbol{R}$. For any $t, \sigma \in I$ and any $G \in L^1(I; L^2(\boldsymbol{R}^n))$ we define the weak integrals $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))G(\tau)d\tau$, $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))\partial_j G(\tau)d\tau$ $(j = 1,\cdots,n)$ in $\mathscr{S}'(\boldsymbol{R}^n)$ as

$$\left\langle \varphi, \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} G(\tau)d\tau \right\rangle = (2\pi)^{-n}\int_\sigma^t \left(\frac{\sin|\cdot|(t-\tau)}{|\cdot|}\hat{\varphi}(\cdot), \hat{G}(\tau,\cdot)\right)_{L^2} d\tau,$$

$$\left\langle \varphi, \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} \partial_j G(\tau)d\tau \right\rangle = -(2\pi)^{-n}\int_\sigma^t \left(\frac{\sin|\cdot|(t-\tau)}{|\cdot|} i\xi_j\hat{\varphi}(\cdot), \hat{G}(\tau,\cdot)\right)_{L^2} d\tau$$

for $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$, respectively. Here $(\cdot,\cdot)_{L^2}$ is the inner product in L^2. The integral $\int_\sigma^t (\cos\omega(t-\tau))G(\tau)d\tau$ is defined similarly. Since $\mathscr{S}(\boldsymbol{R}^n)$ is dense in $L^2(\boldsymbol{R}^n)$, these tempered distributions $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))G(\tau)d\tau$, $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))\partial_j G(\tau)d\tau$ and $\int_\sigma^t (\cos\omega(t-\tau))G(\tau)d\tau$ can be uniquely extended to elements of the dual space of $L^2(\boldsymbol{R}^n)$, which are identified with functions in $L^2(\boldsymbol{R}^n)$ by the Riesz representation theorem. By $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))G(\tau)d\tau$, $\int_\sigma^t (\omega^{-1}\sin\omega(t-\tau))\partial_j G(\tau)d\tau$ and $\int_\sigma^t (\cos\omega(t-\tau))G(\tau)d\tau$

we denote these functions likewise. As is easily checked, they are actually in $C(I;L^2(\boldsymbol{R}^n))$ for every σ. If $u,\ v\in L^1(I;L^2(\boldsymbol{R}^n))$ satisfy

$$\frac{d}{dt}\langle u(t),\varphi\rangle=\langle v(t),\varphi\rangle\quad\text{in } \mathscr{D}'(I)$$

for all $\varphi\in\mathscr{S}(\boldsymbol{R}^n)$, then we denote v by $\partial_t u$. In particular, it is verified that

$$\partial_t\int_\sigma^t(\omega^{-1}\sin\omega(t-\tau))G(\tau)d\tau=\int_\sigma^t(\cos\omega(t-\tau))G(\tau)d\tau \tag{2.4}$$

when $G\in L^1(I;L^2(\boldsymbol{R}^n))$. Moreover, after the redefinition of G on a set of measure 0 in time so that $G\in C(I;L^2(\boldsymbol{R}^n))$, we have

$$\partial_t\int_\sigma^t\frac{\sin\omega(t-\tau)}{\omega}G(\tau)d\tau=\frac{\sin\omega(t-\sigma)}{\omega}G(\sigma)+\int_\sigma^t\frac{\sin\omega(t-\tau)}{\omega}\partial_\tau G(\tau)d\tau \tag{2.5}$$

when $G\in L^1(I;H^1(\boldsymbol{R}^n))$, $\partial_t G\in L^1(I;L^2(\boldsymbol{R}^n))$.

We put $\Sigma_\delta=\{(f_1,f_2)\,|\,\|(f_1,f_2)\|_\Sigma<\delta\}$. Now we can state our theorems. The first problem could be termed "Cauchy problem at $\pm\infty$".

THEOREM 1. *Let $n=4$ and $\rho>2$. There exists a $\delta>0$ depending on λ, ρ with the following properties (I), (II):*

(I) For any $(f_-,g_-)\in\Sigma_\delta$ the equation (1.1) has a unique solution $u=u(t,x)$ satisfying

(2.6) $\Gamma^\alpha u\in BC((-\infty,0];L^2(\boldsymbol{R}^4))$ *for any α with $|\alpha|\le 2$,*

(2.7) $\partial_k\Gamma^\alpha u\in L^\infty((-\infty,0);L^2(\boldsymbol{R}^4))$ *for any α with $|\alpha|=2$ and $k=0,\cdots,4$,*

(2.8) $\|u(t,\cdot)-u_-(t,\cdot)\|_e\to 0$ *as $t\to-\infty$,*

where $u_-(t)=(\cos\omega t)f_-+(\omega^{-1}\sin\omega t)g_-$. Moreover, this solution u satisfies

(2.9) $\partial_k\Gamma^\alpha u\in BC((-\infty,0];L^2(\boldsymbol{R}^4))$ *for any α with $|\alpha|=2$ and $k=0,\cdots,4$,*

(2.10) $(u,\partial_t u)\in C((-\infty,0];\Sigma)$,

(2.11) $\displaystyle\sup_{t<0}\|u(t,\cdot)\|_{\Gamma,2,2}+\sum_{|\alpha|=2}\sup_{t<0}\|D\Gamma^\alpha u(t,\cdot)\|_2 \le C_1\|(f_-,g_-)\|_\Sigma$ *for some constant $C_1>0$,*

(2.12) $\|(u(0),\partial_t u(0))\|_\Sigma\le C_2\|(f_-,g_-)\|_\Sigma$ *for some constant $C_2>0$,*

(2.13) $\|u(t,\cdot)-u_-(t,\cdot)\|_{\Gamma,2,2}=O(|t|^{-3(\rho-2)/2})$,

(2.14) $\|D\{u(t,\cdot)-u_-(t,\cdot)\}\|_{\Gamma,2,2}=O(|t|^{-3(\rho-1)/2+1})$ *as $t\to-\infty$.*

(II) (Continuous dependence) Let $(f_-^{(j)},g_-^{(j)})\in\Sigma_\delta$ $(j=1,2)$. Let $u^{(j)}=u^{(j)}(t,x)$ be the two corresponding solutions to (1.1) in (I). When $\|(f_-^{(1)}-f_-^{(2)},g_-^{(1)}-g_-^{(2)})\|_\Sigma\to 0$, it

holds that

$$\sup_{t<0} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t<0} \|D\Gamma^{\alpha}\{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}\|_2 \to 0, \tag{2.15}$$

$$\|(u^{(1)}(0,\cdot) - u^{(2)}(0,\cdot), \partial_t u^{(1)}(0,\cdot) - \partial_t u^{(2)}(0,\cdot))\|_{\Sigma} \to 0. \tag{2.16}$$

REMARK 2.1. The same result as Theorem 1 holds for positive time.

The next result is concerned with the ordinary Cauchy problem. This is an improvement of recent result of Y. Zhou on a class of Cauchy data.

THEOREM 2. *Let $n = 4$ and $p > 2$. Then there exists an $\varepsilon > 0$ depending on λ, p with the following properties:*

(I) (Existence and Uniqueness) For any $(f, g) \in \Sigma_\varepsilon$ the equation (1.1) *has a unique solution $u = u(t,x)$ satisfying*

$$(u(0), \partial_t u(0)) = (f, g), \tag{2.17}$$

$$\Gamma^{\alpha} u \in BC(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| \le 2, \tag{2.18}$$

$$\partial_k \Gamma^{\alpha} u \in L^{\infty}(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4. \tag{2.19}$$

Moreover, this solution u satisfies

$$\partial_k \Gamma^{\alpha} u \in BC(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4, \tag{2.20}$$

$$(u, \partial_t u) \in C(\boldsymbol{R}; \Sigma), \tag{2.21}$$

$$\sup_{t\in\boldsymbol{R}} \|u(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t\in\boldsymbol{R}} \|D\Gamma^{\alpha} u(t,\cdot)\|_2 \le C_3 \|(f,g)\|_{\Sigma} \text{ for some constant } C_3 > 0. \tag{2.22}$$

(II) (Asymptotic behavior) There exists a unique pair of functions (f^+, g^+), (f^-, g^-) satisfying

$$(f^{\pm}, g^{\pm}) \in E(\boldsymbol{R}^4), \tag{2.23}$$

$$\|u(t,\cdot) - u^{\pm}(t,\cdot)\|_e \to 0 \text{ as } t \to \pm\infty \text{ (the double sign in the same order)}. \tag{2.24}$$

Here $u^{\pm}(t) = (\cos \omega t) f^{\pm} + (\omega^{-1} \sin \omega t) g^{\pm}$. These $u^{\pm}$ satisfy

$$(u^{\pm}(0), \partial_t u^{\pm}(0)) \in \Sigma, \tag{2.25}$$

$$\|(u^{\pm}(0), \partial_t u^{\pm}(0))\|_{\Sigma} \le C_4 \|(f,g)\|_{\Sigma} \text{ for some constant } C_4 > 0, \tag{2.26}$$

$$\|u(t,\cdot) - u^{\pm}(t,\cdot)\|_{\Gamma,2,2} = O(|t|^{-3(p-2)/2}), \tag{2.27}$$

$$\|D\{u(t,\cdot) - u^{\pm}(t,\cdot)\}\|_{\Gamma,2,2} = O(|t|^{-3(p-1)/2+1}) \text{ as } t \to \pm\infty. \tag{2.28}$$

(III) (Continuous dependence) Let $u^{(j)} = u^{(j)}(t,x)$ $(j = 1, 2)$ be the two solutions to (1.1) *with $(u^{(j)}(0), \partial_t u^{(j)}(0)) = (f^{(j)}, g^{(j)}) \in \Sigma_\varepsilon$. Let $(f^{(j)+}, g^{(j)+})$, $(f^{(j)-}, g^{(j)-})$ be the corresponding pairs of functions in (II). When $\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_{\Sigma} \to 0$, it holds*

that

$$\sup_{t\in\boldsymbol{R}} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t\in\boldsymbol{R}} \|D\Gamma^{\alpha}\{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}\|_2 \to 0, \tag{2.29}$$

$$\|(f^{(1)+} - f^{(2)+}, g^{(1)+} - g^{(2)+})\|_{\Sigma}, \quad \|(f^{(1)-} - f^{(2)-}, g^{(1)-} - g^{(2)-})\|_{\Sigma} \to 0. \tag{2.30}$$

The next theorem follows immediately from Theorem 1 and Remark 2.1.

THEOREM 3. *The wave operators W_+, W_-*

$$W_\pm : (u_\pm(0), \partial_t u_\pm(0)) \mapsto (u(0), \partial_t u(0))$$

can be defined as one-one and continuous mappings from Σ_δ into $\Sigma_{C_2\delta}$.

We take δ small so that $C_2\delta < \varepsilon$ may hold. Combining Theorem 2 with Theorem 3, we conclude

THEOREM 4. *The scattering operator S*

$$S : (u_-(0), \partial_t u_-(0)) \mapsto (u^+(0), \partial_t u^+(0))$$

can be defined as a one-one and continuous mapping from Σ_δ into $\Sigma_{C_2C_4\delta}$.

REMARK 2.2. Theorems 1–4 are valid for the equation (1.1) with the more general nonlinear function $F \in C^2(\boldsymbol{R})$ such that for $|u|$, $|v| \le 1$

$$|F(u)| \le C|u|^\rho, \quad |F'(u)| \le C|u|^{\rho-1} \text{ for some } \rho > 2,$$

$$|F''(u) - F''(v)| \le \begin{cases} C|u-v|^{\rho-2} & \text{if } 2 < \rho < 3, \\ C(|u|+|v|)^{\rho-3}|u-v| & \text{if } \rho \ge 3 \end{cases}$$

with some constants $C > 0$ independent of u, v. These assumptions admit the sum of several nonlinear terms with different powers like $F(u) = \lambda_1|u|^{\rho_1-1}u + \lambda_2|u|^{\rho_2-1}u$ with $\rho_1, \rho_2 > 2$.

3. Preliminary results.

In this section we prepare several lemmas and propositions which will be frequently used in the proof of our theorems. In what follows different constants will be denoted by C. Let $[\cdot,\cdot]$ be a commutator and δ_{jk} be the Kronecher delta. Then we have the commutation relations as follows.

LEMMA 3.1. *It holds that*

$$[\Gamma_j, \Gamma_k] = \sum_{m=0}^{\mu} C_{jkm}\Gamma_m,$$

$$[\Gamma_k, \partial_\ell] = \sum_{m=0}^{\mu} \tilde{C}_{k\ell m}\partial_m \tag{3.1}$$

for $j,k=0,\cdots,\mu$, $\ell=0,\cdots,n$ *with coefficients* C_{jkm}, $\tilde{C}_{k\ell m}\in\{-1,0,1\}$. *In particular,*

$$[L_j,\partial_k]=-\delta_{jk}\partial_t \text{ for } j,\ k=1,\cdots,n, \tag{3.2}$$

$$[\Omega_{jk},\partial_\ell]=-\delta_{j\ell}\partial_k+\delta_{k\ell}\partial_j \text{ for } 1\le j<k\le n, \ell=1,\cdots,n, \tag{3.3}$$

$$[L_0,\partial_k]=-\partial_k \text{ for } k=1,\cdots,n. \tag{3.4}$$

Note that the precise commutation relations (3.2)–(3.4) are necessary to develop the scattering theory by using the operators L_j, Ω_{jk} and L_0. See the proof of Proposition 3.1 below. This lemma is easily verified by direct calculations. Thus we omit the proof. Let $\chi=\chi(t,x)$ be the characteristic function of the set $\{(t,x)||x|\le(1+|t|)/2\}$ and put $\phi:=1-\chi$. Then we have

LEMMA 3.2. *Let I be any (not necessarily bounded) interval. It holds that*

$$\|u(t,\cdot)\|_q\le C\|u(t,\cdot)\|_{W^{s,p}} \quad (a.e.\ t\in I), \tag{3.5}$$

$$0<s\le n/p,\quad 1\le p\le q<\infty,\quad 1/q\ge 1/p-s/n$$

for all $u\in L^\infty(I;W^{s,p}(\boldsymbol{R}^n))$. *Moreover, the following inequalities hold for all functions* $u=u(t,x)$ *for which the norm appearing on the right-hand side below is finite for every* $t\in I$

$$\|u(t,\cdot)\|_{q,\chi}\le C(1+|t|)^{-n(1/p-1/q)}\|u(t,\cdot)\|_{\Gamma,s,p}, \tag{3.6}$$

$$0<s\le n/p,\quad 1\le p\le q<\infty,\quad 1/q\ge 1/p-s/n,$$

$$\|\langle|t|-|\cdot|\rangle^{|\alpha|}\partial^\alpha u(t,\cdot)\|_2\le C\|u(t,\cdot)\|_{\Gamma,|\alpha|,2}. \tag{3.7}$$

Here $\langle|t|-|x|\rangle=\sqrt{1+(|t|-|x|)^2}$, $\partial^\alpha=\partial_0^{\alpha_0}\cdots\partial_n^{\alpha_n}$ *and all the constants C are independent of u, t and I.*

PROOF. The inequality (3.5) is well-known. Taking account of the relation

$$\partial_j=\frac{tL_j+\sum_{k=1}^n x_k(x_j\partial_k-x_k\partial_j)-x_jL_0}{t^2-|x|^2} \quad (j=1,\cdots,n), \tag{3.8}$$

we can prove (3.6) by a simple scaling argument. For the proof see Hidano and Tsutaya [4] or Zhou [21]. The inequality (3.7) also follows from (3.8) and

$$\partial_t=\left(tL_0-\sum_{k=1}^n x_kL_k\right)/(t^2-|x|^2).$$

In fact, since $L_j[(t^2-|x|^2)^{-1}]=\Omega_{k\ell}[(t^2-|x|^2)^{-1}]=0$ and $L_0[(t^2-|x|^2)^{-1}]=-2(t^2-|x|^2)^{-1}$, a simple observation shows that

$$|\partial^\alpha u(t,x)|\le C||t|-|x||^{-|\alpha|}|\Lambda^\alpha u(t,x)| \text{ for every } \alpha,$$

where $\Lambda^\alpha u$ represents the vector formed by all $\Lambda_1 \cdots \Lambda_m u$ $(1 \le m \le |\alpha|)$ with $\Lambda_1, \cdots, \Lambda_m$ any of operators L_j, $\Omega_{k\ell}$, L_0. This inequality implies

$$(1 + ||t| - |x||)^{|\alpha|} |\partial^\alpha u(t,x)| \le C(|\partial^\alpha u(t,x)| + |\Lambda^\alpha u(t,x)|). \tag{3.9}$$

Hence we get (3.7) by integrating the square of (3.9) over $\boldsymbol{R}^n$. Q.E.D.

