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On finitary shifts

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1. Introduction.

In [5] A. Heller introduced a class of stochastic processes called finitary processes ($\mathcal{F}$-processes). He proved that every Markov chain and every functional of a Markov chain is finitary. J. B. Robertson ([9]) investigated mixing properties of $\mathcal{F}$-processes. He has shown that every mixing $\mathcal{F}$-process is a Kolmogoroff one. This result has been sharpened by M. Binkowska in [2]. It is proved in [2] that every weakly mixing $\mathcal{F}$-process is weak Bernoulli. An example of a weak Bernoulli process which is not finitary is given in [3].

A concept of a finitary shift ($\mathcal{F}$-shift), i.e., a measure preserving automorphism of a Lebesgue space which has a finite generator forming together with this automorphism an $\mathcal{F}$-process, is introduced in [4]. It follows at once from the above remark that Markov shifts are $\mathcal{F}$-shifts.

An extension of the well known Adler-Shields-Smorodinsky classification of irreducible Markov shifts has been proved in [4]. Namely, it has been shown that two ergodic $\mathcal{F}$-shifts are isomorphic iff they have the same number of eigenvalues and the same entropy.

It would be interesting to extend the classification results of Kubo, Murata and Totoki (cf. [6]) to one-sided finitary shifts.

The main purpose of this paper is to give a probabilistic description of $\mathcal{F}$-processes and to show that the class of $\mathcal{F}$-shifts is closed under factors, products and roots. A characterization of the inverse limits of $\mathcal{F}$-shifts which are $\mathcal{F}$-shifts is also given.

Section 2 contains the definitions and results used in the next sections.

In Section 3 using the Robertson spectral representation we give a probabilistic characterization of $\mathcal{F}$-processes and we discuss it on the class of Bernoulli and Markov processes.

In Section 4 we show that sums and products of $\mathcal{F}$-processes are again $\mathcal{F}$-processes.

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In Section 5 using results of Section 4 we prove that the class of $\mathcal{F}$-shifts is closed under taking sums, products, roots and factors. We give an example showing that in general the class of $\mathcal{F}$-shifts is not closed under taking inverse limits. We also give a necessary and sufficient condition for an inverse limit of $\mathcal{F}$-shifts to be an $\mathcal{F}$-shift.

2. Preliminaries.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, T be a measure preserving automorphism of $(X, \mathcal{B}, \mu)$ and U_T be the unitary operator defined on $L^2(X, \mu)$ by the formula

$$U_T f = f \circ T, \quad f \in L^2(X, \mu).$$

We denote by $\sigma(T)$ the set of eigenvalues of U_T.

Let $P=\{P_i, i \in I\}$ be a finite measurable partition of X. The pair (T, P) will be called a process.

Considering the problem of functionals of Markov chains, Heller introduced in [5] a class of processes, called finitary processes, defined as follows.

A triple (L, q, l_0) is said to be an algebraic representation of a process (T, P), $P=\{P_i, i \in I\}$ if

(H_1) L is an A_I-module where A_I is the free associative algebra over the field $\boldsymbol{R}$ of real numbers generated by I,

(H_2) $l_0 \in L$,

(H_3) $q: L \to \boldsymbol{R}$ is a linear functional with $q(l_0)=1$,

(H_4) $q(\xi(\sigma-\emptyset)l_0)=0$, $\xi \in A_I$, $\sigma=\sum_{i \in I} i$, ϕ is the empty word,

(H_5) $q(i_0, \cdots, i_n l_0)=\mu(P_{i_0} \cap T^{-1}P_{i_1} \cap \cdots \cap T^{-n}P_{i_n})$, $i_0, \cdots, i_n \in I$, $n \geqq 0$.

An algebraic representation (L, q, l_0) is said to be reduced if

(H_6) $A_I l_0 = L$,

(H_7) L does not contain a nonzero submodule L' such that $q(x)=0$, $x \in L'$.

The process (T, P) is said to be finitary ($\mathcal{F}$-process) if $\dim_{\boldsymbol{R}} L < \infty$ where (L, q, l_0) is a reduced algebraic representation of (T, P).

The automorphism T is said to be an $\mathcal{F}$-shift if there exists a finite generator P such that the process (T, P) is an $\mathcal{F}$-process.

In the sequel we shall also use a spectral representation of a process introduced by Robertson in [8].

A triple $(H, e, \{W_i, i \in I\})$ is said to be a spectral representation of (T, P) if

(R_1) H is a complex Hilbert space,

(R_2) $e \in H$, $\|e\|=1$,

(R_3) for every $J \subset I$ the operator $\sum_{i \in J} W_i$ is a contraction on H,

(R_4) $W_T e = W_T^* e = e$ where $W_T = \sum_{i \in I} W_i$,

(R_5) $(W_\alpha e, e) = \mu(P_{i_0} \cap \cdots \cap T^{-n} P_{i_n})$ where $\alpha = (i_0, \cdots, i_n)$, $W_\alpha = W_{i_0} \cdots W_{i_n}$.

