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K-spherical representations for Gelfand pairs associated to solvable Lie groups

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Introduction.

Let S be a connected and simply connected unimodular solvable Lie group, K a connected compact group acting on S as automorphisms. We call the pair $(K;S)$ a Gelfand pair if the Banach $*$-algebra $L^1(K\backslash K\ltimes S/K)$ of all K-biinvariant integrable functions on $K\ltimes S$ is commutative. The assumption that $(K;S)$ is a Gelfand pair prescribes the structure of S. For example, if $(K;S)$ is a Gelfand pair, then S is of type R ([**BJR**], Corollary 7.4) and thus S has polynomial growth ([**J**], Theorem 1.4). In this paper we first show that the nilradical N of S splits in S if $(K;S)$ is a Gelfand pair. Let $\mathfrak{s}$ be the Lie algebra of S.

THEOREM A. *If* $(K;S)$ *is a Gelfand pair, then there exists a K-invariant abelian subspace* $\mathfrak{a}$ *of* $\mathfrak{s}$ *on which* K *acts trivially. Moreover putting* $A=\exp\mathfrak{a}$, *one has* $S=A\ltimes N$ *and* $K\ltimes S=(K\times A)\ltimes N$.

Suppose that $(K;S)$ is a Gelfand pair. Since S has polynomial growth, the Banach $*$-algebra $L^1(S)$ is symmetric ([**L**], Lemma 1). This fact tells us that all bounded K-spherical functions on S are positive definite (cf. [**BJR**], Lemma 8.2). Thus to each bounded K-spherical function on S there corresponds an irreducible K-spherical representation of $K\ltimes S$ (cf. [**H**], Chapter IV). Let $\hat{N}$ be the unitary dual of N and K_π the stabilizer of $\pi\in\hat{N}$ in K. As an immediate consequence of Theorem A, we see that bounded K-spherical functions ϕ on S are parametrized as $\phi_{\pi,\alpha,a}$ with $(\pi,\alpha,a)\in\hat{N}/K\times\hat{K}_\pi\times\mathfrak{a}^*$. This parametrization improves that of [**BJR**], Theorem 8.11 a little in the sense that we make an explicit use of the subgroup $A=\exp\mathfrak{a}$.

Our second purpose of this paper is to realize the irreducible K-spherical representations $\tilde{U}_{\pi,\alpha,a}$ of $K\ltimes S$ by induction using the structure of $K\ltimes S=(K\times A)\ltimes N$, to which the parameters (π,α,a) are closely related. Though S need not be of type I, Theorem A makes it possible to get all irreducible unitary representations of $K\ltimes S$ from those of the nilradical N which is CCR. To carry out this, we need to know that N is regularly imbedded in $K\ltimes S$ and what is the structure of the stabilizers like. Our Proposition 3.2 says that the $(K\times A)$-orbit space $\hat{N}/(K\times A)$ is equal to the K-orbit space $\hat{N}/K$ as Borel spaces, which ensures the regularity of the imbedding of N in $K\ltimes S$. We denote by $(K\times A)_\pi$ the stabilizer of $\pi\in\hat{N}$ in $K\times A$. Making use of the theory of maximally almost periodic groups, we obtain

PROPOSITION B. *There exists a vector group V isomorphic to A such that $(K\times A)_\pi=K_\pi\times V$, the direct product of K_π and V.*

With these preparations it is straightforward to construct irreducible K-spherical representations of $K\ltimes S$. Continuing to suppose that $(K;S)$ is a Gelfand pair, we know that $(K;N)$ is also a Gelfand pair. Let $\pi\in\hat{N}$. We have a multiplicity-free decomposition $H_\pi=\bigoplus_\alpha V_\alpha$ of the representation space H_π of π under the intertwining representation W_π of K_π by [C]. For the intertwining representation $\widetilde{W}_\pi$ of $(K\times A)_\pi$, there exists $a_\alpha\in\mathfrak{a}^*$ such that $\widetilde{W}_\pi(k_x, x)|_{V_\alpha}$ is a scalar operator $\chi_{a_\alpha}(x)$ for every $(k_x, x)\in V$, where $\chi_a(\exp X)=e^{\sqrt{-1}\langle a, X\rangle}$ for $a\in\mathfrak{a}^*$, $X\in\mathfrak{a}$. Here it should be noted that we may assume $W_\pi=\widetilde{W}_\pi|_{K_\pi}$ (Proposition 5.1). Putting $T_\alpha=W_\pi|_{V_\alpha}$, we have an irreducible unitary representation $U_{\pi,\alpha,a}$ $(a\in\mathfrak{a}^*)$ of $(K\times A)_\pi\ltimes N$ given by

$$U_{\pi,\alpha,a}(kk_x, x, n)=\chi_{a-a_\alpha}(x)\overline{T}_\alpha(k)\otimes\pi(n)\widetilde{W}_\pi(kk_x, x),$$

where $k\in K_\pi$, $(k_x, x)\in V$ and $n\in N$. Combining the above with parametrization of K-spherical functions on S, we obtain

THEOREM C. *The irreducible K-spherical representation of $K\ltimes S$ corresponding to $\phi=\phi_{\pi,\alpha,a}$ is given by $\tilde{U}_{\pi,\alpha,a}=\mathrm{Ind}_{(K\times A)_\pi\ltimes N}^{(K\times A)\ltimes N}U_{\pi,\alpha,a}$.*

We will make a further observation on the particular case where the following condition (C) is satisfied:

(C) there exists a continuous homomorphism $\varphi: A\rightarrow K$ such that $x\cdot n=\varphi(x)\cdot n$ for all $x\in A$ and $n\in N$.

In this case the irreducible K-spherical representations $\tilde{U}_{\pi,\alpha,a}$ are canonically constructed. In fact putting $V_0=\{(\varphi(x)^{-1}, x)\,|\,x\in A\}$, we have $K\ltimes S=(K\ltimes N)\times V_0$.

PROPOSITION D. *One has $\tilde{U}_{\pi,\alpha,a}(k\varphi(x)^{-1}, x, n)=\tilde{U}_{\pi,\alpha}(k, n)\otimes\chi_a(x)$, where $\tilde{U}_{\pi,\alpha}$ is the irreducible K-spherical representation of $K\ltimes N$ corresponding to the K-spherical function $\phi_{\pi,\alpha,a}|_N$.*

We conclude this paper by giving three examples of Gelfand pairs. The first is the pair $(\boldsymbol{T}^2; S)$, where S is the Mautner group, a simple example of non-type I solvable Lie group. This pair satisfies the condition (C). The second is the pair $(\mathrm{U}(2); \boldsymbol{R}\ltimes(H_2\times\boldsymbol{C}))$, where H_2 is the Heisenberg Lie group homeomorphic to $\boldsymbol{C}^2\times\boldsymbol{R}$. The group U(2) acts on $\boldsymbol{C}^2$ naturally and on $\boldsymbol{C}$ by scalar multiplications of the square of determinants. This pair satisfies the condition

(C) but the stabilizer of some representation is not connected. The last is the pair $(\mathrm{SU}(2); \boldsymbol{R}\ltimes H_2)$, where the action of SU(2) on H_2 is as in the second example. This pair does not satisfy the condition (C).

1. Preliminaries.

Let S be a connected and simply connected unimodular solvable Lie group with Haar measure $d\mu$, and K a connected compact group acting on S as automorphisms. By taking a factor group if necessary, we may assume that K is a connected compact Lie group. A bounded continuous function ϕ on S is called a K-spherical function if

$$\tag{1.1} \begin{aligned} &\int_K \phi(x(k\cdot y))dk = \phi(x)\phi(y) \quad \text{for } x,\, y \in S, \\ &\phi(1_S) = 1, \end{aligned}$$

where dk is the normalized Haar measure on K and 1_S the unit element of S. The Banach space $L^1(S)$ of integrable functions on S has a structure of Banach $*$-algebra with convolution and involution defined respectively by

$$\begin{aligned} (f*g)(x) &:= \int_S f(y)g(y^{-1}x)d\mu(y), \\ f^*(x) &:= \overline{f(x^{-1})}. \end{aligned}$$

K acts also on $L^1(S)$ as automorphisms by $f^k(x)=f(k^{-1}\cdot x)$ for $x\in S$, $k\in K$. Denote by $L_K^1(S)$ the closed $*$-subalgebra of $L^1(S)$ of all K-invariant functions. For a bounded continuous function ϕ on S, we define a linear functional λ_ϕ on $L^1(S)$ by $\lambda_\phi(f)=\int_S f(x)\phi(x)d\mu(x)$. Then ϕ is K-spherical if and only if λ_ϕ is multiplicative on $L_K^1(S)$, that is, $\lambda_\phi(f*g)=\lambda_\phi(f)\lambda_\phi(g)$ for $f,\, g\in L_K^1(S)$.

