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The uniton numbers of the harmonic 2-spheres in $Gr_2(\boldsymbol{C}^4)$

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§ 0. Introduction.

In recent years, a considerable number of works have been done on the harmonic maps from compact Riemann surfaces to compact Lie groups or compact symmetric spaces. Among many results we are interested in the following two results;

(1) Uhlenbeck has constructed all the harmonic maps from 2-sphere S^2 into unitary group $U(n)$, at each of which the uniton number $(\leqq n-1)$ is attached ([**U**]).

(2) Burstall and Wood constructed all harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^n)$ (=Grassmann manifold of the set of 2-dimensional subspace of $\boldsymbol{C}^n$) by finitely many times of forward (or backward) replacements to the elementary harmonic maps ([**Bu-Woo**]).

We have to remark about (2). At first, Ramanathan constructed all harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^4)$ ([**R**]). And Burstall and Wood generalized this construction. Chern and Wolfson also constructed all the harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^n)$ by ∂- and $\bar{\partial}$-transform, recrossings and returnings ([**C-Wol 1**], [**C-Wol 2**]). We also remark $Gr_2(\boldsymbol{C}^4)$ is canonically embedded into $U(4)$ as a totally geodesic submanifold. Thus any harmonic map from S^2 to $Gr_2(\boldsymbol{C}^4)$ is also a harmonic map into $U(4)$, and has the uniton number. In this paper we shall calculate the uniton numbers of all the harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^4)$ $\subset U(4)$ from the view point of results [**Bu-Woo**].

First we shall strengthen results (2) to obtain that in results (2), at most one forward replacement is sufficient. Then we calculate the change of uniton number with respect to the forward replacements. We shall obtain the following theorem (the terminology is recalled later):

THEOREM. *All harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^4)$ are classified in terms of isotropy and uniton number as follows*:

i) *isotropic case*;

1) *A constant map has uniton number* 0.

2) *A (anti-)holomorphic map has uniton number* 1.

3) *A map which has uniton number* 2 *is one of the following*:
- *A map* ϕ *which decomposes to the form* $\underline{\phi}$ *or* $\underline{\phi}^{\perp}=\underline{f}\oplus c$, *where* f *is a non (anti-)holomorphic harmonic map to* $\boldsymbol{C}P^2$ *and* c *is a trivial line bundle.*
- *A Frenet pair.*
- *A non (anti-)holomorphic, harmonic map obtained by a forward replacement of a Frenet pair.*
- *A mixed pair* ϕ *defined by* $\underline{\phi}=\underline{f}\oplus\underline{g}$ *where* f *is holomorphic and* g *is anti-holomorphic with* $span_{S^2} Im\, \pi_{\underline{f}} \perp span_{S^2} Im\, \pi_{\underline{g}}$ *namely a strongly isotropic mixed pair.*
- *A non (anti-)holomorphic harmonic map obtained by a forward replacement of a strongly isotropic mixed pair.*

ii) *non-isotropic case*;

1) *A map which has uniton number* 2 *is a mixed pair* ϕ *defined by* $\underline{\phi}=\underline{f}\oplus\underline{g}$ *where* f *is holomorphic and* g *is anti-holomorphic with* $span_{S^2} Im\, \pi_{\underline{f}} \not\perp span_{S^2} Im\, \pi_{\underline{g}}$, *namely a non-strongly isotropic mixed pair.*
2) *The map obtained by a forward replacement from a non-strongly isotropic mixed pair has uniton number* 3.

As a corollary, we also see the following.

COROLLARY. *Let* $\phi: S^2 \rightarrow Gr_2(\boldsymbol{C}^4)$ *be a full harmonic map. Then the uniton number is equal to* 2 *if and only if* ϕ *is isotropic or a mixed pair.*

In § 1, we review the twister theory of harmonic maps from S^2 to $U(n)$ due to Uhlenbeck ([**U**]). In § 2, we also review the twister theory of harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^n)$ due to Burstall and Wood ([**Bu-Woo**]). They found that harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^n)$ are obtained by finite times of certain transformations (called "forward" or "backward" replacements) from holomorphic maps, Frenet pairs or mixed pairs, which are some simple examples of harmonic maps. We also review the notion of "diagrams". In § 3, we observe the more precise information of § 2 in the case when the target is $Gr_2(\boldsymbol{C}^4)$. In § 4 we observe that "forward replacements" appearing in [**Bu-Woo**] is a special case of the transformation "uniton transformations" appearing in [**U**]. In § 5 we describe the diagram for Gauss bundles of holomorphic maps from S^2 to $\boldsymbol{C}P^{n-1}$. The Gauss bundle already introduced in § 2 is a fundamental notion related to forward replacements. The results we shall prove in § 4 and § 5 seem to have been known among the experts in this field, but no explicit proof seems to be available. In § 6 we observe the isotropic conditions. In § 7 we describe the normalized extended solution of Frenet pairs and harmonic maps obtained by a forward replacement from a Frenet pair via the fact in § 4. In § 8 we describe the normalized extended solution of harmonic maps obtained by a forward replacement from a mixed pair by using the fact in § 4. And lastly in § 9 on these grounds above we arrive at the conclusion.

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§ 1. The twister theory of $U(n)$.

Let Σ be a Riemann surface and through this paper we fix this notation. Let $U(n)$ be the unitary group of degree n, namely $U(n)=\{A\in GL(n;\boldsymbol{C});A^*A=I\}$, endowed with the bi-invariant metric. We denote by $\mathfrak{u}(n)$ the Lie algebra of $U(n)$, namely $\mathfrak{u}(n)=\{A\in M_n(\boldsymbol{C});A+A^*=0\}$. We set that $\boldsymbol{C}^*=\boldsymbol{C}\setminus\{0\}$.

Let $\phi:\Sigma\to U(n)$ be a harmonic map. We consider the following system of linear partial differential equations: fix $p\in\Sigma$ as a base point,

$$(1.1)\qquad \begin{cases} E_\lambda^{-1}\bar{\partial}E_\lambda=(1-\lambda)\cdot\dfrac{1}{2}\phi^{-1}\bar{\partial}\phi \\ E_\lambda^{-1}\partial E_\lambda=(1-\lambda^{-1})\cdot\dfrac{1}{2}\phi^{-1}\partial\phi \\ E_1(p)\equiv I. \end{cases}$$

DEFINITION 1.1. If a solution E_λ of (1.1) exists, we call E_λ an *extended solution* of ϕ.

In case Σ is simply connected the integrability condition of (1.1) is the harmonicity of ϕ so that extended solutions exist. The following is the most important result for the case $\Sigma=S^2$ due to Uhlenbeck.

FACT 1.2 ([**U**]). Let $\phi:S^2\to U(n)$ be a harmonic map. Then there exists a map $E:S^2\times\boldsymbol{C}^*\to GL(n;\boldsymbol{C})$ as a solution of (1.1) satisfying the following properties:

(1) There exist $k\in\boldsymbol{N}$ and $T_\alpha:S^2\to\mathfrak{gl}(n;\boldsymbol{C})$ $(\alpha=0,\cdots,k)$ such that $E_\lambda(x):=E(x,\lambda)=\sum_{\alpha=0}^{k}T_\alpha(x)\lambda^\alpha$.

(2) $E_1=I$.

(3) $E_{-1}=Q\phi$ for a constant $Q\in U(n)$.

(4) $(E_{\bar{\lambda}})^*=(E_{\lambda^{-1}})^{-1}$.

If $\Sigma=S^2$, there exist extended solutions for all harmonic maps and by Fact 1.2, we will assume that extended solutions satisfy the properties (1), (2), (3) and

(4). For E in Fact 1.2, we call k *the uniton number* $u(E)$ of E. Note that E is not unique for ϕ. Thus for ϕ, the uniton number k can always be enlarged in a fake way.

DEFINITION 1.3 ([**U**]). For a harmonic map $\phi: \Sigma \to U(n)$ with $E_\lambda = \sum_{\alpha=0}^{k} T_\alpha \lambda^\alpha$ for some $0 \leqq k \leqq \infty$, we call the minimum of k in Fact 1.2(1) the *minimal uniton number* u_ϕ of ϕ and at that time ϕ the *k-uniton*.

Under the condition of $span_{x \in S^2}$Image of $T_0(x) = C^n$, it is known that E is unique for ϕ. We call such E the *normalized extended solution* of ϕ. We know by [**U**, Theorem 13.3] that if E is the normalized extended solution of ϕ, then $u(E) = u_\phi$. We also know that for $\phi: S^2 \to U(n)$ a harmonic map, $u_\phi < n$ holds.

FACT 1.4. Up to the multiplication by constants of $U(n)$, ϕ is 0-uniton (resp. 1-uniton) if and only if ϕ is a constant (resp. (anti-)holomorphic) (see [**U**]). We also know that a harmonic map whose image is included in a Grassmannian in $U(n)$ is 2-uniton if it is isotropic ([**Sak**], see also [**Be-Gu**]). (The definition of 'isotropic' is given in Definition 6.1.)

§ 2. Harmonic maps into $Gr_2(\boldsymbol{C}^n)$ (cf. [Bu-Woo]).

Let M be a Riemann manifold and $Gr_t(\boldsymbol{C}^n)$ a Grassman manifold of the set of t-dimensional subspaces of $\boldsymbol{C}^n$. A smooth map ϕ from M to $Gr_t(\boldsymbol{C}^n)$ can be identified as the subbundle $\underline{\phi}$ of $\underline{C}^n = M \times \boldsymbol{C}^n$ of rank t with fiber $\phi(x)$ at $x \in M$. This bundle is equivalent to the pull back bundle $\phi^{-1}\mathcal{T}$, where $\mathcal{T}$ is the tautological t-plane bundle over $Gr_t(\boldsymbol{C}^n)$. It is easy to see that the holomorphicity of ϕ is equal to that of $\underline{\phi}$ as a subbundle of $\underline{C}^n$. As in [**Bu-Woo**], a rank t subbundle $\underline{\phi}$ of $\underline{C}^n$ is said to be *harmonic* if $\phi: M \to Gr_t(\boldsymbol{C}^n)$ is harmonic.

We define $\phi^\perp: M \to Gr_{n-t}(\boldsymbol{C}^n)$ by $\phi^\perp(x) := \{\phi(x)\}^\perp$ for $x \in M$. Here for a subspace $V \subset \boldsymbol{C}^n$, $V^\perp$ stands for the orthogonal complement of V in $\boldsymbol{C}^n$.

Let $E \to M$ be a vector bundle. We denote by $\Gamma(E)$ the space of C^∞ sections of E over M and by $\Gamma_U(E)$ the space of C^∞ sections of E over an open subset U of M. When E is a subbundle of $\underline{C}^n$, E has a natural hermitian metric and the connection induced from the trivial bundle. We denote these by $\langle\, ,\rangle$ and ∇^E respectively. We set $E^\perp$ the subbundle of $\underline{C}^n$ whose fiber is perpendicular to that of E on each fiber of $\underline{C}^n$. We set π_E (resp. $\pi_E^\perp$): $\underline{C}^n \to E$ (resp. $E^\perp$) the fiberwise hermitian projection.