A spectral representation $(H, e, \{W_i, i \in I\})$ is called reduced if

(R_6) $H = \bar{S}_P\{W_\alpha e, \alpha \in I^\infty\} = \bar{S}_P\{W_\alpha^* e, \alpha \in I^\infty\}$.

In the sequel we shall make use of the following results (cf. [4])

LEMMA A. *The following conditions are equivalent*

(i) *(T, P) is an $\mathcal{F}$-process,*

(ii) *for every reduced spectral representation $(H, e, \{W_i\}_{i \in I})$ of (T, P) we have* $\dim H < \infty$,

(iii) *there exists a spectral representation $(H, e, \{W_i\}_{i \in I})$ of (T, P) with* $\dim H < \infty$.

Let $Y \in \mathcal{B}$ be a T-invariant set with a positive measure. We consider the measure space $(Y, \mathcal{B}_Y, \mu_Y)$ where $\mathcal{B}_Y = \{A \cap Y\ ;\ A \in \mathcal{B}\}$ and μ_Y is the conditional measure on Y determined by μ. Let $T_Y = T|_Y$ and $P_Y = \{P_i \cap Y\ ;\ i \in I\}$ where $P = \{P_i, i \in I\}$ is a partition of X.

LEMMA B. *If (T, P) is an $\mathcal{F}$-process, then (T_Y, P_Y) is an $\mathcal{F}$-process (in Y).*

LEMMA C. *If T is an ergodic automorphism and the set $\sigma(T)$ is finite, then there exists a natural number n such that*

(i) *T admits an n-tower $\{A, TA, \cdots, T^{n-1}A\}$, i.e. the sets $T^i A$, $0 \leq i \leq n-1$ are pairwise disjoint and $\bigcup_{i=0}^{n-1} T^i A = X$,*

(ii) *the automorphism $T^n|_{T^i A}$, $0 \leq i \leq n-1$ is totally ergodic.*

LEMMA D. *If T is an $\mathcal{F}$-shift, then the set $\sigma(T)$ is finite.*

LEMMA E. *If T is an $\mathcal{F}$-shift, then T^n, $n \geq 1$ is also an $\mathcal{F}$-shift.*

LEMMA F. *Every totally ergodic $\mathcal{F}$-shift is isomorphic to a Bernoulli shift.*

3. Probabilistic characterization of $\mathcal{F}$-processes.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, $P = \{P_i\}_{i \in I}$ a finite measurable partition of X and T a measure preserving automorphism of $(X, \mathcal{B}, \mu)$.

For given natural numbers m, n, $m < n$ we put

$$P[m, n] = \bigvee_{i=m}^{n} T^i P.$$

THEOREM 1. *A process (T, P) is an $\mathcal{F}$-process if there exists a natural number n such that for every $q \geq 1$ and $B \in P[-q, -1]$ there exist numbers $\lambda_C(B)$,*

$C\in P[-n, -1]$ *with the following property*

$$(1) \qquad \forall_{p\geq 0}\forall_{A\in P[0, p]}\mu(B \mid A) = \sum_{C\in P[-n, -1]} \lambda_C(B)\mu(C \mid A).$$

PROOF. Let (T, P) be an $\mathcal{F}$-process and let $(H, e, \{V_i\}_{i\in I})$ be its reduced spectral representation. It follows from Lemma A that $\dim H<\infty$.

Let $\{V_{\alpha_1}e, \cdots, V_{\alpha_s}e\}$ be a basis of H and let $n=n(T, P)$ be the smallest natural number such that $\alpha_i\in I^n$, $i=1, \cdots, s$. The existence of such n follows at once from the equality

$$(2) \qquad V_{\alpha_i}e = V_{\alpha_i}\circ V_T^m e = \sum_{j_1,\cdots,j_m\in I} V_{\alpha_i j_1,\cdots,j_m}e, \quad i=1, 2, \cdots, s, \ m\geqq 1.$$

Let q be an arbitrary natural number and let $B\in P[-q, -1]$, $B=T^{-1}P_{j_1}\cap \cdots \cap T^{-q}P_{j_q}$.

Let $\alpha=(j_1, \cdots, j_q)$ and let $a_1, \cdots, a_s\in \boldsymbol{R}$ be such that

$$(3) \qquad V_\alpha e = a_1 V_{\alpha_1}e+ \cdots +a_s V_{\alpha_s}e.$$

Let p be an arbitrary natural number and let $A\in P[0, p]$, $A=P_{i_0}\cap TP_{i_1}\cap \cdots \cap T^pP_{i_p}$.