Consider the semidirect product group $K\ltimes S$ with product

$$(k_1,\, x)(k_2,\, y) := (k_1k_2,\, x(k_1\cdot y)).$$

Since K is compact and since S is unimodular, the group $K\ltimes S$ is unimodular and has a Haar measure $dkd\mu$. We denote by $L^1(K\backslash K\ltimes S/K)$ the Banach $*$-algebra of all K-biinvariant integrable functions. We know that $L_K^1(S)$ is isometrically isomorphic to $L^1(K\backslash K\ltimes S/K)$.

We observe first the case where S is a nilpotent Lie group N. To do so, we recall that any connected nilpotent Lie group N is symmetric in the sense that the L^1-group algebra $L^1(N)$ is a symmetric Banach $*$-algebra ([**P**]). Suppose that $(K; N)$ is a Gelfand pair. Then the closed $*$-subalgebra $L_K^1(N)$ is a commutative symmetric Banach $*$-algebra. So each homomorphism of $L_K^1(N)$ into $\boldsymbol{C}$ is the restriction of a $*$-representation of $L^1(N)$ on a certain one-dimensional subspace. For an irreducible unitary representation π of N on a Hilbert space H_π and a unit vector v in H_π, we put

$$(1.2)\qquad \phi_{\pi,v}(n):=\int_K \langle \pi(k\cdot n)v, v\rangle dk\,, \quad n\in N.$$

Then we have the following proposition due to [**BJR**], Lemma 8.2.

PROPOSITION 1.1 (Benson-Jenkins-Ratcliff). *Every bounded K-spherical function ϕ on N is of the form $\phi_{\pi,v}$, so that ϕ is positive definite.*

Let $\hat{N}$ be the set of all equivalence classes of irreducible unitary representations of N. For $\pi\in\hat{N}$, we set $\pi_k(n)=\pi(k\cdot n)$ $(k\in K,\ n\in N)$. Then $\pi_k\in\hat{N}$. We denote by K_π the stabilizer of π in K. Then K_π is a closed subgroup of K. For $k\in K_\pi$ there exists a unitary operator $W_\pi(k)$ on the representation space H_π of π such that $\pi_k(n)=W_\pi(k)\pi(n)W_\pi(k)^{-1}$ for $n\in N$. We know that W_π can be chosen to be an ordinary (not merely projective) representation of K_π (cf. for example [**Ki**], Section 2). By Theorem 1 of [**C**], the representation W_π is multiplicity-free. Let $(W_\pi, H_\pi)=\oplus_\alpha(T_\alpha, V_\alpha)$ be the decomposition into irreducibles. Then,

PROPOSITION 1.2 [**BJR**, Theorem 8.7]. (1) *$\phi_{\pi,v}$ is K-spherical if and only if v belongs to some component V_α.*

(2) *Suppose $v\in V_\alpha$. Let π' be another irreducible unitary representation of N on $H_{\pi'}$ and $v'\in H_{\pi'}$. Then, $\phi_{\pi,v}=\phi_{\pi',v'}$ if and only if $\pi'\cong\pi_k$ for some $k\in K$ and $v'\in V_{\alpha'}\cong V_\alpha$.*

In view of Proposition 1.2, we shall put $\phi_{\pi,\alpha}=\phi_{\pi,v}$ by taking $v\in V_\alpha$ in what follows.

2. Gelfand pairs.

Let S be a connected and simply connected unimodular solvable Lie group, K a connected compact Lie group acting on S as automorphisms. In this section we give a description of the structure of S and $K\ltimes S$ when $(K;S)$ is a Gelfand pair. Denote by $\mathfrak{s}=\mathrm{Lie}(S)$ the Lie algebra of S and by $\mathfrak{n}$ the nilradical of $\mathfrak{s}$, that is, the largest nilpotent ideal of $\mathfrak{s}$. By Leptin's theorem in [**BJR**], p. 108, there exists a subspace $\mathfrak{a}$ of $\mathfrak{s}$ on which K acts trivially, and we have a vector space direct sum $\mathfrak{s}=\mathfrak{a}\oplus\mathfrak{n}$. Let $X\in\mathfrak{s}$ and $y\in S$. We define $i_X(y)=(\exp X)y(\exp X)^{-1}$, that is, i_X is the inner automorphism of S determined by $\exp X$. Denote by N the analytic subgroup of S corresponding to $\mathfrak{n}$. Theorem 7.3 of [**BJR**] says that the pair $(K;S)$ is a Gelfand pair if and only if the following two conditions are satisfied: (1) $(K;N)$ is a Gelfand pair. (2) For $X\in\mathfrak{a}$ and $y\in S$, there exists $k\in K$ such that $i_X(y)=k\cdot y$. We first show that these two conditions determine the structure of S and $K\ltimes S$ as split group extensions of the nilradical N of S.

THEOREM 2.1. *Let $\mathfrak{a}$ be as above. Then $\mathfrak{a}$ is an abelian subspace of $\mathfrak{s}$. Moreover, one has $S=A\ltimes N$ and $K\ltimes S=(K\times A)\ltimes N$, where A is the analytic subgroup of S corresponding to $\mathfrak{a}$.*

PROOF. First we show that $\mathfrak{a}$ is an abelian subspace of $\mathfrak{s}$. Theorem 7.3 of [**BJR**] yields that for $X\in\mathfrak{a}$, $Y\in\mathfrak{s}$ and $t\in\boldsymbol{R}$, there exists $k_t'\in K$ such that $(\exp X)(\exp tY)(\exp X)^{-1}=k_t'\cdot(\exp tY)$. Put $k_n=k_{2^{-n}}'$ for $n=1, 2, \cdots$. Then $(\exp X)(\exp tY)(\exp X)^{-1}=k_n\cdot\exp tY$ for $t=q/2^n$ and $q=1, 2, \cdots, 2^n$. Since K is compact, we may assume that $\{k_n\}$ converges to $k\in K$ by taking a subsequence if necessary. Then for all positive integers p, q, n with $n\geqq p$, we get

$$(\exp X)\Big(\exp\frac{q}{2^p}Y\Big)(\exp X)^{-1}=\Big((\exp X)\Big(\exp\frac{qY}{2^n}\Big)(\exp X)^{-1}\Big)^{2^{n-p}}=k_n\cdot\Big(\exp\frac{q}{2^p}Y\Big).$$

Letting $n\to\infty$, we get

$$(\exp X)(\exp tY)(\exp X)^{-1}=k\cdot\exp tY \tag{2.1}$$

for $t=q/2^p$, $p=1, 2, \cdots$ and $q=1, 2, \cdots, 2^p$. By continuity we get (2.1) for all $t\in\boldsymbol{R}$. In particular, if $Y\in\mathfrak{a}$, we have $\mathrm{Ad}(\exp X)(Y)=k\cdot Y=Y$. Hence $[X, Y]=\mathrm{ad}(X)(Y)=0$. Therefore $\mathfrak{a}$ is an abelian subspace of $\mathfrak{s}$. Since S is simply connected, it holds that $S=((\exp\boldsymbol{R}X_1)\ltimes(\cdots\ltimes((\exp\boldsymbol{R}X_m)\ltimes N)\cdots))$, where $\{X_1, \cdots, X_m\}$ is a basis of $\mathfrak{a}$. The commutativity of $\mathfrak{a}$ gives rise to $S=((\exp\boldsymbol{R}X_1)\times\cdots\times(\exp\boldsymbol{R}X_m))\ltimes N=A\ltimes N$. The statement $K\ltimes S=(K\times A)\ltimes N$ is now clear because the action of K on $\mathfrak{a}$ is trivial. □

It should be noted that our solvable Lie group S need not be of type I (see Example 7.1 below). Nevertheless, since N is CCR, the last assertion of Theorem 2.1 (together with the investigations of sections 3 and 4) enables us to realize all irreducible unitary representations of $K\ltimes S$ by using the nilradical N instead of S. This will be done in section 5.

