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Inner amenable semigroups I

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1. Introduction.

Let S be a semigroup. Let $m(S)$ denote the Banach space of bounded real-valued functions on S. A *mean* on $m(S)$ is a bounded linear functional μ on $m(S)$ such that for all $f\in m(S)$, $\inf\{f(s): s\in S\}\leqq\mu(f)\leqq\sup\{f(s): s\in S\}$. An equivalent formulation is that $\|\mu\|=\mu(1)=1$, where 1 denotes the constant function on S with value 1. A mean μ on $m(S)$ is said to be a *left [respectively, right] invariant mean* if $\mu(l_{s}f)$ [*respectively,* $\mu(r_{s}f)$] $=\mu(f)$ for all $s\in S$ and $f\in m(S)$, where $l_{s}f$ and $r_{s}f$ are defined on S by $(l_{s}f)(t)=f(st)$ and $(r_{s}f)(t)=f(ts)$, $t\in S$. A mean that is both a left invariant mean (LIM) and a right invariant mean (RIM) is called an *invariant mean.* A semigroup which admits [respectively left, right] invariant means is called [respectively *left, right*] *amenable.* We refer the reader to [4] for an introductory exposition on amenable semigroups. Many results we use without mention in this article concerning amenable semigroups can be found in this reference.

We say that a mean μ on $m(S)$ is an *inner invariant mean* if $\mu(l_{s}f)=\mu(r_{s}f)$ for all $s\in S$ and $f\in m(S)$. A semigroup S which admits inner invariant means is called *inner amenable.* It follows immediately that every amenable semigroup is inner amenable. In particular, commutative semigroups are inner amenable. Another example of inner amenable semigroups is the class of semigroups with nonempty centres. Indeed, if $a\in S$ commutes with all $s\in S$, then the point mean (or Dirac measure) p_{a} defined by $p_{a}(f)=f(a)$ for all $f\in m(S)$ is an inner invariant mean on $m(S)$. We note that if S is a group, then an inner invariant mean is a mean for which $\mu(\mathfrak{T}_{s}f)=\mu(f)$ for all $s\in S$ and $f\in m(S)$, where $(\mathfrak{T}_{s}f)(t)=f(sts^{-1})=(f\circ\sigma_{s})(t)$ for all $s, t\in S$, where σ_{s} is the inner automorphism defined by s on the group S. Inner invariant means on groups are introduced in [6] and later studied extensively by Akeman [1], H. Choda [2], M. Choda [3], Paschke [10], Pier [12], and Watatani [14], to name a few. Both Paterson [11] and Pier [13] contain an account of the study of inner amenability of groups. In this article, we shall study inner amenable semigroups and show that many classical properties concerning amenability have similar analogues for inner amenability.

2. Analogues of some classical results.

In this section, S denotes a semigroup, and $m(S)$ is defined as in the introduction.

We first prove the basic characterization theorem for inner amenability.

THEOREM 1. *Let*

$$\mathcal{H} = \left\{ \sum_{k=1}^{n} (l_{a_k} f_k - r_{a_k} f_k) : f_k \in m(S),\ a_k \in S \right\}.$$

Then S is inner amenable if and only if

$$\sup\{h(s) : s \in S\} \geqq 0 \quad \textit{for all } h \in \mathcal{H}.$$

PROOF. Suppose that $m(S)$ has an inner invariant mean μ. Then for all $h = \sum_{k=1}^{n}(l_{a_k} f_k - r_{a_k} f_k)$, $f_k \in m(S)$, and $a_k \in S$, we have $\sup\{h(s) : s \in S\} \geqq \mu(h) = \sum_{k=1}^{n}(\mu(l_{a_k} f_k) - \mu(r_{a_k} f_k)) = 0$. Conversely, if $\sup\{h(s) : s \in S\} \geqq 0$ for all $h \in \mathcal{H}$, then the constant zero functional on $\mathcal{H}$ extends to a linear functional μ on $m(S)$ such that $\mu(f) \leqq \sup\{f(s) : s \in S\}$ for all $f \in m(S)$. It follows that μ is a mean on $m(S)$. Since it vanishes on $\mathcal{H}$, it is inner invariant. □

THEOREM 2. *S is inner amenable if and only if*

$$\inf\{\|1-h\|_u : h \in \mathcal{H}\} = 1.$$

If S is either left cancellative (i.e., $t=u$ if $st=su$ for some s) or right cancellative (i.e., $t=u$ if $ts=us$ for some s), then it is inner amenable if and only if $\mathcal{H}$ is not norm dense in $m(S)$.