Lemma 3.3. *It holds that*

$$\|\langle \cdot \rangle^{(n-1)(1/p-1/q)} v\|_q \le C \sum_{|\gamma|+|\theta| \le s} \|\partial_x^\gamma \Omega^\theta v\|_p, \tag{3.10}$$

$$0 < s \le n/p, \quad 1 \le p \le q < \infty, \quad 1/q \ge 1/p - s/n$$

for all functions $v = v(x)$ for which the norm on the right-hand side is finite. Here $\Omega^\theta = \Omega_{12}^{\theta_{12}} \cdots \Omega_{n-1n}^{\theta_{n-1n}}$ for a multi-index $\theta = (\theta_{12}, \cdots, \theta_{n-1n})$. Let I be any (not necessarily bounded) interval. The following inequalities hold for all functions $u = u(t,x)$ defined in $I \times \boldsymbol{R}^n$ for which the norm appearing on the right-hand side below is finite for every $t \in I$

$$|u(t,x)| \le C(1 + |t| + |x|)^{-(n-1)/p} (1 + ||t| - |x||)^{-1/p} \|u(t,\cdot)\|_{\Gamma,s,p}, \tag{3.11}$$

$$s > n/p, \quad p \ge 1,$$

$$\|u(t,\cdot)\|_q \le C(1 + |t|)^{-(n-1)(1/p-1/q)} \|u(t,\cdot)\|_{\Gamma,s,p}, \tag{3.12}$$

$$0 < s \le n/p, \quad 1 \le p \le q < \infty, \quad 1/q \ge 1/p - s/n.$$

Moreover, for all $u \in L^\infty(I; H^1(\boldsymbol{R}^n))$

$$\|u(t,\cdot)\|_{q,2,\phi} \le C(1 + |t|)^{-(n-1)(1/2-1/q)} \|u(t,\cdot)\|_{H^1} \quad a.e.\ t \in I \tag{3.13}$$

if $2 \le q \le \infty$. Here all the constants C are independent of u, v, t and I.

Proof. The inequality (3.10) is proved in Section 3 of Hörmander [**5**]. The inequality (3.11) is a Klainerman's one in [**8**]. The inequality (3.12) immediately follows from (3.6) and (3.10) (see Hidano and Tsutaya [**4**] for the proof). The proof of (3.13) can be found in Zhou [**21**]. Q.E.D.

We have the Sobolev inequality on the unit sphere as follows.

Lemma 3.4.

$$|v(x)| \le C \sum_{|\theta| \le s} \left(\int_{S^{n-1}} |(\Omega^\theta v)(r\zeta)|^2 d\zeta \right)^{1/2}, \quad s > (n-1)/2, \tag{3.14}$$

$$\left(\int_{S^{n-1}} |v(r\zeta)|^q d\zeta \right)^{1/q} \le C \sum_{|\theta| \le s} \left(\int_{S^{n-1}} |(\Omega^\theta v)(r\zeta)|^p d\zeta \right)^{1/p}, \tag{3.15}$$

$$0 < s \le (n-1)/p, \quad 1 \le p \le q < \infty, \quad 1/q \ge 1/p - s/(n-1).$$

LEMMA 3.5. *Let $n \geq 3$. If $\langle x\rangle v \in L^2(\boldsymbol{R}^n)$, then there holds $v \in \dot{H}^{-1}(\boldsymbol{R}^n)$ and*

$$\|v\|_{\dot{H}^{-1}} \leq C\| \, |\cdot| v \,\|_2. \tag{3.16}$$

PROOF. This inequality is well-known and has been used in the study of nonlinear wave equations. See, e.g., Ginibre and Velo [1] on page 252 or Remark 1.1 in Ozawa and Tsutsumi [14]. See also (3.22) below, from which (3.16) follows immediately.

Q.E.D.

LEMMA 3.6. *Let I be any (not necessarily bounded) interval and let $h \in L^\infty(I; L^2(\boldsymbol{R}^n))$. Suppose that the norms appearing on the right-hand side below are finite for a.e. $\sigma \in I$. Then it holds that*

$$\left\|\frac{\sin\omega(t-\sigma)}{\omega}h(\sigma,\cdot)\right\|_2 \leq C\|h(\sigma,\cdot)\|_{q,\chi} + \frac{C}{(1+|\sigma|)^{(n-2)/2}}\|h(\sigma,\cdot)\|_{1,2,\phi} \text{ for all } t \in \boldsymbol{R} \tag{3.17}$$

if $n \geq 3$. Moreover, if $n \geq 4$ and $\langle x\rangle h \in L^\infty(I; L^2(\boldsymbol{R}^n))$, then for $j = 1, \cdots, n$

$$\left\|\frac{\sin\omega(t-\sigma)}{\omega}[x_j h(\sigma,\cdot)]\right\|_2 \leq C\|x_j h(\sigma,\cdot)\|_{q,\chi} + \frac{C}{(1+|\sigma|)^{(n-4)/2}}\|h(\sigma,\cdot)\|_{1,2,\phi} \text{ for all } t \in \boldsymbol{R}. \tag{3.18}$$

Here $1/q = 1/n + 1/2$ and all the positive constants C are independent of t, h, σ and I.

PROOF. (3.17) is essentially due to Li and Yu [11]. But in [21] Zhou has slightly changed their original estimate into (3.17). We give the proof of (3.18). Making the change of variables $x = (t-\sigma)y$, $\xi = \eta/(t-\sigma)$ and then proceeding as in the proof for (3.17) due to Li and Yu (see [11] on page 916), we see without any difficulty that

$$\begin{aligned}
\left\|\frac{\sin\omega(t-\sigma)}{\omega}[x_j h(\sigma,\cdot)]\right\|_2 &= (2\pi)^{-n/2}\left\|\frac{\sin|\xi|(t-\sigma)}{|\xi|}\widehat{x_j h}(\sigma,\cdot)\right\|_2 \\
&\leq C|t-\sigma|^{n/2+1}\sup_{\substack{v\in\mathscr{S}\\ v\neq 0}}\left[\frac{\|\chi(\sigma,\cdot)x_j h(\sigma,\cdot)\|_{L^q(\boldsymbol{R}^n_x)}\|v\|_{L^p(\boldsymbol{R}^n_y)}}{\|v\|_{H^1(\boldsymbol{R}^n_y)}}|t-\sigma|^{-n/q}\right] \\
&\quad + C|t-\sigma|^{n/2+1} \\
&\quad \times \sup_{\substack{v\in\mathscr{S}\\ v\neq 0}}\left[\frac{\|h(\sigma,\cdot)\|_{1,2,\phi}\|(1-\psi(\sigma,\cdot))(t-\sigma)y_j v\|_{L^{\infty,2}}}{\|v\|_{H^1(\boldsymbol{R}^n_y)}}|t-\sigma|^{-n}\right] \\
&\equiv C|t-\sigma|^{n/2+1}(M_1+M_2),
\end{aligned} \tag{3.19}$$

where $1/p = 1/2 - 1/n$, $\psi(\sigma,y)(=\chi(\sigma,x))$ is a characteristic function of the set $\{(\sigma,y) \,|\, |y| \leq (1+|\sigma|)/2|t-\sigma|\}$. Since $\|v\|_p \leq C\|v\|_{H^1}$ by (3.5), we obtain, recalling

$1/q = 1/2 + 1/n$,

$$M_1 \le C\|\chi(\sigma,\cdot)x_j h(\sigma,\cdot)\|_q = C\|x_j h(\sigma,\cdot)\|_{q,\chi}. \tag{3.20}$$

On the other hand, taking account of the inequality

$$\begin{aligned}\int_{S^{n-1}} r^2 v^2(r\zeta)d\zeta &= -\int_{S^{n-1}}\int_r^\infty \frac{d}{d\lambda}[\lambda^2 v^2(\lambda\zeta)]d\lambda d\zeta \\ &\le 2\int_{S^{n-1}}\int_r^\infty \lambda v^2(\lambda\zeta)d\lambda d\zeta + 2\int_{S^{n-1}}\int_r^\infty \lambda^2|v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|d\lambda d\zeta,\end{aligned}$$

we get

$$\begin{aligned}&\|(1-\psi(\sigma,\cdot))(t-\sigma)y_j v(\cdot)\|_{L^{\infty,2}} \\ &\le C|t-\sigma| \sup_{r\in\left(\frac{1+|\sigma|}{2|t-\sigma|},\infty\right)} \left(\int_{S^{n-1}}\int_r^\infty \frac{1}{\lambda^{n-4}} v^2(\lambda\zeta)\lambda^{n-3}d\lambda d\zeta\right)^{1/2} \\ &\quad + C|t-\sigma| \sup_{r\in\left(\frac{1+|\sigma|}{2|t-\sigma|},\infty\right)} \left(\int_{S^{n-1}}\int_r^\infty \frac{1}{\lambda^{n-4}} |v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|\lambda^{n-2}d\lambda d\zeta\right)^{1/2} \\ &\equiv C|t-\sigma|M_3 + C|t-\sigma|M_4.\end{aligned} \tag{3.21}$$

Note that integration by part gives

$$\begin{aligned}&\int_{S^{n-1}}\int_0^\infty |v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|\lambda^{n-2}d\lambda d\zeta \\ &\le \left(\int_{S^{n-1}}\int_0^\infty v^2(\lambda\zeta)\lambda^{n-3}d\lambda d\zeta\right)^{1/2}\left(\int_{S^{n-1}}\int_0^\infty |(\nabla v)(\lambda\zeta)|^2\lambda^{n-1}d\lambda d\zeta\right)^{1/2} \\ &\le C\left(\int_{S^{n-1}}\int_0^\infty |v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|\lambda^{n-2}d\lambda d\zeta\right)^{1/2} \\ &\quad \times \left(\int_{S^{n-1}}\int_0^\infty |(\nabla v)(\lambda\zeta)|^2\lambda^{n-1}d\lambda d\zeta\right)^{1/2},\end{aligned}$$

that is,

$$\left(\int_{S^{n-1}}\int_0^\infty |v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|\lambda^{n-2}d\lambda d\zeta\right)^{1/2} \le C\|v\|_{\dot H^1}. \tag{3.22}$$

Therefore, we obtain in view of the assumption $n \ge 4$

$$M_4 \le C\left(\frac{|t-\sigma|}{1+|\sigma|}\right)^{(n-4)/2}\|v\|_{\dot H^1}. \tag{3.23}$$

Moreover, integration by part together with (3.22) gives

$$M_3 \le C\left(\frac{|t-\sigma|}{1+|\sigma|}\right)^{(n-4)/2}\left(\int_{S^{n-1}}\int_0^\infty v^2(\lambda\zeta)(\lambda^{n-2})' d\lambda d\zeta\right)^{1/2} \tag{3.24}$$

$$\le C\left(\frac{|t-\sigma|}{1+|\sigma|}\right)^{(n-4)/2}\left(\int_{S^{n-1}}\int_0^\infty |v(\lambda\zeta)||(\nabla v)(\lambda\zeta)|\lambda^{n-2} d\lambda d\zeta\right)^{1/2}$$

$$\le C\left(\frac{|t-\sigma|}{1+|\sigma|}\right)^{(n-4)/2}\|v\|_{\dot H^1}.$$

Combining (3.21), (3.23) with (3.24), we find

$$M_2 \le \left(\frac{|t-\sigma|}{1+|\sigma|}\right)^{(n-4)/2}|t-\sigma|^{-n+1}\|h(\sigma,\cdot)\|_{1,2,\phi}. \tag{3.25}$$

Substituting (3.20) and (3.25) into (3.19), we have completed the proof of (3.18).

Q.E.D.

Proposition 3.1. *Let I be any interval. Suppose that a function $h = h(t,x)$ satisfies*

$$\Gamma^\alpha h \in L^1(I; L^2(\boldsymbol{R}^n)) \quad \text{for } |\alpha| \le 1. \tag{3.26}$$

Set

$$I_\sigma[h](t) := \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} h(\tau) d\tau,$$

$$J_\sigma[h](t) := \int_\sigma^t (\cos\omega(t-\tau)) h(\tau) d\tau$$

for any $\sigma \in I$. Then for every $\sigma \in I$ the following equalities (3.29)–(3.30), (3.32)–(3.36) *hold in $C(I; L^2(\boldsymbol{R}^n))$. Moreover, the equalities* (3.28), (3.31) *also hold in the same space if in addition*

$$\langle\,\cdot\,\rangle h(\sigma,\cdot) \in L^2(\boldsymbol{R}^n) \quad \textit{for every } \sigma \in I. \tag{3.27}$$

$$L_j I_\sigma[h](t) = \frac{\sin\omega(t-\sigma)}{\omega}[x_j h(\sigma)] + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}[(L_j h)(\tau)] d\tau \quad (j = 1, \cdots, n), \tag{3.28}$$

$$\Omega_{jk} I_\sigma[h](t) = \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\Omega_{jk} h(\tau) d\tau \quad (1 \le j < k \le n), \tag{3.29}$$

$$L_0 I_\sigma[h](t) = \sigma\frac{\sin\omega(t-\sigma)}{\omega}h(\sigma) + 2\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}h(\tau)d\tau + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}[(L_0 h)(\tau)] d\tau, \tag{3.30}$$

$$L_j\partial_t I_\sigma[h](t) = L_j J_\sigma[h](t) \tag{3.31}$$
$$= (\cos\omega(t-\sigma))\left[x_j h(\sigma)\right] - \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_j h(\tau)d\tau$$
$$+ \int_\sigma^t (\cos\omega(t-\tau))\left[\left(L_j h\right)(\tau)\right]d\tau \quad (j = 1,\cdots,n),$$

$$\Omega_{jk}\partial_t I_\sigma[h](t) = \Omega_{jk} J_\sigma[h](t) \tag{3.32}$$
$$= \int_\sigma^t (\cos\omega(t-\tau))\Omega_{jk}h(\tau)d\tau \quad (1 \le j < k \le n),$$

$$L_0\partial_t I_\sigma[h](t) = L_0 J_\sigma[h](t) = \sigma(\cos\omega(t-\sigma))h(\sigma) + \int_\sigma^t (\cos\omega(t-\tau))h(\tau)d\tau \tag{3.33}$$
$$+ \int_\sigma^t (\cos\omega(t-\tau))[(L_0 h)(\tau)]d\tau,$$

$$L_j\partial_k I_\sigma[h](t) = \frac{\sin\omega(t-\sigma)}{\omega}\partial_k\left(x_j h(\sigma)\right) - \delta_{jk}\int_\sigma^t (\cos\omega(t-\tau))h(\tau)d\tau \tag{3.34}$$
$$+ \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\left[\left(\partial_k L_j h\right)(\tau)\right]d\tau \quad (j,k = 1,\cdots,n),$$

$$\Omega_{jk}\partial_\ell I_\sigma[h](t) = \delta_{k\ell}\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_j h(\tau)d\tau - \delta_{j\ell}\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_k h(\tau)d\tau \tag{3.35}$$
$$+ \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_\ell\Omega_{jk}h(\tau)d\tau \quad (1 \le j < k \le n, \ell = 1,\cdots,n),$$

$$L_0\partial_k I_\sigma[h](t) = \sigma\frac{\sin\omega(t-\sigma)}{\omega}\partial_k h(\sigma) + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\partial_k h(\tau)d\tau \tag{3.36}$$
$$+ \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}[(\partial_k L_0 h)(\tau)]d\tau \quad (k = 1,\cdots,n).$$

PROOF. All these equalities are verified by direct calculations. See Hidano and Tsutaya [**4**]. Note that we have used the precise commutation relations (3.2)–(3.4) to cancel undesirable terms when proving this proposition. Q.E.D.