It follows from (2) that

$$(4) \qquad \begin{aligned} \mu(A\cap B) &= \mu(T^{-p}(A\cap B)) \\ &= \mu(P_{i_p}\cap T^{-1}P_{i_{p-1}}\cap \cdots \cap T^{-p}P_{i_0}\cap T^{-p-1}P_{j_1}\cap \cdots \cap T^{-p-q}P_{j_q}) \\ &= (V_{i_p\cdots i_0 j_1\cdots j_q}e, e) = (V_{i_p\cdots i_0}V_\alpha e, e) = \sum_{i=1}^s a_i(V_{i_p\cdots i_0}V_{\alpha_i}e, e). \end{aligned}$$

Let $\alpha_i=(k_{1,i}, \cdots, k_{n,i})\in I^n$, $1\leqq i\leqq s$. Introducing the notation

$$C_i = T^{-1}P_{k_{1,i}}\cap \cdots \cap T^{-n}P_{k_{n,i}}, \quad i=1, \cdots, .$$

we may write (3) in the form

$$\begin{aligned} \mu(A\cap B) &= \textstyle\sum_{i=1}^s a_i\mu(T^{-p}(A\cap C_i)) \\ &= \textstyle\sum_{i=1}^s a_i\mu(A\cap C_i). \end{aligned}$$

Putting

$$\lambda_C(B) = \begin{cases} a_i, & \text{if } C = C_i,\ 1\leqq i\leqq s \\ 0, & \text{if } C\cap\bigcup_{i=1}^s C_i = \emptyset \end{cases}$$

we get

$$\mu(A\cap B) = \sum_{C\in P[-n, -1]} \lambda_C(B)\cdot\mu(A\cap C)$$

which proves the necessity.

In order to prove the sufficiency let us suppose that there exists a number n for which the property (1) is satisfied.

Let $(H, e, \{W_i\}_{i\in I})$ be a reduced representation of the process (T, P). We shall show that $\dim H<\infty$.

Let $I^n=\{\alpha_1, \cdots, \alpha_s\}$. We shall prove that the vectors $W_{\alpha_1}e, \cdots, W_{\alpha_s}e$ form a basis of H.

Let $\alpha, \beta\in I^\infty$, $\beta=(j_1, \cdots, j_q)$ be arbitrary and $\alpha=(i_0, \cdots, i_p)\in I^{p+1}$, $p\geqq 1$. We put

$$A = P_{i_0}\cap TP_{i_1}\cap \cdots \cap T^pP_{i_p},$$

$$B = T^{-1}P_{j_1}\cap T^{-2}P_{j_2}\cap \cdots \cap T^{-q}P_{j_q}.$$

$$(W_\alpha W_\beta e, e) = \mu(A\cap B) = \sum_{i=1}^{s}\lambda_i\mu(A\cap C_i) \tag{5}$$

$$= \sum_{i=1}^{s}\lambda_i(W_\alpha W_{\alpha_i}e, e)$$

where $C_i=T^{-1}P_{k_{1,i}}\cap \cdots \cap T^{-n}P_{k_{n,i}}$ if $\alpha_i=(k_{1,i}, \cdots, k_{n,i})$ and $\lambda_i=\lambda_{C_i}(B)$, $i=1, 2, \cdots, s$.

We may rewrite (5) in the form

$$(W_\beta e, W_\alpha^* e) = \left(\sum_{i=1}^{s}\lambda_i W_{\alpha_i}e, W_\alpha^* e\right), \quad \alpha, \beta \in I^\infty.$$

Since our spectral representation is reduced we have

$$W_\beta e = \sum_{i=1}^{s}\lambda_i W_{\alpha_i}e, \quad \beta \in I^\infty.$$

Using again the reducity of the representation we see that the vectors form a basis of H, i.e. $\dim H<\infty$. □

COROLLARY 1. *A process (T, P) is an $\mathcal{F}$-process iff there exist natural numbers n, q_0 such that for every $q\geqq q_0$, $B\in P[-q, -1]$ there exist numbers $\lambda_C(B)$, $C\in P[-n, -1]$ with the following property*:

$$\forall_{p\geqq 0}\forall_{A\in P[0, p]}\mu(B \mid A) = \sum_{C\in P[-n, -1]}\lambda_C(B)\mu(C \mid A) \tag{6}$$

PROOF. It is enough to show the sufficiency. Let n and q_0 satisfy (6) and let $q<q_0$. For every $B\in P[-q, -1]$ there exist sets $B_j\in P[-q_0, -1]$, $1\leqq j\leqq s$ such that $B=\bigcup_{j=1}^{s} B_j$.

Let $p\geqq 0$ and $A\in P[0, p]$ be arbitrary. It follows from (6) that

$$\mu(A\cap B_j) = \sum_{C\in P[-n, -1]}\lambda_C(B_j)\mu(A\cap C).$$

Putting

$$\lambda_C(B)=\sum_{j=1}^{s}\lambda_C(B_j)$$

we get

$$\mu(A\cap B)=\sum_{C\in P[-n,-1]}\lambda_C(B)\mu(A\cap C),$$

i.e. the number n satisfies (1). □

EXAMPLE 1. Let (T, P) be a Bernoulli process on $(X, \mathcal{B}, \mu)$, $n=n(P, T)=q_0=1$ and $\lambda_C(B)=\mu(B)$, $B\in P[-q, -1]$, $C\in T^{-1}P$, $q\geqq 1$. Since every $A\in P[0, p]$, $p\geqq 0$ is independent of B, the condition (1) is of course satisfied, i.e. (T, P) is an $\mathcal{F}$-process.

