J. Math. Soc. Japan
Vol. 49, No. 3, 1997

Existence of curves of genus three on a product of two elliptic curves

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(Received June 13, 1995)

1. Introduction.

Let E be an elliptic curve over the field of complex numbers, and let A be the abelian surface $E\times E$. It seems interesting to study if A contains a smooth curve of genus g. In the case when $g=2$, Hayashida and Nishi [**3**] studied this subject. Their aim was to determine if a product of two elliptic curves can be a Jacobian variety of some curve. In this note we will consider the case when $g=3$. Our first aim is to determine if A has a (1, 2)-polarization which is not a product one ([**1**]). Second one is as follows: for an algebraic variety V, the degree of irrationality $d_r(V)$ has been introduced in [**4**] or [**7**]. Especially we take an interest in the value $d_r(A)$ for an abelian surface A. Concerning this we have shown that $d_r(A)=3$ if an abelian surface A contains a smooth curve of genus 3 ([**5**]).

On the other hand the following assertion has been obtained ([**8**]):

Let n be a positive square free integer. Put $\omega=\sqrt{-n}$ [resp. $\{1+\sqrt{-n}\}/2$] if $-n\equiv 2$ or 3 (mod 4) [resp. $-n\equiv 1$ (mod 4)]. Let $K=\boldsymbol{Q}(\sqrt{-n})$ be an imaginary quadratic field. For each $\xi\in K\setminus\boldsymbol{Q}$, let $a\xi^2+b\xi+c=0$ be the equation of ξ satisfying that $a, b, c\in\boldsymbol{Z}$, $a>0$ and $(a, b, c)=1$. Let L be the lattice generated by $\{1, \xi\}$ and let E be the elliptic curve $\boldsymbol{C}/L$.

PROPOSITION 1. *Under the situation above, suppose that at least one of a, b, c is an even number. Then there exist two elliptic curves E_1 and E_2 on $A=E\times E$ satisfying $(E_1, E_2)=2$, where (E_1, E_2) denotes the intersection number of E_1 and E_2. Especially there exists a nonsingular curve of genus* 3 *on A, hence $d_r(A)=3$.*

REMARK 2. Of course there are many elliptic curves E satisfying the condition in this proposition. In fact, if $-n\equiv 2$ or 3 (mod 4), then b is even, because $a\xi$ becomes an integer. Hence every ξ enjoys the condition. For the remainder case, letting k and l ($\neq 0$) be rational integers, we have the following.

This research was partially supported by Grant-Aid for Scientific Research (No. 07640025), Ministry of Education, Science and Culture, Japan.

(i) If $-n\equiv 1 \pmod 8$, then $\xi=k+l\omega$ and $1/2+l\omega$ are the suitable ones.

(ii) If $-n\equiv 5 \pmod 8$, then $\xi=k+2l\omega$ and $1/2+l\omega$ are the suitable ones.

Moreover we will consider if A has an infinitely many smooth curves of genus 3 modulo birational equivalence.

We would like to thank the referee for suggesting a simple proof of Theorem 6.

2. Statement of results.

Let m be 0 or a square free positive integer and put $K=\boldsymbol{Q}(\sqrt{-m})$. Let $\mathfrak{o}$ be the principal order of K. When $m=0$, we understand that K and $\mathfrak{o}$ coincide with $\boldsymbol{Q}$ and $\boldsymbol{Z}$, respectively. Let E be an elliptic curve with the ring of endomorphisms isomorphic to $\mathfrak{o}$ and let A be the abelian surface $E\times E$. Then our result is stated as follows:

THEOREM 3. *If $m\neq 0$ and $\neq 3$, then there exists a smooth curve of genus 3 on A. On the contrary if $m=0$ or 3, then there exists no such a curve.*

REMARK 4. If $m=1$, 7 or 15, then there exists no smooth genus-2 curve, but exists a genus-3 curve in each case.

REMARK 5. If E has complex multiplications, then $d_r(E\times E)=3$. Because, in case $m=3$, there is an automorphism φ of order 3. Since $A/\varphi\times\varphi$ is a rational surface, we conclude that $d_r(E\times E)=3$ (cf. [**5**]).

Similarly as in [**3**] we feel an interest to know whether there are infinitely many smooth curves of genus 3 on A. Contrary to the case of genus 2 the result is as follows.