We conclude this section by describing bounded K-spherical functions on $S=A\ltimes N$ using those on the nilradical N. Although such a description is given in Theorem 8.11 in [**BJR**], our emphasis here is a parametrization given by an explicit use of the subgroup $A=\exp\mathfrak{a}$, which their parametrization lacks. Let ϕ be a K-spherical function on S. For $(x, n)\in A\ltimes N$, we have

$$\begin{aligned}\phi(x, n)=\phi((0, n)(x, 1_N))&=\int_K\phi((0, n)(k\cdot(x, 1_N)))dk\\&=\phi(0, n)\phi(x, 1_N),\end{aligned}$$

where 1_N is the unit element of N. The function $\phi(0, n)$ is a K-spherical function on N by (1.1). On the other hand, one has

$$\phi(xy, 1_N) = \phi((x, 1_N)(y, 1_N)) = \int_K \phi((x, 1_N)(k\cdot(y, 1_N)))dk$$
$$= \phi(x, 1_N)\phi(y, 1_N).$$

Since ϕ is bounded, there exists $a\in\mathfrak{a}^*$ such that $\phi(x, 1_N)=\chi_a(x)$, where $\chi_a(\exp X)=e^{\sqrt{-1}\langle a, X\rangle}$ $(X\in\mathfrak{a})$. Thus we have

PROPOSITION 2.2. *Suppose that $(K; S)$ is a Gelfand pair. Then every bounded K-spherical function on S is of the form $\phi_{\pi,\alpha,a}(x, n)=\chi_a(x)\psi_{\pi,\alpha}(n)$ for some $a\in\mathfrak{a}^*$, where $\psi_{\pi,\alpha}$ is the function defined by* (1.2).

3. Orbit spaces.

From now on we suppose that $(K; S)$ is a Gelfand pair. In this section we investigate the smoothness of $(K\times A)$-orbits space in $\hat{N}$. Denote by $\mathfrak{n}^*$ the dual space of $\mathfrak{n}$. The automorphism group $\mathrm{Aut}(\mathfrak{n})$ of $\mathfrak{n}$ acts on $\mathfrak{n}^*$ from the right by $(l\cdot\varphi)(X)=l(\varphi(X))$, where $l\in\mathfrak{n}^*$, $\varphi\in\mathrm{Aut}(\mathfrak{n})$, $X\in\mathfrak{n}$. Take a K-invariant real inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$. Theorem 7.3 in [**BJR**] says that for $x\in A$, $X\in\mathfrak{n}$, there exists $k\in K$ such that $x\cdot X=k\cdot X$. We have $\|x\cdot X\|=\|k\cdot X\|=\|X\|$, so that $\langle\cdot,\cdot\rangle$ is $(K\times A)$-invariant.

LEMMA 3.1. *Let $l\in\mathfrak{n}^*$. Then the K-orbit $l\cdot K$ is equal to the $(K\times A)$-orbit $l\cdot(K\times A)$ as sets.*

PROOF. It suffices to show that $l\cdot A\subset l\cdot K$. Using the inner product $\langle\cdot,\cdot\rangle$, we identify $\mathfrak{n}$ with $\mathfrak{n}^*$ by $l_X(Y)=\langle X, Y\rangle$ for $X, Y\in\mathfrak{n}$. Let $x\in A$. Then

$$(l_X\cdot x)(Y) = l_X(x\cdot Y) = \langle X, x\cdot Y\rangle = \langle x^{-1}\cdot X, Y\rangle.$$

By Theorem 7.3 in [**BJR**] there exists $k\in K$ such that $x^{-1}\cdot X=k^{-1}\cdot X$. So

$$\langle x^{-1}\cdot X, Y\rangle = \langle k^{-1}\cdot X, Y\rangle = \langle X, k\cdot Y\rangle = l_X(k\cdot Y) = (l_X\cdot k)(Y).$$

Hence we arrive at $l_X\cdot x=l_X\cdot k\in l_X\cdot K$. □

Noting that $\mathrm{Aut}(N)=\mathrm{Aut}(\mathfrak{n})$ acts on $\hat{N}$ from the right by $(\pi\cdot\varphi)(n)=\pi(\varphi(n))$, where $\pi\in\hat{N}$, $\varphi\in\mathrm{Aut}(N)$, $n\in N$, we have

PROPOSITION 3.2. *Let $\pi\in\hat{N}$. Then the K-orbit $\pi\cdot K$ is equal to the $(K\times A)$-orbit $\pi\cdot(K\times A)$, so that $\hat{N}/K$ can be identified with $\hat{N}/(K\times A)$ as Borel spaces.*

PROOF. Take $l\in\mathfrak{n}^*$ for which π is equivalent to π_l via Kirillov's theory. For $x\in N$ and $\varphi\in\mathrm{Aut}(\mathfrak{n})$, we have

$$\begin{aligned}\mathrm{Ad}^*(x)(l\cdot\varphi)(X) &= (l\cdot\varphi)(\mathrm{Ad}(x^{-1})X) = l(\mathrm{Ad}(\varphi(x^{-1}))\varphi(X)) \\ &= (\mathrm{Ad}^*(\varphi(x))l)(\varphi(X)), \quad X\in\mathfrak{n}.\end{aligned}$$

Thus $\mathrm{Ad}^*(N)(l\cdot\varphi)=(\mathrm{Ad}^*(N)l)\cdot\varphi$. Since $l\cdot K=l\cdot(K\times A)$ by Lemma 3.1, we get $\pi\cdot K=\pi\cdot(K\times A)$ by virtue of Kirillov's theory. □

Since N is CCR and since K is compact, the Borel space $\hat{N}/K$ is smooth. Hence $\hat{N}/(K\times A)$ is also smooth ([**G1**], Theorem 2 and [**G2**], Theorem 1). Therefore all irreducible unitary representations of $K\ltimes S=(K\times A)\ltimes N$ are obtainable by the Mackey machine.

4. Stabilizers.

In order to study the structure of the stabilizers with which the Mackey machine works, we need the theory of maximally almost periodic groups. Let us summarize it here briefly. Our reference is the book [**D**], Chapter 16, where the term "injectable in a compact group" is used instead. Let G be a topological group, $C_b(G)$ the Banach space of all bounded continuous functions on G with uniform convergence topology. For $x\in G$, we set $f^x(y)=f(x^{-1}y)$ for $y\in G$. The function $f\in C_b(G)$ is said to be almost periodic if $\{f^x \mid x\in G\}$ is relatively compact in $C_b(G)$. We call G a maximally almost periodic group if for distinct x, $y\in G$, there exists an almost periodic function f such that $f(x)\neq f(y)$. This condition is equivalent to the condition that there exist a compact group K and a continuous injective homomorphism $\varphi: G\to K$.

We know by [**D**], Theorem 16.4.6 that

(4.1) *a connected locally compact group G is maximally almost periodic if and only if it is the direct product of a compact group K and a vector group V.*

We note that (4.1) is no longer true if G is not connected. But under a certain compactness condition, the structure of G is still knowable.