EXAMPLE 2. Let (T, P) be an r-fold Markov process $r\geqq 1$ on $(X, \mathcal{B}, \mu)$, i.e.

$$\mu(A \mid B)=\mu(A \mid \tilde{B}), \tag{7}$$

$$A\in P[0, m],\quad B\in P[-l-r, -1],\quad \tilde{B}\in P[-r, -1],$$
$$B\subset\tilde{B},\quad m\geqq 0,\ l\geqq 0.$$

We put $n(P, T)=q_0=r$. For $B\in P[-q, -1]$, $q\geqq r$ we fix $C_0\in P[-r, -1]$ with $B\subset C_0$ and we define

$$\lambda_C(B)=\begin{cases}\dfrac{\mu(B)}{\mu(C_0)}; & C=C_0,\\ 0\quad ; & C\neq C_0.\end{cases}$$

The condition (1) easily follows from (7).

4. Algebraic properties of finitary processes.

In this section we shall show that sums and products of $\mathcal{F}$-processes are $\mathcal{F}$-processes.

Let (T, P) and (S, Q), $P=\{P_1, \cdots, P_m\}$, $Q=\{Q_1, \cdots, Q_n\}$ be processes on Lebesgue probability spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ respectively, where $X\cap Y=\emptyset$. Let $s, t\in(0, 1)$ be such that $s+t=1$.

We denote by $(Z, \mathcal{C}, \lambda)$ the sum of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, i.e. $Z=X\cup Y$, $A\in\mathcal{C}$ iff $A\cap X\in\mathcal{A}$, $A\cap Y\in\mathcal{B}$ and $\lambda=s\mu+t\nu$.

We equip Z with the automorphism U defined by $U|_X=T$, $U|_Y=S$ and the partition $R=\{P_1, \cdots, P_m, Q_1, \cdots, Q_n\}$.

The process (U, R) is called the sum of processes (T, P) and (Q, S), and is denoted by $(P, T)\oplus(Q, S)$.

THEOREM 2. *The process $(P, T)\oplus(Q, S)$ is an $\mathcal{F}$-process iff (P, T) and (Q, S) are $\mathcal{F}$-processes.*

PROOF. The necessity follows at once from Lemma B.

In order to prove the sufficiency we consider spectral representations $(H_1, e_1, \{W_i^1\}_{1\le i\le m})$ and $(H_2, e_2, \{W_i^2\}_{1\le i\le n})$ of (P, T) and (Q, S) respectively such that $\dim H_i<\infty$, $i=1, 2$.

In the view of Lemma A it is enough to construct a spectral representation $(H, e, \{W_i\}_{1\le i\le m+n})$ of $(P, T)\oplus(Q, S)$ with $\dim H<\infty$.

Let $(H, (\cdot, \cdot))$ be the product of spaces $(H_i, (\cdot, \cdot)_i)$ where $(\cdot, \cdot)_i$ denotes the inner product in H_i, $i=1, 2$, i.e. $H=H_1\times H_2$,

$$(h, \bar{h}) = s\cdot(h_1, \bar{h}_1)_1+t\cdot(h_2, \bar{h}_2)_2,$$

$h=(h_1, h_2)$, $\bar{h}=(\bar{h}_1, \bar{h}_2)$.

It is clear that the dimension of the Hilbert space H is finite.

We put $e=(e_1, e_2)$ and we define $W_i: H\to H$ as follows

$$W_1(h) = (W_1^1(h_1), 0), \cdots, W_m(h) = (W_m^1(h_1), 0),$$

$$W_{m+1}(h) = (0, W_1^2(h_2)), \cdots, W_{m+n}(h) = (0, W_n^2(h_2)).$$

It follows at once from the definition of the inner product in H that $\|e\|=1$. For every set $I\subset\{1, 2, \cdots, n+m\}$ the operator $\sum_{i\in I}W_i$ is a contraction because the operators W_i^1 and W_j^2, $1\leqq i\leqq m$, $1\leqq j\leqq n$ have, by the assumption, the same property.

Since $W_T^1e_1=e_1$, $W_S^2e_2=e_2$ we get

$$\begin{aligned} W_Ue &= \sum_{i=1}^{m+n} W_ie = \sum_{i=1}^{m}(W_i^1(e_1), 0)+\sum_{i=1}^{n}(0, W_i^2(e_2)) \\ &= (W_T^1e_1, 0)+(0, W_S^2e_2) = e \end{aligned}$$

Similarly $W_U^*e=e$.