PROPOSITION 3.2. *Suppose that* $(f,g) \in \Sigma$. *Put* $u_0(t) := (\cos\omega t)f + (\omega^{-1}\sin\omega t)g$. *Then* u_0 *satisfies*

$$\Gamma^\alpha u_0, \quad \partial_k\Gamma^\alpha u_0 \in C(\mathbf{R}; L^2(\mathbf{R}^4)) \quad \textit{for any } \alpha \textit{ with } |\alpha| \le 2 \textit{ and } k = 0,\cdots,4, \tag{3.37}$$

$$\sup_{t\in\mathbf{R}}\|u_0(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2}\sup_{t\in\mathbf{R}}\|D\Gamma^\alpha u_0(t,\cdot)\|_2 \le C_5\|(f,g)\|_\Sigma. \tag{3.38}$$

PROOF. Applying L_j, Ω_{jk} and L_0 directly to the formula $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$, we may find with the help of the Fourier transform that the following three equalities hold in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$:

$$\begin{aligned} L_j u_0(t) &= -\frac{\sin \omega t}{\omega} \partial_j f - \frac{\sin \omega t}{\omega} \omega^2 (x_j f) + (\cos \omega t)(x_j g) \\ &= \frac{\sin \omega t}{\omega} \partial_j f + \frac{\sin \omega t}{\omega} [x_j \Delta f] + (\cos \omega t)(x_j g) \quad (j = 1, \cdots, 4), \end{aligned}$$

$$\Omega_{jk} u_0(t) = (\cos \omega t)(\Omega_{jk} f) + \frac{\sin \omega t}{\omega} \Omega_{jk} g \quad (1 \le j < k \le 4),$$

$$L_0 u_0(t) = \sum_{j=1}^{4} (\cos \omega t)(x_j \partial_j f) - 3 \frac{\sin \omega t}{\omega} g + \sum_{j=1}^{4} \frac{\sin \omega t}{\omega} \partial_j (x_j g).$$

Then, this proposition can be proved without any difficulty in view of (3.1), (3.16), a simple equality $x_j \partial_k g = \partial_k (x_j g) - \delta_{jk} g$ and the strong continuity with respect to t variable of the operators $\omega^{-1} \sin \omega t$, $\cos \omega t$ in H^s for any $s \in \boldsymbol{R}$. Thus, we omit the details of the proof.

4. Proof of Theorem 2.

Throughout this section we confine ourselves to the case of $n = 4$. For any $(f, g) \in \Sigma$ let us consider the integral equation

$$u(t) = u_0(t) + I_0[F(u)](t), \quad t \in \boldsymbol{R}, x \in \boldsymbol{R}^4. \tag{4.1}$$

Here $u_0(t) = (\cos \omega t)f + (\omega^{-1} \sin \omega t)g$, $I_0[F(u)](t) = \int_0^t (\omega^{-1} \sin \omega(t - \tau))F(u(\tau))d\tau$ as before. We introduce the sets of functions Y_δ $(\delta > 0)$, Z as follows.

$$\begin{aligned} Y_\delta = \Big\{ & u = u(t, x) | \Gamma^\alpha u \in C(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \quad \text{for any } \alpha \text{ with } |\alpha| \le 2, \\ & \partial_k \Gamma^\alpha u \in L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \quad \text{for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4, \\ & u(0, x) = f(x), \partial_t u(0, x) = g(x), \\ & \|u\|_Y \equiv \sup_{t \in \boldsymbol{R}} \|u(t, \cdot)\|_{\Gamma, 2, 2} + \sum_{|\alpha| = 2} \text{ess} \cdot \sup_{t \in \boldsymbol{R}} \|D\Gamma^\alpha u(t, \cdot)\|_2 \le \delta \Big\}, \end{aligned}$$

$$Z = \Big\{ u = u(t, x) | u \in C(\boldsymbol{R}; L^2(\boldsymbol{R}^4)), \|u\|_Z \equiv \sup_{t \in \boldsymbol{R}} \|u(t, \cdot)\|_2 < \infty \Big\}.$$

Observe that $\|u\|_Y$ is equivalent to $\sup_{t \in \boldsymbol{R}} \|u(t, \cdot)\|_{\Gamma, 2, 2} + \text{ess} \cdot \sup_{t \in \boldsymbol{R}} \|Du(t, \cdot)\|_{\Gamma, 2, 2}$. Note that Y_δ is nonempty if $\|(f, g)\|_\Sigma$ is sufficiently small relatively to δ. Z is a Banach space with the norm $\|u\|_Z$. We shall employ a simpler iteration scheme than that in Zhou [**21**] to point out that such a modified iteration method as in John [**6**], which was used in [**21**], is not necessary to construct solutions. Not only our scheme simplifies the proof of the existence of solutions, but also it is of help to relax the smallness condition on the data. On the other hand, the method in [21] will take effect when we prove the continuous dependence results. See Proposition 4.8 below.

LEMMA 4.1. *Y_δ is a closed subset of Z for any positive δ.*

PROOF. Let $\{u_m\} \subset Y_\delta$ converges to v in Z. Then $v(0) = f$. We show $v \in Y_\delta$. Since $\|u_m\|_Y \le \delta$, it follows from the sequentially weak-$*$ compactness of closed balls in L^∞ that there exists a subsequence $\{u_{m'}\}$ and a function w_α (resp. $w_{k\alpha}$) for every multi-index α with $|\alpha| \le 2$ (resp. $k = 0, \cdots, 4$ and α with $|\alpha| = 2$) such that

$$\Gamma^\alpha u_{m'} \rightharpoonup w_\alpha \quad \text{weak-}* \text{ in } L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4)),$$

$$\partial_k \Gamma^\alpha u_{m'} \rightharpoonup w_{k\alpha} \quad \text{weak-}* \text{ in } L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$$

with

$$\|w_\alpha\|_{L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))} \le \liminf_{m' \to \infty} \|\Gamma^\alpha u_{m'}\|_{L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))},$$

$$\|w_{k\alpha}\|_{L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))} \le \liminf_{m' \to \infty} \|\partial_k \Gamma^\alpha u_{m'}\|_{L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))}.$$

On the other hand, it is easily checked that

$$\Gamma^\alpha u_{m'} \to \Gamma^\alpha v \quad \text{in } \mathscr{D}'(\boldsymbol{R} \times \boldsymbol{R}^4),$$

$$\partial_k \Gamma^\alpha u_{m'} \to \partial_k \Gamma^\alpha v \quad \text{in } \mathscr{D}'(\boldsymbol{R} \times \boldsymbol{R}^4)$$

because $u_{m'} \to v$ in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$. Thus, we find that $w_\alpha = \Gamma^\alpha v$, $w_{k\alpha} = \partial_k \Gamma^\alpha v$. Since $\Gamma^\alpha v$, $\partial_t \Gamma^\alpha v \in L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ for $1 \le |\alpha| \le 2$, we see with the eventual modifications on a set of measure 0 in time that $\Gamma^\alpha v \in C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ for $|\alpha| \le 2$. Moreover, $\|v\|_Y \le \liminf_{m' \to \infty} \|u_{m'}\|_Y \le \delta$.

It remains to show $\partial_t v(0, x) = g(x)$. Note that there holds that

$$(\partial_t u_{m'}(t), \varphi)_{L^2} = (g, \varphi)_{L^2} + \int_0^t (\partial_t^2 u_{m'}(\tau), \varphi)_{L^2} \, d\tau$$

for all $\varphi \in L^2(\boldsymbol{R}^4)$. Let $\psi = \psi(\tau)$ be the characteristic function of the interval $[0, t]$ (or $[t, 0]$ if $t < 0$). Since $\psi\varphi \in L^1(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$, it holds that for any $t \in \boldsymbol{R}$

$$\int_0^t (\partial_t^2 u_{m'}(\tau), \varphi)_{L^2} d\tau = \int_{\boldsymbol{R}} (\partial_t^2 u_{m'}(\tau), \varphi)_{L^2} \psi(\tau) d\tau$$

$$\to \int_{\boldsymbol{R}} (\partial_t^2 v(\tau), \varphi)_{L^2} \psi(\tau) d\tau = \int_0^t (\partial_t^2 v(\tau), \varphi)_{L^2} d\tau$$

as $m' \to \infty$. On the other hand, since $\{(\partial_t u_{m'}, \varphi)_{L^2}\}$ is uniformly bounded and equivalently continuous on $\boldsymbol{R}$ for any fixed $\varphi \in L^2(\boldsymbol{R}^4)$, we can apply the Ascoli-Arzelà theorem to the subsequence $\{(\partial_t u_{m'}, \varphi)_{L^2}\}$ if we restrict the domain of the definition to any compact interval of $\boldsymbol{R}$. With the help of the diagonal method, we can extract a subsequence $\{(\partial_t u_{m''}, \varphi)_{L^2}\}$ which converges to a continuous function $w = w(t)$, defined on $\boldsymbol{R}$ and depending on φ, uniformly on every compact interval of $\boldsymbol{R}$. Let I be any compact interval of $\boldsymbol{R}$. Then there holds

$$\int_I (\partial_t u_{m''}(t), \varphi)_{L^2} dt \to \int_I w(t) dt.$$

On the other hand, since $\partial_t u_{m''} \rightharpoonup \partial_t v$ weak-$*$ in $L^\infty(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$, we see that

$$\int_I (\partial_t u_{m''}(t), \varphi)_{L^2} dt \to \int_I (\partial_t v(t), \varphi)_{L^2} dt.$$

By the uniqueness of the limit we find

$$\int_I [w(t) - (\partial_t v(t), \varphi)_{L^2}] dt = 0.$$

Since I is arbitrary, it follows that $w(t) = (\partial_t v(t), \varphi)_{L^2}$ for all $t \in \boldsymbol{R}$. Therefore, it holds that

$$(\partial_t v(t), \varphi)_{L^2} = (g, \varphi)_{L^2} + \int_0^t (\partial_t^2 v(\tau), \varphi)_{L^2} d\tau$$

for any $t \in \boldsymbol{R}$. Since φ is arbitrary in $L^2(\boldsymbol{R}^4)$, we find

$$\partial_t v(t) = g + \int_0^t \partial_t^2 v(\tau) d\tau \quad \text{in } L^2(\boldsymbol{R}^4)$$

for any $t \in \boldsymbol{R}$. This implies $\partial_t v(0) = g$. Thus we have completed the proof of Lemma 4.1. Q.E.D.

PROPOSITION 4.1. *Let t_0 be any fixed finite number. For any $(f, g) \in H^2(\boldsymbol{R}^4) \times H^1(\boldsymbol{R}^4)$ the equation* (1.1) *has at most one solution in*

$$\Theta_{t_0} = \left\{ u = u(t,x) | u \in \bigcap_{j=0}^{2} C^j(\boldsymbol{R}; H^{2-j}(\boldsymbol{R}^4)), \right.$$
$$\left. u(t_0, x) = f(x), \partial_t u(t_0, x) = g(x) \right\}.$$

PROOF. Without loss of generality we may take $t_0 = 0$ because (1.1) is invariant under the translation in the time variable. Moreover, we have only to prove the uniqueness for $t > 0$ because (1.1) is invariant under the change of variable $t \to -t$.

Let u, v be solutions to (1.1) in Θ_0. Let I be any interval of $(0, \infty)$. Since u and v satisfy the equation (1.1) as an equality in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$, we have by the usual energy inequality

$$\begin{aligned} \|u(t,\cdot) - v(t,\cdot)\|_e &\le C \int_0^t (\|u(\tau,\cdot)\|_{4(\rho-1)} + \|v(\tau,\cdot)\|_{4(\rho-1)})^{\rho-1} \|u(\tau,\cdot) - v(\tau,\cdot)\|_4 d\tau \\ &\le C \int_0^t \left(\sup_I \|u(\tau,\cdot)\|_{H^2} + \sup_I \|v(\tau,\cdot)\|_{H^2} \right)^{\rho-1} \|u(\tau,\cdot) - v(\tau,\cdot)\|_{H^1} d\tau \end{aligned}$$

for all $t \in I$. Moreover, in view of $u(0,x) = v(0,x)$ it follows from an elementary identity $u(t) = u(0) + \int_0^t \partial_\tau u(\tau)d\tau$ that

$$\|u(t,\cdot) - v(t,\cdot)\|_2 \le \int_0^t \|\partial_\tau u(\tau,\cdot) - \partial_\tau v(\tau,\cdot)\|_2 d\tau.$$

Hence we conclude by Gronwall's inequality that $u = v$ on I. Since I is arbitrary, we have got the desirable uniqueness result. Q.E.D.

In what follows we simply denote $\|(f,g)\|_\Sigma$ by Λ. Since u_0 satisfies (3.37)–(3.38), u_0 belongs to $Y_{C_5\Lambda}$. We shall show through the following propositions that a sequence $\{u_m\}$ can be defined in $Y_{2C_5\Lambda}$ for small Λ inductively by

$$u_{m+1}(t) = u_0(t) + I_0[F(u_m)](t). \tag{4.2}$$

PROPOSITION 4.2. *$C_0^\infty(\boldsymbol{R}^4) \times C_0^\infty(\boldsymbol{R}^4)$ is dense in Σ.*

PROOF. Observe that the norm $\|(f,g)\|_\Sigma$ is equivalent to

$$\|f\|_{H^3} + \sum_{|\alpha|=1}^{2} \sum_{|\beta|=|\alpha|} \|x^\beta \partial_x^\alpha f\|_2 + \sum_{\substack{|\alpha|=3\\|\beta|=2}} \|x^\beta \partial_x^\alpha f\|_2$$
$$+ \|g\|_{H^2} + \sum_{|\alpha|=0}^{1} \sum_{|\beta|=|\alpha|+1} \|x^\beta \partial_x^\alpha g\|_2 + \sum_{\substack{|\alpha|=2\\|\beta|=2}} \|x^\beta \partial_x^\alpha g\|_2.$$

Here we have set $x^\beta = x_1^{\beta_1} \cdots x_4^{\beta_4}$. Then, by the usual regularizing and cutting, together with an elementary equality $x = (x-y) + y$, we can prove this proposition without any difficulty. Thus we omit the details. Q.E.D.