Let $\phi: \Sigma \to Gr_t(\boldsymbol{C}^n)$ be a smooth map. From now through this paper we endow $\underline{\phi}$ and $\underline{\phi}^\perp$ with the Koszul-Malgrange holomorphic structure ([**K-M**]) as in [**Bu-Woo**, § 1B].

Let ϕ, ψ be smooth maps from Σ to Grassman manifolds such that $\underline{\phi}$ and $\underline{\psi}$ are mutually orthogonal subbundles of $\underline{C}^n = \Sigma \times \boldsymbol{C}^n$. The second fundamental forms $A'_{\phi,\psi}$, $A''_{\phi,\psi}$ are sections of $Hom(\underline{\phi}, \underline{\psi})\otimes(T^C\Sigma)^*$ given by $A'_{\phi,\psi}(s)=\pi_{\underline{\psi}}\partial s$ and $A''_{\phi,\psi}(s)=\pi_{\underline{\psi}}\bar{\partial}s$ for $s\in\Gamma(\underline{\phi})$. We put $A'_\phi=A'_{\phi,\phi^\perp}$ and $A''_\phi=A''_{\phi,\phi^\perp}$.

Let $\underline{\psi}_1, \cdots, \underline{\psi}_l$ be a set of mutually orthogonal subbundles of $\underline{C}^n$ whose sum is $\underline{C}^n$. By a *diagram* ([**Sal**]) *of* $\{\underline{\psi}_1, \cdots, \underline{\psi}_l\}$ we mean an oriented graph such that

- vertices are $\{\underline{\psi}_1, \cdots, \underline{\psi}_l\}$,
- $A'_{\psi_i,\psi_j}=0$ if the edge from $\underline{\psi}_i$ to $\underline{\psi}_j$ is absent.

REMARK 2.1. In general, for holomorphic vector bundles V_1 and V_2 over a Riemann surface Σ and a holomorphic section A of $Hom(V_1, V_2)$, there is a unique holomorphic subbundle $\underline{Im}\,A$ (resp. $\underline{Ker}\,A$) with $(\underline{Im}\,A)_x=Im\,A_x$ (resp. $(\underline{Ker}\,A)_x=Ker\,A_x$) for all $x\in U$ where A achieve its maximal rank. For details, see [**Bu-Woo**, Proposition 2.2]. We apply this for A'_ϕ and also for A''_ϕ. The equivalence between the holomorphicity of A'_ϕ and the harmonicity of ϕ is well-known. For example, see [**Bu-Woo**, Lemma 1.3].

We call $\underline{Im}\,A'_\phi$ the *∂-Gauss bundle* of ϕ and denote it by $G'(\phi)$. Similarly we call $\underline{Im}\,A''_\phi$ the *∂̄-Gauss bundle* of ϕ and denote it by $G''(\phi)$. If $\phi:\Sigma\to Gr_l(\boldsymbol{C}^n)$ is harmonic, then the Gauss bundles $G'(\phi)$ and $G''(\phi)$ are harmonic [**Bu-Woo**, Proposition 2.3]. Denote by $G^{(k)}(\phi)$ the k-th *∂-Gauss bundle* of ϕ defined by $G^{(1)}(\phi)=G'(\phi)$ and $G^{(k+1)}(\phi)=G'(G^{(k)}(\phi))$. Similarly we define the k-th *∂̄-Gauss bundle* $G^{(-k)}(\phi)$ by $G^{(-1)}(\phi)=G''(\phi)$ and $G^{(-k-1)}(\phi)=G''(G^{(-k)}(\phi))$. We put $G^{(0)}(\phi)=\underline{\phi}$. We say that ϕ is *strongly isotropic* (in the sense of [**Bu-Woo**]) if $\underline{\phi}\perp G^{(i)}(\phi)$ for any integer i. A harmonic map $\phi:\Sigma\to Gr_l(\boldsymbol{C}^n)$ is said to be *∂-irreducible* if rank of $\underline{\phi}$=rank of $G'(\phi)$, and *∂-reducible* otherwise.

Let $\underline{\phi}_1$ and $\underline{\phi}_2$ be subbundles of $\boldsymbol{C}^n$ over the same base space M. If $\underline{\phi}_1$ and $\underline{\phi}_2$ are orthogonal, we denote by $\underline{\phi}_1\oplus\underline{\phi}_2$ the usual Whitney sum of $\underline{\phi}_1$ and $\underline{\phi}_2$. If $\underline{\phi}_2\subset\underline{\phi}_1$, we define the subbundle of $\underline{C}^n$, $\underline{\phi}_1\ominus\underline{\phi}_2$ by

$$(\underline{\phi}_1\ominus\underline{\phi}_2)_x = (\underline{\phi}_1)_x\cap(\underline{\phi}_2)_x^\perp \quad \text{for } x\in M.$$

FACT 2.2 ([**Bu-Woo**, Theorem 2.4]). Let $\underline{\phi}$ be a harmonic subbundle of $\underline{C}^n$.

(1) Let α be a holomorphic subbundle of $\underline{\phi}$ such that $\alpha\subset Ker\,A'_{\phi^\perp}\circ A'_\phi$. Then, denoting $\underline{Im}(A'_\phi|\alpha)$ by $G'(\alpha)$, the bundle $\underline{\mathcal{F}(\phi, \alpha)}$ given by $\underline{\mathcal{F}(\phi, \alpha)}:=(\underline{\phi}\ominus\alpha)\oplus G'(\alpha)$ is harmonic.

(2) Let β be an anti-holomorphic subbundle of $\underline{\phi}$ such that $\beta\subset Ker\,A''_{\phi^\perp}\circ A''_\phi$. Then, denoting $\underline{Im}(A''_\phi|\beta)$ by $G''(\beta)$, the bundle $\underline{\mathcal{B}(\phi, \beta)}$ given by $\underline{\mathcal{B}(\phi, \beta)}:=(\underline{\phi}\ominus\beta)\oplus G''(\beta)$ is harmonic.

The procedure which produces the new harmonic map $\mathcal{F}(\phi, \alpha)$ (resp. $\mathcal{B}(\phi, \beta)$) from ϕ in Fact 2.2 is called the *forward* (resp. *backward*) replacement of ϕ with respect to α (resp. β). The backward replacement is generally inverse to the forward replacement, which was demonstrated in [**Bu-Woo**, Proposition 2.5].

Now we recall the construction of all harmonic maps from the standard 2-sphere S^2 to $Gr_2(\boldsymbol{C}^n)$ ([**Bu-Woo**]). For this we introduce two examples of harmonic maps from Riemann surfaces to $Gr_2(\boldsymbol{C}^n)$.

EXAMPLE 2.3 ([**Bu-Woo**]). Let $h: \Sigma \to CP^{n-1}$ be a holomorphic map. Then $\underline{\phi}=G^{(r)}(h) \oplus G^{(r+1)}(h)$ becomes a harmonic bundle. If $G^{r+1}(h) \neq 0$, then we have a harmonic map $\phi: \Sigma \to Gr_2(\boldsymbol{C}^n)$. We say that ϕ is a *Frenet pair* associated with h (more precisely *r-th Frenet pair* associated with h).

EXAMPLE 2.4 ([**Bu-Woo**]). Let $f: \Sigma \to \boldsymbol{C}P^{n-1}$ be a holomorphic map and $g: \Sigma \to \boldsymbol{C}P^{n-1}$ an anti-holomorphic map. If these satisfy $\underline{f} \perp \underline{g}$ and $G'(f) \perp \underline{g}$, we see that $\underline{\phi}=\underline{f} \oplus \underline{g}$ becomes a harmonic bundle. We say that ϕ is a *mixed pair*.

FACT 2.5 ([**Bu-Woo**, Theorem 3.3]). Let $\phi: S^2 \to Gr_2(\boldsymbol{C}^n)$ be a harmonic map. Then there is a sequence of harmonic maps $\phi_0, \cdots, \phi_N: S^2 \to Gr_2(\boldsymbol{C}^n)$ such that

(1) ϕ_N is anti-holomorphic, a Frenet pair associated with a holomorphic map $h: S^2 \to \boldsymbol{C}P^{n-1}$, or a mixed pair.

(2) $\phi_0=\phi$.

(3) For each $i<N$, there is a holomorphic line subbundle L_i of $\underline{\phi}_i$ such that $\underline{\phi}_{i-1}$ is obtained by forward replacement of ϕ_i with respect to L_i or backward replacement of ϕ_i with respect to $\underline{\phi}_i \ominus L_i$.

REMARK 2.6. For the later purpose we have to recall the sketch of proof of Fact 2.5 due to [**Bu-Woo**] when $\phi: S^2 \to Gr_2(\boldsymbol{C}^n)$ is ∂-reducible. In this case, rank of $G'(\phi)=0$ or 1. We try to find suitable successive forward or backward replacements from ϕ to achieve a holomorphic map, a Frenet pair or a mixed pair.

(i) If rank of $G'(\phi)=0$, we see that ϕ is anti-holomorphic. We put $\phi_N=\phi$.

(ii) If rank of $G'(\phi)=1$ and $A''_\phi((\underline{Ker}\ A'_\phi)^\perp)=0$, we see that ϕ is a Frenet pair or a mixed pair ([**Bu-Woo**, Proposition 3.7]). We put $\phi_N=\phi$.

(iii) In the rest of case, i.e., rank of $G'(\phi)=1$ and $A''_\phi((\underline{Ker}\ A'_\phi)^\perp) \neq 0$, we do the backward replacements with $\beta=\underline{\phi} \ominus \underline{Ker}\ A'_\phi$ of ϕ in succession. Finally these procedures produce the harmonic map ϕ_N which satisfies $A''_{\phi_N}((\underline{Ker}\ A'_{\phi_N})^\perp)=0$ and that is in case (ii).

We call the set of harmonic maps $\{\phi_0, \cdots, \phi_N\}$ *the Burstall-Wood's sequence* of ϕ.

§3. The case $Gr_2(\boldsymbol{C}^4)$.

FACT 3.1 ([**R**, Lemma 3.4]). Either $\phi: S^2 \to Gr_2(\boldsymbol{C}^4)$ or $\phi^\perp: S^2 \to Gr_2(\boldsymbol{C}^4)$ is ∂-reducible.

By Fact 3.1, the harmonic map $\phi: S^2 \to Gr_2(\boldsymbol{C}^4)$ or $\phi^\perp$ is obtained simply by the procedure in Remark 2.6.

LEMMA 3.2. *Let* $\phi: S^2 \to Gr_2(\boldsymbol{C}^4)$ *be a harmonic map and we consider* $\{\phi_0, \cdots, \phi_N\}$ *the Burstall-Wood's sequence of* ϕ. *Then we have* $N \leqq 1$ *for* ϕ *or* $\phi^\perp$.