Now, let $\alpha=(i_0, i_1, \cdots, i_s)\in I^\infty$. Let us observe that

$$W_\alpha e = W_{i_0}W_{i_1}\cdots W_{i_s}e$$

is equal to

$$\begin{array}{ll} (W_{i_0}^1W_{i_1}^1\cdots W_{i_s}^1e_1, 0) & \text{if all } i_k\in\{1, \cdots, m\}, \\ (0, W_{i_0}^2W_{i_1}^2\cdots W_{i_s}^2 0) & \text{if all } i_k\in\{m+1, \cdots, n\},\ 1\leqq k\leqq s, \\ (0, 0) & \text{otherwise.} \end{array}$$

Similarly

$$\lambda(R_{i_0}\cap U^{-1}R_{i_1}\cap\cdots\cap U^{-s}R_{i_s})$$

is equal to

$$\begin{array}{ll} s\mu(P_{i_0}\cap T^{-1}P_{i_1}\cap\cdots\cap T^{-s}P_{i_s}) & \text{if all } i_k\in\{1,\cdots,m\}, \\ t\nu(Q_{i_0}\cap S^{-1}Q_{i_1}\cap\cdots\cap S^{-s}Q_{i_s}) & \text{if all } i_k\in\{m+1,\cdots,m+n\}_{1\le k\le s}, \\ 0 & \text{otherwise.} \end{array}$$

Therefore we get

$$(W_\alpha e, e)=\lambda(R_{i_0}\cap U^{-1}R_{i_1}\cap\cdots\cap U^{-s}R_{i_s}).$$

i.e. the triple $(H, e, \{W_i\}_{1\le i\le n+m})$ is a spectral representation of $(P, T)\oplus(Q, S)$ with $\dim H<\infty$, which completes the proof. □

One can easily extend Theorem 2 to the case of an arbitrary finite number of processes.

In order to consider other operations on $\mathcal{F}$-processes we introduce the concept of an euclidean representation of a process.

Let (T, P), $P=\{P_i, i\in I\}$ be a process on a Lebesgue space $(X, \mathcal{B}, \mu)$.

DEFINITION. An euclidean representation of (T, P) is a triple $(r, e, \{A_i\}_{i\in I})$ where $e\in\boldsymbol{R}^s$, $r:\boldsymbol{R}^s\to\boldsymbol{R}$, $A_i:\boldsymbol{R}^s\to\boldsymbol{R}^s$ are linear transformations, $s\geqq1$, $i\in I$ such that

(i) $r(e)=1$,

(ii) $\mu(P_{i_0}\cap T^{-1}P_{i_1}\cap\cdots\cap T^{-k}P_{i_k})=r(A_{i_0}A_{i_1}\cdots A_{i_k}e)$, $i_s\in I$, $0\leqq s\leqq k$, $k\geqq0$.

PROPOSITION. *A process (T, P) is finitary iff it has an euclidean representation.*

PROOF. If (T, P) is finitary it has a Heller algebraic representation (L, q, l_0) with $\dim_{\boldsymbol{R}} L<\infty$. Let $\dim_{\boldsymbol{R}} L=s$ and let $\varphi: L\to\boldsymbol{R}^s$ be a linear isomorphism.

We put

$$r=q_0\varphi^{-1},\quad e=\varphi(l_0),\quad A_i x=\varphi(i\varphi^{-1}(x)),\quad i\in I,\ x\in\boldsymbol{R}^s.$$

It is clear that the triple $(r, e, \{A_i\}_{i\in I})$ is an euclidean representation.

Conversely, let $(r, e, \{A_i\}_{i\in I})$ be an euclidean representation of (T, P). Let $L=\boldsymbol{R}^s$, $l_0=e$ and $q(x)=r(x)$, $x\in L$. For every $i_s\in I$, $0\leqq s\leqq k$ we define

$$i_0 i_1\cdots i_k x=A_{i_0}A_{i_k}x,\quad x\in\boldsymbol{R}^s.$$

It is easy to see that (L, q, l_0) is an algebraic representation of (T, P). □

Let (T, P), $P=\{P_i\}_{i\in I}$ and (S, Q), $Q=\{Q_j\}_{j\in J}$ be processes on Lebesgue spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ respectively.

We consider the partition $R=\{R_{ij}\}$ of $X\times Y$ defined as follows

$$R_{ij}=P_i\times Q_j,\quad i\in I,\ j\in J.$$

The process $(R, T\times S)$ is called the product of the processes (T, P) and (S, Q) and is denoted by $(T, P)\times(S, Q)$.

THEOREM 3. *If (T, P) and (S, Q) are $\mathcal{F}$-processes then their product $(T, P)\times(S, Q)$ is also an $\mathcal{F}$-process.*

PROOF. Let $(r, e, \{A_i\}_{i\in I})$ and $(\bar{r}, \bar{e}, \{\bar{A}_j\}_{j\in J})$ be euclidean representations of (T, P) and (S, Q), respectively, and let $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ be the corresponding euclidean spaces.