PROPOSITION 4.1 [**Ku**, Lemma 2]. *Let G be a maximally almost periodic Lie group and G_0 the connected component of unit element. If G/G_0 is compact, there exist a compact group K and a vector group V such that $G=K\ltimes V$ and $G_0=K_0\times V$, where K_0 is the connected component of unit element.*

Now we will apply Proposition 4.1 to the case where G is the stabilizer $(K\times A)_\pi$ of $\pi\in\hat{N}$ in $K\times A$ and reveal its structure. Obviously the groups K and A are maximally almost periodic and $(K\times A)_\pi$ is closed in $K\times A$. So $(K\times A)_\pi$ is also maximally almost periodic. Denote by $(K_\pi)_0$ and $((K\times A)_\pi)_0$ the connected component of K_π and $(K\times A)_\pi$ respectively. Then $((K\times A)_\pi)_0$ is maximally almost periodic and $(K_\pi)_0$ is a compact subgroup of $((K\times A)_\pi)_0$. Let

$p_A: K\times A\to A$ be the canonical projection. Clearly $(K\times A)_\pi\cap(\ker p_A)=(K\times A)_\pi\cap K=K_\pi$. Since $\pi\cdot(K\times A)=\pi\cdot K$ by Proposition 3.2, we have $p_A((K\times A)_\pi)=A$.

LEMMA 4.2. *$((K\times A)_\pi)_0=(K_\pi)_0\times V'$, where V' is a vector group isomorphic to A.*

PROOF. By (4.1), there exist a connected compact group K' and a vector group V' such that $((K\times A)_\pi)_0=K'\times V'$. Since $(K\times A)_\pi$ is second countable, there exists a complete system of representatives $\{y_i\}_{i=0}^\infty$ in $(K\times A)_\pi$ such that $(K\times A)_\pi=\bigcup_{i=0}^\infty y_i((K\times A)_\pi)_0$ and y_0 is the unit element of $K\times A$. Hence

$$A=p_A((K\times A)_\pi)=\bigcup_{i=0}^\infty \overline{p_A(y_i((K\times A)_\pi)_0)}\,.$$

Now Baire's category theorem ensures one of $\overline{p_A(y_i((K\times A)_\pi)_0)}$ has an interior point. By translation we see that the unit element of A is an interior point of $\overline{p_A(((K\times A)_\pi)_0)}$. Hence $p_A(((K\times A)_\pi)_0)$ is dense in A. On the other hand, since K' is compact, $p_A(K')$ is trivial. Hence $p_A(((K\times A)_\pi)_0)=p_A(V')$, a Lie subgroup of A. Therefore $p_A(V')$ coincides with A. Moreover the subgroup $V'\cap(\ker p_A)=V'\cap K$ of V' is compact. Hence $V'\cap K$ is trivial, so that $V'\cong A$. Thus it remains to show $K'=(K_\pi)_0$. Since $(K_\pi)_0$ is a compact subgroup of $((K\times A)_\pi)_0$, we have $(K_\pi)_0\subset K'$. On the other hand K' is connected, compact and included in $(K\times A)_\pi$. Hence K' is included in the connected component of $(K\times A)_\pi\cap K=K_\pi$, that is, in the subgroup $(K_\pi)_0$. Therefore $K'=(K_\pi)_0$, which completes the proof. □

LEMMA 4.3. *There exists a vector group V isomorphic to A such that*

(1) *$(K\times A)_\pi$ normalizes V,*
(2) *$(kk_x, x)=(k_xk, x)$ for any $k\in(K_\pi)_0$ and $(k_x, x)\in V$,*
(3) *$(K\times A)_\pi=K_\pi\ltimes V$.*

PROOF. Let $(k, x)\in(K\times A)_\pi$. Then there exists $k'\in K$ such that $(k', x)\in((K\times A)_\pi)_0$. So $(k', x)^{-1}(k, x)\in(K\times A)_\pi$. Hence $(k')^{-1}k\in K_\pi$, which says that $((K\times A)_\pi)_0\backslash(K\times A)_\pi$ is homeomorphic to $(K_\pi)_0\backslash K_\pi$ and is compact. By Proposition 4.1, there exist a compact subgroup $\tilde{K}$ of $(K\times A)_\pi$ with the connected component $(\tilde{K})_0$ equal to $(K_\pi)_0$ and a vector group V such that $(K\times A)_\pi=\tilde{K}\ltimes V$ and $((K\times A)_\pi)_0=(\tilde{K})_0\times V$. By an argument similar to the case of V' in Lemma 4.2, we see that V is isomorphic to A. Since $\tilde{K}$ is a compact subgroup of $(K\times A)_\pi$, we have $\tilde{K}\subset(K\times A)_\pi\cap K=K_\pi$. Comparing the numbers of connected components, we get $\tilde{K}=K_\pi$. Hence $(K\times A)_\pi=K_\pi\ltimes V$. □

PROPOSITION 4.4. *$(K\times A)_\pi=K_\pi\times V$, the direct product of K_π and V.*

PROOF. Let $(k_x, x)\in V$ and $k\in K_\pi$. Then $(k, 0)(k_x, x)(k, 0)^{-1}=(kk_xk^{-1}, x)\in V$. Since p_A is injective on V, we have $kk_xk^{-1}=k_x\in K_\pi$. Hence $(k, 0)(k_x, x)=(k_x, x)(k, 0)$. □

5. *K*-spherical representations.

With the preparations made in the previous sections we now construct all irreducible K-spherical representations of $K\ltimes S=(K\times A)\ltimes N$. Let $l\in\mathfrak{n}^*$ $(l\neq 0)$ and $\pi=\pi_l$ the irreducible unitary representation of N corresponding to l. Since $(K; N)$ is a Gelfand pair, N is at most 2-step ([**BJR**]). Denote by B_l the alternating form on $\mathfrak{n}$ defined by $B_l(X, Y)=l([X, Y])$ $(X, Y\in\mathfrak{n})$ and by $\mathfrak{n}(l)$ the radical of B_l. Put $\mathfrak{b}(l)=\mathfrak{n}(l)\cap(\ker l)$. Then $\mathfrak{b}(l)$ is an ideal of $\mathfrak{n}$. We put $\mathrm{Aut}(\mathfrak{n})_\pi=\{\varphi\in\mathrm{Aut}(\mathfrak{n})\,|\,\pi\circ\varphi\cong\pi\}$. Recall the K-invariant real inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ taken in section 3. If $(k, x)\in(K\times A)_\pi$, the action of (k, x) on $\mathfrak{n}$ is in $\mathrm{Aut}(\mathfrak{n})_\pi\cap\mathrm{O}(\langle\cdot,\cdot\rangle)$, where $\mathrm{O}(\langle\cdot,\cdot\rangle)$ is the orthogonal group for $\langle\cdot,\cdot\rangle$.

PROPOSITION 5.1. *The intertwining representations W_π and $\widetilde{W}_\pi$ of K_π and $(K\times A)_\pi$ respectively can be taken in such a way that $W_\pi=\widetilde{W}_\pi|_{K_\pi}$.*

PROOF. If $\dim\mathfrak{n}/\mathfrak{b}(l)=1$, the proposition is trivial. Suppose that $\dim\mathfrak{n}/\mathfrak{b}(l)>1$. Put $G=\mathrm{Aut}(\mathfrak{n})_\pi\cap\mathrm{O}(\langle\cdot,\cdot\rangle)$ for simplicity. Since G is compact, there exists a G-invariant subspace V of $\mathfrak{n}/\mathfrak{b}(l)$ complementary to $\mathfrak{n}(l)/\mathfrak{b}(l)$, the center of $\mathfrak{n}/\mathfrak{b}(l)$. Moreover a complex inner product $(\cdot,\cdot)$ on V can be defined so that G is included in $\mathrm{U}(V, (\cdot,\cdot))$, the unitary group for $(\cdot,\cdot)$ (see for example [**Ki**], Section 2). The representation π is stable in $\hat{N}$ under $\mathrm{U}(V, (\cdot,\cdot))$, and the intertwining representation W of $\mathrm{U}(V, (\cdot,\cdot))$ is a unitary. Thus it suffices to take $W_\pi=W|_{K_\pi}$ and $\widetilde{W}_\pi=W|_{(K\times A)_\pi}$. □

Recall that we have $(K\times A)_\pi=K_\pi\times V$ for some vector group V isomorphic to A by Proposition 4.4. We regard $(K\times A)_\pi^\wedge$ as $\hat{K}_\pi\times\mathfrak{a}^*$ by

$$(T, b)(kk_x, x)=\chi_b(x)T(k), \tag{5.1}$$

where $T\in\hat{K}_\pi$, $b\in\mathfrak{a}^*$, $k\in K_\pi$, $(k_x, x)\in V\cong A$. We denote by $U_{(T,b),\pi}$ the irreducible unitary representation of $(K\times A)_\pi\ltimes N$ defined by

$$\begin{aligned} U_{(T,b),\pi}(kk_x, x, n) &= (\overline{(T, b)}\otimes\pi\widetilde{W}_\pi)(kk_x, x, n) \\ &= \overline{\chi_b(x)}\,\overline{T}(k)\otimes\pi(n)\widetilde{W}_\pi(kk_x, x). \end{aligned} \tag{5.2}$$

Then the induced representation $\tilde{U}_{(T,b),\pi}=\mathrm{Ind}_{(K\times A)_\pi\ltimes N}^{(K\times A)\ltimes N}U_{(T,b),\pi}$ is an irreducible unitary representation of $K\ltimes S$.