PROPOSITION 4.3. *All $(f,g) \in \Sigma$ satisfy the estimates*

$$\|\langle\cdot\rangle^2 F(f)\|_2 \le C\Lambda^p, \tag{4.3}$$
$$\|\langle\cdot\rangle^2 F'(f)\partial_k f\|_2,\quad \|\langle\cdot\rangle^2 F'(f) g\|_2 \le C\Lambda^p \quad (k = 1,\cdots,4). \tag{4.4}$$

It also holds that

$$\sup_{t\in\boldsymbol{R}} \left\| \frac{\sin\omega t}{\omega}[x_j F(f)] \right\|_2,\quad \sup_{t\in\boldsymbol{R}} \left\| \frac{\sin\omega t}{\omega}[x_k x_j F'(f)\partial_k f] \right\|_2, \tag{4.5}$$
$$\sup_{t\in\boldsymbol{R}} \left\| \frac{\sin\omega t}{\omega}[x_k x_j F'(f) g] \right\|_2 \le C\Lambda^p$$

for $j,\ k = 1,\cdots,4$.

PROOF. Note that the following formal calculations can be justified by virtue of Proposition 4.2. By applying L_k $(k = 1,\cdots,4)$ to (3.28) with $\sigma = 0$, there appears the term $(\cos\omega t)[x_k x_j h(0)]$. This is the reason why we need to show (4.3). Let us begin with verifying (4.3). It follows from the inequality $\|v\|_4 \le C\|\nabla v\|_2$ in four space dimensions that

$$\|\langle\cdot\rangle^2 F(f)\|_2 \le C\|f\|_\infty^{p-2}\|\langle\cdot\rangle f\|_4^2 \le C\|f\|_{H^3}^{p-2}(\|f\|_2 + \|\langle\cdot\rangle\nabla f\|_2)^2 \le CA^p.$$

Next we show (4.4). Applying the Schwarz inequality, we have

$$\|\langle\cdot\rangle^2 F'(f)\partial_k f\|_2 \le C\|f\|_\infty^{p-2}\|\langle\cdot\rangle f\|_4\|\langle\cdot\rangle\partial_k f\|_4 \le CA^p.$$

Quite similarly it follows that $\|\langle\cdot\rangle^2 F'(f)g\|_2 \le CA^p$.

Finally we verify (4.5). Applying (3.18) with $\sigma = 0$, we obtain

$$\left\|\frac{\sin\omega t}{\omega}[x_k x_j F'(f)g]\right\|_2 \le C\big\||f|^{p-1}g\big\|_{4/3} + C\big\||f|^{p-1}x_j g\big\|_{1,2}. \tag{4.6}$$

Applying the Hölder inequality, we get

$$\big\||f|^{p-1}g\big\|_{4/3} \le \|f\|_{4(p-1)}^{p-1}\|g\|_2 \le C\|f\|_{H^2}^{p-1}\|g\|_2. \tag{4.7}$$

Moreover, by virtue of (3.15) we see that

$$\begin{aligned}\big\||f|^{p-1}x_j g\big\|_{1,2} &\le \|f\|_\infty^{p-2}\int_0^\infty\left(\int_{|\zeta|=1} f^4(r\zeta)d\zeta\right)^{1/4}\left(\int_{|\zeta|=1}(r\zeta_j)^4 g^4(r\zeta)d\zeta\right)^{1/4} r^3 dr \\ &\le C\|f\|_\infty^{p-2}\int_0^\infty\left(\sum_{|\theta|\le 1}\int_{|\zeta|=1}(\Omega^\theta f)^2(r\zeta)d\zeta\right)^{1/2} \\ &\quad\times\left(\sum_{|\theta|\le 1}\int_{|\zeta|=1}(\Omega^\theta(x_j g))^2(r\zeta)d\zeta\right)^{1/2} r^3 dr \\ &\le C\|f\|_{H^3}^{p-2}(\|f\|_2 + \|\langle\cdot\rangle\nabla f\|_2)(\|\langle\cdot\rangle g\|_2 + \|\langle\cdot\rangle^2\nabla g\|_2).\end{aligned} \tag{4.8}$$

Combining (4.6)–(4.8), we find that one of the estimates in (4.5) is true. The others can be proved in quite the same way. Q.E.D.

REMARK 4.1. We can prove (4.5) in a different way. That is, (4.5) also follows from the combination of the Hardy–Littlewood–Sobolev inequality and the Sobolev inequality $\|v\|_4 \le C\|\nabla v\|_2$. See Lemma 4.2 in Ginibre and Velo **[1]**. See also Remark 4.2 below.

PROPOSITION 4.4. *Let u be any function in Y_{2C_5A}, where C_5 is the same constant as in* (3.38). *Then, it holds that*

$$\|F(u(\tau,\cdot))\|_{\Gamma,2,2} \le C(2C_5A)^p(1+|\tau|)^{-3(p-1)/2}, \tag{4.9}$$

$$(4.10)\qquad \sup_{t\in\boldsymbol{R}}\left\|\frac{\sin\omega(t-\tau)}{\omega}\Gamma^\alpha F(u(\tau,\cdot))\right\|_2 \le C(2C_5A)^p(1+|\tau|)^{-1-3(p-2)/2}\ (|\alpha|\le 2)$$

for a.e. $\tau\in\boldsymbol{R}$, *and*

$$(4.11)\qquad \Gamma^\alpha I_0[F(u)]\in C(\boldsymbol{R};L^2(\boldsymbol{R}^4))\quad \textit{for any } \alpha \textit{ with } |\alpha|\le 2,$$

$$(4.12)\qquad \partial_k\Gamma^\alpha I_0[F(u)]\in C(\boldsymbol{R};L^2(\boldsymbol{R}^4))\quad \textit{for any } \alpha \textit{ with } |\alpha|=2 \textit{ and } k=0,\cdots,4,$$

$$(4.13)\qquad \|I_0[F(u)]\|_Y\le C_6A^p+C_7(2C_5A)^p\quad \textit{for some positive constants } C_6, C_7.$$

PROOF. We start the proof with showing (4.9). By chain rule and the Schwarz inequality together with (3.12) we get

$$(4.14)\qquad \begin{aligned}\|F(u(\tau,\cdot))\|_{\Gamma,1,2} &\le C\|u(\tau,\cdot)\|_{4(p-1)}^{p-1}\|u(\tau,\cdot)\|_{\Gamma,1,4}\\ &\le C(1+|\tau|)^{-3(p-1)/2}(2C_5A)^p.\end{aligned}$$

To show the corresponding estimate for $\|F(u(\tau,\cdot))\|_{\Gamma,2,2}$ we utilize some devices which have been already employed implicitly in Zhou [**21**]. Note that the estimate

$$(4.15)\qquad \|u(\tau,\cdot)\|_{\infty,\chi}\le C(1+|\tau|)^{-n/p}\|u(t,\cdot)\|_{\Gamma,s,p},\quad s>n/p, p\ge 1$$

holds for the same reason as (3.6) holds. Then, combining (4.15) with (3.5), we have the estimate

$$(4.16)\qquad \|u(t,\cdot)\|_{\infty,\chi}\le C(1+|t|)^{-n/\nu}\sum_{|\alpha|\le 2}\|\Gamma^\alpha u(t,\cdot)\|_{H^1}\quad \text{with } 2<\nu\le 4$$

for a.e. $t\in\boldsymbol{R}$. Then

$$(4.17)\qquad \begin{aligned}&\sum_{|\alpha|,|\beta|=1}\|F''(u(\tau,\cdot))\Gamma^\alpha u(\tau,\cdot)\Gamma^\beta u(\tau,\cdot)\|_2\\ &\le C\sum_{|\alpha|=1}\|u(\tau,\cdot)\|_{\infty,\chi}^{p-2}\|\Gamma^\alpha u(\tau,\cdot)\|_{4,\chi}^2\\ &\quad + C\sum_{|\alpha|=1}\left(\int_{(1+|\tau|)/2}^\infty\int_{|\zeta|=1}|u(\tau,r\zeta)|^{2(p-2)}(\Gamma^\alpha u)^4(\tau,r\zeta)d\zeta r^3dr\right)^{1/2}.\end{aligned}$$

The first term on the right-hand side of (4.17) is estimated from above by

$$(4.18)\qquad C(1+|\tau|)^{-2-4(p-2)/\nu}(2C_5A)^p\quad \text{for a.e. } \tau\in\boldsymbol{R}$$

with the help of (3.6) and (4.16). On the other hand, the second term can be estimated from above by virtue of (3.14) and the Hölder inequality as follows.

$$(4.19)\quad C\sum_{|\alpha|=1}\left(\int_{(1+|\tau|)/2}^{\infty}\left(\sup_{|\zeta|=1}|u(\tau,r\zeta)|\right)^{2(\rho-2)}\int_{|\zeta|=1}(\Gamma^{\alpha}u)^{4}(\tau,r\zeta)d\zeta r^{3}dr\right)^{1/2}$$

$$\leq C\sum_{|\alpha|=1}\left[\int_{(1+|\tau|)/2}^{\infty}\left(\sum_{|\theta|\leq 2}\int_{|\zeta|=1}(\Omega^{\theta}u)^{2}(\tau,r\zeta)d\zeta\right)^{\rho-2}\right.$$

$$\left.\times\left(\sum_{|\theta|\leq 1}\int_{|\zeta|=1}(\Omega^{\theta}\Gamma^{\alpha}u)^{2}(\tau,r\zeta)d\zeta\right)^{2}r^{3}dr\right]^{1/2}$$

$$\leq C\|u(\tau,\cdot)\|_{\Gamma,2,2\rho,2,\phi}^{\rho}\leq C(1+|\tau|)^{-3(\rho-1)/2}(2C_{5}\Lambda)^{\rho}\quad\text{for a.e. }\tau\in\boldsymbol{R}.$$

We have employed (3.13) at the last inequality. It is easily checked that $3(\rho-1)/2<2+4(\rho-2)/\nu$ for all $\rho>2$ if $2<\nu<8/3$. Thus, combining (4.17) with (4.18)–(4.19), we obtain

$$(4.20)\quad \sum_{|\alpha|,|\beta|=1}\|F''(u(\tau,\cdot))\Gamma^{\alpha}u(\tau,\cdot)\Gamma^{\beta}u(\tau,\cdot)\|_{2}$$

$$\leq C(1+|\tau|)^{-3(\rho-1)/2}(2C_{5}\Lambda)^{\rho}\quad\text{for a.e. }\tau\in\boldsymbol{R}.$$

Moreover, by employing (3.14), the Schwarz inequality and (3.13), we have

$$(4.21)\quad \sum_{|\alpha|=2}\|F'(u(\tau,\cdot))\Gamma^{\alpha}u(\tau,\cdot)\|_{2}$$

$$\leq C\|u(\tau,\cdot)\|_{\infty,\chi}^{\rho-1}\|u(\tau,\cdot)\|_{\Gamma,2,2}$$

$$+\sum_{|\alpha|=2}\left(\int_{(1+|\tau|)/2}^{\infty}\left(\sup_{|\zeta|=1}|u(\tau,r\zeta)|\right)^{2(\rho-1)}\int_{|\zeta|=1}(\Gamma^{\alpha}u)^{2}(\tau,r\zeta)d\zeta r^{3}dr\right)^{1/2}$$

$$\leq C(1+|\tau|)^{-4(\rho-1)/\nu}(2C_{5}\Lambda)^{\rho}+C\|u(\tau,\cdot)\|_{\Gamma,2,4(\rho-1),2,\phi}^{\rho-1}\|u(\tau,\cdot)\|_{\Gamma,2,4,2,\phi}$$

$$\leq C(1+|\tau|)^{-3(\rho-1)/2}(2C_{5}\Lambda)^{\rho}\quad\text{for a.e. }\tau\in\boldsymbol{R}.$$

At the last inequality we have used the fact that $4(\rho-1)/\nu>3(\rho-1)/2$ for all $\rho>2$ if $2<\nu<8/3$. Combining (4.14) with (4.20)–(4.21) yields (4.9).

We next prove (4.10) by sharpening the corresponding estimate in Zhou [**21**]. Employing the Hölder inequality first and then (3.6), we have

$$(4.22)\quad \|F(u(\tau,\cdot))\|_{\Gamma,1,4/3,\chi}\leq C\|u(\tau,\cdot)\|_{4(\rho-1),\chi}^{\rho-1}\|u(\tau,\cdot)\|_{\Gamma,1,2}$$

$$\leq C(1+|\tau|)^{-2\rho+3}(2C_{5}\Lambda)^{\rho}.$$

Applying the Schwarz inequality to the integration on the unit sphere first, then (3.15), finally the Schwarz inequality to the integration over $(0, \infty)$, we obtain

$$\text{(4.23)} \quad \|F(u(\tau,\cdot))\|_{\Gamma,1,1,2,\phi}$$

$$\leq C\int_0^\infty \left(\int_{|\zeta|=1} |u(\tau,r\zeta)|^{4(\rho-1)}d\zeta\right)^{1/4}\left(\sum_{|\alpha|\leq 1}\int_{|\zeta|=1}(\Gamma^\alpha u)^4(\tau,r\zeta)d\zeta\right)^{1/4} r^3dr$$

$$\leq C\int_0^\infty \left(\sum_{|\theta|\leq 1}\int_{|\zeta|=1}(|u|^{\rho-2}\Omega^\theta u)^2(\tau,r\zeta)d\zeta\right)^{1/2}$$

$$\times\left(\sum_{|\alpha|,|\theta|\leq 1}\int_{|\zeta|=1}(\Omega^\theta\Gamma^\alpha u)^2(\tau,r\zeta)d\zeta\right)^{1/2} r^3dr$$

$$\leq C\left(\sum_{|\theta|\leq 1}\int_0^\infty\int_{|\zeta|=1}|(\Omega^\theta u)(\tau,r\zeta)|^{2(\rho-1)}r^3d\zeta dr\right)^{1/2}\|u(\tau,\cdot)\|_{\Gamma,2,2}.$$

At the last inequality we have used the Hölder inequality. Hence, if $2 < \rho \leq 3$, it follows from (3.12)

$$\text{(4.24)} \qquad \|F(u(\tau,\cdot))\|_{\Gamma,1,1,2,\phi} \leq C(1+|\tau|)^{-3(\rho-2)/2}(2C_5A)^\rho.$$