We denote by $\{e_1, \cdots, e_n\}$ and $\{\bar{e}_1, \cdots, \bar{e}_m\}$ the standard basis of $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$, respectively. Let $\{e_{11}, e_{12}, \cdots, e_{1m}, e_{21}, \cdots, e_{2m}, \cdots, e_{n1}, \cdots, e_{nm}\}$ be the standard basis in $R^{n\cdot m}$.

Let

$$e=\sum_{i=1}^{n}\alpha_i e_i \quad\text{and}\quad \bar{e}=\sum_{j=1}^{m}\beta_j\bar{e}_j .$$

We put

$$\bar{\bar{e}}=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j e_{ij} .$$

Next we define linear transformations

$$\bar{\bar{r}}:\boldsymbol{R}^{mn}\longrightarrow\boldsymbol{R}\quad\text{and}\quad \bar{\bar{A}}_{ij}:\boldsymbol{R}^{mn}\longrightarrow\boldsymbol{R}^{mn},\quad i\in I,\ j\in J$$

by the formulas

$$\bar{\bar{r}}(e_{st})=r(e_s)\bar{r}(\bar{e}_t),\quad s\in I,\ t\in J \tag{8}$$

$$\bar{\bar{A}}_{ij}(e_{st})=\sum_{p=1}^{n}\sum_{q=1}^{m}\gamma_{ps}\beta_{qt}e_{pq} \tag{9}$$

where

$$A_i(e_s)=\sum_{p=1}^{n}\gamma_{ps}e_p,\quad \bar{A}_j(\bar{e}_t)=\sum_{q=1}^{m}\beta_{qt}\bar{e}_q,\quad i\in I,\ j\in J.$$

We shall prove that the triple $(\bar{\bar{r}}, \bar{\bar{e}}, \{A_{ij}\}_{i\in I, j\in J})$ is an $n\cdot m$-dimensional euclidean representation of $(T, P)\times(S, Q)$.

It is clear that

$$\bar{\bar{r}}(\bar{\bar{e}})=1 .$$

It follows from (9), by induction, that if

$$A_{i_1}\cdots A_{i_s}e=\sum_{i=1}^{n}\alpha_i e_i \quad\text{and}\quad \bar{A}_{j_1}\cdots\bar{A}_{j_s}\bar{e}=\sum_{j=1}^{m}\beta_j\bar{e}_j$$

then

$$\bar{\bar{A}}_{i_1j_1}\cdots\bar{\bar{A}}_{i_sj_s}(\bar{\bar{e}})=\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\beta_j e_{ij} .$$

Hence, by the assumption

$$\bar{\bar{r}}(\bar{\bar{A}}_{i_1} \cdots \bar{\bar{A}}_{i_s j_s} e\bar{e}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j r(e_i) \cdot \bar{r}(\bar{e}_j) = r(A_{i_1} \cdots A_{i_s} e) \cdot r(\bar{A}_{j_1} \cdots \bar{A}_{j_s} \bar{e})$$
$$= \mu(P_{i_1} \cap T^{-1} P_{i_2} \cap \cdots \cap T^{-s+1} P_{i_s}) \cdot \nu(Q_{j_1} \cap S^{-1} Q_{j_2} \cap \cdots \cap S^{-s+1} Q_{j_s})$$
$$= (\mu \times \nu)(R_{i_1 j_1} \cap (T \times S)^{-1} R_{i_2 j_2} \cap \cdots \cap (T \times S)^{-s+1} R_{i_s j_s}).$$

This means that $(\bar{\bar{r}}, \bar{\bar{e}}, \{\bar{\bar{A}}_{ij}\}_{i \in I, j \in J})$ is an euclidean representation of $(T, P) \times (S, Q)$. □

5. Applications to finitary shifts.

We have proved in [4] that the class of $\mathcal{F}$-shifts is closed under taking powers and passing to subsystems. In this section we shall show that this class is closed under taking sums, products, roots and factors. We also consider inverse limits of $\mathcal{F}$-shifts.

An immediate consequence of Theorems 2 and 3 is the following

REMARK. The class of $\mathcal{F}$-shifts is closed under taking sums and products.

Let T be an automorphism on a Lebesgue space $(X, \mathcal{B}, \mu)$.

THEOREM 4. *If T^n, $n \geqq 1$ is an $\mathcal{F}$-shift, then T is also an $\mathcal{F}$-shift.*

PROOF. First suppose T is ergodic. It follows from Lemma C that T admits a k-tower $\{A, TA, \cdots, T^{k-1}A\}$ such that $T^k|_{T^i A}$ is totally ergodic, $0 \leqq i \leqq k-1$. Hence the automorphism $T^{nk}|_{T^i A}$ is also totally ergodic. Since T^n is an $\mathcal{F}$-shift Lemmas E and B imply $T^{nk}|_{T^i A}$ is also an $\mathcal{F}$ shift, $0 \leqq i \leqq k-1$. By Lemma F the automorphism $T^{nk}|_{T^i A}$ is isomorphic to a Bernoulli shift, $0 \leqq i \leqq k-1$. It follows from Corollary to Theorem 13 ([7]) that the automorphism $T^k|_{T^i A}$ as a root of $T^{nk}|_{T^i A}$ is also isomorphic to a Bernoulli shift.