PROOF. By Fact 3.1, changing ϕ to $\phi^\perp$ if necessary, we may assume that ϕ is ∂-reducible.

It is sufficient to prove that $\phi_1^\perp$ is strongly conformal, i.e., $A'_{\phi_1} \circ A'_{\phi_1^\perp} \equiv 0$, which provide by [**Bu-Woo**, Proposition 5.3] that ϕ_1 is (anti-)holomorphic, a Frenet pair or a mixed pair.

At first, we have a diagram

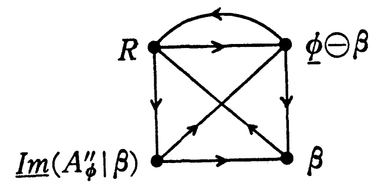


Diagram 3.1.

where $R = \underline{\phi}^\perp \ominus \underline{Im}(A''_\phi | \beta)$, which is proved as the same way for Diagram 4.1.

By definition of ϕ_1, $\underline{\phi}_1 = \underline{\phi} \ominus \beta \perp \underline{Im}(A''_\phi | \beta)$ so that by Diagram 3.1, we see $A'_{\phi_1} \circ A'_{\phi_1^\perp} = A'_{\underline{\phi} \ominus \beta, R} \circ A_{R, \underline{\phi} \ominus \alpha} + A'_{\underline{\phi} \ominus \beta, \alpha} \circ A'_{R, \phi \ominus \alpha} + A'_{\underline{Im}(A''_\phi | \beta), \beta} \circ A'_{R, \underline{Im}(A''_\phi | \beta)}$. The backward replacement of ϕ appearing in Burstall-Wood's sequence is in case $\beta = \underline{\phi} \ominus \underline{Ker}\, A'_\phi$ so that the edge from $\underline{\phi} \ominus \alpha$ to R in Diagram 3.1 disappears. Thus we have

$$A'_{\phi_1} \circ A'_{\phi_1^\perp} = A'_{\mathscr{B}(\phi, \beta), \beta} \circ A'_{R, \mathscr{B}(\phi, \beta)}. \tag{3.1}$$

By the way, $A'_{\beta, R}$, $A'_{\mathscr{B}(\phi, \beta), \beta}$ and $A'_{R, \mathscr{B}(\phi, \beta)}$ are holomorphic by [**Bu-Woo**, Proposition 1.5] thus also $A := A'_{\mathscr{B}(\phi, \alpha), \beta} \circ A'_{R, \mathscr{B}(\phi, \beta)} \circ A'_{R, (\phi, \beta)}: \beta \to \beta$ is, so that A is nilpotent (see [**Bu-Woo**, Proposition 1.8]). Since $A'_{\beta, R}$ is of full rank, $A'_{\mathscr{B}(\phi, \alpha), \beta} \circ A'_{R, \mathscr{B}(\phi, \beta)}$ is not of full rank so $A'_{\mathscr{B}(\phi, \alpha), \beta} \circ A'_{R, \mathscr{B}(\phi, \beta)} = 0$. Therefore by (3.1), $A'_{\phi_1} \circ A'_{\phi_1^\perp} \equiv 0$ and thus $\phi_1^\perp$ is strongly conformal. □

§4. Forward replacements and extended solutions.

The Grassmann manifold $Gr_t(\boldsymbol{C}^n)$ $(1 \leqq t \leqq n)$ can be embedded in $U(n)$ by

$$Gr_t(\boldsymbol{C}^n) \ni V \longmapsto \pi_V - \pi_V^\perp \in U(n)$$

and it is known that this embedding is totally geodesic. In general, if ϕ is harmonic and ι is totally geodesic, then $\iota \circ \phi$ is harmonic. So we regard harmonic maps to $Gr_t(\boldsymbol{C}^n)$ as harmonic maps to $U(n)$. We prove the following proposition, appeared essentially in [**Bu-Sal**, Proposition 3].

PROPOSITION 4.1. *Let* $\phi: \Sigma \to Gr_t(\boldsymbol{C}^n) \hookrightarrow U(n)$ *be a harmonic map and* E_λ^ϕ *an extended solution of* ϕ. *Let* $\mathcal{F}(\phi, \alpha)$ *be a harmonic map as in Fact 2.2. Then*

$$E_\lambda^{\mathcal{F}(\phi,\alpha)} := E_\lambda^\phi(\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} + \lambda\pi^\perp_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)})$$

is an extended solution of $\mathcal{F}(\phi, \alpha)$.

PROOF. By [**U**, Theorem 12.1], it is sufficient to prove the following two conditions:

(a) $\pi^\perp_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} A_z \pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} = 0$

(b) $\pi^\perp_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} (\bar{\partial}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} + A_{\bar{z}}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}) = 0$

satisfying $E_{-1}^{\mathcal{F}(\phi,\alpha)} = Q\mathcal{F}(\phi, \alpha)$ for some constant $Q \in U(n)$, where $A_{\bar{z}} = (1/2)(\pi_\phi - \pi_\phi^\perp)^{-1}\bar{\partial}(\pi_\phi - \pi_\phi^\perp) = -(A_z)^*$.

(a):

$$\begin{aligned}
&\pi^\perp_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} A_z \pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} \\
&= \frac{1}{2}\pi^\perp_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}(\pi_{\underline{\phi}} - \pi_{\underline{\phi}}^\perp)\{\partial(\pi_{\underline{\phi}} - \pi_{\underline{\phi}}^\perp)\}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)} \\
&= \frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha} - \pi_{\underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha)}) \\
&\quad \times [\partial\{(\pi_{\underline{\phi}} - \pi_{\underline{\phi}}^\perp)\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}\} - (\pi_{\underline{\phi}} - \pi_{\underline{\phi}}^\perp)\partial\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}] \\
&= \frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha} - \pi_{\underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha)})\partial(\pi_\alpha - \pi_{\underline{Im}(A'_\phi|\alpha)}) \\
&\quad -\frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha} + \pi_{\underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha)})\partial(\pi_\alpha + \pi_{\underline{Im}(A'_\phi|\alpha)}) \\
&= -\pi_{\underline{\phi}\ominus\alpha}\partial\pi_{\underline{Im}(A'_\phi|\alpha)} - \pi_{\underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha)}\partial\pi_\alpha .
\end{aligned}$$

We see that

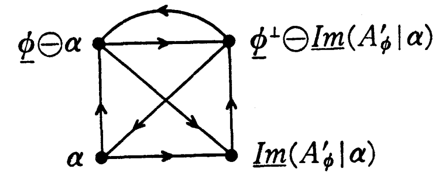


Diagram 4.1.

is a diagram of $\{\alpha, \underline{\phi}\ominus\alpha, \underline{Im}(A'_\phi|\alpha), \underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha)\}$. Here we explain how to obtain Diagram 4.1. Because α, $\underline{Im}(A'_\phi|\alpha)$ are holomorphic subbundles of $\underline{\phi}$, $\underline{\phi}^\perp$ respectively, we have $A'_{\underline{\phi}\ominus\alpha,\alpha} = 0$ and $A'_{\underline{\phi}^\perp\ominus\underline{Im}(A'_\phi|\alpha),\underline{Im}(A'_\phi|\alpha)} = 0$. And because $\alpha \subset Ker\, A'_{\phi^\perp}\circ A'_\phi$, we have $A'_{\underline{Im}(A'_\phi|\alpha),\underline{\phi}} = A'_{\phi^\perp}|(\underline{Im}(A'_\phi|\alpha)) = 0$. Further $Im\, A'_{\alpha,\underline{\phi}^\perp\ominus}$

${}_{\underline{Im}(A'_{\phi}|\alpha)}\subset Im\,A'_{\alpha,\underline{\phi}^{\perp}}=\underline{Im}(A'_{\phi}|\alpha)$ and $Im\,A'_{\alpha,\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\subset(\underline{Im}(A'_{\phi}|\alpha))^{\perp}$ so $Im\,A'_{\alpha,\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}=0$. Thus we obtain the above diagram.

By this diagram, we see that

$$\pi_{\underline{\phi}\ominus\alpha}\partial\pi_{\underline{Im}(A'_{\phi}|\alpha)}=A'_{\underline{Im}(A'_{\phi}|\alpha),\underline{\phi}\ominus\alpha}=0$$

$$\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\partial\pi_{\alpha}=A'_{\alpha,\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}=0.$$

(b): α, $\underline{Im}(A'_{\phi}|\alpha)$ are holomorphic subbundle of $\underline{\phi}$, $\underline{\phi}^{\perp}$ respectively so we have

$$\text{(4.1)}\qquad\begin{cases}\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\alpha}=0\\ \pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}=0.\end{cases}$$

On the other hand, we have

$$\begin{aligned}&\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}(\bar{\partial}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}+A_{\bar{z}}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)})\\ &=(\pi_{\underline{\phi}\ominus\alpha}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)})\\ &\quad\times\left[\bar{\partial}(\pi_{\alpha}+\pi_{\underline{Im}(A'_{\phi}|\alpha)})+\frac{1}{2}(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})^{-1}\{\bar{\partial}(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})\}(\pi_{\alpha}+\pi_{\underline{Im}(A'_{\phi}|\alpha)})\right]\\ &=\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\alpha}+\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\alpha}\\ &\quad+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}\\ &\quad+\frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)})(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})\\ &\quad\times[\bar{\partial}\{(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})(\pi_{\alpha}+\pi_{\underline{Im}(A'_{\phi}|\alpha)})\}-(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})\bar{\partial}(\pi_{\alpha}+\pi_{\underline{Im}(A'_{\phi}|\alpha)})].\end{aligned}$$

By (4.1), the first term and the fourth term of last side of the equation vanish. Thus continuing this calculation, we see

$$\begin{aligned}&\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}(\bar{\partial}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}A_{\bar{z}}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)})\\ &=\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\alpha}\\ &\quad+\frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha}-\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)})\bar{\partial}(\pi_{\alpha}-\pi_{\underline{Im}(A'_{\phi}|\alpha)})\\ &\quad-\frac{1}{2}(\pi_{\underline{\phi}\ominus\alpha}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)})\bar{\partial}(\pi_{\alpha}+\pi_{\underline{Im}(A'_{\phi}|\alpha)})\\ &=\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\alpha}-\pi_{\underline{\phi}\ominus\alpha}\bar{\partial}\pi_{\underline{Im}(A'_{\phi}|\alpha)}-\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_{\phi}|\alpha)}\bar{\partial}\pi_{\alpha}\\ &=0.\end{aligned}$$

(c): Since $E^{\phi}_{-1}=Q(\pi_{\underline{\phi}}-\pi^{\perp}_{\underline{\phi}})$ for some constant $Q\in U(n)$, we have

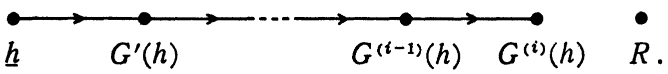
$$E^{\mathcal{F}(\phi,\alpha)}_{-1}=E^{\phi}_{-1}(\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}-\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)})$$

$$
\begin{aligned}
&= Q(\pi_{\underline{\phi}} - \pi_{\underline{\phi}}^{\perp})(\pi_{\alpha \oplus \underline{Im}(A'_\phi | \alpha)} - \pi_{\alpha \oplus \underline{Im}(A'_\phi | \alpha)}^{\perp}) \\
&= Q(\pi_\alpha - \pi_{\underline{\phi} \ominus \alpha} - \pi_{\underline{Im}(A'_\phi | \alpha)} + \pi_{\underline{\phi}^{\perp} \ominus \underline{Im}(A'_\phi | \alpha)}) \\
&= -Q(\pi_{(\underline{\phi} \ominus \alpha) \oplus \underline{Im}(A'_\phi | \alpha)} - \pi_{(\underline{\phi} \ominus \alpha) \oplus \underline{Im}(A'_\phi | \alpha)}^{\perp}) \\
&= -Q\mathcal{F}(\phi, \alpha). \qquad \square
\end{aligned}
$$

§ 5. Holomorphic map.