On the other hand, if $\rho > 3$, then we obtain

$$\text{(4.25)} \quad \|F(u(\tau,\cdot))\|_{\Gamma,1,1,2,\phi}$$

$$\leq C\left(\sum_{|\theta|\leq 1}\int_0^\infty\int_{|\zeta|=1}|u(\tau,r\zeta)|^{2(\rho-2)}(\Omega^\theta u)^2(\tau,r\zeta)r^3d\zeta dr\right)^{1/2}\|u(\tau,\cdot)\|_{\Gamma,2,2}$$

$$\leq C\|u(\tau,\cdot)\|_{4(\rho-2)}^{\rho-2}\|u(\tau,\cdot)\|_{\Gamma,1,4}\|u(\tau,\cdot)\|_{\Gamma,2,2}$$

$$\leq C(1+|\tau|)^{-3(\rho-2)/2}(2C_5A)^\rho.$$

Combining (3.17) with (4.22)–(4.25) leads us to (4.10) for $|\alpha| \leq 1$. To prove (4.10) for $|\alpha| = 2$ we proceed as follows. By chain rule we have

$$\text{(4.26)} \quad \sum_{|\alpha|=2}\left\|\frac{\sin\omega(t-\tau)}{\omega}\Gamma^\alpha F(u(\tau,\cdot))\right\|_2$$

$$\leq C\sum_{|\alpha|=1}\left\||u(\tau,\cdot)|^{\rho-2}(\Gamma^\alpha u)^2(\tau,\cdot)\right\|_{4/3,\chi} + C\|u(\tau,\cdot)\|_{4(\rho-1),\chi}^{\rho-1}\|u(\tau,\cdot)\|_{\Gamma,2,2}$$

$$+ C(1+|\tau|)^{-1}\sum_{|\alpha|=1}\left\||u(\tau,\cdot)|^{\rho-2}(\Gamma^\alpha u)^2(\tau,\cdot)\right\|_{1,2,\phi}$$

$$+ C(1+|\tau|)^{-1}\sum_{|\alpha|=2}\left\||u(\tau,\cdot)|^{\rho-1}\Gamma^\alpha u(\tau,\cdot)\right\|_{1,2,\phi}.$$

By the Hölder inequality we have for $|\alpha| = 1$

$$(4.27)\qquad \left\| |u(\tau,\cdot)|^{p-2}(\Gamma^\alpha u)^2(\tau,\cdot)\right\|_{4/3,\chi} \le \|u(\tau,\cdot)\|_{4(p-1),\chi}^{p-2}\|u(\tau,\cdot)\|_{\Gamma,1,8(p-1)/(2p-1),\chi}^2 \le C(1+|\tau|)^{-2p+3}(2C_5A)^p.$$

Moreover, it easily follows that

$$(4.28)\qquad \|u(\tau,\cdot)\|_{4(p-1),\chi}^{p-1}\|u(\tau,\cdot)\|_{\Gamma,2,2} \le C(1+|\tau|)^{-2p+3}(2C_5A)^p.$$

We next estimate the third and fourth terms on the right-hand side of (4.26). For $|\alpha| = 1$ we get by (3.14), (3.15) and (3.13)

$$(4.29)\qquad \left\| |u(\tau,\cdot)|^{p-2}(\Gamma^\alpha u)^2(\tau,\cdot)\right\|_{1,2,\phi}$$
$$\le \int_{(1+|\tau|)/2}^\infty \left(\sup_{|\zeta|=1}|u(\tau,r\zeta)|\right)^{p-2}\left(\int_{|\zeta|=1}(\Gamma^\alpha u)^4(\tau,r\zeta)d\zeta\right)^{1/2} r^3dr$$
$$\le \int_{(1+|\tau|)/2}^\infty \left(\sum_{|\theta|\le 2}\int_{|\zeta|=1}(\Omega^\theta u)^2(\tau,r\zeta)d\zeta\right)^{(p-2)/2}$$
$$\times\left(\sum_{|\theta|\le 1}\int_{|\zeta|=1}(\Omega^\theta\Gamma^\alpha u)^2(\tau,r\zeta)d\zeta\right) r^3dr$$
$$\le C\|u(\tau,\cdot)\|_{\Gamma,2,p,2,\phi}^p \le C(1+|\tau|)^{-3(p-2)/2}(2C_5A)^p$$

for a.e. $\tau\in\boldsymbol{R}$. Finally, by (3.14), (3.13) we obtain for $|\alpha| = 2$

$$(4.30)\qquad \left\| |u(\tau,\cdot)|^{p-1}\Gamma^\alpha u(\tau,\cdot)\right\|_{1,2,\phi}$$
$$\le \int_{(1+|\tau|)/2}^\infty \left(\sup_{|\zeta|=1}|u(\tau,r\zeta)|\right)^{p-1}\left(\int_{|\zeta|=1}(\Gamma^\alpha u)^2(\tau,r\zeta)d\zeta\right)^{1/2} r^3dr$$
$$\le C\|u(\tau,\cdot)\|_{\Gamma,2,2(p-1),2,\phi}^{p-1}\|u(\tau,\cdot)\|_{\Gamma,2,2}$$
$$\le C(1+|\tau|)^{-3(p-2)/2}(2C_5A)^p \quad \text{for a.e. } \tau\in\boldsymbol{R}.$$

Therefore, combining (3.17) with (4.29)–(4.30) leads us to (4.10) for $|\alpha| = 2$.

There remains to show (4.11)–(4.13). (4.11)–(4.12) follow from Propositions 3.1, 4.3 and (4.9). In view of Propositions 3.1, 4.3 and (4.9)–(4.10) we see that (4.13) certainly holds. Thus we have completed the proof of Proposition 4.4. Q.E.D.

Let A be small so that $C_6A^p + C_7(2C_5A)^p \le C_5A$ may hold. Then we see that a sequence $\{u_m\}$ can be defined in Y_{2C_5A} inductively by (4.2).

PROPOSITION 4.5. *The sequence* $\{u_m\}\subset Y_{2C_5A}$ *defined just above satisfies*

$$(4.31)\qquad \|u_{m+1}-u_m\|_Z \le C_8(\|u_m\|_Y + \|u_{m-1}\|_Y)^{p-1}\|u_m - u_{m-1}\|_Z$$

for all $m = 1, 2, \cdots$. *Here* C_8 *is a positive constant independent of* m.

PROOF. For simplicity we put $u_{m+1}^* := u_{m+1} - u_m$, $u_m^* := u_m - u_{m-1}$. It follows from (3.6) that

$$(4.32)\quad \|F(u_m(\tau,\cdot)) - F(u_{m-1}(\tau,\cdot))\|_{4/3,\chi}$$
$$\le C(\|u_m(\tau,\cdot)\|_{4(\rho-1),\chi} + \|u_{m-1}(\tau,\cdot)\|_{4(\rho-1),\chi})^{\rho-1}\|u_m^*(\tau,\cdot)\|_2$$
$$\le C(1+|\tau|)^{-2\rho+3}(\|u_m(\tau,\cdot)\|_{\Gamma,2,2} + \|u_{m-1}(\tau,\cdot)\|_{\Gamma,2,2})^{\rho-1}\|u_m^*(\tau,\cdot)\|_2.$$

Moreover,

$$(4.33)\quad \|F(u_m(\tau,\cdot)) - F(u_{m-1}(\tau,\cdot))\|_{1,2,\phi}$$
$$\le C\int_{(1+|\tau|)/2}^{\infty}\left(\int_{|\zeta|=1}(|u_m(\tau,r\zeta)| + |u_{m-1}(\tau,r\zeta)|)^{2(\rho-1)}u_m^{*2}(\tau,r\zeta)d\zeta\right)^{1/2} r^3 dr$$
$$\le C\int_{(1+|\tau|)/2}^{\infty}\left(\sup_{|\zeta|=1}|u_m(\tau,r\zeta)| + \sup_{|\zeta|=1}|u_{m-1}(\tau,r\zeta)|\right)^{\rho-1}$$
$$\times\left(\int_{|\zeta|=1}u_m^{*2}(\tau,r\zeta)d\zeta\right)^{1/2} r^3 dr$$
$$\le C(\|u_m(\tau,\cdot)\|_{\Gamma,2,2(\rho-1),2,\phi} + \|u_{m-1}(\tau,\cdot)\|_{\Gamma,2,2(\rho-1),2,\phi})^{\rho-1}\|u_m^*(\tau,\cdot)\|_2$$
$$\le C(1+|\tau|)^{-3(\rho-2)/2}(\|u_m\|_Y + \|u_{m-1}\|_Y)^{\rho-1}\|u_m^*(\tau,\cdot)\|_2 \quad \text{for a.e. } \tau \in \boldsymbol{R}.$$

Combining (3.17) with (4.32)–(4.33), we have proved (4.31). Q.E.D.

Let us take Λ still smaller so that

$$(4.34)\qquad C_6\Lambda^\rho + C_7(2C_5\Lambda)^\rho \le C_5\Lambda \quad \text{and} \quad C_8(4C_5\Lambda)^{\rho-1} \le \frac{1}{2}$$

may hold. Then it follows from Lemma 4.1 and (4.31) that $\{u_m\} \subset Y_{2C_5\Lambda}$ is a Cauchy sequence in Z and there exists a unique function $u \in Y_{2C_5\Lambda}$ such that $u_m \to u$ in Z. For this u the integral $\int_0^t(\omega^{-1}\sin\omega(t-\tau))F(u(\tau))d\tau$ is in $L^2(\boldsymbol{R}^4)$ for any finite t because $F(u(\tau))$ is a bounded function with values in $L^2(\boldsymbol{R}^4)$. Let I be any interval. Then, as is easily seen, there holds

$$\int_0^t \frac{\sin\omega(t-\tau)}{\omega}F(u_m(\tau))d\tau \to \int_0^t \frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))d\tau \quad \text{in } C(I; L^2(\boldsymbol{R}^4))$$

as $m \to \infty$ because $u_m \to u$ in Z. Hence u satisfies the integral equation

$$(4.35)\qquad u(t) = u_0(t) + \int_0^t \frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))d\tau$$

as an equality in $C(I; L^2(\boldsymbol{R}^4))$. In fact, (4.35) holds in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ because I is arbitrary.

Next let us investigate the strong continuity with respect to t for the derivatives of u. Because u belongs to $Y_{2C_5\Lambda}$ and satisfies (4.35), we see from (4.11)–(4.12) that $\Gamma^\alpha u$,

$\partial_k \Gamma^\alpha u \in BC(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ for $|\alpha| \le 2$ and $k = 0, \cdots, 4$. It follows from (3.7), (2.18), (2.20) and an elementary inequality $\langle x \rangle \le \langle |t| - |x| \rangle + \langle t \rangle$ that $(u(t), \partial_t u(t)) \in \Sigma$ for any $t \in \boldsymbol{R}$. Taking account of the equivalence of the norms (see the proof of Proposition 4.2), we may derive the continuity of $(u, \partial_t u)$ with respect to t in Σ by using (4.35) itself and carrying out rather long, but very elementary computations. Clearly, u satisfies the differential equation (1.1) as an equality in $\bigcap_{j=0}^{1} C^j(\boldsymbol{R}; H^{1-j}(\boldsymbol{R}^4))$. Thus, letting ε be the supremum of Λ satisfying (4.34) and setting $C_3 := 2C_5$, we have proved the part (*I*) of Theorem 2.

In the rest of this section we prove (*II*), (*III*) of Theorem 2.

PROPOSITION 4.6. *There exists at most one pair of functions* (f^+, g^+) (*resp.* (f^-, g^-)) *satisfying* (2.23)–(2.24) *as* $t \to +\infty$ (*resp.* $-\infty$).

PROOF. The proof is standard. Thus we omit it. Q.E.D.

Define

$$u^+(t) := u(t) + \int_t^\infty \frac{\sin \omega(t - \tau)}{\omega} F(u(\tau)) d\tau, \tag{4.36}$$

$$u^-(t) := u(t) - \int_{-\infty}^t \frac{\sin \omega(t - \tau)}{\omega} F(u(\tau)) d\tau \tag{4.37}$$

for the solution u obtained just above. Since u satisfies (4.9)–(4.10), $u^\pm(t)$ are well-defined in $H^3(\boldsymbol{R}^4)$ for every $t \in \boldsymbol{R}$. $u^\pm$ actually belong to $\bigcap_{j=0}^{3} C^j(\boldsymbol{R}; H^{3-j}(\boldsymbol{R}^4))$ and solve the linear equation $\Box u = 0$. Because $u^\pm(0) \in H^1(\boldsymbol{R}^4)$, we see from the Sobolev imbedding theorem that $u^\pm(0) \in L^4(\boldsymbol{R}^4)$ and thus $(u^\pm(0), \partial_t u^\pm(0)) \in E(\boldsymbol{R}^4)$. Moreover, in view of (4.9), u^+ (resp. u^-) satisfies (2.24) as $t \to +\infty$ (resp. $-\infty$).

It remains to prove that $u^\pm$ defined by (4.36)–(4.37) have the properties (2.25)–(2.28).

PROPOSITION 4.7. *For any* α *with* $|\alpha| \le 2$ *the equality*

$$\Gamma^\alpha u^-(t) = \Gamma^\alpha u(t) + \sum_{|\beta| \le |\alpha|} C_{\alpha\beta} \int_{-\infty}^t \frac{\sin \omega(t - \tau)}{\omega} [(\Gamma^\beta F(u))(\tau)] d\tau \tag{4.38}$$

holds in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$. *Moreover, for any* α *with* $|\alpha| = 2$ *and* $k = 0, \cdots, 4$ *the equality*

$$\begin{aligned} \partial_k \Gamma^\alpha u^-(t) = \partial_k \Gamma^\alpha u(t) &+ \sum_{\substack{|\beta| \le 2 \\ |\gamma| = 1}} C_{k\alpha\beta\gamma} \int_{-\infty}^t \frac{\sin \omega(t - \tau)}{\omega} [(\partial_x^\gamma \Gamma^\beta F(u))(\tau)] d\tau \\ &+ \sum_{|\beta| \le 2} C_{k\alpha\beta} \int_{-\infty}^t \cos \omega(t - \tau) [(\Gamma^\beta F(u))(\tau)] d\tau \end{aligned} \tag{4.39}$$

also holds in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$. *Here* $C_{k\alpha\beta}$ (*resp.* $C_{k\alpha\beta\gamma}$) *is a not necessarily positive constant depending only on* k, α *and* β (*resp.* k, α, β *and* γ).