PROOF. Suppose that S is inner amenable. Then for any $h \in \mathcal{H}$,

$$0 \leqq \sup\{-h(s) : s \in S\} = -\inf\{h(s) : s \in S\},$$

by Theorem 1. Thus, $\inf\{h(s) : s \in S\} \leqq 0$. Hence, for any $\varepsilon > 0$, there exists $s_0 \in S$ such that $h(s_0) < \varepsilon$, and so $1 - h(s_0) > 1 - \varepsilon$. Therefore, $\|1-h\|_u \geqq 1$ for any $h \in \mathcal{H}$. But since $0 \in \mathcal{H}$,

$$\inf\{\|1-h\|_u : h \in \mathcal{H}\} \leqq \|1-0\|_u = 1.$$

Conversely, suppose that $\inf\{\|1-h\|_u : h \in \mathcal{H}\} = 1$. Then an application of the Hahn-Banach extension theorem shows that there exists a bounded linear functional $\mu \in m(S)^*$ such that $\mu(\mathcal{H}) = \{0\}$, $\mu(1) = 1$ and $\|\mu\| = 1/\inf\{\|1-h\|_u : h \in \mathcal{H}\} = 1$. The last two identities imply that μ is a mean on $m(S)$, whereas $\mu(\mathcal{H}) = \{0\}$ means that μ is inner invariant.

We note that the preceeding proof shows also that if S is inner amenable, then $\mathcal{H}$ is not norm dense in $m(S)$. This is true without even assuming any cancellative properties.

Now, for the converse, suppose that $\mathscr{H}$ is not norm dense in $m(S)$. Then by Hahn-Banach extension theorem again, there exists $\mu\in m(S)^*$ such that $\mu(\mathscr{H})=\{0\}$, while $\mu\neq 0$. By a general result of decomposing a bounded linear functional, [7] (B.37), we may write $\mu=\mu^+-\mu^-$, where

$$\mu^+(f)=\max\{\mu,0\}(f)=\sup\{\mu(g):0\leqq g\leqq f\}$$

and

$$\mu^-(f)=-\min\{\mu,0\}(f)=-\inf\{\mu(g):0\leqq g\leqq f\}$$

for all $f\in m(S)$ with $f\geqq 0$. We note that μ^+ and μ^- are nonnegative bounded linear functionals on $m(S)$, and that μ^+ has the following properties:

1) $\mu^+(f)\geqq\mu(f)$ for all $f\in m(S)$ with $f\geqq 0$;
2) If ν is any other nonnegative linear functional on $m(S)$ such that $\nu(f)\geqq\mu(f)$ for all $f\in m(S)$ with $f\geqq 0$, then $\nu\geqq\mu^+$, i.e., $\nu(f)\geqq\mu^+(f)$ for all $f\in m(S)$ with $f\geqq 0$.

Since $\mu^+\geqq\mu$, we have $l_s^*(\mu^+)\geqq l_s^*(\mu)$. Also, as $\mu^+\geqq 0$, we have $l_s^*(\mu^+)\geqq 0$. It follows from 2) for the functional $l_s^*(\mu)$ that $l_s^*(\mu^+)\geqq(l_s^*(\mu))^+$. It thus follows that

$$\begin{aligned}\|l_s^*(\mu^+)-(l_s^*(\mu))^+\| &= (l_s^*(\mu^+)-(l_s^*(\mu))^+)(1)\\ &= \mu^+(l_s1)-(l_s^*(\mu))^+(1)\\ &= \mu^+(1)-(l_s^*(\mu))^+(1),\end{aligned}$$

where we note that

$$\mu^+(1)=\sup\{\mu(g):0\leqq g\leqq 1\}$$

and

$$(l_s^*(\mu))^+(1)=\sup\{(l_s^*(\mu))(g):0\leqq g\leqq 1\}=\sup\{\mu(l_sg):0\leqq g\leqq 1\}.$$

We claim that if S is either left cancellative or right cancellative, then $\mu^+(1)=(l_s^*(\mu))^+(1)$.

Case 1. Suppose that S is left cancellative. Then given any $g\in m(S)$ with $0\leqq g\leqq 1$, one can define $h\in m(S)$ with $0\leqq h\leqq 1$ such that $g=l_sh$. In fact for any $x\in sS$, $x=st$ for some uniquely defined $t\in S$, and we put $h(x)=g(t)$, while for any $x\notin sS$, put $h(x)=0$. This shows that $\{\mu(l_sg):0\leqq g\leqq 1\}=\{\mu(g):0\leqq g\leqq 1\}$, and consequently, $\mu^+(1)=(l_s^*(\mu))^+(1)$.

Case 2. Suppose that S is right cancellative. Then every $g\in m(S)$ with $0\leqq g\leqq 1$ can be written as $g=r_sh$ for some $h\in m(S)$ with $0\leqq h\leqq 1$. But then since $\mu(\mathscr{H})=0$, we have $\mu(r_sg)=\mu(l_sg)$ for all $s\in S$. Consequently,

$$\{\mu(g):0\leqq g\leqq 1\}=\{\mu(r_sg):0\leqq g\leqq 1\}=\{\mu(l_sg):0\leqq g\leqq 1\}.$$

Hence, $\mu^+(1)=(l_s^*(\mu))^+(1)$.

So, in both cases, we get $\mu^+(1)=(l_s^*(\mu))^+(1)$. Therefore, $l_s^*(\mu^+)=(l_s^*(\mu))^+$.

Next, we can repeat the whole argument above for $r_s^*(\mu^+)$ and $(r_s^*(\mu))^+$ and we conclude that $r_s^*(\mu^+)=(r_s^*(\mu))^+$.