Iterating the method in [**Bu-Woo**, § 1D], we have the following lemma.

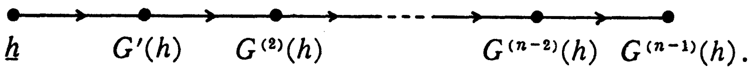
LEMMA 5.1. *Let $h: S^2 \to \boldsymbol{C}P^{n-1}$ be a holomorphic map. Then the following is a diagram of Gauss bundles of h:*



Here i is some integer and $R = (\sum_{k=0}^{i} G^{(k)}(h))^{\perp}$. (It is possible that $R = 0$.)

REMARK 5.2. It is known by [**D-Z 1**], [**D-Z 2**] and [**Gl-St**] that holomorphic maps from S^2 to $\boldsymbol{C}P^{n-1}$ are isotropic. (For the definition of isotropic, see Definition 6.1.) So by Proposition 6.4 below, holomorphic maps from S^2 to $\boldsymbol{C}P^{n-1}$ are strongly isotropic. Lemma 5.1 gives the another proof of this fact.

COROLLARY 5.3. *Let $h: S^2 \to \boldsymbol{C}P^{n-1}$ be a full holomorphic map. Then the following is a diagram of Gauss bundles of h:*



PROOF. In general the diagram in Lemma 5.1 is that of Gauss bundles of h. We assume that $R \not\equiv 0$. We can see that R is a holomorphic subbundle of $\underline{\boldsymbol{C}}^n$ and also an anti-holomorphic subbundle of $\underline{\boldsymbol{C}}^n$ by [**Bu-Woo**, Proposition 1.4]. Thus R is a trivial bundle. This contradicts the fullness of h. $\square$

§ 6. Isotropic conditions.

First we recall the notion of *isotropic* ([**E-Woo**]). Let $\phi: \Sigma \to Gr_t(\boldsymbol{C}^n)$ be a smooth map. It is well known that

$$\phi^{-1}T^{1,0}Gr_t(\boldsymbol{C}^n) \cong \phi^{-1}(\mathcal{T}^* \otimes \mathcal{T}^{\perp}).$$

From now on we keep this identification. In this bundle, there is the Hermitian metric and connection ∇ induced by the subbundle metric of $\mathcal{T} \subset \underline{\boldsymbol{C}}^n$.

DEFINITION 6.1 [**E-Woo**]. A smooth map $\phi: \Sigma \to Gr_t(\boldsymbol{C}^n)$ is *isotropic* if

$$\text{Image of } ((\nabla_{\partial/\partial z})^\alpha \phi)_x \perp \text{Image of } ((\nabla_{\partial/\partial \bar{z}})^\beta \phi)_x$$

for all $x\in\Sigma$ and $\alpha, \beta\in\boldsymbol{N}$. Here $d\phi=\partial^{1,0}\phi+\partial^{0,1}\phi$ is the decomposition with respect to the complex structure of $Gr_t(\boldsymbol{C}^n)$, $\nabla\phi:=\partial^{1,0}\phi$ and $\nabla^a:=\nabla(\nabla^{a-1})$.

FACT 6.2 ([**E-Woo**, Theorem 1.1]). Let (V, X) be a pair of holomorphic subbundles of the trivial bundle $\underline{\boldsymbol{C}}^n=\Sigma\times\boldsymbol{C}^n$ with

(1) $V\subset X$

(2) $rank(X)-rank(V)=t$

(3) $\partial\Gamma(V)\subset\Gamma(X\otimes T^{1,0*}\Sigma)$.

Then the map $\phi:\Sigma\to Gr_t(\boldsymbol{C}^n)$ defined by $\underline{\phi}(x)=X_x\ominus V_x$ $(x\in M)$ is harmonic. Further ϕ is isotropic. All isotropic harmonic maps from Σ to $Gr_t(\boldsymbol{C}^n)$ are obtained in this way.

It is well known that any harmonic maps from S^2 to $\boldsymbol{C}P^{n-1}$ are isotropic. Now we mention the relation between the isotropic condition and the strongly isotropic condition.

LEMMA 6.3. *Let $\phi:\Sigma\to Gr_t(\boldsymbol{C}^n)$ be a harmonic map. Then on any holomorphic coordinate (U, z) and for any non-negative integer k,*

$$\begin{cases}\sum_{i=1}^k\{\text{Image of }(\nabla_{\partial/\partial z})^i\phi\}=\sum_{i=1}^k G^{(i)}(\phi)\cap\underline{\phi}^\perp & \text{a.e. over }\Sigma\\ \sum_{i=1}^k\{\text{Image of }(\nabla_{\partial/\partial\bar z})^i\phi\}=\sum_{i=1}^k G^{(-i)}(\phi)\cap\underline{\phi}^\perp & \text{a.e. over }\Sigma.\end{cases}$$

PROOF. We shall prove only the first equation. The proof for the second equation is similar. We use the mathematical induction on k.

In case $k=1$, we have

$$\begin{aligned}\text{(6.1)}\qquad \text{Image of }(\nabla_{\partial/\partial z}\phi)&=\text{Image of }\left[\left\{(d\phi)\left(\frac{\partial}{\partial z}\right)\right\}^{(1,0)\text{-part}}\right]\\&=\underline{Im}\left[\left\{(d\phi)\left(\frac{\partial}{\partial z}\right)\right\}^{(1,0)\text{-part}}\right]\quad\text{a.e.}\\&=\underline{Im}\ A'_\phi\\&=G'(\phi).\end{aligned}$$

For an integer l, suppose that $\sum_{i=1}^l\{\text{Image of }(\nabla_{\partial/\partial z})^i\phi\}=\sum_{i=1}^l G^{(i)}(\phi)\cap\underline{\phi}^\perp$, a.e. over Σ. Then we have

$$\begin{aligned}\text{(6.2)}\qquad &\sum_{i=1}^{l}\{\text{Image of }\nabla_{\partial/\partial z}((\nabla_{\partial/\partial z})^i\phi)\}\\&=\operatorname*{span}_{i=1,\cdots,l}\ \operatorname*{span}_{s_i\in\Gamma(\underline{\phi})}[\nabla^{\phi^\perp}_{\partial/\partial z}\{((\nabla_{\partial/\partial z})^i\phi)s_i\}-((\nabla_{\partial/\partial z})^i\phi)(\nabla^{\phi}_{\partial/\partial z}s_i)]\\&\subset\sum_{i=1}^{l+1}G^{(i)}(\phi)\cap\underline{\phi}^\perp\quad\text{a.e..}\end{aligned}$$

On the other hand, we can see

$$(6.3)\qquad \sum_{i=0}^{l}\{\text{Image of } \nabla_{\partial/\partial z}((\nabla_{\partial/\partial\bar z})^{i}\phi)\}+\sum_{i=1}^{l}\{\text{Image of } (\nabla_{\partial/\partial z})^{i}\phi\}$$

$$\supset \operatorname*{span}_{i=1,\cdots,l}\ \operatorname*{span}_{s_i\in\Gamma(\underline{\phi})} \nabla^{\phi^{\perp}}_{\partial/\partial z}\{((\nabla_{\partial/\partial z})^{i}\phi)s_i\}$$

$$=\operatorname*{span}_{s}\left\{\nabla^{\phi^{\perp}}_{\partial/\partial z}s \mid s\in\Gamma\left(\sum_{i=1}^{l}G^{(i)}(\phi)\cap\underline{\phi}^{\perp}\right)\right\}\quad \text{a.e.}$$

$$\supset G^{(l+1)}(\phi)\cap\underline{\phi}^{\perp}.$$

By (6.1), (6.2) and (6.3), we have

$$\sum_{i=1}^{l+1}G^{(i)}(\phi)\cap\underline{\phi}^{\perp}\subset\sum_{i=1}^{l+1}\{\text{Image of } (\nabla_{\partial/\partial z})^{i}\phi\}\quad \text{a.e.}$$

$$\subset\sum_{i=1}^{l+1}G^{(i)}(\phi)\cap\underline{\phi}^{\perp}\quad \text{a.e.}$$

which conclude the proof. □

PROPOSITION 6.4. *Let $\phi:\Sigma\to Gr_l(\boldsymbol{C}^n)$ be a harmonic map. Then ϕ is strongly isotropic if and only if ϕ is isotropic.*

PROOF. Let ϕ be strongly isotropic. Then from Lemma 6.3 we can see immediately that ϕ is isotropic.

Conversely suppose that ϕ is isotropic. We use the mathematical induction. We have $\underline{\phi}\perp G'(\phi)$ clearly. Suppose that $\underline{\phi}\perp G^{(k)}(\phi)$ for all k with $1\leqq k\leqq l$. We must show that $\underline{\phi}\perp G^{(k)}(\phi)$ for all k with $1\leqq k\leqq l+1$, which is equivalent to $G^{(i)}(\phi)\perp G^{(j)}(\phi)$ for all i, j with $-1\leqq i<j\leqq l$ (see [**Bu-Woo**, Lemma 3.1]). By hypothesis of the mathematical induction, we have $G^{(i)}(\phi)\perp G^{(j)}(\phi)$ for all i, j with $0\leqq i<j\leqq l$. So it remains to show that

$$(6.4)\qquad G''(\phi)\perp\textstyle\bigoplus_{i=0}^{l}G^{(i)}(\phi).$$

Because ϕ isotropic, Image of $\nabla_{\partial/\partial\bar z}\phi\perp\sum_{i=1}^{l}\{\text{Image of } (\nabla_{\partial/\partial\bar z})^{i}\phi\}$ by Lemma 6.3 so that $G''(\phi)\perp\bigoplus_{i=1}^{l}G^{(i)}(\phi)$. Thus (6.4) is proved. □

PROPOSITION 6.5. *Let $h:S^2\to \boldsymbol{C}P^{n-1}$ be a holomorphic map. Then for any integer r, l with $r\geqq 0$ and $l\geqq 1$, the map $\phi:S^2\to Gr_{l+1}(\boldsymbol{C}^n)$ defined by*

$$\underline{\phi}:=\sum_{k=r}^{r+l}G^{(k)}(h)$$

is an isotropic harmonic map.