PROOF. It is impossible to prove this proposition by direct calculations. Hence we are forced to employ the following simple, but careful limiting procedures as in Hidano and Tsutaya [4]. For any $\sigma \in \boldsymbol{R}$ we put

$$u_\sigma^-(t) := u(t) - \int_\sigma^t \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau))d\tau. \tag{4.40}$$

Obviously, $u_\sigma^- \in \bigcap_{j=0}^3 BC^j(\boldsymbol{R}; H^{3-j}(\boldsymbol{R}^4))$. Moreover, it follows from Proposition 3.1 that

$$L_j u_\sigma^-(t) = L_j u(t) - \frac{\sin \omega(t-\sigma)}{\omega}[x_j F(u(\sigma))] - I_\sigma[L_j F(u)](t) \quad (j = 1, \cdots, 4), \tag{4.41}$$

$$\Omega_{jk} u_\sigma^-(t) = \Omega_{jk} u(t) - I_\sigma[\Omega_{jk} F(u)](t) \quad (1 \le j < k \le 4), \tag{4.42}$$

$$L_0 u_\sigma^-(t) = L_0 u(t) - \sigma \frac{\sin \omega(t-\sigma)}{\omega} F(u(\sigma)) - 2I_\sigma[F(u)](t) - I_\sigma[L_0 F(u)](t). \tag{4.43}$$

Observe that (3.10) implies that $x_j F(u(\sigma)) \in L^2(\boldsymbol{R}^4)$ for any $\sigma \in \boldsymbol{R}$. Hence all of the terms on the right-hand sides of (4.41)–(4.43) are in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ for any $\sigma \in \boldsymbol{R}$. It follows from Lemma 3.6, (4.22)–(4.25) that

$$\frac{\sin \omega(t-\sigma)}{\omega}[x_j F(u(\sigma))],\ \sigma \frac{\sin \omega(t-\sigma)}{\omega} F(u(\sigma)) \to 0 \quad \text{in } C(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \text{ as } \sigma \to -\infty.$$

Hence, it holds that

$$L_j u_\sigma^-(t) \to L_j u(t) - I_{-\infty}[L_j F(u)](t),$$

$$\Omega_{jk} u_\sigma^-(t) \to \Omega_{jk} u(t) - I_{-\infty}[\Omega_{jk} F(u)](t),$$

$$L_0 u_\sigma^-(t) \to L_0 u(t) - 2I_{-\infty}[F(u)](t) - I_{-\infty}[L_0 F(u)](t)$$

in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ as $\sigma \to -\infty$. Here we have set

$$I_{-\infty}[h](t) := \int_{-\infty}^t (\omega^{-1} \sin \omega(t-\tau)) h(\tau) d\tau.$$

On the other hand, since $u_\sigma^- \to u^-$ in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$ as $\sigma \to -\infty$, it follows immediately that $L_j u_\sigma^- \to L_j u^-$, $\Omega_{jk} u_\sigma^- \to \Omega_{jk} u^-$, $L_0 u_\sigma^- \to L_0 u^-$ in $\mathscr{D}'(\boldsymbol{R}^{1+4})$ as $\sigma \to -\infty$. Thus we conclude that the equalities

$$L_j u^-(t) = L_j u(t) - I_{-\infty}[L_j F(u)](t) \quad (j = 1, \cdots, 4), \tag{4.44}$$

$$\Omega_{jk} u^-(t) = \Omega_{jk} u(t) - I_{-\infty}[\Omega_{jk} F(u)](t) \quad (1 \le j < k \le 4), \tag{4.45}$$

$$L_0 u^-(t) = L_0 u(t) - 2I_{-\infty}[F(u)](t) - I_{-\infty}[L_0 F(u)](t) \tag{4.46}$$

hold in $C(\boldsymbol{R}; L^2(\boldsymbol{R}^4))$. Thus, we find that (4.38) is true for $|\alpha| = 1$. To check (4.38) for $|\alpha| = 2$ we proceed as follows. Put for any $\sigma \in \boldsymbol{R}$,

$$(L_j u^-)_\sigma(t) := L_j u(t) - I_\sigma[L_j F(u)](t),$$

$$(\Omega_{jk} u^-)_\sigma(t) := \Omega_{jk} u(t) - I_\sigma[\Omega_{jk} F(u)](t),$$

$$(L_0 u^-)_\sigma(t) := L_0 u(t) - 2I_\sigma[F(u)](t) - I_\sigma[L_0 F(u)](t).$$

Then, observing $x_j \Gamma^\alpha F(u(\sigma)) \in L^2(\boldsymbol{R}^4)$ ($|\alpha| = 1$) by (3.10), we may prove by repeating quite the same argument as above that (4.38) certainly holds for $|\alpha| = 2$.

We turn to the proof of (4.39). The proof of (4.39) is quite the same as that of (4.38) except that it is necessary to show

$$\|\langle \cdot \rangle \Gamma^\alpha F(u(\sigma, \cdot))\|_2 \to 0 \quad \text{as } \sigma \to -\infty$$

for $|\alpha| \le 1$. This is easily derived from (3.6) and (3.10). In fact,

$$\begin{aligned}
&\sum_{|\alpha| \le 1} \|\langle \cdot \rangle \Gamma^\alpha F(u(\sigma, \cdot))\|_2 \\
&\quad \le \sum_{|\alpha| \le 1} \|\langle \cdot \rangle^{1/4(\rho-1)} u(\sigma, \cdot)\|_{4(\rho-1)}^{\rho-1} \|\langle \cdot \rangle^{3/4} \Gamma^\alpha u(\sigma, \cdot)\|_4 \\
&\quad \le C\Big(\|\langle \cdot \rangle^{1/4(\rho-1)} u(\sigma, \cdot)\|_{4(\rho-1),\chi}^{\rho-1} + \|\langle \cdot \rangle^{1/4(\rho-1)} u(\sigma, \cdot)\|_{4(\rho-1),\phi}^{\rho-1}\Big) \|u(\sigma, \cdot)\|_{\Gamma,2,2} \\
&\quad \le C(1 + |\sigma|)^{-3(\rho-1)/2+1} \|u(\sigma, \cdot)\|_{\Gamma,2,2}^{\rho}.
\end{aligned}$$

We omit the details of the proof of (4.39). Q.E.D.

REMARK 4.2. The crucial step to prove this proposition was deriving

$$\frac{\sin \omega(t - \sigma)}{\omega}[x_j F(u(\sigma))] \to 0 \quad \text{in } C(\boldsymbol{R}; L^2(\boldsymbol{R}^4)) \quad \text{as } \sigma \to -\infty.$$

Note that it no longer follows from the combination of the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality $\|v\|_4 \le C\|\nabla v\|_2$. If $\rho > 13/6$, it would follow immediately from the combination of the Hardy-Littlewood-Sobolev inequality and (3.6), (3.10). These facts imply what an important role (3.18) plays.

Now we are ready to show that u^- defined in (4.37) have the properties (2.25)–(2.28). It follows immediately from (4.9)–(4.10) and Proposition 4.7 that (2.27)–(2.28) are true. Since $\|u\|_Y \le C_3 \Lambda$, we also find from (4.9)–(4.10) and Proposition 4.7 that

$$\|u^-\|_Y \le C\Lambda + C\Lambda^\rho \le C\Lambda. \tag{4.47}$$

Since $\Gamma^\alpha u^-$ and $\partial_k \Gamma^\alpha u^-$ are continuous with values in $L^2(\boldsymbol{R}^4)$ for $|\alpha| \le 2$ and $k = 0, \cdots, 4$, both $(\Gamma^\alpha u^-)(0)$ and $(\partial_k \Gamma^\alpha u^-)(0)$ make sense in $L^2(\boldsymbol{R}^4)$. Then, applying (3.7) to u^-, we find from (4.47) that u^- has the properties (2.25)–(2.26). In quite the same argument it can be checked that u^+ also satisfies (2.25)–(2.28). Thus we have completed the proof of (*II*) of Theorem 2.

To complete the proof of Theorem 2 there remains to check (*III*). We need the following proposition part of which has been essentially proved in Zhou [**21**]. But harder calculations and more delicate estimates are necessary to complete the proof.

PROPOSITION 4.8. *Let ν be any positive number with $2 < \nu \le 4$. Let $u^{(j)}$, $(f^{(j)}, g^{(j)})$, $(f^{(j)\pm}, g^{(j)\pm})$ $(j = 1, 2)$ be functions described in (III) of Theorem 2. Set $u^{(j)\pm}(t) := (\cos\omega t) f^{(j)\pm} + (\omega^{-1}\sin\omega t) g^{(j)\pm}$. Then they satisfy the following estimates.*

$$\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \le C\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_{\Sigma} + C\int_I (1+|\tau|)^{-1-3(\rho-2)/\nu} \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2} d\tau, \tag{4.48}$$

$$\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2} \|D\Gamma^{\alpha}\{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}\|_2 \tag{4.49}$$
$$\le \begin{cases} C\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_{\Sigma} + C\|f^{(1)} - f^{(2)}\|_{H^3}^{\rho-2} \\ \quad + C\left[\sup_{t\in\mathbf{R}} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{\rho-2} \\ \quad + C\int_I (1+|\tau|)^{-1-3(\rho-2)/\nu}\Big[\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2} \\ \quad + \sum_{|\alpha|=2} \|D\Gamma^{\alpha}\{u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\}\|_2\Big] d\tau \quad \text{if } 2 < \rho < 3, \\ C\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_{\Sigma} \\ \quad + C\int_I (1+|\tau|)^{-1-3(\rho-2)/\nu}\Big[\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2} \\ \quad + \sum_{|\alpha|=2} \|D\Gamma^{\alpha}\{u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\}\|_2\Big] d\tau \quad \text{if } \rho \ge 3 \end{cases}$$

for all $t \in \mathbf{R}$, where $I = [0, t]$ if $t \ge 0$ or $[t, 0]$ if $t < 0$. Moreover

$$\|u^{(1)\pm} - u^{(2)\pm}\|_Y \le C\left[\sup_{t\in\mathbf{R}} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{\rho-2} + C\|u^{(1)} - u^{(2)}\|_Y. \tag{4.50}$$

All the constants C appearing (4.48)–(4.50) *depend only on ε, λ, ν and ρ.*

PROOF. To begin with, we must show that the differential equation (1.1) is equivalent to the integral equation (4.1). Set $u_0^{(j)}(t) = (\cos\omega t) f^{(j)} + (\omega^{-1}\sin\omega t) g^{(j)}$. Then $u^{(j)}$ satisfies

$$u^{(j)}(t) = u_0^{(j)}(t) + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} F(u^{(j)}(\tau)) d\tau \tag{4.51}$$

as an equality at least in $\bigcap_{k=0}^{3} C^k(\boldsymbol{R}; H^{3-k}(\boldsymbol{R}^4))$. This can be easily verified. In fact, put

$$\tilde{u}^{(j)}(t) = u_0^{(j)}(t) + \int_0^t \frac{\sin \omega(t-\tau)}{\omega} F(u^{(j)}(\tau)) d\tau$$

for $u^{(j)}$. Since $\square(u^{(j)} - \tilde{u}^{(j)}) = 0$ and $(u^{(j)}(0), \partial_t u^{(j)}(0)) = (\tilde{u}^{(j)}(0), \partial_t \tilde{u}^{(j)}(0))$, it holds by the usual energy equality that $\|u^{(j)}(t,\cdot) - \tilde{u}^{(j)}(t,\cdot)\|_e = 0$ for all $t \in \boldsymbol{R}$. Thus, for every $t \in \boldsymbol{R}$, $u^{(j)}(t) - \tilde{u}^{(j)}(t)$ is independent of a.e. $x \in \boldsymbol{R}^4$. Since $L^2(\boldsymbol{R}^4)$ has no nontrivial constant function, we find that $u^{(j)}(t) = \tilde{u}^{(j)}(t)$ in $L^2(\boldsymbol{R}^4)$ for all $t \in \boldsymbol{R}$. Hence, (4.51) certainly holds.

It is sufficient to prove (4.48)–(4.49) for $t > 0$ because in the other case they can be shown in quite the same way. Noting (3.28)–(3.30) and applying (3.17)–(3.18) to (4.51), we have

$$\begin{aligned} (4.52) \qquad & \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \\ & \le \|u_0^{(1)}(t,\cdot) - u_0^{(2)}(t,\cdot)\|_{\Gamma,1,2} \\ & \quad + C\|F(f^{(1)}) - F(f^{(2)})\|_{4/3} + C\|F(f^{(1)}) - F(f^{(2)})\|_{1,2} \\ & \quad + C\int_0^t \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,1,4/3,\chi} d\tau \\ & \quad + C\int_0^t (1+\tau)^{-1} \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,1,1,2,\phi} d\tau \end{aligned}$$

for all $t > 0$. It follows from (3.6), (4.16) that

$$\begin{aligned} (4.53) \quad & \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,1,4/3,\chi} \\ & \le C(\|u^{(1)}(\tau,\cdot)\|_{4(\rho-1),\chi} + \|u^{(2)}(\tau,\cdot)\|_{4(\rho-1),\chi})^{\rho-1} \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2} \\ & \quad + C(\|u^{(1)}(\tau,\cdot)\|_{\infty,\chi} + \|u^{(2)}(\tau,\cdot)\|_{\infty,\chi})^{\rho-2} \\ & \quad \times \|u^{(2)}(\tau,\cdot)\|_{\Gamma,1,4,\chi} \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_2 \\ & \le C(1+\tau)^{-1-4(\rho-2)/\nu} (\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1} \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}. \end{aligned}$$

Note that we have the estimate

$$(4.54) \qquad \|u^{(j)}(t,\cdot)\|_\infty \le C(1+|t|)^{-(n-1)/\nu} \sum_{|\alpha| \le 2} \|\Gamma^\alpha u^{(j)}(t,\cdot)\|_{H^1} \quad \text{with } 2 < \nu \le 4$$

for the same reason as (4.16) holds. Then we obtain by employing (3.14), (3.15) and (4.54)

$$(4.55)\qquad \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,1,1,2,\phi}$$
$$\leq C(\|u^{(1)}(\tau,\cdot)\|_{\infty} + \|u^{(2)}(\tau,\cdot)\|_{\infty})^{\rho-2}$$
$$\times (\|u^{(1)}(\tau,\cdot)\|_{2,\infty} + \|u^{(2)}(\tau,\cdot)\|_{2,\infty})\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}$$
$$+ C(\|u^{(1)}(\tau,\cdot)\|_{\infty} + \|u^{(2)}(\tau,\cdot)\|_{\infty})^{\rho-2}\|u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}$$
$$\times \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}$$
$$\leq C(1+\tau)^{-3(\rho-2)/\nu}(\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}.$$

It is easy to show

$$(4.56)\qquad \|F(f^{(1)}) - F(f^{(2)})\|_{4/3}, \|F(f^{(1)}) - F(f^{(2)})\|_{1,2}$$
$$\leq C(\|(f^{(1)}, g^{(1)})\|_{\Sigma} + \|(f^{(2)}, g^{(2)})\|_{\Sigma})^{\rho-1}\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_{\Sigma}.$$

Combining (4.52)–(4.56) yields (4.48).