THEOREM 5.2. *Let $T\in\hat{K}_\pi$. If the intertwining number $c(T, W_\pi)$ equals 1, the irreducible representation $\tilde{U}_{(T,b),\pi}$ is a K-spherical representation of $K\ltimes S$.*

PROOF. We denote by 1_K the trivial representation of K. Then we have

$$c(1_K, (\mathrm{Ind}_{(K\times A)_\pi\ltimes N}^{(K\times A)\ltimes N}(\overline{(T, b)}\otimes\pi\widetilde{W}_\pi))|_K) = c(1_K, \mathrm{Ind}_{K_\pi}^{K}((\overline{(T, b)}\otimes\pi\widetilde{W}_\pi)|_{K_\pi}))$$

$$= c(1_K, \mathrm{Ind}_{K_\pi}^{K}(\bar{T}\otimes W_\pi)) = c(T, W_\pi).$$

Since $(K; N)$ is a Gelfand pair, we have $c(T, W_\pi)\leqq 1$ ([**C**]). If $c(T, W_\pi)=1$, we have $c(1_K, \tilde{U}_{(T,b),\pi})=1$, so that $\tilde{U}_{(T,b),\pi}$ is K-spherical. □

We denote by $B(l)$ the analytic subgroup of N corresponding to $\mathfrak{b}(l)$ and by $p_l: N\to B(l)\backslash N$ the canonical projection. Define $\Phi_\pi: (K\times A)_\pi\to \mathrm{Aut}(B(l)\backslash N)$ by $\Phi_\pi(kk_x, x)p_l(n)=p_l((kk_x, x)\cdot n)$ for $(kk_x, x)\in(K\times A)_\pi$, $n\in N$. Let $H_\pi=\bigoplus_\alpha V_\alpha$ be the multiplicity-free decomposition of W_π. Since elements of $\Phi_\pi(K_\pi)$ commute with elements of $\Phi_\pi(V)$ and since $(K; N)$ is a Gelfand pair, there exists $a_\alpha\in\mathfrak{a}^*$ for each α such that

$$\widetilde{W}_\pi(k_x, x)v = \chi_{a_\alpha}(x)v, \quad (k_x, x)\in V,\ v\in V_\alpha. \tag{5.3}$$

Put $T_\alpha=W_\pi|_{V_\alpha}$ and define

$$\tilde{U}_{\pi,\alpha,a} = \tilde{U}_{(T_\alpha, a_\alpha-a),\pi} = \mathrm{Ind}_{(K\times A)_\pi\ltimes N}^{(K\times A)\ltimes N}(\overline{(T_\alpha, a_\alpha-a)}\otimes\pi\widetilde{W}_\pi).$$

By Theorem 5.2, $\tilde{U}_{\pi,\alpha,a}$ is a K-spherical representation of $K\ltimes S=(K\times A)\ltimes N$. We recall that every bounded K-spherical function ϕ is expressed as $\phi=\phi_{\pi,\alpha,a}$.

THEOREM 5.3. *The irreducible representation $\tilde{U}_{\pi,\alpha,a}$ is the K-spherical representation of $K\ltimes S$ corresponding to $\phi_{\pi,\alpha,a}$.*

PROOF. Let $\{v_1, \cdots, v_l\}$ be an orthonormal basis for V_α. Put $v=(1/\sqrt{l})\sum_i \bar{v}_i\otimes v_i\in\bar{V}_\alpha\otimes H_\pi$. We define $U_{\pi,\alpha,a}$ by

$$U_{\pi,\alpha,a}(kk_x, x, n) = \chi_{a-a_\alpha}(x)\bar{T}_\alpha(k)\otimes\pi(n)\widetilde{W}_\pi(kk_x, x), \tag{5.4}$$

where $k\in K_\pi$, $(k_x, x)\in V$, $n\in N$. Then it holds that

$$U_{\pi,\alpha,a}(k)v = v \quad \text{for all } k\in K_\pi. \tag{5.5}$$

Set

$$f(k, x, n) = \chi_a(x)(1\otimes\pi(n))v, \quad (k, x, n)\in(K\times A)\ltimes N.$$

Then we have for $k'\in K_\pi$, $(k_{x'}, x')\in V$,

$$\begin{aligned} f((k'k_{x'}, x', n')(k, x, n)) &= f(k'k_{x'}k, x'x, n'((k'k_{x'}, x')\cdot n)) \\ &= \chi_a(x'x)(1\otimes\pi(n')\pi((k'k_{x'}, x')\cdot n))v. \end{aligned} \tag{5.6}$$

On the other hand one has

$$
\begin{aligned}
&U_{\pi,\alpha,a}(k'k_{x'}, x', n')f(k, x, n) \\
&= U_{\pi,\alpha,a}(k'k_{x'}, x', n')(\chi_a(x)(1\otimes\pi(n))v) \\
&= \chi_a(xx')\overline{\chi_{a_\alpha}(x')}(\bar{T}_\alpha(k')\otimes\pi(n')\widetilde{W}_\pi(k'k_{x'}, x')\pi(n))v \\
&= \chi_a(xx')\overline{\chi_{a_\alpha}(x')}(1\otimes\pi(n')\pi((k'k_{x'}, x')\cdot n))(\bar{T}_\alpha(k')\otimes\widetilde{W}_\pi(k'k_{x'}, x'))v.
\end{aligned}
$$

Here the formulas (5.3)~(5.5) lead us to $\overline{\chi_{a_\alpha}(x')}(\bar{T}_\alpha(k')\otimes\widetilde{W}_\pi(k'k_{x'}, x'))v=v$, which in turn implies

$$U_{\pi,\alpha,a}(k'k_{x'}, x', n')f(k, x, n) = \chi_a(xx')(1\otimes\pi(n')\pi((k'k_{x'}, x')\cdot n))v.$$

This together with (5.6) says that f is an element of the representation space $H_{\tilde{U}_{\pi,\alpha,a}}$ of $\tilde{U}_{\pi,\alpha,a}$.

The function f is K-invariant. In fact we have for $k\in K$

$$
\begin{aligned}
\tilde{U}_{\pi,\alpha,a}(k, 0, 1_N)f(k', x', n') &= f(k'k, x', n') = \chi_a(x')(1\otimes\pi(n'))v \\
&= f(k', x', n'),
\end{aligned}
$$

where 1_N is the unit element of N.