THEOREM 6. *If an abelian surface B contains a smooth curve of genus 3, then it contains infinitely many such curves modulo birational equivalence. Hence in case $m\neq 0$ and $\neq 3$, $E\times E$ contains infinitely many smooth curves of genus 3.*

3. Proof of Theorems.

In this section we use the same notation as in [**3**]. First we enumerate several lemmas.

LEMMA 7. *Let X be an effective divisor on an abelian surface with $X^2=4$. Then X is one of the following, where E', E'' and F are elliptic curves:*

(i) *X is a smooth genus-3 curve.*

(ii) *X is an irreducible curve with one double point and the genus of the normalization of X is 2.*

(iii) *$X=E'+E''$ and $(E', E'')=2$.*

(iv) $X=F+E'+E''$ *and* $(F, E')=(F, E'')=1$, $(E', E'')=0$.

PROOF. See (1.2) in [**1**].

LEMMA 8. *Let* X *be a divisor as in Lemma* 7. *Then* X *is not of type* (iv) *if and only if* $(X, E_{\lambda,\mu})>1$ *for all elliptic curves* $E_{\lambda,\mu}$ *on* A.

PROOF. If X is of type (iv), i.e., $X=F+E'+E''$, then $(X, E')=(X, E'')=1$. Note that E' and E'' can be expressed as translations of $E_{\alpha,\beta}$ for some α, $\beta\in\mathfrak{o}$ (cf. Lemma 1 in [**3**]). Suppose that X is not of type (iv) and that $(X, E_{\lambda,\mu})=1$ for some $E_{\lambda,\mu}$. Then we have a contradiction as follows: in case X is irreducible, we have a birational mapping $E\times E\rightarrow X\times E_{\lambda,\mu}$, i.e., $E\times E$ and $\tilde{X}\times E_{\lambda,\mu}$ are birational (cf. Cor. 2, Th. 4 in [**6**]), where $\tilde{X}$ is the normalization of X. This means that the irregularity of $\tilde{X}$ must be 1. In the case when X is reducible, put $X=E'+E''$. We may assume that $(E', E_{\lambda,\mu})=1$ and $(E'', E_{\lambda,\mu})=0$. This means that $E_{\lambda,\mu}$ is a translation of E'', hence $(E_{\lambda,\mu}, E'')$ must be 2, which is a contradiction. □

LEMMA 9. *If there is an effective divisor* X *in Lemma* 7, *which is not of type* (iv), *then there is a smooth genus*-3 *curve on* A.

PROOF. Since the pencil $|X|$ has no fixed components, its general member is irreducible and smooth (see, (1.5) in [**1**]). □

We will prove the theorem in a similar way as in [**3**]. Let D be a divisor on A. Note that the Néron-Severi group of A is generated by $E_{1,1}$, $E_{1,\omega}$, $E_{1,0}$ and $E_{0,1}$, where we regard $E_{1,\omega}$ as 0 in case $m=0$. Hence we have a unique expression

$$D\equiv aE_{1,1}+bE_{1,\omega}+cE_{1,0}+dE_{0,1},$$

where a, b, c, $d\in\boldsymbol{Z}$.

Therefore we obtain that

$$(D, E_{\xi,\eta})=(k\xi\bar{\xi}+l\eta\bar{\eta}-\alpha\xi\bar{\eta}-\bar{\alpha}\bar{\xi}\eta)/N(\xi, \eta),$$

where $k=a+b\omega\bar{\omega}+d$, $\alpha=a+b\omega$, $l=a+b+c$.

Hence we have that

$$(D, D)=2(kl-\alpha\bar{\alpha})\quad\text{and}\quad(D, E_{1,0})=k\,.$$

Now let X be a divisor as in Lemma 7. Since X is effective and $X^2=4$, X is ample and hence $k>0$. Conversely, let D be a divisor on A with $D^2=4$. If $k>0$, then $l(D)>0$. So we may assume that D is effective. Combining the lemmas above, we obtain the following criterion:

LEMMA 10 (CRITERION). *Let* D *be a divisor on* A *satisfying that*

$$k>0,\quad kl-\alpha\bar{\alpha}=2. \tag{1}$$

If the equation

$$k\xi\bar{\xi}+l\eta\bar{\eta}-\alpha\xi\bar{\eta}-\bar{\alpha}\bar{\xi}\eta=N(\xi,\ \eta) \tag{2}$$

has a non-trivial solution $(\xi,\ \eta)\neq(0,\ 0)$ *in* $\mathfrak{o}$, *then* X *is of type* (iv); *and otherwise there exists a smooth genus*-3 *curve on* A.