PROOF. For any i, $\sum_{k=0}^{i}G^{(k)}(h)$ is a holomorphic subbundle of $\underline{\boldsymbol{C}}^n$. Thus ϕ is an isotropic harmonic map by Fact 6.2. □

Putting $l=1$ in Proposition 6.5, ϕ is a Frenet pair.

§7. Frenet pairs, forward replacements and extended solutions.

Let $\Omega U(n)$ be the based loop group defined by $\Omega U(n)=\{\gamma: S^1\to U(n)\mid\gamma(1)=I\}$ and $\mathcal{F}$ a subgroup of $\Omega U(n)$ defined by $\mathcal{F}=\{\gamma\in\Omega U(n)\mid\gamma(\lambda_1\lambda_2)=\gamma(\lambda_1)\gamma(\lambda_2)$ for $\lambda_1, \lambda_2\in S^1\}$. Then we have the following lemma.

LEMMA 7.1. *For $\gamma\in\mathcal{F}$, there are non-negative integers l, m such that $\gamma(\lambda)=\sum_{\alpha=-l}^{m}\xi_\alpha\lambda^\alpha$ satisfying* (i), (ii) *and* (iii):

(i) $\xi_\alpha^2=\xi_\alpha=\xi_\alpha^*$,

(ii) $\xi_\alpha\xi_\beta=0$ *for* $\alpha\neq\beta$,

(iii) $\sum_{\alpha=-l}^{m}\xi_\alpha=I$.

PROOF. We put $\gamma(\lambda)=\sum_{\alpha=-\infty}^{\infty}\xi_\alpha\lambda^\alpha$. The base point condition $\gamma(1)=I$ implies $I=\gamma(1)=\gamma(\lambda\lambda^{-1})=\gamma(\lambda)\gamma(\lambda^{-1})$ so that

$$I=(\sum\xi_\alpha\lambda^\alpha)(\sum\xi_\beta\lambda^{-\beta})=\sum\xi_\alpha\xi_\beta\lambda^{\alpha-\beta}.$$

So we have (ii).

Comparing the coefficients of $\gamma(\lambda)^*=\gamma(\lambda)^{-1}=\gamma(\lambda^{-1})$ (respectively $\gamma(\lambda^2)=\gamma(\lambda)^2$), we have $\xi_\alpha=\xi_\alpha^*$ (respectively $\xi_\alpha=\xi_\alpha^2$) which is (i).

By (i) and (ii), $\{\xi_\alpha\}$ are hermitian projections whose images are mutually orthogonal. On the other hand, by $\gamma(1)=I$ we have $\sum_{\alpha=-\infty}^{\infty}\xi_\alpha=I$ namely $\bigoplus_{\alpha=-\infty}^{\infty}\{\text{Image of }\xi_\alpha\}=\boldsymbol{C}^n$ so that $\sum_{\alpha=-\infty}^{\infty}\xi_\alpha\lambda^\alpha$ must be of finite terms. □

By Lemma 7.1, we can regard that the connected components of $\mathcal{F}$ are flag manifolds.

Let $E_\lambda:\Sigma\to\mathcal{F}\subset\Omega U(n)$ be a smooth map of the form $E_\lambda=\sum_{\alpha=0}^{k}T_\alpha\lambda^\alpha$. By Lemma 7.1, $\{\text{Image of }T_\alpha\}_{\alpha=0,\dots,k}$ is a set of mutually orthogonal subbundles of $\underline{\boldsymbol{C}}^n=\Sigma\times\boldsymbol{C}^n$ whose sum is $\underline{\boldsymbol{C}}^n$.

PROPOSITION 7.2. *E_λ is an extended solution if and only if E_λ satisfies*

$$(7.1)\qquad\begin{cases}\partial\Gamma(\text{Image of }T_\alpha)\subset\Gamma((\text{Image of }T_\alpha\oplus\text{Image of }T_{\alpha+1})\otimes T^{1,0*}\Sigma)\\ \bar\partial\Gamma(\text{Image of }T_\alpha)\subset\Gamma((\text{Image of }T_\alpha\oplus\text{Image of }T_{\alpha-1})\otimes T^{0,1*}\Sigma)\end{cases}$$

for $0\leq\alpha\leq k$.

PROOF. By [**U**, Theorem 2.3], it is sufficient to see that $(1-\lambda^{-1})^{-1}E_\lambda^{-1}\partial E_\lambda$ and $(1-\lambda)^{-1}E_\lambda^{-1}\bar\partial E_\lambda$ do not depend on λ if and only if E_λ satisfies (7.1). By the direct calculation, we can see that $(1-\lambda^{-1})^{-1}E_\lambda^{-1}\partial E_\lambda$ (respectively, $(1-\lambda)^{-1}E_\lambda^{-1}\bar\partial E_\lambda$) is independent in λ if and only if $T_\alpha\partial T_\beta=0$ for all α, β with $\alpha\neq\beta+1$ (respectively, $T_\alpha\bar\partial T_\beta=0$ for all α, β with $\alpha\neq\beta-1$). □

We have the following proposition. It is known also by using [**Sak**, Theorem 4.1].

PROPOSITION 7.3. *We define a harmonic map $\phi: S^2 \to Gr_{l+1}(\boldsymbol{C}^n)$ as in Proposition* 6.5. *Then*

$$E_\lambda = (\pi_{\sum_{k=0}^{r+l} G^{(k)}(h)} + \lambda \pi^{\perp}_{\sum_{k=0}^{r+l} G^{(k)}(h)})(\pi_{\sum_{k=0}^{r-1} G^{(k)}(h)} + \lambda \pi^{\perp}_{\sum_{k=0}^{r-1} G^{(k)}(h)})$$

is an extended solution of ϕ. If h is full, E_λ is the normalized extended solution and in particular $u_\phi=2$.

PROOF. By Lemma 5.1, the diagram of $(\sum_{i=0}^{r-1} G^{(i)}(h), \underline{\phi}(\sum_{i=0}^{r+l} G^{(i)}(h))^\perp)$ is the following:

$$\sum_{i=0}^{r-1} G^{(i)}(h) \longrightarrow \underline{\phi} \longrightarrow \Big(\sum_{i=0}^{r+l} G^{(i)}(h)\Big)^{\perp}.$$

So we can see that E_λ satisfies (7.1). If h is full,

$$\boldsymbol{C}^n = \operatorname*{span}_{x \in S^2} h(x) \subset \operatorname*{span}_{x \in S^2} Im\, \pi_{\sum_{k=0}^{r-1} G^{(k)}(h)}(x) = \boldsymbol{C}^n.$$

The λ^0-order term T_0 of E_λ is $\pi_{\sum_{k=0}^{r-1} G^{(k)}(h)}$ so that E_λ is the normalized extended solution. □

DEFINITION 7.4. We say that a smooth map ϕ from a Riemannian manifold M into $Gr_t(\boldsymbol{C}^n)$ is *strongly full* if

(a) the only subspace of $\boldsymbol{C}^n$ containing each subspace $\phi(x)$, for $x \in M$, is $\boldsymbol{C}^n$ itself.

(b) the only subspace of $\boldsymbol{C}^n$ containing each subspace $\{\phi(x)\}^\perp$, for $x \in M$, is $\boldsymbol{C}^n$ itself.

Now we consider extended solutions of harmonic maps obtained by forward replacements from Frenet pairs. Let $h: S^2 \to \boldsymbol{C}P^{n-1}$ be a *full* holomorphic map and $\phi: S^2 \to Gr_2(\boldsymbol{C}^n)$ the r-th Frenet pair associated with h. (See Example 2.4). We suppose that $\mathcal{F}(\phi, \alpha): S^2 \to Gr_2(\boldsymbol{C}^n)$ becomes a strongly full harmonic map. We define an extended solution E_λ^ϕ of ϕ by

$$E_\lambda^\phi = (\pi_{\sum_{k=0}^{r+1} G^{(k)}(h)} + \lambda \pi^{\perp}_{\sum_{k=0}^{r+1} G^{(k)}(h)})(\pi_{\sum_{k=0}^{r-1} G^{(k)}(h)} + \lambda \pi^{\perp}_{\sum_{k=0}^{r-1} G^{(k)}(h)})$$

as in Proposition 7.3 and also an extended solution $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ of $\mathcal{F}(\phi, \alpha)$ by

$$E_\lambda^{\mathcal{F}(\phi,\alpha)} = E_\lambda^\phi(\pi_{\alpha \oplus \underline{Im}(A'_\phi|\alpha)} + \lambda \pi^{\perp}_{\alpha \oplus \underline{Im}(A'_\phi|\alpha)})$$

as in Proposition 4.1. From the direct computation, we have the following lemma.

LEMMA 7.5.

$$E_\lambda^{\mathcal{F}(\phi,\alpha)} = \pi_{\sum_{k=0}^{r-1} G^{(k)}(h)} \pi_{\alpha \oplus \underline{Im}(A'_\phi|\alpha)}$$
$$+ \lambda(\pi_{\sum_{k=0}^{r-1} G^{(k)}(h)} \pi^{\perp}_{\alpha \oplus \underline{Im}(A'_\phi|\alpha)} + \pi_{\sum_{k=r}^{r+1} G^{(k)}(h)} \pi_{\alpha \oplus \underline{Im}(A'_\phi|\alpha)})$$

$$+\lambda^2(\pi_{\Sigma_{k=r}^{r+1}G^{(k)}(h)}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}+\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)})$$
$$+\lambda^3(\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}).$$

LEMMA 7.6. *The coefficient of λ^0 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ vanishes.*

PROOF. The coefficient of λ^0 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is

$$\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}=\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi_\alpha+\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi_{\underline{Im}(A'_\phi|\alpha)}.$$

Since $\alpha\subset\underline{\phi}=G^{(r)}(h)\oplus G^{(r+1)}(h)$ we have $\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi_\alpha=0$. Furthermore we see $\underline{Im}(A'_\phi|\alpha)\subset Im\, A'_\phi=Im\, A'_{G^{(r)}(h)\oplus G^{(r+1)}(h)}$ and this is included in $G^{(r+2)}(h)$ by the diagram in Corollary 5.3. In particular $\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi_{\underline{Im}(A'_\phi|\alpha)}=0$. □

By Lemma 7.6 we know that $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ has only positive powers respect to λ.