Since now $\mu(\mathcal{K})=0$, we have $l_s^*(\mu)=r_s^*(\mu)$, and so $l_s^*(\mu^+)=(l_s^*(\mu))^+=(r_s^*(\mu))^+=r_s^*(\mu^+)$. So, $\mu^+(\mathcal{K})=0$, and since $\mu^-=\mu^+-\mu$, $\mu^-(\mathcal{K})=0$ as well. Note that $0\neq\mu=\mu^+-\mu^-$, at least one of μ^+ and μ^- must be nonzero. If $\mu^+\neq 0$, then $\nu=\mu^+/\|\mu^+\|$ will be an inner invariant mean on $m(S)$, and if $\mu^-\neq 0$, then $\nu=\mu^-/\|\mu^-\|$ will be an inner invariant mean on $m(S)$. In any case, $m(S)$ admits an inner invariant mean. □

The next theorem says that increasing union of a family of inner amenable semigroups is inner amenable.

THEOREM 3. *Let $\mathfrak{F}$ be a family of subsemigroups of S with the following properties.*

(a) *For any F_1, $F_2\in\mathfrak{F}$, there exists $F_3\in\mathfrak{F}$ such that $F_1\cup F_2\subseteq F_3$.*

(b) $\cup\mathfrak{F}=S$.

If $m(F)$ has an inner invariant mean for all $F\in\mathfrak{F}$, then so does $m(S)$.

PROOF. Let $h=\sum_{k=1}^n(l_{a_k}f_k-r_{a_k}f_k)$, $f_k\in m(S)$, and $a_k\in S$. Then by our given conditions, there exists an $F\in\mathfrak{F}$ such that $a_k\in F$ for all $k=1, 2, \cdots, n$. Since $m(F)$ has an inner invariant mean, it follows from theorem 1 that

$$\sup\left\{\sum_{k=1}^n(l_{a_k}f_k-r_{a_k}f_k)(s): s\in F\right\}\geqq 0.$$

In particular,

$$\sup\left\{\sum_{k=1}^n(l_{a_k}f_k-r_{a_k}f_k)(s): s\in S\right\}\geqq 0.$$

By theorem 1 again, $m(S)$ has an inner invariant mean. □

Let T be a subsemigroup of S. We denote the characteristic function of T in $m(S)$ by ξ_T. A mean μ on $m(S)$ is said to be *inner T-invariant* if $\mu(l_tf)=\mu(r_tf)$ for all $t\in T$, $f\in m(S)$. We have the following theorem connecting the inner amenability of a subsemigroup with the existence of a T-invariant mean which "concentrates" on T.

THEOREM 4. *Let T be a subsemigroup of S. Then $m(T)$ has an inner invariant mean if and only if $m(S)$ has an inner T-invariant mean μ such that $\mu(\xi_T)=1$.*

PROOF. Let $\eta: T\to S$ be the embedding of T into S. Then it induces $\bar{\eta}: m(S)\to m(T)$ via $\bar{\eta}(f)=f|_T$. Then $\bar{\eta}$ is bounded linear. Consider $\bar{\eta}^*: m(T)^*\to m(S)^*$.

Suppose that $\mu\in m(T)^*$ is an inner invariant mean. Since $\bar{\eta}(f)\geqq 0$ whenever $f\geqq 0$, and since $\bar{\eta}(1)=1$, it follows that $\bar{\eta}^*(\mu)$ is a mean on $m(S)$. Also, for any $f\in m(S)$, $t\in T$, it is easy to see that

$$\bar{\eta}(l_t f)=(l_t f)|_T=l_t(f|_T)=l_t(\bar{\eta}(f))$$

and

$$\bar{\eta}(r_t f)=(r_t f)|_T=r_t(f|_T)=r_t(\bar{\eta}(f)).$$

Therefore, for all $f\in m(S)$, $t\in T$,

$$\begin{aligned}(\bar{\eta}^*(\mu))(l_t f)&=\mu(\bar{\eta}(l_t f))=\mu(l_t(\bar{\eta}(f)))=\mu(r_t(\bar{\eta}(f)))\\&=\mu(\bar{\eta}(r_t f))=(\bar{\eta}^*(\mu))(r_t f).\end{aligned}$$

So, $\bar{\eta}^*(\mu)$ is an inner T-invariant mean on $m(S)$. Finally,

$$(\bar{\eta}^*(\mu))(\xi_T)=\mu(\bar{\eta}(\xi_T))=\mu(\xi_T|_T)=\mu(1)=1.$$

For the converse, consider the function $\zeta: m(T)\to m(S)$ defined by

$$\zeta(f)|_T=f,\quad \zeta(f)|_{S\setminus T}=0,\quad \text{for all } f\in m(T).$$

Then ζ is bounded linear. Consider $\zeta^*: m(S)^*\to m(T)^*$.