We now divide the proof of Theorem 3 into several cases according to the value m.

(I) The case $m=0$.

In this case we may assume that $b=0$. Then the criterion becomes as follows:

$$a+d>0,\quad (a+d)(a+c)-a^2=2 \tag{3}$$

$$(a+d)x^2-2axy+(a+c)y^2=1 \tag{4}$$

Put $q(x,\ y)=(a+d)x^2-2axy+(a+c)y^2$. By the condition (3) this quadratic form is primitive, i.e., $(a+d,\ 2a,\ a+c)=1$. The discriminant δ of q is -8, hence the class number of the discriminant $h^+(\delta)$ is 1. Thus we infer that the equation $q(x,\ y)=1$ has a primitive solution. Namely, there is no smooth genus-3 curve on $E\times E$.

(II) The case $m>0$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathfrak{o}$ satisfying $(\xi,\ \eta)\mathfrak{a}=\eta$ and $(\xi,\ \eta)\mathfrak{b}=(k\xi-\bar{\alpha}\eta)$. In case $\eta=0$, we see that $k=1$ if $\xi\neq0$. Hence for our purpose we may assume that $k\neq1$ hereafter. Thus $\eta\neq0$. Putting $\gamma=\mathfrak{a}\bar{\mathfrak{a}}/\eta$, we obtain that

$$\begin{cases}(\gamma\xi,\ \gamma\eta)=\bar{\mathfrak{a}}\\ \gamma\eta=\mathfrak{a}\bar{\mathfrak{a}}=N(\mathfrak{a}).\end{cases}$$

Putting further $\zeta=\gamma\xi\in\mathfrak{o}$ and $n=\gamma\eta\in\boldsymbol{N}$, we infer that the equation (2) becomes

$$k\zeta\bar{\zeta}-\alpha\zeta n-\bar{\alpha}\bar{\zeta}n+ln^2=n.$$

Multiplying k on both sides of this equation and using (1), we obtain that

$$N(k\zeta-\alpha n)=n(k-2n). \tag{5}$$

We want to find $(k,\ l,\ \alpha)$ satisfying (1) such that (5) has no non-trivial solutions. By Proposition 1 we have only to consider the case when $-m\equiv1 \pmod 4$.

(II-1) The case $m\equiv7 \pmod 8$.

Let $a=b=1$, i.e., $\alpha=1+\omega$, then we let $k=2$. In this case the equation (5) becomes

$$N(2\zeta-\alpha n)=n(2-2n).$$

In case $n=0$, the solution is trivial, but in case $n=1$, we have $2\zeta=1+\omega$, hence $\zeta\notin\mathfrak{o}$. So that there is no non-trivial solution.

(II-2) The case $m\equiv 3 \pmod 8$.

CLAIM 1. *Suppose that $m=3$. Then the simultaneous equations* (1) *and* (2) *have always solutions.*

PROOF. In the equation (2) put $\xi=x+y\omega$ and $\eta=s+t\omega$. Then we can regard the left hand side of (2) as a quadratic form Q of x, y, s and t over $\boldsymbol{Z}$. By a simple calculation we infer that Q is positive definite if $m=3$, and its determinant is 9/4. Since the minimum value of Q is not greater than $\sqrt[4]{9}$ (cf. Appendix in [2]), the minimum value must be 1. Hence the equation (2) is always satisfied when ξ and η give the minimum value of Q. Therefore there is no smooth genus-3 curve on A. □

CLAIM 2. *Suppose that $m\neq 3$. Then for a suitable value (k, l, α) satisfying* (1), *the equation* (5) *has no non-trivial solution.*

PROOF. Let us express m as $8m_1+3$.

(a). If $m_1\equiv 0$ or $2 \pmod 3$, then let $k=3$ and $\alpha=\omega$ or $1+\omega$, respectively. The equation (5) becomes $N(3\zeta-\alpha n)=n(3-2n)$. If $n=0$, then $\zeta=0$, which yields a trivial solution. Hence $n=1$, this means that $3\zeta-\alpha$ must be a unit in $\mathfrak{o}$, i.e., $3\zeta-\alpha=\pm 1$, since $m_1\neq 0$. Then we have that $\zeta\notin\mathfrak{o}$.