LEMMA 7.7. *$\lambda^{-1}E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is the normalized extended solution of $\mathcal{F}(\phi, \alpha)$.*

PROOF. We must observe the coefficient of λ^1 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$, that is

$$\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}+\pi_{\Sigma_{k=r}^{r+1}G^{(k)}(h)}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}. \tag{7.2}$$

The first terms of (7.2) is $\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}=\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}$ since $\alpha\oplus Im\, A'_\phi\subset\sum_{k=r}^{r+2}G^{(k)}(h)$ by Corollary 5.3.

The second term of (7.2) is

$$\begin{aligned}\pi_{\Sigma_{k=r}^{r+1}G^{(k)}(h)}\pi_{\alpha\oplus\underline{Im}(A'_\phi|\alpha)}&=\pi_{\Sigma_{k=r}^{r+1}G^{(k)}(h)}\pi_\alpha+\pi_{\Sigma_{k=r}^{r+1}G^{(k)}(h)}\pi_{\underline{Im}(A'_\phi|\alpha)}\\&=\pi_\alpha\end{aligned}$$

since $\underline{Im}(A'_\phi|\alpha)\subset G^{(r+2)}(h)$ (so that $\underline{Im}(A'_\phi|\alpha)=G^{(r+2)}(h)$) and also $\alpha\subset\sum_{k=r}^{r+1}G^{(k)}(h)$ by Corollary 5.3. Thus the coefficient of λ^1 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is $\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}+\pi_\alpha=\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)\oplus\alpha}$.

Now h is full so that

$$\boldsymbol{C}^n=\operatorname*{span}_{x\in S^2} Im\,\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)}(x)\subset\operatorname*{span}_{x\in S^2} Im\,\pi_{\Sigma_{k=0}^{r-1}G^{(k)}(h)\oplus\alpha}\subset\boldsymbol{C}^n.$$

Thus we see that $\lambda^{-1}E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is the normalized extended solution of $\mathcal{F}(\phi, \alpha)$. □

We consider the following lemma under the same situation as Lemma 7.5, 7.6 and 7.7.

LEMMA 7.8. *The following three conditions are mutually equivalent:*

(i) *The coefficient of λ^3 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ vanishes.*

(ii) $G^{(r+3)}(h)=0$

(iii) $r=n-3$.

PROOF. First we prove the equivalence of (i) and (ii). The coefficient of λ^3 in $E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is:

$$\begin{aligned}(7.3)\qquad &\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi^{\perp}_{\underline{\alpha}\oplus \underline{Im}(A'_\phi|\alpha)}\\ &=\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}(\pi_{\underline{\phi}\ominus\alpha}+\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_\phi|\alpha)})\\ &=\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi_{\underline{\phi}\ominus\alpha}+\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_\phi|\alpha)}\,.\end{aligned}$$

The first term of (7.3) vanishes since $\underline{\phi}\ominus\alpha\subset\underline{\phi}=G^{(r)}(h)\oplus G^{(r+1)}(h)$.

Now $\underline{Im}(A'_\phi|\alpha)\subset G^{(r+2)}(h)$ by Corollary 5.3 so that $\underline{Im}(A'_\phi|\alpha)=G^{(r+2)}(h)$ because the rank of both sides is 1. So we have

$$\underline{\phi}^{\perp}\ominus\underline{Im}(A'_\phi|\alpha)=\sum_{k=0}^{r-1}G^{(k)}(h)\oplus\sum_{k\geq r+3}G^{(k)}(h)$$

which implies the second term of (7.3) is

$$\pi^{\perp}_{\Sigma_{k=0}^{r+1}G^{(k)}(h)}\pi_{\underline{\phi}^{\perp}\ominus\underline{Im}(A'_\phi|\alpha)}=\pi_{\Sigma_{k\geq r+3}G^{(k)}(h)}\,.$$

Thus we see the equivalence of (i) and (ii).

We will see the equivalence of (ii) and (iii). We assume (ii). By this assumption and the diagram of Corollary 5.3, we have $n-1<r+3$ so

$$(7.4)\qquad n\leq r+3\,.$$

On the other hand, $\mathcal{F}(\phi,\alpha)$ is the map to $Gr_2(\boldsymbol{C}^n)$ so that the rank of $\underline{\mathcal{F}(\phi,\alpha)}=(\underline{\phi}\ominus\alpha)\oplus\underline{Im}(A'_\phi|\alpha)$ is 2. Hence $\underline{Im}(A'_\phi|\alpha)\not\equiv 0$. And we see that $\underline{Im}(A'_\phi|\alpha)\subset G^{(r+2)}(h)$ so we have $G^{(r+2)}(h)\not\equiv 0$. Thus again by Corollary 5.3, we have $r+2\leqq n-1$ so

$$(7.5)\qquad r+3\leq n\,.$$

Therefore (7.4) and (7.5) imply (iii).

We can see easily that (iii) implies (ii) by Corollary 5.3. □

By Lemma 7.6, 7.7 and 7.8 we have the following proposition.

PROPOSITION 7.9. *Let $h:S^2\to \boldsymbol{C}P^{n-1}$ be a full holomorphic map and ϕ: $S^2\to Gr_2(\boldsymbol{C}^n)$ the r-th Frenet pair associated with h. Let α be a holomorphic subbundle of $\underline{\phi}$ and let $\mathcal{F}(\phi,\alpha):S^2\to Gr_2(\boldsymbol{C}^n)$ be as in Fact 2.10. Then $\mathcal{F}(\phi,\alpha)$ satisfies the following:*

(i) *If $r=n-3$, then $u_{\mathcal{F}(\phi,\alpha)}\leqq 1$.*

(ii) *If $r\neq n-3$, then $u_{\mathcal{F}(\phi,\alpha)}=2$.*

We shall confine our attention to non strongly full harmonic maps.

LEMMA 7.10. *Let $f:\Sigma\to Gr_k(\boldsymbol{C}^l)$ be a harmonic map. Via the standard inclusion $\underline{f}\hookrightarrow\underline{\boldsymbol{C}}^l\hookrightarrow\underline{\boldsymbol{C}}^{l+m}$ we define a map $\phi:\Sigma\to Gr_{k+m}(\boldsymbol{C}^{l+m})$ by $\underline{\phi}=\underline{f}\oplus c$ where $c=\underline{\boldsymbol{C}}^{l+m}\ominus\underline{\boldsymbol{C}}^l$. Then ϕ is a harmonic map.*

Conversely for a harmonic map $\phi:\Sigma\to Gr_{k+m}(\boldsymbol{C}^{l+m})$ such that $\underline{\phi}=\underline{f}\oplus c$ and c a trivial bundle over Σ of rank m, $f:\Sigma\to Gr_k(\boldsymbol{C}^{l+m}\ominus c)\cong Gr_k(\boldsymbol{C}^l)$ is a harmonic map.

PROOF. Let $f:\Sigma\to Gr_k(\boldsymbol{C}^l)$ and $\phi:\Sigma\to Gr_{k+m}(\boldsymbol{C}^{l+m})$ be a smooth maps such that $\underline{\phi}=\underline{f}\oplus c$, where $\underline{f}\hookrightarrow\underline{\boldsymbol{C}}^l\hookrightarrow\underline{\boldsymbol{C}}^{l+m}$ is the standard inclusion and $c=\underline{\boldsymbol{C}}^{l+m}\ominus\underline{\boldsymbol{C}}^l$.

For a section s of $\underline{\phi}$, we have by the triviality of c, $A'_\phi(s)=\pi_c^\perp A'_f(\pi_{\underline{f}}s)$ and $\nabla^{\phi}_{\partial/\partial\bar z}s=\pi_\phi\bar\partial(\pi_{\underline{f}}s)+\pi_c\bar\partial s$, which provide the equality

$$
\begin{aligned}
(7.6)\qquad (\nabla^{Hom(\underline{\phi},\underline{\phi}^\perp)\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}A'_\phi)(s) &= \nabla^{\underline{\phi}\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}(A'_\phi(s))-A'_\phi(\nabla^{\phi}_{\partial/\partial\bar z}s)\\
&=(\nabla^{\underline{\phi}^\perp\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}\circ\pi_c^\perp A'_f-A'_f\circ\nabla^{\underline{f}}_{\partial/\partial\bar z})(\pi_{\underline{f}}s).
\end{aligned}
$$

On the other hand, again by the triviality of c, we see

$$
\begin{aligned}
(7.7)\qquad &(\nabla^{Hom(\underline{f},\underline{f}^\perp)\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}A'_f)(\pi_{\underline{f}}s)\\
&=(\nabla^{c\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}\circ\pi_c A'_f+\nabla^{\underline{\phi}^\perp\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}\circ\pi_{\underline{\phi}^\perp}A'_f-A'_f\circ\nabla^{\underline{f}}_{\partial/\partial\bar z})(\pi_{\underline{f}}s)\\
&=(\nabla^{\underline{\phi}^\perp\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}\circ\pi_c^\perp A'_f-A'_f\circ\nabla^{\underline{f}}_{\partial/\partial\bar z})(\pi_{\underline{f}}s).
\end{aligned}
$$

By (7.6) and (7.7), we have for any $s\in\Gamma(\underline{\phi})$,

$$(\nabla^{Hom(\underline{\phi},\underline{\phi}^\perp)\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}A'_\phi)(s)=(\nabla^{Hom(\underline{f},\underline{f}^\perp)\otimes T^{*1,0}\Sigma}_{\partial/\partial\bar z}A'_f)(\pi_{\underline{f}}s)$$

so that by Remark 2.1 the result follows immediately. □

Let $\phi:\Sigma\to Gr_2(\boldsymbol{C}^n)$ be a harmonic map such that there exist a trivial line bundle c of $\underline{\boldsymbol{C}}^n$ and $\underline{\phi}=\underline{f}\oplus c$. Then by Lemma 7.10 f is a harmonic map from Σ to $\boldsymbol{C}P^{n-2}$, which is isotropic. Thus by Fact 6.2 there is a pair of holomorphic subbundles (V, X) of $\underline{\boldsymbol{C}}^{n-1}$ such that $V\subset X$, $\partial\Gamma(V)\subset\Gamma(X)$ and $\underline{f}=X\ominus V$. Then it is easy to see that E_λ satisfies (7.1). Thus by Proposition 7.2 or by [**Sak**, Theorem 4.1], we have the following.

PROPOSITION 7.11. *Let $\phi:\Sigma\to Gr_2(\boldsymbol{C}^n)$ be a harmonic map such that there exist a trivial line bundle c of $\underline{\boldsymbol{C}}^n$ and $\underline{\phi}=\underline{f}\oplus c$. Then $(V, X\oplus c)$ is a pair of holomorphic subbundles of $\underline{\boldsymbol{C}}^n$ in Fact 6.2 for ϕ. In particular ϕ is isotropic and*

$$E^{\phi}_\lambda:=(\pi_{X\oplus c}+\lambda\pi^\perp_{X\oplus c})(\pi_V+\lambda\pi^\perp_V)$$

is an extended solution of ϕ.