Suppose that ν is an inner T-invariant mean on $m(S)$ such that $\nu(\xi_T)=1$. Since $\zeta(f)\geqq 0$ whenever $f\geqq 0$, and since $\zeta(1)=\xi_T$, it is easy to see that $\zeta^*(\nu)$ is a mean on $m(T)$. Also, for any $f\in m(T)$, t, $t'\in T$, we have

$$(\zeta(l_t f)-l_t(\zeta(f)))(t')=(l_t f)(t')-(\zeta(f))(tt')=f(tt')-f(tt')=0.$$

So, $(\zeta(l_t f)-l_t(\zeta(f)))|_T=0$, and

$$|\zeta(l_t f)-l_t(\zeta(f))|\leqq \|\zeta(l_t f)-l_t(\zeta(f))\|_u\xi_{S\setminus T}.$$

This implies that $\nu(\zeta(l_t f)-l_t(\zeta(f)))=0$, or, $\nu(\zeta(l_t f))=\nu(l_t(\zeta(f)))$. Similarly, one can show that $\nu(\zeta(r_t f))=\nu(r_t(\zeta(f)))$. Therefore,

$$\begin{aligned}(\zeta^*(\nu))(l_t f)&=\nu(\zeta(l_t f))=\nu(l_t(\zeta(f)))=\nu(r_t(\zeta(f)))\\&=\nu(\zeta(r_t f))=(\zeta^*(\nu))(r_t f).\end{aligned}$$

So, $\zeta^*(\nu)$ is an inner invariant mean on $m(T)$. □

Next, we prove the characterization theorem by "convergence to inner invariance". A mean is called a *finite mean* if it is a (finite) convex combination of the point means. See Day [4] and [5] for more details.

THEOREM 5. *The following statements are equivalent.*

(a) *S is inner amenable.*

(b) *There exists a net φ_α of finite means such that*

$$s\varphi_\alpha - \varphi_\alpha s \longrightarrow 0 \quad \textit{weakly in } l_1(S), \quad \textit{for all } s \in S.$$

(c) *There exists a net φ_α of finite means such that*

$$\|s\varphi_\alpha - \varphi_\alpha s\|_1 \longrightarrow 0 \quad \textit{for all } s \in S.$$

PROOF. (a)$\Rightarrow$(b). Suppose that μ is an inner invariant mean on $m(S)$. Since the finite means are weak* dense in the set of means, there is a net φ_α of finite means in $l_1(S)$ which converges to μ weakly* in $m(S)^*$. Then for all $f \in m(S)$, and $s \in S$,

$$f(s\varphi_\alpha - \varphi_\alpha s) = \varphi_\alpha(l_s f) - \varphi_\alpha(r_s f) \longrightarrow \mu(l_s f) - \mu(r_s f) = 0.$$

(b)$\Rightarrow$(c). Consider the product $E = (l_1(S))^S$. Then E is a locally convex topological vector space under the product of the norm topologies. Define $T: l_1(S) \to E$ as follows: $(T(\varphi))(s) = s\varphi - \varphi s$, $\varphi \in l_1(S)$, $s \in S$. Then T is well-defined and linear. Let ψ_α be a net of finite means satisfying the convergence in (b). Then $(T(\psi_\alpha))(s) \to 0$ weakly in $l_1(S)$ for all $s \in S$. Since the weak topology of E is the product of the weak topologies of $l_1(S)$, (see, for example, [8] (17.13)), $T(\psi_\alpha) \to 0$ in the weak topology of E. But then 0 lies in the weak closure of the convex set $T(\Phi)$ in E, where Φ denotes the set of all finite means. It follows that 0 lies in the norm closure of $T(\Phi)$ in the original topology for E, i.e., the product of the norm topologies. Thus, there exists a net φ_β of finite means such that $T(\varphi_\beta) \to 0$ in E, i.e., $(T(\varphi_\beta))(s) \to 0$ in norm in $l_1(S)$ for all $s \in S$.

(c)$\Rightarrow$(b). Trivial.

(b)$\Rightarrow$(a). Let φ_α be a net satisfying the convergence in (b). By Alaoglu's theorem, it has a weak* convergent subnet. By passing to such a subnet if necessary, we may assume that $\varphi_\alpha \to \mu$ weakly* in $m(S)^*$. Then μ is a mean on $m(S)$, and for all $s \in S$, $f \in m(S)$,

$$\mu(l_s f) - \mu(r_s f) = \lim_\alpha (\varphi_\alpha(l_s f) - \varphi_\alpha(r_s f)) = \lim_\alpha (s\varphi_\alpha - \varphi_\alpha s)(f) = 0. \qquad \square$$

REMARK. If $S = G$ is a group, then we can say more about the nets in (b) and (c). If $f \in m(G)$, let $f^* \in m(G)$ be defined by $f^*(s) = f(s^{-1})$, $s \in G$. We say that a mean μ on $m(G)$ is *symmetric*, or *inversion invariant*, if $\mu(f^*) = \mu(f)$ for all $f \in m(G)$. Let Φ_S denote the set of all symmetric finite means on $m(G)$. It is easy to see that Φ_S is convex. Namioka has proved that Φ_S is weak* dense in the set of all symmetric means on $m(G)$, [9]. Using this fact, one can prove the following result.

THEOREM 6. *Let G be a group. The following are equivalent.*

(a) *G is inner amenable.*

(b) *There exists a net* φ_α *of symmetric finite means such that* $s\varphi_\alpha-\varphi_\alpha s\to 0$ *weakly in* $l_1(G)$ *for all* $s\in G$.

(c) *There exists a net* φ_α *of symmetric finite means such that* $\|s\varphi_\alpha-\varphi_\alpha s\|_1\to 0$ *for all* $s\in G$.