(b). If $m_1\equiv 1 \pmod 3$, then put $m_1=3m_2+1$, i.e., $m=11+24m_2$. If $m_2\equiv 1 \pmod 5$, then $2+\alpha\bar{\alpha}$ can be a multiple of 5 for suitable values of a and b, so let $k=5$. Consider the equation (5); $N(5\zeta-\alpha n)=n(5-2n)$. Clearly n must be odd. So let $n=1$, then we have $N(5\zeta-\alpha)=3$. This equation has solutions only if $m_2=0$. Hence we consider the case when $m=11$. Take $a=0$ and $b=5$, i.e., $\alpha=5\omega$ and let $k=11$. Then $N(11\zeta-\alpha n)=n(11-2n)$. If we put $11\zeta-\alpha n=x+y\omega$, then this equation becomes

$$x^2+xy+3y^2=n(11-2n),$$

where $1\leqq n\leqq 5$.

Clearly n must be odd, so the right hand side takes the values 9, 15 and 5. By checking each case $n=1$, 3 and 5, we conclude that there are no solutions.

Lastly we consider the case when $m_2\equiv 1 \pmod 5$. Put $m_2=5m_3+1$ and $m_3=11m_4+r$, where $0\leqq r\leqq 10$. Then the equation (1) becomes

$$kl=2+a^2+ab+(9+30r)b^2+330m_4b^2. \tag{6}$$

Note that for each value r, there exist a, $b\in\boldsymbol{Z}$ satisfying $b\equiv 0 \pmod{11}$ and the right hand side of (6) is a multiple of 11. For example we can take as follows:

$$(r, a, b) = (0, 0, 1), (1, 6, 3), (2, 0, 5), (3, 1, 8),$$
$$= (4, 1, 6), (5, 0, 2), (6, 2, 5), (7, 4, 2),$$
$$= (8, 1, 1), (9, 0, 4), (10, 0, 3).$$

Then we consider the equation (5): $N(11\zeta-\alpha n)=n(11-2n)$. Putting $11\zeta-\alpha n=x+y\omega$, we see that this equation becomes

$$x^2+xy+(9+30r)y^2+330m_4y^2 = n(11-2n).$$

Clearly n must be odd, hence the right hand side of this equation takes the values 9, 15 and 5. If $r\neq 0$ or $m_4\neq 0$, then $y=0$, $x=\pm 3$. Hence $n=1$ and $11\zeta-\alpha=\pm 3$. Thus we see that $\zeta\notin\mathfrak{o}$ in view of the above list of (r, a, b). If $r=m_4=0$, then take $a=0$ and $b=8$, and let $k=17$. Similarly we infer that the equation $N(17\zeta-\alpha n)=n(17-2n)$ has no solutions. □

Thus we complete the proof of Theorem 3. We note the following.

REMARK 11. In the classification of (1, 2)-polarization in Lemma 7, the singularity of the curve of type (ii) is a node.

PROOF. By the genus formula we infer that the double point is a node or a (simple) cusp. Let $\tilde{C}$ be the normalization of C, then there is a finite unramified covering $\lambda: J(\tilde{C})\to A$ satisfying $\lambda(\tilde{C})=C$, where $J(\tilde{C})$ is the Jacobian variety of $\tilde{C}$. This implies that the singularity cannot be locally irreducible, i.e., it is a node. □

Let C be a smooth curve of genus 3 on an abelian surface B. The complete linear system $|C|$ has four base points. By blowing-up these points, we obtain a morphism $f: S\to \boldsymbol{P}^1$. Let $\omega_{S/\boldsymbol{P}^1}$ be the dualising sheaf of f. Then, since $\deg f_*\omega_{S/\boldsymbol{P}^1}>0$, f is locally non-trivial. Hence Theorem 6 is clear. Note that f has singular fibers, each of which is of type (ii), (iii) or (iv) in Lemma 7. Finally we mention a problem concerning d_r.

PROBLM. We do not know the value $d_r(E\times E)$ when E has no complex multiplications. Moreover we conjecture that $d_r(E_1\times E_2)=4$ if E_1 and E_2 are not isogenous.

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