Let $\psi: \Sigma \to Gr_2(\boldsymbol{C}^4)$ be a non strongly full harmonic map. Then ψ (or $\psi^{\perp}$) decompose to the form $\underline{\psi}$ (or $\underline{\psi}^{\perp}$)$=\underline{f}\oplus c$, where f is a harmonic map from Σ to $\boldsymbol{C}P^2$ and c is a trivial line bundle over Σ by Lemma 7.10. If f is not (anti-) holomorphic, ψ is isotropic and $u_{\psi}=2$ by Proposition 7.11.

§ 8. Mixed pairs, forward replacements and extended solutions.

PROPOSITION 8.1. *Let $f: \Sigma \to \boldsymbol{C}P^{n-1}$ be a holomorphic map and $g: \Sigma \to \boldsymbol{C}P^{n-1}$ an anti-holomorphic map which satisfy $G'(f) \perp g$. Let $\underline{\phi}=\underline{f}\oplus\underline{g}$, i.e., $\phi: \Sigma \to Gr_2(\boldsymbol{C}^n)$ a mixed pair. Then*

$$E_{\lambda}^{\phi} = (\pi_{\underline{g}}^{\perp}+\lambda\pi_{\underline{g}})(\pi_{\underline{f}}+\lambda\pi_{\underline{f}}^{\perp})$$

is an extended solution of ϕ.

PROOF. Now $\underline{f}$ and $\underline{g}^{\perp}$ are holomorphic subbundles of $\underline{\boldsymbol{C}}^n$. By the definition of mixed pair, we see $G'(f) \perp \underline{g}$ so that $\partial\Gamma(\underline{f}) \subset \Gamma(\underline{g}^{\perp})$. Thus we have the diagram

$$\underline{f} \longrightarrow (\underline{f}\oplus\underline{g})^{\perp} \longrightarrow \underline{g}.$$

So by Proposition 7.2, E_{λ}^{ϕ} is an extended solution of $\phi^{\perp}$. It is easily seen that E^{ϕ} is an extended solution of $\phi^{\perp}$ if and only if E^{ϕ} is an extended solution of ϕ. □

REMARK 8.2. For a mixed pair ϕ defined by $\underline{\phi}=\underline{f}\oplus\underline{g}$ as in Proposition 8.1, $(\underline{f}, \underline{g}^{\perp})$ is a pair of holomorphic subbundles in Fact 6.2 for $\phi^{\perp}$. Hence $\phi^{\perp}$ is isotropic. And also by [**Sak**, Theorem 4.1] we can see Proposition 8.1.

Now we compute the uniton number of the harmonic maps obtained by the forward replacements from mixed pairs. Let $\phi: S^2 \to Gr_2(\boldsymbol{C}^n)$ be a mixed pair as in Proposition 8.1. And $\mathcal{F}(\phi, \alpha): S^2 \to Gr_2(\boldsymbol{C}^n)$ is a harmonic map defined by Fact 2.2.

We set

$$E_{\lambda}^{\phi} = (\pi_{\underline{g}}^{\perp}+\lambda\pi_{\underline{g}})(\pi_{\underline{f}}+\lambda\pi_{\underline{f}}^{\perp})$$

which is an extended solution of ϕ by Proposition 8.1, and also set

$$E_{\lambda}^{\mathcal{F}(\phi,\alpha)} = E_{\lambda}^{\phi}(\pi_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)}+\lambda\pi_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)}^{\perp}),$$

which is an extended solution of $\mathcal{F}(\phi, \alpha)$ by Proposition 4.1. From the direct computation, we have

LEMMA 8.3.

$$E_{\lambda}^{\mathcal{F}(\phi,\alpha)} = \pi_{\underline{f}}\pi_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)} \tag{8.1}$$

$$+\lambda(\pi_{\underline{f}}\pi^{\perp}_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)}+\pi_{(\underline{f}\oplus\underline{g})^{\perp}}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)})$$
$$+\lambda^2(\pi_{(\underline{f}\oplus\underline{g})^{\perp}}\pi^{\perp}_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)}+\pi_{\underline{g}}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)})$$
$$+\lambda^3\pi_{\underline{g}}\pi^{\perp}_{\alpha\oplus \underline{Im}(A'_{\phi}|\alpha)}.$$

LEMMA 8.4. *Assume that f is full. Then $E_{\lambda}^{\mathcal{F}(\phi,\alpha)}$ is the normalized extended solution if and only if $\alpha\not\equiv\underline{g}$.*

PROOF. The coefficient of λ^0 in $E_{\lambda}^{\mathcal{F}(\phi,\alpha)}$ is

$$\pi_{\underline{f}}\pi_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}=\pi_{\underline{f}}\pi_{\alpha}+\pi_{\underline{f}}\pi_{\underline{Im}(A'_{\phi}|\alpha)}.$$

Since $\underline{Im}(A'_{\phi}|\alpha)\subset\underline{\phi}^{\perp}\subset\underline{f}^{\perp}$, we have $\pi_{\underline{f}}\pi_{\underline{Im}(A'_{\phi}|\alpha)}=0$.

Assume that $\alpha\not\equiv\underline{g}$. Both α and $\underline{g}$ are holomorphic subbundles of $\underline{\phi}$, so we have $\alpha\neq\underline{g}$ a.e. over S^2, that is to say $\alpha\not\perp\underline{f}$ a.e. over S^2. Thus we have

$$\pi_{\underline{f}}\pi_{\alpha}\neq 0 \quad \text{a.e. over } S^2.$$

Observing that $Im\,\pi_{\underline{f}}\pi_{\alpha}\subset Im\,\pi_{\underline{f}}=\underline{f}$ and that the both sides are generically rank 1, we have $Im\,\pi_{\underline{f}}\pi_{\alpha}=\underline{f}$ a.e. over S^2. By the fullness of f, we have

$$\operatorname*{span}_{x\in S^2} Im\,\pi_{\underline{f}}\pi_{\alpha}(x)=\operatorname*{span}_{x\in S^2}\underline{f}_x=\boldsymbol{C}^n,$$

which conclude the lemma. □

LEMMA 8.5. *The coefficient of λ^3 in $E_{\lambda}^{\mathcal{F}(\phi,\alpha)}$ does not vanish if and only if $\alpha\not\equiv\underline{g}$.*

PROOF. The coefficient of λ^3 in $E_{\lambda}^{\mathcal{F}(\phi,\alpha)}$ is $\pi_{\underline{g}}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}$ by Lemma 8.3. We assume that $\pi_{\underline{g}}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}\equiv0$. Then $\underline{g}\subset\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)$ so that $\underline{g}\subset\alpha$ since $\underline{g}\subset\underline{\phi}$ and $\underline{Im}(A'_{\phi}|\alpha)\subset\underline{\phi}^{\perp}$. Because the rank of both sides of $\underline{g}\subset\alpha$ is 1, we have $\underline{g}\equiv\alpha$. Conversely $\underline{g}\equiv\alpha$ implies $\pi_{\underline{g}}\pi^{\perp}_{\alpha\oplus\underline{Im}(A'_{\phi}|\alpha)}\equiv0$. □

REMARK 8.6. If $\alpha\equiv\underline{g}$, since g is an anti-holomorphic map we have $\underline{Im}(A'_{\phi}|\alpha)=0$ so that the rank of $\mathcal{F}(\phi,\alpha)=(\underline{\phi}\ominus\underline{g})\oplus\underline{Im}(A'_{\phi}|\alpha)=\underline{f}$ is 1. In this paper we consider only the case $\mathcal{F}(\phi,\alpha)$ is the full map to $Gr_2(\boldsymbol{C}^n)$. So in our case we may assume $\alpha\not\equiv\underline{g}$.

By Lemma 8.3, 8.4, 8.5 and Remark 8.6, we obtain the following proposition.

PROPOSITION 8.7. *Let $\phi: S^2\to Gr_2(\boldsymbol{C}^n)$ be a mixed pair as in Proposition 8.1. Assume that f is full. Then the strongly full harmonic map $\mathcal{F}(\phi,\alpha): S^2\to Gr_2(\boldsymbol{C}^n)$ is a 3-uniton.*

From now we do not assume the fullness of f.

PROPOSITION 8.8. *Let $\phi: S^2\to Gr_2(\boldsymbol{C}^2)$ be a mixed pair as in Proposition 8.1. Assume that f is not full. Then the strongly full harmonic map $\mathcal{F}(\phi,\alpha): S^2\to$*

$Gr_2(\boldsymbol{C}^n)$ is a 3-uniton if and only if $span_{S^2} Im\, \pi_{\underline{f}} \not\perp span_{S^2} Im\, \pi_{\underline{g}}$.

PROOF. We compute (8.1) consecutively. Using the relations $\underline{f} \oplus \underline{g} \perp \underline{Im}(A'_\phi | \alpha)$, $\underline{Im}(A'_\phi | \alpha) \perp \underline{\phi}$ and $\alpha \subset \underline{\phi}$, we obtain

$$\begin{aligned} E_\lambda^{\mathcal{F}(\phi,\alpha)} = &\ \pi_{\underline{f}} \pi_\alpha \\ &+ \lambda(\pi_{\underline{f}} \pi_\alpha^\perp + \pi_{\underline{Im}(A'_\phi|\alpha)}) \\ &+ \lambda^2(\pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_{\underline{g}} \pi_\alpha) \\ &+ \lambda^3 \pi_{\underline{g}} \pi_\alpha^\perp . \end{aligned}$$

We set a subspace V_1 of $\boldsymbol{C}^n$ by $V_1 := \mathrm{span}_{x \in S^2} Im\, \pi_{\underline{f}}(x)$. Because f is not full, V_1 is a proper subspace in $\boldsymbol{C}^n$. We also set $\pi_1 := \pi_{V_1}$. $\pi_1 + \lambda^{-1}\pi_1^\perp$ is a constant of $\Omega U(n)$ so that $(\pi_1 + \lambda^{-1}\pi_1^\perp) E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is also an extended solution of $\mathcal{F}(\phi, \alpha)$. We have

$$\begin{aligned} (\pi_1 + \lambda^{-1}\pi_1^\perp) E_\lambda^{\mathcal{F}(\phi,\alpha)} = &\ \lambda^{-1} \pi_1^\perp \pi_{\underline{f}} \pi_\alpha \\ &+ \pi_1 \pi_{\underline{f}} \pi_\alpha + \pi_1^\perp \pi_{\underline{f}} \pi_\alpha^\perp + \pi_1^\perp \pi_{\underline{Im}(A'_\phi|\alpha)} \\ &+ \lambda(\pi_1 \pi_{\underline{f}} \pi_\alpha^\perp + \pi_1 \pi_{\underline{Im}(A'_\phi|\alpha)} + \pi_1^\perp \pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_1^\perp \pi_{\underline{g}} \pi_\alpha) \\ &+ \lambda^2(\pi_1 \pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_1 \pi_{\underline{g}} \pi_\alpha + \pi_1^\perp \pi_{\underline{g}} \pi_\alpha^\perp) \\ &+ \lambda^3 \pi_1 \pi_{\underline{g}} \pi_\alpha^\perp . \end{aligned}$$