We now calculate $\langle\tilde{U}_{\pi,\alpha,a}(1_K, x, n)f, f\rangle$, where 1_K is the unit element of K. First we have

$$
\begin{aligned}
\tilde{U}_{\pi,\alpha,a}(1_K, x, n)f(k', x', n') &= f((k', x', n')(1_K, x, n)) = f(k', x'x, n'((k', x')\cdot n)) \\
&= \chi_a(x'x)(1\otimes\pi(n')\pi((k', x')\cdot n))v.
\end{aligned}
$$

So denoting by the same symbol $d\nu$ the quasi-invariant measures on $(K\times A)_\pi\ltimes N\backslash(K\times A)\ltimes N$, $(K\times A)_\pi\backslash K\times A$ and $K_\pi\backslash K$, we get

$$
\begin{aligned}
&\langle\tilde{U}_{\pi,\alpha,a}(1_K, x, n)f, f\rangle \\
&= \int_{(K\times A)_\pi\ltimes N\backslash(K\times A)\ltimes N}\langle\tilde{U}_{\pi,\alpha,a}(1_K, x, n)f(k', x', n'), f(k', x', n')\rangle d\nu(k', x', n') \\
&= \chi_a(x)\int_{(K\times A)_\pi\backslash K\times A}\langle(1\otimes\pi(n')\pi((k', x')\cdot n))v, (1\otimes\pi(n'))v\rangle d\nu(k', x') \\
&= \chi_a(x)\int_{(K\times A)_\pi\backslash K\times A}\langle(1\otimes\pi((k', x')\cdot n))v, v\rangle d\nu(k', x').
\end{aligned}
$$

Here in the last integrand, we substitute the definition $v=(1/\sqrt{l})\sum_i\bar{v}_i\otimes v_i$ to obtain

$$\langle(1\otimes\pi((k', x')\cdot n))v, v\rangle = \frac{1}{l}\sum_i\langle\pi((k', x')\cdot n)v_i, v_i\rangle.$$

In particular if $(k', x')\in(K\times A)_\pi$, we have

$$\sum_i \langle \pi((k', x')\cdot n)v_i, v_i\rangle = \sum_i \langle \widetilde{W}_\pi(k', x')\pi(n)\widetilde{W}_\pi(k', x')^{-1}v_i, v_i\rangle$$
$$= \sum_i \langle \pi(n)v_i, v_i\rangle .$$

Hence we arrive at

$$\langle \tilde{U}_{\pi,\alpha,a}(1_K, x, n)f, f\rangle = \frac{\chi_a(x)}{l} \sum_i \int_{K_\pi \backslash K} \langle \pi(k'\cdot n)v_i, v_i\rangle d\nu(k'),$$
$$= \frac{\chi_a(x)}{l} \sum_i \psi_{\pi,v_i}(n) = \chi_a(x)\psi_{\pi,w}(n),$$

where $w=(1/\sqrt{l})\sum_i v_i \in V_\alpha$. Therefore $\langle \tilde{U}_{\pi,\alpha,a}(1_K, x, n)f, f\rangle = \chi_a(x)\psi_{\pi,\alpha}(n)$. □

6. Special cases.

In this section, we use the notation in section 2 and suppose that $(K; S)$ is a Gelfand pair. As we remarked before, the condition (2) in section 2 does not necessarily imply that the action of A on N can be described through a homomorphism of A into K. This being so, we take here a closer look at the particular case where the following condition (C) is satisfied:

(C) there exists a continuous homomorphism $\varphi: A \to K$ such that $x\cdot n = \varphi(x)\cdot n$ for all $x \in A$ and $n \in N$.

Under the condition (C), we can take a specific group as V in Proposition 4.4, namely we put $V_0 = \{(\varphi(x)^{-1}, x) \mid x \in A\}$. Then we have

Lemma 6.1. *Suppose that K acts on N effectively. Then*

(1) *The image of φ is in the center of K.*

(2) *One has $K\times A = K\times V_0$, so that $K \ltimes S = (K \ltimes N)\times V_0$.*

Proof. (1) For $x\in A$, $k\in K$, $n\in N$, we have

$$(k\varphi(x))\cdot n = k\cdot(x\cdot n) = x\cdot(k\cdot n) = (\varphi(x)k)\cdot n ,$$

whence $k\varphi(x) = \varphi(x)k$, because of the effectiveness of the K-action on N.

(2) The former is obvious, and for the latter note that V_0 acts on N trivially. □

Lemma 6.2. *Let $\pi = \pi_l$ be the irreducible unitary representation of N corresponding to $l \in \mathfrak{n}^*$. Then*

(1) *$l\cdot V_0 = l$, so that $l\cdot(K\times A) = l\cdot K$,*

(2) *$(K\times A)_\pi = K_\pi \times V_0$.*

Proof. (1) For $x\in A$, $X\in\mathfrak{n}$, we have $(l\cdot(\varphi(x)^{-1}, x))(X) = l((\varphi(x)^{-1}, x)\cdot X) = l(X)$. Hence $l\cdot V_0 = l$. The second assertion follows from Lemma 6.1 (2).

(2) For $x\in A$, $n\in N$, one has $\pi_{(\varphi(x)^{-1},x)}(n)=\pi((\varphi(x)^{-1}, x)\cdot n)=\pi(n)$. Therefore we get $(K\times A)_\pi=(K\times V_0)_\pi=K_\pi\times V_0$. □

Now we construct irreducible K-spherical representations of $K\ltimes S$. By Lemma 6.1, $K\ltimes S$ is the direct product of $K\ltimes N$ and a vector group V_0. Hence each irreducible K-spherical representation of $K\ltimes S$ is the tensor product of the representations of $K\ltimes N$ and V_0.

PROPOSITION 6.3. *The irreducible K-spherical representation of* $K\ltimes S$ *corresponding to the K-spherical function* $\phi_{\pi,\alpha,a}$ *is given by* $\tilde{U}_{\pi,\alpha,a}=\mathrm{Ind}_{(K_\pi\ltimes N)\times V_0}^{(K\ltimes N)\times V_0}U_{\pi,\alpha,a}$, *where*

$$U_{\pi,\alpha,a}(k\varphi(x)^{-1}, x, n)=\chi_a(x)\bar{T}_\alpha(k)\otimes\pi(n)W_\pi(k) \tag{6.1}$$

for $k\in K_\pi$, $x\in A$, $n\in N$. *In particular, any irreducible K-spherical representation of* $K\ltimes S$ *is the tensor product of an irreducible K-spherical representation of* $K\ltimes N$ *and a unitary character of* V_0.

PROOF. Comparing (6.1) with (5.4), we have only to show that $a_\alpha=0$ in (5.3), where $(k_x, x)\in V$ is now replaced by $(\varphi(x)^{-1}, x)\in V_0$. Take $l\in\mathfrak{n}^*$ such that the corresponding irreducible unitary representation π_l is equivalent to π. With the same notation as in section 5, we have for $x\in A$ and $n\in N$

$$\Phi_\pi(\varphi(x)^{-1}, x)(p_l(n))=p_l((\varphi(x)^{-1}, x)\cdot n)=p_l(n).$$

This means that $\Phi_\pi(V_0)$ is trivial. Hence $a_\alpha=0$ for any α. □

REMARK. Of course a_α depends on V in Proposition 4.4 and the representative $\tilde{W}_\pi$ of intertwining representation. For example, if $K_\pi=K$, we can take A as V in Proposition 4.4. In this case, we see that $a_\alpha\neq 0$ in general.

7. Examples.

EXAMPLE 1. *Mautner group.* The Mautner group S is by definition the semidirect product of $\boldsymbol{R}$ and $\boldsymbol{C}^2$ with the product

$$(x, z_1, z_2)(x', z_1', z_2'):=(x+x', z_1+e^{\sqrt{-1}\alpha_1 x}z_1', z_2+e^{\sqrt{-1}\alpha_2 x}z_2'), \tag{7.1}$$

where $\alpha_1, \alpha_2\in\boldsymbol{R}$ linearly independent over $\boldsymbol{Q}$. It is well known that the solvable Lie group S is *not* of type I. The nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{s}$ of S is equal to $\boldsymbol{C}^2$. We denote by N the analytic subgroup of S corresponding to $\mathfrak{n}$. We identify N with $\boldsymbol{C}^2$. Let $K=\boldsymbol{T}^2$ the 2-dimensional torus act on S by

$$(u_1, u_2)\cdot(x, z_1, z_2):=(x, u_1z_1, u_2z_2),\quad (u_i\in\boldsymbol{C}, |u_i|=1 \text{ for } i=1, 2).$$

Putting $A=\boldsymbol{R}$, we have $S=A\ltimes N$ and this agrees with the notation that we have used until now. Obviously $(\boldsymbol{T}^2; N)$ is a Gelfand pair. Moreover, defining a continuous homomorphism $\varphi: \boldsymbol{R}\to\boldsymbol{T}^2$ by $\varphi(x)=(e^{\sqrt{-1}\alpha_1 x}, e^{\sqrt{-1}\alpha_2 x})$, we see easily by (7.1) that the pair $(K; S)$ satisfies the condition (C).