PROOF. (a)$\Rightarrow$(b). Let μ be an inner invariant mean on $m(G)$. Then it is easy to see that

$$\nu(f)=\frac{1}{2}(\mu(f)+\mu(f^*)),\quad f\in m(G),$$

defines a symmetric mean on $m(G)$. Now, a direct calculation shows that

$$l_s(f^*)=(r_{s^{-1}}f)^*,\quad \text{and}\quad r_s(f^*)=(l_{s^{-1}}f)^*.$$

Therefore,

$$\begin{aligned}\nu(l_sf)&=\frac{1}{2}(\mu(l_sf)+\mu((l_sf)^*))\\&=\frac{1}{2}(\mu(r_sf)+\mu(r_{s^{-1}}(f^*)))\\&=\frac{1}{2}(\mu(r_sf)+\mu(l_{s^{-1}}(f^*)))\\&=\frac{1}{2}(\mu(r_sf)+\mu((r_sf)^*))\\&=\nu(r_sf).\end{aligned}$$

This show that ν is a symmetric inner invariant mean on $m(G)$. By weak* density of Φ_S in the means, we get a net φ_α of symmetric finite means converging weakly* to ν in $m(G)^*$. The remaining argument is identical to the one given in the semigroup case.

(b)$\Rightarrow$(c) and (c)$\Rightarrow$(a) are easy. □

3. On Følner's condition.

We now turn to a characterization theorem analogous to the Følner's characterization of amenability. The original result for amenability is basically combinatorial in nature. When we investigate the corresponding question for inner amenability, we find that most of the combinatorial lemmas to the final result require working out all over again. The spirit behind the calculations here is the same as in the case of amenability. But the details in the calculations requires many adjustments. In particular, cancellative properties play an important role in this situation.

In the theorems and discussions below, S denotes a semigroup and A is a nonempty subset of S. If $s\in S$, we write $As^{-1}=\{t\in S: ts\in A\}$ and $s^{-1}A=$

$\{t\in S: st\in A\}$. We also note that

$$\mu_A(s)=\frac{1}{|A|}\xi_A=\begin{cases}\frac{1}{|A|} & \text{if } s\in A\\ 0 & \text{if } s\notin A\end{cases}$$

defines an element in $l_1(S)$.

LEMMA 7. *We have*

$$\begin{aligned}(\mu_A s-s\mu_A)(t)&=\frac{|A\cap\{t\}s^{-1}|-|A\cap s^{-1}\{t\}|}{|A|}\\ &=\begin{cases}\frac{|A\cap\{t\}s^{-1}|}{|A|}\geqq\frac{1}{|A|} & \textit{if } t\in As\backslash sA\\ -\frac{|A\cap s^{-1}\{t\}|}{|A|}\leqq-\frac{1}{|A|} & \textit{if } t\in sA\backslash As\\ \frac{|A\cap\{t\}s^{-1}|-|A\cap s^{-1}\{t\}|}{|A|} & \textit{if } t\in sA\cap As\\ 0 & \textit{if } t\notin sA\cup As.\end{cases}\end{aligned}$$

PROOF. By a direct calculation, we get

$$(s\mu_A)(t)=\sum_{ss'=t}\mu_A(s')=\sum_{s'\in s^{-1}\{t\}}\mu_A(s')=\frac{|A\cap s^{-1}\{t\}|}{|A|}.$$

Similarly, one finds that $(\mu_A s)(t)=|A\cap\{t\}s^{-1}|/|A|$. The second equality comes from rewriting the general formula in different cases. □

LEMMA 8. *If S is right cancellative, then*

$$(\mu_A s-s\mu_A)(t)=\begin{cases}\frac{1}{|A|} & \textit{if } t\in As\backslash sA\\ -\frac{|A\cap s^{-1}\{t\}|}{|A|}\leqq-\frac{1}{|A|} & \textit{if } t\in sA\backslash As\\ \frac{1-|A\cap s^{-1}\{t\}|}{|A|}\leqq 0 & \textit{if } t\in sA\cap As\\ 0 & \textit{if } t\notin sA\cup As.\end{cases}$$

If S is left cancellative, then

$$(\mu_A s-s\mu_A)(t)=\begin{cases}\frac{|A\cap\{t\}s^{-1}|}{|A|}\geqq\frac{1}{|A|} & \textit{if } t\in As\backslash sA\\ -\frac{1}{|A|} & \textit{if } t\in sA\backslash As\\ \frac{|A\cap\{t\}s^{-1}|-1}{|A|}\geqq 0 & \textit{if } t\in sA\cap As\\ 0 & \textit{if } t\notin sA\cup As.\end{cases}$$

PROOF. If S is right cancellative, then $t\in As\Rightarrow|A\cap\{t\}s^{-1}|=1$, and if S is left cancellative, then $t\in sA\Rightarrow|A\cap s^{-1}\{t\}|=1$. The result follows from the preceeding lemma. □

LEMMA 9. *If S is right cancellative, then $\|\mu_A s-s\mu_A\|_1=2|As\backslash sA|/|A|$. If S is left cancellative, then $\|\mu_A s-s\mu_A\|_1=2|sA\backslash As|/|A|$.*