Next we show (4.49). It is enough to prove (4.49) in the case of $2 < \rho < 3$ because in the other case it is proved with much ease. Note that

$$|F''(u^{(1)}) - F''(u^{(2)})| \leq C|u^{(1)} - u^{(2)}|^{\rho-2}$$

because $|u^{(j)}|$ is small. Taking account of Proposition 3.1 and employing (3.17)–(3.18), we have

$$(4.57)\qquad \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2}$$
$$\leq \|u_0^{(1)}(t,\cdot) - u_0^{(2)}(t,\cdot)\|_{\Gamma,2,2} + C\|\langle\cdot\rangle^2[F(f^{(1)}) - F(f^{(2)})]\|_2$$
$$+ C\|F(f^{(1)}) - F(f^{(2)})\|_{4/3} + C\|F(f^{(1)}) - F(f^{(2)})\|_{1,2}$$
$$+ C\|F'(f^{(1)})\nabla f^{(1)} - F'(f^{(2)})\nabla f^{(2)}\|_{4/3}$$
$$+ C\|\langle\cdot\rangle[F'(f^{(1)})\nabla f^{(1)} - F'(f^{(2)})\nabla f^{(2)}]\|_{1,2}$$
$$+ C\|F'(f^{(1)})g^{(1)} - F'(f^{(2)})g^{(2)}\|_{4/3}$$
$$+ C\|\langle\cdot\rangle[F'(f^{(1)})g^{(1)} - F'(f^{(2)})g^{(2)}]\|_{1,2}$$
$$+ C\int_0^t \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,4/3,\chi}\,d\tau$$
$$+ C\int_0^t (1+\tau)^{-1}\|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,1,2,\phi}\,d\tau.$$

It follows that

$$(4.58)\qquad \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,4/3,\chi}$$
$$\leq C\|u^{(1)}(\tau,\cdot)\|^2_{\Gamma,1,8/(5-\rho),\chi}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|^{\rho-2}_{4,\chi}$$
$$+ C\|u^{(2)}(\tau,\cdot)\|^{\rho-2}_{\infty,\chi}(\|u^{(1)}(\tau,\cdot)\|_{\Gamma,1,4,\chi} + \|u^{(2)}(\tau,\cdot)\|_{\Gamma,1,4,\chi})$$

$$\times \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}$$

$$+ C(\|u^{(1)}(\tau,\cdot)\|_{\infty,\chi} + \|u^{(2)}(\tau,\cdot)\|_{\infty,\chi})^{\rho-2}\|u^{(1)}(\tau,\cdot)\|_{\Gamma,2,2}$$

$$\times \|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{4,\chi}.$$

Since $2 < \rho < 3$, we have by (3.6)

$$\|u^{(1)}(\tau,\cdot)\|_{\Gamma,1,8/(5-\rho),\chi} \leq C(1+\tau)^{-(\rho-1)/2}\|u^{(1)}(\tau,\cdot)\|_{\Gamma,2,2}. \tag{4.59}$$

Thus, employing (3.6) and (4.16), we find that the right-hand side of (4.58) can be estimated from above by

$$C(1+\tau)^{-2\rho+3}\|u^{(1)}\|_Y^2\left[\sup_{t\in \boldsymbol{R}}\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{\rho-2} \tag{4.60}$$

$$+ C(1+\tau)^{-1-4(\rho-2)/\nu}(\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}.$$

Moreover, making use of the Sobolev inequality on the sphere S^3 and (3.13), we have

$$\|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,1,2,\phi} \tag{4.61}$$

$$\leq C\|u^{(1)}(\tau,\cdot)\|^2_{\Gamma,1,4/(4-\rho),12/(5-\rho),\phi}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|^{\rho-2}_{\Gamma,1,2}$$

$$+ C(\|u^{(1)}(\tau,\cdot)\|_\infty + \|u^{(2)}(\tau,\cdot)\|_\infty)^{\rho-2}$$

$$\times (\|u^{(1)}(\tau,\cdot)\|_{\Gamma,2,2} + \|u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2})\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}$$

$$\leq C(1+\tau)^{-3(\rho-2)/2}\|u^{(1)}\|_Y^2\left[\sup_{t\in \boldsymbol{R}}\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{\rho-2}$$

$$+ C(1+\tau)^{-3(\rho-2)/\nu}(\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}.$$

Repeating essentially the same argument as in (3.38), (4.53)–(4.56), we easily find that all of the terms except for the second and the last two on the right-hand side of (4.57) are estimated from above by $C\|(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})\|_\Sigma$. Since $|F(u) - F(v)| \leq C(|u| + |v|)^{\rho-1}|u - v| \leq C(|u| + |v|)^2|u - v|^{\rho-2}$ for any $2 < \rho < 3$, the second term on the right-hand side of (4.57) are estimated as

$$\|\langle\cdot\rangle^2[F(f^{(1)}) - F(f^{(2)})]\|_2 \leq C\|f^{(1)} - f^{(2)}\|_\infty^{\rho-2}(\|\langle\cdot\rangle f^{(1)}\|_4 + \|\langle\cdot\rangle f^{(2)}\|_4)^2$$

$$\leq C\|f^{(1)} - f^{(2)}\|_{H^3}^{\rho-2}\left(\sum_{j=1}^{2}(\|f^{(j)}\|_2 + \|\langle\cdot\rangle\nabla f^{(j)}\|_2)\right)^2.$$

Finally, we must carry out the following estimate to complete the proof of (4.49). In view of Proposition 3.1 we have

$$(4.62)\qquad \sum_{|\alpha|=2} \|D\Gamma^\alpha[u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)]\|_2$$
$$\le \sum_{|\alpha|=2} \|D\Gamma^\alpha[u_0^{(1)}(t,\cdot)-u_0^{(2)}(t,\cdot)]\|_2 + C\|\langle\cdot\rangle[F(f^{(1)})-F(f^{(2)})]\|_2$$
$$+ C\|\langle\cdot\rangle^2[F'(f^{(1)})\nabla f^{(1)} - F'(f^{(2)})\nabla f^{(2)}]\|_2$$
$$+ C\|\langle\cdot\rangle^2[F'(f^{(1)})g^{(1)} - F'(f^{(2)})g^{(2)}]\|_2$$
$$+ C\int_0^t \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,2}d\tau.$$

It is not difficult to estimate all of the terms except for the last one on the right-hand side of (4.62) from above by $C\|(f^{(1)}-f^{(2)},g^{(1)}-g^{(2)})\|_\Sigma$. Thus we have only to estimate the last term. Note that the combination of (3.12) and (3.5) produces the estimate

$$(4.63)\qquad \|u^{(1)}(\tau,\cdot)\|_{8/(4-\rho)} \le C(1+\tau)^{-3/4}\sum_{|\alpha|\le 1}\|\Gamma^\alpha u^{(1)}(\tau,\cdot)\|_{H^1}.$$

Then, we get by (3.12), (4.54) and (4.63)

$$(4.64)\quad \|F(u^{(1)}(\tau,\cdot)) - F(u^{(2)}(\tau,\cdot))\|_{\Gamma,2,2}$$
$$\le C\|u^{(1)}(\tau,\cdot)\|^2_{\Gamma,1,8/(4-\rho)}\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_4^{\rho-2}$$
$$+ C(\|u^{(1)}(\tau,\cdot)\|_\infty + \|u^{(2)}(\tau,\cdot)\|_\infty)^{\rho-2}$$
$$\times(\|u^{(1)}(\tau,\cdot)\|_{\Gamma,1,4} + \|u^{(2)}(\tau,\cdot)\|_{\Gamma,1,4})\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_{\Gamma,1,4}$$
$$+ C(\|u^{(1)}(\tau,\cdot)\|_\infty + \|u^{(2)}(\tau,\cdot)\|_\infty)^{\rho-2}\|u^{(1)}(\tau,\cdot)\|_{\Gamma,2,2}\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_\infty$$
$$+ C\|u^{(2)}(\tau,\cdot)\|_\infty^{\rho-1}\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}$$
$$\le C(1+\tau)^{-3\rho/4}\|u^{(1)}\|_Y^2\left[\sup_{t\in\boldsymbol{R}}\|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{\rho-2}$$
$$+ C(1+\tau)^{-3(\rho-2)/\nu-3/2}(\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1}\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}$$
$$+ C(1+\tau)^{-3(\rho-1)/\nu}(\|u^{(1)}\|_Y + \|u^{(2)}\|_Y)^{\rho-1}$$
$$\times\left[\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)]\|_2\right].$$

Combining (4.57), (4.60)–(4.62) and (4.64) leads us to (4.49) for $2<\rho<3$.

It remains to show (4.50). We have only to prove (4.50) for $u^{(1)-}-u^{(2)-}$ because it can be proved similarly for $u^{(1)+}-u^{(2)+}$. Recall that $u^{(j)-}$ has been determined by

$$(4.65)\qquad u^{(j)-}(t) = u^{(j)}(t) - I_{-\infty}[F(u^{(j)})](t)\quad (j=1,2).$$

Then, employing Proposition 4.7 and repeating the same calculations as we have done to obtain (4.48)–(4.49), we can get (4.50) without any difficulty. We omit the details. Thus, the proof of Proposition 4.8 has been completed. Q.E.D.

Now we are ready to show (*III*) of Theorem 2. Applying Gronwall's inequality to (4.48), we have

$$\sup_{t\in\boldsymbol{R}}\|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\le C\|(f^{(1)}-f^{(2)},g^{(1)}-g^{(2)})\|_{\Sigma}. \tag{4.66}$$

We apply Gronwall's inequality again together with (4.66) to (4.49) and obtain

$$\begin{aligned}\sup_{t\in\boldsymbol{R}}\|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,2,2}&+\sum_{|\alpha|=2}\sup_{t\in\boldsymbol{R}}\|D\Gamma^{\alpha}[u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)]\|_2\\ &\le C\|(f^{(1)}-f^{(2)},g^{(1)}-g^{(2)})\|_{\Sigma}+C\|f^{(1)}-f^{(2)}\|_{H^3}^{\rho-2}\\ &\quad+C\|(f^{(1)}-f^{(2)},g^{(1)}-g^{(2)})\|_{\Sigma}^{\rho-2}.\end{aligned} \tag{4.67}$$

This implies (2.29). The remaining part (2.30) follows from (3.7), (4.50) and (4.67). Thus, we have completed the proof of Theorem 2.

5. Proof of Theorem 1.

Let us introduce Banach spaces X, Ξ as follows.

$$\begin{aligned}X=\Big\{u=u(t,x)\,|\,&\Gamma^{\alpha}u\in C((-\infty,0];L^2(\boldsymbol{R}^4))\text{ for any }\alpha\text{ with }|\alpha|\le 2,\\ &\partial_k\Gamma^{\alpha}u\in L^{\infty}((-\infty,0);L^2(\boldsymbol{R}^4))\text{ for any }\alpha\text{ with }|\alpha|=2\text{ and }k=0,\cdots,4,\\ &\|u\|_X\equiv\sup_{t<0}\|u(t,\cdot)\|_{\Gamma,2,2}+\sum_{|\alpha|=2}\operatorname{ess\cdot sup}_{t<0}\|D\Gamma^{\alpha}u(t,\cdot)\|_2<\infty\Big\},\end{aligned}$$

$$\Xi=\Big\{u=u(t,x)\,|\,u\in C((-\infty,0];L^2(\boldsymbol{R}^4)),\ \|u\|_{\Xi}\equiv\sup_{t<0}\|u(t,\cdot)\|_2<\infty\Big\}.$$

We put $X_{\delta}:=\{u=u(t,x)\,|\,u\in X,\|u\|_X\le\delta\}$ for any positive δ. As in the proof of Lemma 4.1, we can easily show the following lemma.

LEMMA 5.1. *X_{δ} is a closed subset of Ξ for any positive δ.*

PROPOSITION 5.1. *In X the equation* (1.1) *has at most one solution satisfying* (2.8).

PROOF. Let u, v (u, $v\in X$) be solutions to (1.1) satisfying (2.8). Then u satisfies

$$u(t)=u_-(t)+\int_{-\infty}^{t}\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))d\tau \tag{5.1}$$

as an equality at least in $\bigcap_{j=0}^{3}C^j((-\infty,0];H^{3-j}(\boldsymbol{R}^4))$. So does v. This is easily checked. In fact, put

$$\tilde{u}(t) := u_-(t) + \int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega} F(u(\tau))d\tau$$

for u. Note that $\tilde{u}$ satisfies $\|\tilde{u}(t,\cdot) - u_-(t,\cdot)\|_e \to 0$ $(t \to -\infty)$. Since $\Box(u - \tilde{u}) = 0$, there holds $\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_e = \|u(\sigma,\cdot) - \tilde{u}(\sigma,\cdot)\|_e$ for all $t, \sigma \leq 0$. Because $\|u(\sigma,\cdot) - \tilde{u}(\sigma,\cdot)\|_e \leq \|u(\sigma,\cdot) - u_-(\sigma,\cdot)\|_e + \|u_-(\sigma,\cdot) - \tilde{u}(\sigma,\cdot)\|_e \to 0$ as $\sigma \to -\infty$, $\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_e = 0$ for all $t \leq 0$. Hence, for every $t \leq 0$, $u(t) - \tilde{u}(t)$ is independent of a.e. $x \in \boldsymbol{R}^4$. Since $L^2(\boldsymbol{R}^4)$ has no nontrivial constant function, $u(t) = \tilde{u}(t)$ in $L^2(\boldsymbol{R}^4)$ for every $t \leq 0$. Thus, (5.1) certainly holds.

Repeating essentially the same argument as in (4.32)–(4.33), we get

$$\|u(t,\cdot) - v(t,\cdot)\|_2 \leq C_9(1+|t|)^{-3(\rho-2)/2}(\|u\|_X + \|v\|_X)^{\rho-1} \sup_{\tau<t} \|u(\tau,\cdot) - v(\tau,\cdot)\|_2.$$

Put $T_0 := \sup\{t \,|\, t < 0, C_9(1+|t|)^{-3(\rho-2)/2}(\|u\|_X + \|v\|_X)^{\rho-1} \leq 1/2\}$. Then we see that $u = v$ on $(-\infty, T_0]$. To show $u = v$ on the whole half line $(-\infty, 0]$ we proceed as follows. Set $T_1 = 2T_0$. Note that $u = v$, $\partial_t u = \partial_t v$ at $t = T_1$. The standard energy inequality gives us

$$\begin{aligned} \|u(t,\cdot) - v(t,\cdot)\|_e &\leq \int_{T_1}^{t} \|F(u(\tau,\cdot)) - F(v(\tau,\cdot))\|_2 d\tau \\ &\leq C(\|u\|_X + \|v\|_X)^{\rho-1} \int_{T_1}^{t} \|u(\tau,\cdot) - v(\tau,\cdot)\|_2 d\tau \end{aligned} \tag{5.2}$$

for all t with $T_1 \leq t \leq 0$. Moreover, an elementary identity $u(t) = u(T_1) + \int_{T_1}^{t} \partial_\tau u(\tau) d\tau$ yields

$$\|u(t,\cdot) - v(t,\cdot)\|_2 \leq \int_{T_1}^{t} \|\partial_\tau u(\tau,\cdot) - \partial_\tau v(\tau,\cdot)\|_2 d\tau \quad \text{for } T_1 \leq t \leq 0. \tag{5.3}$$

Combining (5.2) with (5.3), we conclude by Gronwall's inequality that $u(t) = v(t)$ in $L^2(\boldsymbol{R}^4)$ for $T_1 \leq t \leq 0$. Thus we have got the desirable uniqueness result. Q.E.D.

Set $\Lambda = \|(f_-, g_-)\|_\Sigma$. Since u_- has the properties (3.37)–(3.38), u_- belongs to $X_{C_5\Lambda}$. We shall show through the following proposition that a sequence $\{u_m\}$ can be defined in $X_{2C_5\Lambda}$ for small Λ by

$$u_{m+1}(t) = u_-(t) + I_{-\infty}[F(u_m)](t). \tag{5.4}$$

In what follows we simply denote $I_{-\infty}[h](t)$ by $I[h](t)$.