Now, arguing similarly as in the proof of Theorem ([1]) we may show that T is isomorphic to an automorphism which is a product of a rotation on the group Z_k and a Bernoulli shift. By Remark the automorphism T is an $\mathcal{F}$-shift.

Let us now suppose T is an arbitrary automorphism of $(X, \mathcal{B}, \mu)$. Since T^n is an $\mathcal{F}$-shift the set $\sigma(T^n)$ is finite. Hence the set $\sigma(T)$ is also finite. Therefore the partition of X on ergodic components with respect to T is finite. We denote it by $\{X_1, \cdots, X_s\}$.

It follows from the first part of the proof that the automorphisms T_{X_i}, $1 \leqq i \leqq s$ are $\mathcal{F}$-shifts. Applying Theorem 2 we see that T is an $\mathcal{F}$-shift, which finishes the proof. □

LEMMA. *If T is an $\mathcal{F}$-shift, $\{X_1, \cdots, X_m\}$ is a partition of X on ergodic components with respect to T and S is an ergodic factor of T, then S is also a* *factor of* $T|_{X_i}$, $1 \leqq i \leqq m$.

PROOF. Suppose S acts on a Lebesgue space $(Y, \mathcal{C}, \nu)$. Let $\varphi: X \to Y$ be a measure preserving transformation such that $\varphi \circ T = S \circ \varphi$.

Let $1 \leqq i \leqq m$ be fixed. We define a measure ν_i on $\mathcal{C}$ by the formula

$$\nu_i(B) = \mu(\varphi^{-1}(B) \mid X_i), \quad B \in \mathcal{C}.$$

Since $TX_i = X_i$ the measure ν_i is S-invariant. The obvious equality

$$\nu = \sum_{k=1}^{m} \mu(X_k)\nu_k$$

implies $\nu_i \ll \nu$. The ergodicity of S implies that $\nu_i = \nu$, $1 \leqq i \leqq m$. This means that S is a factor of $T|_{X_i}$, $1 \leqq i \leqq m$.

THEOREM 5. *The class of $\mathcal{F}$-shifts is closed under taking factors.*

PROOF. Let T be an $\mathcal{F}$-shift and let S be a factor of T acting on Lebesgue spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$, respectively.

Let $\{Y_1, \cdots, Y_k\}$ be the partition of Y on ergodic components with respect to S and let $\varphi: X \to Y$ be a measure preserving transformation such that $\varphi \circ T = S \circ \varphi$.

For $1 \leqq i \leqq k$ we put $X_i = \varphi^{-1}(Y_i)$. It is clear that the elements of the partition $\{X_1, \cdots, X_k\}$ are T-invariant sets.

Equipping X_i and Y_i with the conditional measures μ_i and ν_i, respectively, one easily verifies that the automorphism $S_i = S|_{Y_i}$ is a factor of $T_i = T|_{X_i}$ given by the transformation $\varphi_i = \varphi|_{X_i}$, $1 \leqq i \leqq k$.

Let $1 \leqq i \leqq k$ be fixed. It follows from the assumption and Lemma B that T_i is an $\mathcal{F}$-shift. Let $\{X_1, \cdots, X_{il_i}\}$ be the partition of X_i on ergodic components with respect to T_i.

Let μ_{ij} be the conditional measure on X_{ij} defined by μ_i and let $T_{ij} = T_i|_{X_j}$, $1 \leqq j \leqq l_i$.

It follows from Lemma that for every $1 \leqq j \leqq l_i$ the automorphism S_i is a factor of T_{ij}. Applying Lemma C to T_{ij} and S_i we see that there exists a measurable set $B \subset Y_i$ and a natural number $m = m_i$ such that the sets B, $S_i B, \cdots, S_i^{m-1}B$ form a partition of Y_i and the automorphism $S_i^m|_{S_i^p B}$ is totally ergodic, $0 \leqq p \leqq m-1$.

It follows from Corollary 4 ([4]) that $S_i^m|_{S_i^p B}$ is a Bernoulli shift and so is an $\mathcal{F}$-shift, $0 \leqq p \leqq m-1$. Using Theorem 4 and Theorem 2 we see that S is an $\mathcal{F}$-shift. □

The well known Ornstein theorem says that the class of Bernoulli shifts is closed under taking inverse limits. It is a natural question whether this property holds for $\mathcal{F}$-shifts.

The following example shows that this question has a negative answer.

EXAMPLE 3. Let X_1 be the group of 2-adic numbers, $\mathcal{B}_1$ the σ-algebra of Borel sets of X_1 equipped with the Haar measure μ_1 and let $T_1: X_1 \to X_1$ be the adding machine, i.e.

$$T_1 x = x + e, \quad x \in X_1$$

where $e=(1, 0, \cdots)$.