PROOF. Suppose that S is right cancellative. Then by the formula in the preceeding lemma,

$$\begin{aligned}\|\mu_A s-s\mu_A\|_1 &=\sum_{t\in As\backslash sA}\frac{1}{|A|}+\sum_{t\in sA\backslash As}\frac{|A\cap s^{-1}\{t\}|}{|A|}+\sum_{t\in sA\cap As}\frac{|A\cap s^{-1}\{t\}|-1}{|A|}\\ &=\frac{|As\backslash sA|}{|A|}+\sum_{t\in(sA\backslash As)\cup(sA\cap As)}\frac{|A\cap s^{-1}\{t\}|}{|A|}-\frac{|sA\cap As|}{|A|}\\ &=\frac{1}{|A|}\Big(|As\backslash sA|+\sum_{t\in sA}|A\cap s^{-1}\{t\}|-|sA\cap As|\Big)\\ &=\frac{1}{|A|}(|As\backslash sA|+|A|-|sA\cap As|)\\ &=\frac{1}{|A|}(|As\backslash sA|+|As|-|sA\cap As|),\\ &\qquad\qquad\text{since } S \text{ is right cancellative}\\ &=\frac{1}{|A|}(|As\backslash sA|+|As\backslash sA|)\\ &=\frac{2|As\backslash sA|}{|A|}.\end{aligned}$$

The proof for the left cancellative case is similar. □

THEOREM 10. *A right cancellative semigroup S is inner amenable if it satisfies the following condition.*

($\sharp$) *For any finite set $F\subseteqq S$ and any $\varepsilon>0$, there exists a finite nonempty set $A\subseteqq S$ such that $|As\backslash sA|<\varepsilon|A|$ for all $s\in F$.*

A left cancellative semigroup S is inner amenable if it satisfies the following condition.

($\flat$) *For any finite set $F\subseteqq S$ and any $\varepsilon>0$, there exists a finite nonempty set $A\subseteqq S$ such that $|sA\backslash As|<\varepsilon|A|$ for all $s\in F$.*

PROOF. Let $\mathfrak{F}$ be the family of all finite subsets of S directed upward by inclusion, and let $E=(0, 1)$ be directed downward by its usual order. Let D be the product directed set $\mathfrak{F}\times E$.

If S is right cancellative, then ($\sharp$) says that for any $\alpha\in D$, $\alpha=(F, \varepsilon)$, there exists a nonempty set $A_\alpha\subseteqq S$ such that

$$\|\mu_{A_\alpha}s-s\mu_{A_\alpha}\|_1=\frac{2|A_\alpha s\backslash sA_\alpha|}{|A_\alpha|}<2\varepsilon.$$

It follows that the net μ_{A_α} in $l_1(S)$ converges in norm to inner invariance, and hence S is inner amenable.

The case for left cancellative semigroups is similar. □

Observing that $|As\backslash sA|+|sA\backslash As|=|As\Delta sA|$, we have the following result immediately.

COROLLARY 11. *Suppose that S is either right cancellative or left cancellative. Then S is inner amenable if*

($\star$) *For any finite set $F\subseteqq S$ and any $\varepsilon>0$, there exists a finite nonempty set $A\subseteqq S$ such that $|As\Delta sA|<\varepsilon|A|$ for all $s\in F$.*

It turns out that for semigroups that are both right and left cancellative, ($\star$) is also a necessary condition for inner amenability. We first need a lemma just like above, and its proof should be evident now.

LEMMA 12. *Suppose that S is both right and left cancellative. Then*

$$(\mu_A s-s\mu_A)(t)=\begin{cases}\frac{1}{|A|} & \text{if } t\in As\backslash sA\\ -\frac{1}{|A|} & \text{if } t\in sA\backslash As\\ 0 & \text{otherwise}.\end{cases}$$

Consequently,

$$\|\mu_A s-s\mu_A\|_1=\frac{|As\Delta sA|}{|A|}.$$

We need another technical lemma. We first note by [5] that every finite mean φ in $l_1(S)$ can be written in the form

$$\varphi=\sum_{i=1}^{n}\lambda_i\mu_{A_i},$$

where $A_1\supsetneqq A_2\supsetneqq\cdots\supsetneqq A_n$ are finite nonempty subsets of S, $\lambda_i>0$ for $1\leqq i\leqq n$, and $\sum_{i=1}^{n}\lambda_i=1$.

LEMMA 13. *Suppose that S is both left and right cancellative, and φ is a finite mean in $l_1(S)$ as expressed in the above form. Then*

$$\|\varphi s-s\varphi\|_1=\sum_{i=1}^{n}\lambda_i\|\mu_{A_i}s-s\mu_{A_i}\|_1=\sum_{i=1}^{n}\lambda_i\frac{|A_i s\Delta sA_i|}{|A_i|},$$

for any $s\in S$.