The unitary dual $\hat{N}$ of N is identified with $\boldsymbol{R}^4$ by

$$\pi_{(a_1, b_1, a_2, b_2)}(x+\sqrt{-1}\, y_1,\ x_2+\sqrt{-1}\, y_2) := e^{\sqrt{-1}\Sigma_{i=1}^2(a_i x_i + b_i y_i)}.$$

Then the elements of $\hat{N}/\boldsymbol{T}^2=\hat{N}/(\boldsymbol{T}^2\times\boldsymbol{R})$ are of the form

$$\mathcal{O}_{r_1, r_2} := \{\pi_{(a_1, b_1, a_2, b_2)} \mid a_i^2+b_i^2=r_i^2 \text{ for } i=1, 2\}, \quad r_1, r_2 \geqq 0.$$

Thus putting $\pi_{r_1, r_2}=\pi_{(r_1, 0, r_2, 0)}$ for simplicity, we take a system of representatives by $\{\pi_{r_1, r_2} \mid r_i \geqq 0 \text{ for } i=1, 2\}$.

The $\boldsymbol{T}^2$-spherical function $\phi_{r_1, r_2, a}$ on S are given by

$$\phi_{r_1, r_2, a}(x, z_1, z_2) = e^{\sqrt{-1}ax}\int_{\boldsymbol{T}^2}\pi_{r_1, r_2}((u_1, u_2)\cdot(z_1, z_2))du_1 du_2, \quad (a\in\boldsymbol{R}),$$

where du_i is the normalized Haar measure on $\boldsymbol{T}$ for $i=1, 2$. To realize the corresponding irreducible $\boldsymbol{T}^2$-spherical representation of $\boldsymbol{T}^2\ltimes S$, we take first the irreducible unitary representation $\tilde{U}_r := \mathrm{Ind}_{\boldsymbol{C}}^{\boldsymbol{T}\ltimes\boldsymbol{C}}\chi_r$ $(r>0)$ of the euclidean motion group $\boldsymbol{T}\ltimes\boldsymbol{C}$ on the plane, where $\chi_r(x+\sqrt{-1}\, y)=e^{\sqrt{-1}rx}$. When $r=0$, we take $\tilde{U}_0$ to be the trivial one-dimensional representation. Then the irreducible $\boldsymbol{T}^2$-spherical representation $\tilde{U}_{r_1, r_2}$ of $\boldsymbol{T}^2\ltimes\boldsymbol{C}^2=(\boldsymbol{T}\ltimes\boldsymbol{C})^2$ is $\tilde{U}_{r_1}\otimes\tilde{U}_{r_2}$. Therefore the irreducible $\boldsymbol{T}^2$-spherical representation of $\boldsymbol{T}^2\ltimes S$ corresponding to $\phi_{r_1, r_2, a}$ is

$$\tilde{U}_{r_1, r_2, a}(u_1 e^{-\sqrt{-1}\alpha_1 x}, u_2 e^{-\sqrt{-1}\alpha_2 x}, x, z_1, z_2) = e^{\sqrt{-1}ax}\tilde{U}_{r_1}(u_1, z_1)\otimes\tilde{U}_{r_2}(u_2, z_2).$$

EXAMPLE 2. We next given an example in which the stabilizer of some representation is not connected. Let H_2 be the 5-dimensional Heisenberg Lie group with Lie algebra $\mathfrak{h}_2$. We shall identify the underlying manifold of H_2 with $\boldsymbol{C}^2\times\boldsymbol{R}$. We put $A=\boldsymbol{R}$ and make A act on $H_2\times\boldsymbol{C}$ by

$$x\cdot((z_1, z_2), t, z_3) := ((e^{\sqrt{-1}x}z_1, e^{\sqrt{-1}x}z_2), t, e^{4\sqrt{-1}x}z_3),$$

where $z_i\in\boldsymbol{C}$ $(i=1, 2, 3)$ and $x, t\in\boldsymbol{R}$. With this action we form the semidirect product $S=A\ltimes(H_2\times\boldsymbol{C})$. Denote by $\mathfrak{s}$ the Lie algebra of S. The nilradical $\mathfrak{n}$ of $\mathfrak{s}$ is $\mathfrak{h}_2\times\boldsymbol{C}$. The corresponding analytic subgroup of S will be denoted by N. Let the action of $K=\mathrm{U}(2)$ on S be

$$k\cdot(x, (z_1, z_2), t, z_3) := (x, k\cdot(z_1, z_2), t, (\det k)^2 z_3),$$

where $k\cdot(z_1, z_2)$ stands for the usual linear action of U(2) on $\boldsymbol{C}^2$.

PROPOSITION 7.1. *The pair $(K; S)$ is a Gelfand pair.*

PROOF. First of all we show that $(K; N)$ is a Gelfand pair. Let $\mathfrak{n}^*$ be the dual space of $\mathfrak{n}$. For a non-zero real number α and $r>0$, we define the element $l_{\alpha,r}\in\mathfrak{n}^*$ by $l_{\alpha,r}((z_1, z_2), t, z_3)=\alpha t+r(\mathrm{Re}\, z_3)$. Then we see that the union of the family of orbits $\{\mathrm{Ad}^*(N)l_{\alpha,r}\cdot K \mid \alpha\neq 0, r>0\}$ is dense in $\mathfrak{n}^*$. The radical $\mathfrak{n}(l_{\alpha,r})$ of the alternating form $l_{\alpha,r}([\cdot, \cdot])$ is $(0\times\boldsymbol{R})\times\boldsymbol{C}$. Put $\mathfrak{b}(l_{\alpha,r})=\mathfrak{n}(l_{\alpha,r})\cap(\ker l_{\alpha,r})$ and denote by $B(l_{\alpha,r})$ the analytic subgroup of N corresponding to $\mathfrak{b}(l_{\alpha,r})$. Then $B(l_{\alpha,r})$ is a normal subgroup of N and $B(l_{\alpha,r})\backslash N$ is isomorphic to the 5-dimensional Heisenberg Lie group H_2. Let us denote by $\pi_{\alpha,r}$ the irreducible unitary representation of N corresponding to $l_{\alpha,r}$. Then we have $\pi_{\alpha,r}=U\circ p_{\alpha,r}$, where $p_{\alpha,r}: N\to B(l_{\alpha,r})\backslash N$ is the canonical projection and U an infinite-dimensional irreducible unitary representation of $B(l_{\alpha,r})\backslash N\cong H_2$. Let $K_{\alpha,r}$ be the stabilizer of $\pi_{\alpha,r}$ in K. Using Lemma 2.4 in [**Ki**], we have

$$\begin{aligned} K_{\alpha,r} &= \{k\in \mathrm{U}(2) \mid l_{\alpha,r}(k\cdot X)=l_{\alpha,r}(X) \text{ for all } X\in\mathfrak{n}(l_{\alpha,r})\} \\ &= \{k\in \mathrm{U}(2) \mid \det k=\pm 1\}. \end{aligned}$$

We define a map $\Phi_{\pi_{\alpha,r}}: K_{\alpha,r}\to \mathrm{Aut}(B(l_{\alpha,r})\backslash N)$ by $\Phi_{\pi_{\alpha,r}}(k)(p_{\alpha,r}(n))=p_{\alpha,r}(k\cdot n)$. Then $\Phi_{\pi_{\alpha,r}}(K_{\alpha,r})$ is isomorphic to $K_{\alpha,r}$. Since $(\mathrm{SU}(2); H_2)$ is a Gelfand pair, so is $(\Phi_{\pi_{\alpha,r}}(K_{\alpha,r}); B(l_{\alpha,r})\backslash N)$, too. Theorem 2.6 of [**Ki**] then yields that $(K; N)$ is a Gelfand pair.