It is well known that T_1 is ergodic and has a discrete spectrum.

For a fixed natural number k and $i_0, i_1, \cdots, i_k \in \{0, 1\}$ we put

$$C(i_0, \cdots, i_k) = \{x \in X;\ x_0 = i_0, \cdots, x_k = i_k\}.$$

Let C_k be the algebra generated by the partition

$$\{C(i_0, i_1, \cdots, i_k);\ i_0, i_1, \cdots, i_k \in \{0, 1\}\}.$$

We have

$$T_1 C_k = C_k, \quad k \geqq 1,\ C_k \nearrow \mathcal{B}_1. \tag{10}$$

Let $(X_2, \mathcal{B}_2, \mu_2, T_2)$ be a Bernoulli shift and let $(X, \mathcal{B}, \mu, T)$ be the direct product of $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i=1, 2$. It is clear that T is ergodic and has a positive entropy.

Let $\mathcal{A}_k = C_k \otimes \mathcal{B}_2$, $k \geqq 1$. It follows at once from (10) that

$$T\mathcal{A}_k = \mathcal{A}_k, \quad k \geqq 1,\ \mathcal{A}_k \nearrow \mathcal{B}.$$

It is easy to see that for every $k \geqq 1$ the factor automorphism $T_{\mathcal{A}_k}$ is isomorphic to the product of the rotation on the group $Z_{2^{k+1}}$ and the Bernoulli shift. Therefore, by Remark, it is an $\mathcal{F}$-shift.

On the other hand the set $\sigma(T)$ is infinite because it contains the set $\sigma(T_1)$ which is infinite. In view of Lemma D T is not an $\mathcal{F}$-shift.

Let now T be an automorphism of a Lebesgue space $(X, \mathcal{B}, \mu)$.

THEOREM 6. *Let $(\mathcal{A}_n)$ be a sequence of σ-algebras such that $T\mathcal{A}_n = \mathcal{A}_n$ and let every factor automorphism $T_{\mathcal{A}_n}$ be an $\mathcal{F}$-shift, $n \geqq 1$ and $\mathcal{A}_n \nearrow \mathcal{B}$.*

T is an $\mathcal{F}$-shift iff the set $\sigma(T)$ is finite and every $\lambda \in \sigma(T)$ has a finite multiplicity.

PROOF. In view of Lemma D it is enough to prove the sufficiency.

First we consider the case when T is ergodic. Since $\sigma(T)$ is finite there exists a natural number n_0 such that

$$\sigma(T_{\mathcal{A}_n}) = \sigma(T_{\mathcal{A}_{n_0}}), \quad n \geqq n_0.$$

Let $n \geqq n_0$ be fixed. In view of Lemma C there exists a natural number k and a set $A \in \mathcal{A}_n$ such that the sets $T^i A$, $i=0, 1, \cdots, k-1$ are pairwise disjoint, $\bigcup_{i=0}^{k-1} T^i A = X$ and $T^k|_{T^i A}$, $i=0, 1, \cdots, k-1$ is totally ergodic.

Let us observe that due to the ergodicity of T all towers $\{A, TA, \cdots, T^{k-1}A\}$ are the same (mod μ). Hence for every set $T^i A$, $0 \leqq i \leqq k-1$ the automorphism $T^k|_{T^i A}$ is an inverse limit of $T^k_{\mathcal{A}_n}|_{T^i A}$, $n \geqq 1$ and $T^k_{\mathcal{A}_n}|_{T^i A}$ is a totally ergodic $\mathcal{F}$-shift. It follows from [4] that $T^k_{\mathcal{A}_n}|_{T^i A}$ is Bernoulli and so $T^k|_{T^i A}$ is Bernoulli by [7].

Adapting some arguments from the proof of Theorem 2 in [1] we see that T is isomorphic to the product of a rotation on the group Z_k and a Bernoulli shift. Therefore, by Remark, T is an $\mathcal{F}$-shift.

Now, let us suppose T is an arbitrary $\mathcal{F}$-shift. It follows from Lemma D that the partition of X on ergodic components with respect to T is finite. Hence our assumption implies that there exists n_0 such that for all $n \geqq n_0$ the partition of X on ergodic components with respect to the factor automorphism $T_{\mathcal{A}_n}$ is the same as this for $T_{\mathcal{A}_{n_0}}$. We denote it by $\{X_1, X_2, \cdots, X_k\}$. It is clear that for every $1 \leqq i \leqq k$ the automorphism $T|_{X_i}$ is an inverse limit of $T_{\mathcal{A}_n}|_{X_i}$. Since $T_{A_n}|_{X_i}$ is ergodic for $n \geqq n_0$, $T|_{X_i}$ is also ergodic. It follows from the first part of the proof that $T|_{X_i}$ is an $\mathcal{F}$-shift and so Theorem 2 implies T is an $\mathcal{F}$-shift. □

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