PROPOSITION 5.2. *Let u be any function in $X_{2C_5\Lambda}$, where C_5 is the same constant as in* (3.38). *Then it holds that*

$$\|F(u(\tau,\cdot))\|_{\Gamma,2,2} \leq C(2C_5\Lambda)^\rho (1+|\tau|)^{-3(\rho-1)/2}, \tag{5.5}$$

$$\sup_{t<0} \left\| \frac{\sin \omega(t-\tau)}{\omega} \Gamma^\alpha F(u(\tau,\cdot)) \right\|_2 \leq C(2C_5\Lambda)^\rho (1+|\tau|)^{-1-3(\rho-2)/2} \quad (|\alpha| \leq 2) \tag{5.6}$$

for a.e. $\tau < 0$, *and*

(5.7) $\Gamma^{\alpha} I[F(u)] \in C((-\infty, 0]; L^2(\boldsymbol{R}^4))$ *for any* α *with* $|\alpha| \leq 2$,

(5.8) $\partial_k \Gamma^{\alpha} I[F(u)] \in C((-\infty, 0]; L^2(\boldsymbol{R}^4))$ *for any* α *with* $|\alpha| = 2$ *and* $k = 0, \cdots, 4$,

(5.9) $\|I[F(u)]\|_Y \leq C_{10}(2C_5 \Lambda)^p$ *for some positive constant* C_{10}.

PROOF. Repeating the same calculations as in the proof of Proposition 4.4 and employing the limiting procedure in the proof of Proposition 4.7, we can prove this proposition without any difficulty. Thus we omit the details. Q.E.D.

Let Λ be small so that $C_{10}(2C_5 \Lambda)^p \leq C_5 \Lambda$ may hold. Then we see that a sequence $\{u_m\}$ can be defined in $X_{2C_5\Lambda}$ inductively by (5.4).

PROPOSITION 5.3. *The sequence* $\{u_m\} \subset X_{2C_5\Lambda}$ *defined just above satisfies*

$$\|u_{m+1} - u_m\|_{\Xi} \leq C_{11}(\|u_m\|_X + \|u_{m-1}\|_X)^{p-1} \|u_m - u_{m-1}\|_{\Xi} \tag{5.10}$$

for all $m = 1, 2, \cdots$. *Here* C_{11} *is a positive constant independent of* m.

PROOF. We have only to repeat essentially the same argument as in (4.32)–(4.33). Thus we omit the details. Q.E.D.

Let us take Λ still smaller so that

$$C_{10}(2C_5 \Lambda)^p \leq C_5 \Lambda \quad \text{and} \quad C_{11}(4C_5 \Lambda)^{p-1} \leq \frac{1}{2} \tag{5.11}$$

may hold. Then it follows from Lemma 5.1 and (5.10) that $\{u_m\} \subset X_{2C_5\Lambda}$ is a Cauchy sequence in Ξ and there exists a unique function $u \in X_{2C_5\Lambda}$ such that $u_m \to u$ in Ξ. For this u the integral $\int_{-\infty}^{t} (\omega^{-1} \sin \omega(t - \tau)) F(u(\tau)) d\tau$ is in $L^2(\boldsymbol{R}^4)$ for any $t \leq 0$ because of (5.6). As is easily seen, $I[F(u_m)] \to I[F(u)]$ in $C((-\infty, 0]; L^2(\boldsymbol{R}^4))$ as $m \to \infty$. Hence u satisfies the integral equation

$$u(t) = u_-(t) + \int_{-\infty}^{t} \frac{\sin \omega(t - \tau)}{\omega} F(u(\tau)) d\tau \tag{5.12}$$

as an equality at least in $C((-\infty, 0]; L^2(\boldsymbol{R}^4))$.

Next let us investigate the strong continuity with respect to t for the derivatives of u. Since $u \in X$ and u satisfies (5.12), we find from (5.7)–(5.9) that $\Gamma^{\alpha} u$, $\partial_k \Gamma^{\alpha} u$ are bounded and continuous on $(-\infty, 0]$ with values in $L^2(\boldsymbol{R}^4)$ for $|\alpha| \leq 2$ and $k = 0, \cdots, 4$. (3.7), (2.6), (2.9) and (2.11) imply that $(u(0), \partial_t u(0)) \in \Sigma$ and $\|(u(0), \partial_t u(0))\|_{\Sigma} \leq C_2 \|(f_-, g_-)\|_{\Sigma}$. Since u satisfies

$$u(t) = u_0(t) + \int_0^t \frac{\sin \omega(t - \tau)}{\omega} F(u(\tau)) d\tau, \quad t \leq 0, x \in \boldsymbol{R}^4$$

for $u_0(t) \equiv (\cos\omega t)u(0) + (\omega^{-1}\sin\omega t)\partial_t u(0)$ (see (4.51) and the subsequent discussion there), the continuity of $(u, \partial_t u)$ with respect to t in Σ follows (see the end of the proof of (I) of Theorem 2). Therefore, u satisfies the differential equation (1.1) as an equality in $\bigcap_{j=0}^{1} C^j((-\infty,0]; H^{1-j}(\boldsymbol{R}^4))$.

It remains to show (2.13)–(2.14). Since u satisfies (5.12), we can prove (2.13)–(2.14) without any difficulty by employing the limiting argument as in the proof of Proposition 4.7 and using the estimates (5.5)–(5.6). Thus, letting δ be the supremum of Λ satisfying (5.11) and putting $C_1 := 2C_5$, we have completed the proof of (I) of Theorem 1.

Next we prove (II). Let $(f_-^{(j)}, g_-^{(j)}) \in \Sigma_\delta$ $(j = 1,2)$ and let $u^{(j)}$ be the two corresponding solutions to (1.1) satisfying (2.6)–(2.8). Set $u_-^{(j)}(t) = (\cos\omega t)f_-^{(j)} + (\omega^{-1}\sin\omega t)g_-^{(j)}$. Then, as we have shown in the proof of Proposition 5.1, $u^{(j)}$ satisfies

$$u^{(j)}(t) = u_-^{(j)}(t) + \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u^{(j)}(\tau))d\tau \tag{5.13}$$

as an equality at least in $\bigcap_{j=0}^{3} C^j((-\infty,0]; H^{3-j}(\boldsymbol{R}^4))$. Employing the limiting procedure in the proof of Proposition 4.7 and repeating the same calculations as in the proof of Proposition 4.8, we obtain

$$\begin{aligned} &\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \\ &\quad \le C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_\Sigma \\ &\qquad + C_{12}\delta^{p-1}\int_{-\infty}^{t}(1+|\tau|)^{-1-3(p-2)/\nu}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}d\tau, \end{aligned} \tag{5.14}$$

$$\begin{aligned} &\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)]\|_2 \\ &\le \begin{cases} C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_\Sigma + C\delta^2\left[\sup_{t<0}\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{p-2} \\ \quad +C_{13}\delta^{p-1}\int_{-\infty}^{t}(1+|\tau|)^{-1-3(p-2)/\nu}\Big[\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2} \\ \quad + \sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)]\|_2\Big]d\tau \quad \text{if } 2 < p < 3, \\ C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_\Sigma \\ \quad + C\delta^{p-1}\int_{-\infty}^{t}(1+|\tau|)^{-1-3(p-2)/\nu}\Big[\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2} \\ \quad + \sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)]\|_2\Big]d\tau \quad \text{if } p \ge 3 \end{cases} \end{aligned} \tag{5.15}$$

for all $t \le 0$. Set $T_1 := \sup\{t < 0 | C_{12}\delta^{p-1} \int_{-\infty}^{t} (1+|\tau|)^{-1-3(p-2)/\nu} \le 1/2\}$. Then we have from (5.14)

$$\sup_{t \le T_1} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \le C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_{\Sigma}. \tag{5.16}$$

Note that there holds

$$\begin{aligned} u^{(j)}(t) &= (\cos\omega(t-T_1))u^{(j)}(T_1) + (\omega^{-1}\sin\omega(t-T_1))\partial_t u^{(j)}(T_1) \\ &\quad + \int_{T_1}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u^{(j)}(\tau))d\tau. \end{aligned} \tag{5.17}$$

Repeating the same calculations as we have done in obtaining (4.48), we get

$$\begin{aligned} &\|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \\ &\quad \le C(|T_1| + \delta^{p-1})(\|u^{(1)}(T_1,\cdot) - u^{(2)}(T_1,\cdot)\|_2 \\ &\qquad + \|\langle\,\cdot\,\rangle[\nabla u^{(1)}(T_1,\cdot) - \nabla u^{(2)}(T_1,\cdot)]\|_2 \\ &\qquad + \|\langle\,\cdot\,\rangle[\partial_t u^{(1)}(T_1,\cdot) - \partial_t u^{(2)}(T_1,\cdot)]\|_2) \\ &\qquad + C\delta^{p-1}\int_{T_1}^{t} (1+|\tau|)^{-1-3(p-2)/\nu}\|u^{(1)}(\tau,\cdot) - u^{(2)}(\tau,\cdot)\|_{\Gamma,1,2}d\tau \\ &\qquad \text{for } T_1 \le t \le 0. \end{aligned} \tag{5.18}$$

Applying Gronwall's inequality, (3.7) and (5.16), together with an elementary inequality $\langle x\rangle \le \sqrt{1+(|T_1|-|x|)^2} + \sqrt{1+T_1^2}$, to (5.18), we get

$$\sup_{T_1 < t < 0} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \le C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_{\Sigma} \tag{5.19}$$

for a constant C depending only on δ, λ, ν, p and T_1. Combining (5.16) with (5.19), we get

$$\sup_{t<0} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,1,2} \le C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_{\Sigma}. \tag{5.20}$$

Next we set $T_2 := \sup\{t < 0 | C_{13}\delta^{p-1} \int_{-\infty}^{t} (1+|\tau|)^{-1-3(p-2)/\nu} d\tau \le 1/2\}$. Then it follows from (5.15) and (5.20) that

$$\begin{aligned} &\sup_{t \le T_2} \|u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\|_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t \le T_2} \|D\Gamma^{\alpha}[u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)]\|_2 \\ &\quad \le C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_{\Sigma} + C\|(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})\|_{\Sigma}^{p-2}. \end{aligned} \tag{5.21}$$

Repeating the same calculations as we have done in obtaining (4.49), we get from (5.17) with T_1 replaced by T_2

$$(5.22)\quad \|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,2,2}+\sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)]\|_2$$
$$\le C(T_2^2+|T_2|+\delta^{p-1})$$
$$\times\|(u^{(1)}(T_2,\cdot)-u^{(2)}(T_2,\cdot),\partial_t u^{(1)}(T_2,\cdot)-\partial_t u^{(2)}(T_2,\cdot))\|_\Sigma$$
$$+C\delta^2\left[\sup_{t<0}\|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,1,2}\right]^{p-2}$$
$$+C\delta^{p-1}\int_{T_2}^t(1+|\tau|)^{-1-3(p-2)/v}$$
$$\times\left[\|u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)\|_{\Gamma,2,2}+\sum_{|\alpha|=2}\|D\Gamma^\alpha[u^{(1)}(\tau,\cdot)-u^{(2)}(\tau,\cdot)]\|_2\right]d\tau$$

for $T_2\le t\le 0$. Applying Gronwall's inequality and (5.20) together with (3.7) and (5.21) to (5.22), we obtain

$$(5.23)\quad \sup_{T_2\le t\le 0}\|u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)\|_{\Gamma,2,2}+\sum_{|\alpha|=2}\sup_{T_2\le t\le 0}\|D\Gamma^\alpha[u^{(1)}(t,\cdot)-u^{(2)}(t,\cdot)]\|_2$$
$$\le C\|(f_-^{(1)}-f_-^{(2)},g_-^{(1)}-g_-^{(2)})\|_\Sigma+C\|(f_-^{(1)}-f_-^{(2)},g_-^{(1)}-g_-^{(2)})\|_\Sigma^{p-2}.$$

Combining (5.21) with (5.23), we have shown (2.15) of Theorem 1 in the case of $2<p<3$. In the case of $p\ge 3$ it can be proved more easily. (2.16) follows immediately from (2.15) and (3.7). Therefore the proof of (*II*) of Theorem 1 has been completed.

6. Proof of Theorems 3 and 4.

We have only to prove that S is one-one. Let $(f_-^{(j)},g_-^{(j)})\in\Sigma_\delta$ $(j=1,2)$. Set $u_-^{(j)}(t):=(\cos\omega t)f_-^{(j)}+(\omega^{-1}\sin\omega t)g_-^{(j)}$. Let $u^{(j)}$ be the solution, which is determined in (*I*) of Theorem 1, to the integral equation

$$(6.1)\quad u(t)=u_-^{(j)}(t)+\int_{-\infty}^t\frac{\sin\omega(t-\tau)}{\omega}F(u(\tau))d\tau\quad\text{in }(-\infty,0]\times\mathbf{R}^4.$$

Let $v^{(j)}$ be the solution, which is determined in Theorem 2, to the integral equation

$$(6.2)\quad v(t)=(\cos\omega t)u^{(j)}(0)+(\omega^{-1}\sin\omega t)\partial_t u^{(j)}(0)$$
$$+\int_0^t\frac{\sin\omega(t-\tau)}{\omega}F(v(\tau))d\tau\quad\text{in }\mathbf{R}\times\mathbf{R}^4.$$

Set

$$(6.3)\quad v^{(j)+}(t)=v^{(j)}(t)+\int_t^\infty\frac{\sin\omega(t-\tau)}{\omega}F(v^{(j)}(\tau))d\tau.$$

Note that $u^{(j)}(t)=v^{(j)}(t)$ for $t\le 0$ by Proposition 4.1. We assume that $(v^{(1)+}(0), \partial_t v^{(1)+}(0))=(v^{(2)+}(0), \partial_t v^{(2)+}(0))$. Repeating the same calculations as we have done in obtaining (5.10) and noting $v^{(1)+} = v^{(2)+}$, we get from (6.3)

$$(6.4) \qquad \|v^{(1)}(t,\cdot) - v^{(2)}(t,\cdot)\|_2 \le C \int_t^{\infty} (1+\tau)^{-1-3(\rho-2)/\nu} \|v^{(1)}(\tau,\cdot) - v^{(2)}(\tau,\cdot)\|_2 d\tau$$

for $t \ge 0$. Thus, we find that there exists $T_3 > 0$ such that $v^{(1)}(t) = v^{(2)}(t)$ in $H^3(\boldsymbol{R}^4)$, $\partial_t v^{(1)}(t) = \partial_t v^{(2)}(t)$ in $H^2(\boldsymbol{R}^4)$ for all $t \ge T_3$. Next we solve the equation (1.1) by giving data $(v^{(1)}(T_3), \partial_t v^{(1)}(T_3))$ at $t = T_3$. Then it follows from the uniqueness of the solutions (Proposition 4.1) that $v^{(1)}(t) = v^{(2)}(t)$ in $H^3(\boldsymbol{R}^4)$ for any $t \in \boldsymbol{R}$. Recalling that $u^{(j)}(t) = v^{(j)}(t)$ for $t \le 0$, we see that $u^{(1)}(t) = u^{(2)}(t)$ for $t \le 0$. Thus we may conclude from (6.1) that $(u_-^{(1)}(0), \partial_t u_-^{(1)}(0)) = (u_-^{(2)}(0), \partial_t u_-^{(2)}(0))$. This implies that S is one-one.

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