PROOF. For $1\leqq i,\ j\leqq n$, we have either $A_i\subseteqq A_j$, or $A_j\subseteqq A_i$. Now,

$$A_i\subseteqq A_j \Longrightarrow sA_i\backslash A_is\subseteqq sA_i\subseteqq sA_j\Longrightarrow (sA_i\backslash A_is)\cap(A_js\backslash sA_j)=\emptyset\,;$$

and

$$A_j\subseteqq A_i \Longrightarrow A_js\backslash sA_j\subseteqq A_js\subseteqq A_is\Longrightarrow (sA_i\backslash A_is)\cap(A_js\backslash sA_j)=\emptyset\,.$$

Thus, in any case, we have $(sA_i\backslash A_is)\cap(A_js\backslash sA_j)=\emptyset$. Put $A=\bigcup_{i=1}^{n}(sA_i\backslash A_is)$. Then for all $t\in A$, $t\in sA_i\backslash A_is$ for some i, and so, $t\notin A_js\backslash sA_j$ for any j. Consequently, $(\mu_{A_j}s-s\mu_{A_j})(t)\leqq 0$ for all j. On the other hand, if $t\notin A$, then $t\notin sA_i\backslash A_is$ for any i, and hence by the preceeding lemma, $(\mu_{A_j}s-s\mu_{A_j})(t)\geqq 0$ for all j. In other words, given $t\in S$, the numbers $(\mu_{A_j}s-s\mu_{A_j})(t)$, $1\leqq j\leqq n$, are all $\leqq 0$ or all $\geqq 0$ according as $t\in A$ or $t\notin A$. In both cases, we have

$$|(\varphi s-s\varphi)(t)|=\left|\sum_{i=1}^{n}\lambda_i(\mu_{A_i}s-s\mu_{A_i})(t)\right|=\sum_{i=1}^{n}\lambda_i|(\mu_{A_i}s-s\mu_{A_i})(t)|\,.$$

Therefore,

$$\begin{aligned}\|\varphi s-s\varphi\|_1&=\sum_{t\in S}|(\varphi s-s\varphi)(t)|\\&=\sum_{t\in S}\sum_{i=1}^{n}\lambda_i|(\mu_{A_i}s-s\mu_{A_i})(t)|\\&=\sum_{i=1}^{n}\lambda_i\sum_{t\in S}|(\mu_{A_i}s-s\mu_{A_i})(t)|\\&=\sum_{i=1}^{n}\lambda_i\|\mu_{A_i}s-s\mu_{A_i}\|_1\,.\end{aligned}$$ □

THEOREM 14. *Suppose that S is both left and right cancellative. If S is inner amenable, then it satisfies* (★).

PROOF. Let $F=\{s_1, s_2, \cdots, s_k\}\subseteqq S$, and let $\varepsilon>0$. By convergence to inner invariance, there exists a finite mean φ in $l_1(S)$ such that $\|\varphi s_i-s_i\varphi\|_1<\varepsilon/k$, for $1\leqq i\leqq k$. Write $\varphi=\sum_{j=1}^{n}\lambda_j\mu_{A_j}$ as above. Then,

$$\varepsilon>\sum_{i=1}^{k}\|\varphi s_i-s_i\varphi\|_1=\sum_{i=1}^{k}\sum_{j=1}^{n}\lambda_j\frac{|A_js_i\Delta s_iA_j|}{|A_j|}=\sum_{j=1}^{n}\lambda_j\Big(\sum_{i=1}^{k}\frac{|A_js_i\Delta s_iA_j|}{|A_j|}\Big).$$

Since $\lambda_i>0$ for $1\leqq i\leqq n$, and $\sum_{i=1}^{n}\lambda_i=1$, there must exist some index j_0 such that

$$\sum_{i=1}^{k}\frac{|A_{j_0}s_i\Delta s_iA_{j_0}|}{|A_{j_0}|}<\varepsilon\,.$$

Let $A=A_{j_0}$. Then for all $i=1, 2, \cdots, k$,

$$\frac{|As_i \Delta s_i A|}{|A|} < \varepsilon . \qquad \square$$

The following corollary is now obvious. Note also that it applies in particular to groups.

COROLLARY 15. *Let S be a left and right cancellative semigroup. Then S is inner amenable if and only if it satisfies* ($\star$).

We may also formulate the above characterizations of inner amenability using "intersections" instead of "symmetric differences" of sets.

THEOREM 16. *Suppose that S is either left cancellative or right cancellative. Then S is inner amenable if it satisfies the following condition.*

($\dagger$) *For any finite subset $F \subseteq S$, and for any $k \in (0, 1)$, there exists a finite nonempty set $A \subseteq S$ such that $|sA \cap As| > k|A|$ for all $s \in F$.*

PROOF. Suppose that S is left cancellative and that it satisfies ($\dagger$). We shall prove that S satisfies ($\flat$).

Let $F \subseteq S$ be finite and let $\varepsilon > 0$. Without loss of generality, we may assume that $\varepsilon \in (0, 1)$. Let $k = 1 - \varepsilon \in (0, 1)$. Then by ($\dagger$), there exists a nonempty $A \subseteq S$ such that $|sA \cap As| > k|A|$ for all $s \in F$. Now, since S is left cancellative, we have $|sA| = |A|$ for all $s \in S$. Therefore, for all $s \in F$,

$$|sA \backslash As| = |sA| - |sA \cap As| = |A| - |sA \cap As| < |A| - k|A| = \varepsilon |A|.$$

The case for right cancellative semigroups is similar. $\square$

Note again that the following applies in particular to groups.

THEOREM 17. *Suppose that S is both left and right cancellative. If S is inner amenable, then it satisfies* ($\dagger$).