We now define a continuous homomorphism $\varphi: \boldsymbol{R}\to \mathrm{U}(2)$ by

$$\varphi(x) := \begin{pmatrix} e^{\sqrt{-1}x} & 0 \\ 0 & e^{\sqrt{-1}x} \end{pmatrix}.$$

Then for $n\in N$, we have

$$\begin{aligned} x\cdot((z_1, z_2), t, z_3) &= ((e^{\sqrt{-1}x}z_1, e^{\sqrt{-1}x}z_2), t, e^{4\sqrt{-1}x}z_3) \\ &= (\varphi(x)\cdot(z_1, z_2), t, (\det\varphi(x))^2 z_3) = \varphi(x)\cdot((z_1, z_2), t, z_3). \end{aligned}$$

Thus $(K; S)$ satisfies the condition (C) in section 6. Since U(2) acts on A trivially, we conclude that $(K; S)$ is a Gelfand pair by using Theorem 7.3 in [**BJR**]. □

Put $V_0=\{(\varphi(x)^{-1}, x) \mid x\in\boldsymbol{R}\}\subset K\times A$ as in Lemma 6.1. For the stabilizer $(K\times A)_{\alpha,r}=(K\times A)_{\pi_{\alpha,r}}$ of $\pi_{\alpha,r}$ in $K\times A$, we have

$$\begin{aligned} (K\times A)_{\alpha,r} &= \{(k, x) \mid l_{\alpha,r}((k, x)\cdot X) = l_{\alpha,r}(X) \text{ for all } X\in\mathfrak{n}(l_{\alpha,r})\} \\ &= \{(k, x) \mid (\det k)e^{2\sqrt{-1}x}=\pm 1\} = \{(k\varphi(x)^{-1}, x) \mid \det k=\pm 1\} \\ &= K_{\alpha,r}\times V_0. \end{aligned}$$

Hence $(K\times A)_{\alpha,r}$ is a non-connected maximally almost periodic group which is the direct product of a compact group and a vector group.

EXAMPLE 3. Finally we give an example in which the condition (C) is not satisfied. Put $A=\boldsymbol{R}$ and let S be the semidirect product of A and the 5-dimensional Heisenberg Lie group H_2 with the action

$$x\cdot((z_1, z_2), t) := ((e^{\sqrt{-1}x}z_1, e^{\sqrt{-1}x}z_2), t).$$

Let $K=\mathrm{SU}(2)$ and the action of K be given by

$$k\cdot(x, (z_1, z_2), t) := (x, k\cdot(z_1, z_2), t).$$

The pair $(\mathrm{SU}(2); H_2)$ is a Gelfand pair (for example [**BJR**], Theorem 4.6). For $(0, 0)\neq z=(z_1, z_2)\in\boldsymbol{C}^2$ and $x\in\boldsymbol{R}$, we define $k_z, k_x\in\mathrm{SU}(2)$ by

$$k_z := \begin{pmatrix} \dfrac{z_1}{\sqrt{|z_1|^2+|z_2|^2}} & \dfrac{-\bar{z}_2}{\sqrt{|z_1|^2+|z_2|^2}} \\ \dfrac{z_2}{\sqrt{|z_1|^2+|z_2|^2}} & \dfrac{\bar{z}_1}{\sqrt{|z_1|^2+|z_2|^2}} \end{pmatrix}, \quad k_x := \begin{pmatrix} e^{\sqrt{-1}x} & 0 \\ 0 & e^{-\sqrt{-1}x} \end{pmatrix}.$$

Then we see that

$$\begin{aligned} k_z k_x k_z^{-1}\cdot(z_1, z_2) &= k_z k_x\cdot(\sqrt{|z_1|^2+|z_2|^2}, 0) = k_z\cdot(e^{\sqrt{-1}x}\sqrt{|z_1|^2+|z_2|^2}, 0) \\ &= (e^{\sqrt{-1}x}z_1, e^{\sqrt{-1}x}z_2) = x\cdot(z_1, z_2). \end{aligned}$$

Hence $(K; S)$ is a Gelfand pair by triviality of the action of SU(2) on A and Theorem 7.3 in [**BJR**].

The pair $(K; S)$ does not satisfy the condition (C). In fact if there exists a continuous homomorphism $\varphi: \boldsymbol{R}\to\mathrm{SU}(2)$ such that $x\cdot z=\varphi(x)\cdot z$ for any $x\in\boldsymbol{R}$ and $z\in\boldsymbol{C}^2$, we see that $\varphi(\boldsymbol{R})$ is included in the center of SU(2) by Lemma 6.1. Since $\boldsymbol{R}$ is connected, $\varphi(\boldsymbol{R})$ is necessarily trivial, a contradiction.

Denote by $\mathfrak{s}$ the Lie algebra of S. Then the nilradical $\mathfrak{n}$ of $\mathfrak{s}$ is the 5-dimensional Heisenberg Lie algebra $\mathfrak{h}_2$. Denote by $\mathfrak{n}^*$ the dual space of $\mathfrak{n}$. The analytic subgroup N of S corresponding to $\mathfrak{n}$ is the Heisenberg Lie group H_2. The $(\mathrm{Ad}^*(H_2), \mathrm{SU}(2))$-orbits in $\mathfrak{n}^*$ are described as follows:

$$\mathcal{O}_{0,0} := \{0\},$$

$$\mathcal{O}_{r,0} := \left\{l_{a_1,b_1,a_2,b_2} \,\Big|\, \sum_{i=1}^{2}(a_i^2+b_i^2)=r^2\right\}, \quad r>0,$$

$$l_{a_1,b_1,a_2,b_2}((x_1+\sqrt{-1}\,y_1, x_2+\sqrt{-1}\,y_2), t) := \sum_{i=1}^{2}(a_i x_i+b_i y_i),$$

$$\mathcal{O}_s := \mathrm{Ad}^*(H_2)l_s\mathrm{SU}(2), \quad s\neq 0, \quad l_s((z_1, z_2), t) := st.$$

We only treat here the orbit $\mathcal{O}_1$ for simplicity. Let $l\in\mathfrak{n}^*$ be defined by $l((z_1, z_2), t)=t$. In this case we have

$$\mathfrak{n}(l) = 0\times\boldsymbol{R}, \quad K_{\pi_l} = K, \quad (K\times A)_{\pi_l} = K\times A.$$

The irreducible representation π_l corresponding to l is realized on the Hilbert space $\mathfrak{F}$ of entire functions f on C^2 satisfying

$$\|f\|^2 := \int_{C^2} |f(w_1, w_2)|^2 e^{-(|w_1|^2+|w_2|^2)/2} dw_1 dw_2 < \infty .$$

The representation operators are given by

$$\pi_l((z_1, z_2), t) f(w_1, w_2) := \exp\left(\sqrt{-1}\,t - \frac{1}{2}(w_1\bar{z}_1 + w_2\bar{z}_2) - \frac{1}{4}(|z_1|^2 + |z_2|^2)\right) \times f(w_1 + z_1, w_2 + z_2).$$

The intertwining representation $\widetilde{W}_{\pi_l}$ of $(K\times A)_{\pi_l}$ is defined by

$$\widetilde{W}_{\pi_l}(k, x) f(w_1, w_2) := f(k^{-1} e^{-\sqrt{-1}x} 1_K \cdot (w_1, w_2)),$$

where 1_K is the (2×2)-identity matrix. Then $\widetilde{W}_{\pi_l}$ splits into the irreducibles $\mathfrak{F} = \bigoplus_m P_m$, where P_m is the space of homogeneous polynomials of degree m. Since the restriction of $\widetilde{W}_{\pi_l}(1_K, x)$ to P_m is the scalar multiplication by $e^{-\sqrt{-1}mx}$, we see that the irreducible SU(2)-spherical representation $\widetilde{U}_{\pi_l, m, a}$ of $\mathrm{SU}(2) \ltimes S$ corresponding to $\phi_{\pi_l, m, a}$ is given by

$$\widetilde{U}_{\pi_l, m, a}(k, x, (z_1, z_2), t) := e^{\sqrt{-1}(a+m)x} T_m(k) \otimes \pi_l((z_1, z_2), t) \widetilde{W}_{\pi_l}(k, x),$$

where T_m is the $(m+1)$-dimensional irreducible representation of SU(2).

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