PROOF. We know that S satisfies ($\star$). Now, given a finite set $F \subseteq S$, and $k \in (0, 1)$, let $\varepsilon = 1 - k$. By ($\star$), there exists a nonempty set $A \subseteq S$ such that $|sA \Delta As| < 2\varepsilon |A|$ for all $s \in F$. Since S is both left and right cancellative, $|sA| = |As| = |A|$ for all $s \in S$. Therefore, for all $s \in F$, we have

$$\begin{aligned} 2|sA \cap As| &= (|sA| - |sA \backslash As|) + (|As| - |As \backslash sA|) \\ &= 2|A| - |sA \Delta As| \\ &> 2|A| - 2\varepsilon |A| = 2k|A|, \end{aligned}$$

i.e., $|sA \cap As| > k|A|$, and S satisfies ($\dagger$). $\square$

REMARK. In case when S is a group, one can choose the set A in ($\star$) and ($\dagger$) to be symmetric, i.e., $s \in A \Rightarrow s^{-1} \in A$. What we need here is a result claimed and used, but not proved in [**9**]. We shall supply the proof here, and then we mention the final theorem.

LEMMA 18. *Let $S=G$ be a group and let φ be a symmetric finite mean on $m(G)$. Write $\varphi=\sum_{j=1}^{n}\lambda_j\mu_{A_j}$ as before. Then the sets A_j $(1\leqq j\leqq n)$ are symmetric.*

PROOF. Let $s\in A_1$. Since $A_1\supsetneqq A_2\supsetneqq\cdots\supsetneqq A_n$, we have

$$s^{-1}\notin A_1\Longrightarrow\varphi^*(s)=\varphi(s^{-1})=0<\varphi(s)\Longrightarrow\text{contradiction.}$$

Therefore, $s\in A_1\Rightarrow s^{-1}\in A_1$, i.e., $A_1^{-1}=A_1$.

Next, suppose that we have shown that $A_1^{-1}=A_1,\cdots,A_k^{-1}=A_k$. Then for all $s\in A_{k+1}$, we have

$$\begin{aligned}s^{-1}\notin A_{k+1}\Longrightarrow\varphi^*(s)&=\varphi(s^{-1})\\&=\sum_{i=1}^{k}\lambda_i\mu_{A_i}(s^{-1})\\&=\sum_{i=1}^{k}\lambda_i/|A_i|\\&=\sum_{i=1}^{k}\lambda_i\mu_{A_i}(s)\\&<\sum_{i=1}^{k+1}\lambda_i\mu_{A_i}(s)\\&\leqq\varphi(s)\Longrightarrow\text{contradiction.}\end{aligned}$$

Therefore, $s\in A_{k+1}\Rightarrow s^{-1}\in A_{k+1}$, i.e., $A_{k+1}^{-1}=A_{k+1}$. By induction, all the A_j are symmetric. □

We now state the final theorem whose proof should be easy. We therefore omit it.

THEOREM 19. *Let G be a group. Then each of the following conditions is equivalent to G being inner amenable.*

$(\star)_S$ *For any finite set $F\subseteqq G$ and any $\varepsilon>0$, there exists a finite nonempty set $A\subseteqq G$ such that $|As\Delta sA|<\varepsilon|A|$ for all $s\in F$.*

$(\dagger)_S$ *For any finite subset $F\subseteqq G$, and for any $k\in(0,1)$, there exists a finite nonempty set $A\subseteqq G$ such that $|sA\cap As|>k|A|$ for all $s\in F$.*

References

[1] C. A. Akeman, Operator algebras associated with fuschian groups, Houston J. Math., 7 (1981), 295-301.

[2] H. Choda and M. Choda, Fullness, simplicity and inner amenability, Math. Japon., 24 (1979), 235-246.

[3] M. Choda, The factors of inner amenable groups, Math. Japon., **24** (1979), 145–152.
[4] M. M. Day, Amenable semigroups, Illinois J. Math., **1** (1957), 509–544.
[5] M. M. Day, Means for the bounded functions and ergodicity of the bounded representations of semigroups, Trans. Amer. Math. Soc., **69** (1950), 276–291.
[6] E. G. Effros, Property Γ and inner amenability, Proc. Amer. Math. Soc., **47** (1975), 483–486.
[7] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis I, Springer-Verlag, New York, 1963.
[8] J.L. Kelley and I. Namioka, Linear Topological Spaces, D. Van Nostrand, New Jersey, 1963.
[9] I. Namioka, Følner's conditions for amenable semigroups, Math. Scand., **15** (1964), 18–28.
[10] W. L. Paschke, Inner amenability and conjugate operators, Proc. Amer. Math. Soc., **71** (1978), 117–118.
[11] A. Paterson, Amenability, Math. Surveys and Monographs No. 29, Amer. Math. Soc., Rhode Island, 1988.
[12] J.-P. Pier, Quasi-invariance intérieure sur les groupes localement compacts, Actualités mathématiques, (1982), 431–436.
[13] J.-P. Pier, Amenable Locally Compact Groups, John Wiley & Sons, New York, 1984.
[14] Y. Watatani, The character groups of amenable group C^*-algebras, Math. Japon., **24** (1979), 141–144.

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