Since $\pi_1 \pi_{\underline{f}} = \pi_{\underline{f}}$, $\pi_1^\perp \pi_{\underline{f}} = 0$ and $\pi_1^\perp \pi_{\underline{Im}(A'_\phi|\alpha)} = \pi_1^\perp \pi_{G'(f)} = 0$, we have

$$\begin{aligned} (\pi_1 + \lambda^{-1}\pi_1^\perp) E_\lambda^{\mathcal{F}(\phi,\alpha)} = &\ \pi_{\underline{f}} \pi_\alpha \\ &+ \lambda(\pi_{\underline{f}} \pi_\alpha^\perp + \pi_{\underline{Im}(A'_\phi|\alpha)} + \pi_1^\perp \pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_1^\perp \pi_{\underline{g}} \pi_\alpha) \\ &+ \lambda^2(\pi_1 \pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_1 \pi_{\underline{g}} \pi_\alpha + \pi_1^\perp \pi_{\underline{g}} \pi_\alpha^\perp) \\ &+ \lambda^3 \pi_1 \pi_{\underline{g}} \pi_\alpha^\perp . \end{aligned}$$

$(\pi_1 + \lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is also an extended solution and we obtain the following equation after the similar calculation:

$$\begin{aligned} (8.2) \qquad (\pi_1 + \lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)} = &\ \pi_1^\perp \pi_{\underline{\phi}^\perp \ominus \underline{Im}(A'_\phi|\alpha)} + \pi_1^\perp \pi_{\underline{g}} \pi_\alpha + \pi_{\underline{f}} \pi_\alpha \\ &+ \lambda(\pi_1^\perp \pi_{\underline{g}} \pi_\alpha^\perp + \pi_{\underline{f}} \pi_\alpha^\perp + \pi_{\underline{Im}(A'_\phi|\alpha)}) \\ &+ \lambda^2(\pi_1 \pi_{\underline{\phi}^\perp \oplus \underline{Im}(A'_\phi|\alpha)} + \pi_1 \pi_{\underline{g}} \pi_\alpha) \\ &+ \lambda^3 \pi_1 \pi_{\underline{g}} \pi_\alpha^\perp . \end{aligned}$$

Now there are two cases: (1) $(\pi_1 + \lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is the normalized extended solution; (2) or not. Consider the first case. The coefficient of λ^3-order term of $(\pi_1 + \lambda^{-1}\pi_1^\perp)^2 E^{\mathcal{F}(\phi,\alpha)}$ vanishes if and only if

$$Im\, \pi_{\underline{g}} \pi_\alpha^\perp(x) \perp \mathrm{span}_{S^2}\, Im\, \pi_{\underline{f}} \quad \text{for any } x \in S^2$$

namely

$$\text{(8.3)}\qquad \operatorname*{span}_{S^2} Im\ \pi_{\underline{g}}\pi_\alpha^\perp \perp \operatorname*{span}_{S^2} Im\ \pi_{\underline{f}}.$$

On the other hand $\mathcal{F}(\phi, \alpha)$ is a map to $Gr_2(\boldsymbol{C}^n)$ so that $\underline{g}\not\equiv\alpha$ and thus the points x of S^2 which satisfies $\underline{g}_x=\alpha_x$ are discrete. Hence $\mathrm{span}_{S^2} Im\ \pi_{\underline{g}}\pi_\alpha^\perp=\mathrm{span}_{S^2} Im\ \pi_{\underline{g}}$. So we see that λ^3-order term of $(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ vanishes if and only if $\mathrm{span}_{S^2} Im\ \pi_{\underline{g}}\perp\mathrm{span}_{S^2} Im\ \pi_{\underline{f}}$.

There remains the case (2). We set a subspace V_2 of $\boldsymbol{C}^n$.

$$\begin{aligned} V_2 &:= \mathrm{span}\ Im(\pi_1^\perp\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_1^\perp\pi_{\underline{g}}\pi_\alpha+\pi_{\underline{f}}\pi_\alpha) \\ &= \mathrm{span}\ Im(\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_{\underline{g}}\pi_\alpha+\pi_{\underline{f}}\pi_\alpha). \end{aligned}$$

Because we are in case (2), V_2 is a proper subspace in $\boldsymbol{C}^n$. The definition of V_2 implies $V_1\subset V_2$. Set $\pi_2:=\pi_{V_2}$. Since $\pi_2+\lambda^{-1}\pi_2^\perp$ is a constant in $\Omega U(n)$, $(\pi_2+\lambda^{-1}\pi_2^\perp)(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is an extended solution of $\mathcal{F}(\phi, \alpha)$. By the relations $V_1\subset V_2$, we find

$$\text{(8.4)}\qquad \pi_2^\perp\pi_1=0,\quad \pi_2\pi_1=\pi_1\quad\text{and}\quad \pi_2^\perp\pi_{\underline{f}}=0.$$

Using (8.4) and the definition of V_2, we have

$$\begin{aligned} \text{(8.5)}\qquad & (\pi_2+\lambda^{-1}\pi_2^\perp)(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)} \\ &= \pi_2^\perp\pi_{\underline{g}}\pi_\alpha^\perp+\pi_1^\perp\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_1^\perp\pi_{\underline{g}}\pi_\alpha+\pi_{\underline{f}}\pi_\alpha \\ &\quad +\lambda(\pi_2\pi_1^\perp\pi_{\underline{g}}\pi_\alpha^\perp+\pi_{\underline{f}}\pi_\alpha^\perp+\pi_{\underline{Im}(A'_\phi|\alpha)}) \\ &\quad +\lambda^2(\pi_1\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_1\pi_{\underline{g}}\pi_\alpha) \\ &\quad +\lambda^3\pi_1\pi_{\underline{g}}\pi_\alpha^\perp. \end{aligned}$$

Let's consider the normalization of $(\pi_2+\lambda^{-1}\pi_2^\perp)(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$. We have

$$\begin{aligned} & \operatorname*{span}_{S^2} Im(\pi_2^\perp\pi_{\underline{g}}\pi_\alpha^\perp+\pi_1^\perp\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_1^\perp\pi_{\underline{g}}\pi_\alpha+\pi_{\underline{f}}\pi_\alpha) \\ &= \operatorname*{span}_{S^2} Im(\pi_{\underline{g}}\pi_\alpha^\perp+\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_{\underline{g}}\pi_\alpha+\pi_{\underline{f}}\pi_\alpha) \\ &= \operatorname*{span}_{S^2} Im(\pi_{\underline{g}}+\pi_{\underline{\phi}^\perp\ominus \underline{Im}(A'_\phi|\alpha)}+\pi_{\underline{f}}\pi_\alpha) \\ &\supset \operatorname*{span}_{S^2} \underline{\phi}=\boldsymbol{C}^n. \end{aligned}$$

Hence $(\pi_2+\lambda^{-1}\pi_2^\perp)(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ is the normalized extended solution. The coefficient of λ^3-order term of $(\pi_2+\lambda^{-1}\pi_2^\perp)(\pi_1+\lambda^{-1}\pi_1^\perp)^2 E_\lambda^{\mathcal{F}(\phi,\alpha)}$ vanishes if and only if $\mathrm{span}_{S^2} Im\ \pi_{\underline{g}}\pi_\alpha^\perp\perp\mathrm{span}_{S^2} Im\ \pi_{\underline{f}}$ namely $\mathrm{span}_{S^2} Im\ \pi_{\underline{g}}\perp\mathrm{span}_{S^2} Im\ \pi_{\underline{f}}$. □

PROPOSITION 8.9. *Let $\phi: S^2\to Gr_2(\boldsymbol{C}^n)$ be a mixed pair. Suppose that $\mathcal{F}(\phi, \alpha)$ is strongly full into $Gr_2(\boldsymbol{C}^n)$. Then $u_{\mathcal{F}(\phi,\alpha)}=3$ if and only if $\mathrm{span}_{S^2} Im\ \pi_{\underline{f}}\not\perp \mathrm{span}_{S^2} Im\ \pi_{\underline{g}}$.*

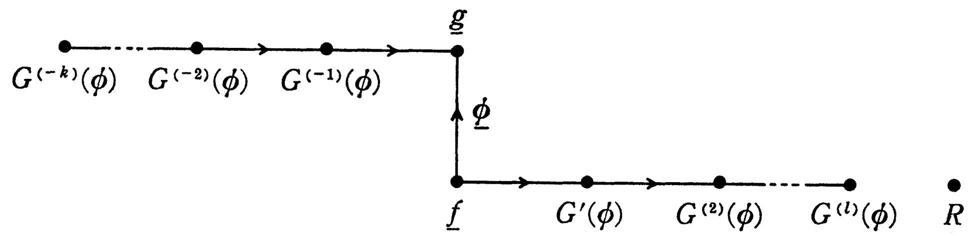
PROOF. Let $f: S^2 \to \boldsymbol{C}P^{n-1}$ be a holomorphic map and $g: S^2 \to \boldsymbol{C}P^{n-1}$ an anti-holomorphic map. Let ϕ be a mixed pair defined by $\underline{\phi} = \underline{f} \oplus \underline{g}$. Then by Proposition 8.7 and Proposition 8.8, we conclude the proposition. □

We observe the condition appearing in Proposition 8.9.

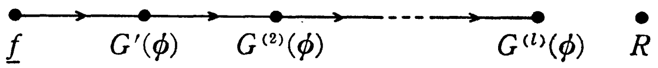
PROPOSITION 8.10. *Let $f: S^2 \to \boldsymbol{C}P^{n-1}$ be holomorphic and $g: S^2 \to \boldsymbol{C}P^{n-1}$ anti-holomorphic. Let $\phi: S^2 \to Gr_2(\boldsymbol{C}^n)$ be a mixed pair defined by $\underline{\phi} = \underline{f} \oplus \underline{g}$. Then ϕ is strongly isotropic if and only if*

$$span_{S^2}\, Im\, \pi_{\underline{f}} \perp span_{S^2}\, Im\, \pi_{\underline{g}}.$$

PROOF. We assume that ϕ is strongly isotropic. Then there exist non-negative integers k and l such that



is a diagram, where $G^{(-i)}(\phi) = G^{(-i)}(g)$ and $G^{(i)}(\phi) = G^{(i)}(f)$ for $i \geqq 1$. On the other hand, by Lemma 5.1,



is a diagram so that $\underline{f} \oplus \bigoplus_{i=1}^{l} G^{(i)}(\phi)$ is a trivial bundle. So we have $span_{S^2}\, Im\, \pi_{\underline{f}} \subset span_{S^2} \underline{f} \oplus \bigoplus_{i=1}^{l} G^{(i)}(\phi) = \underline{f} \oplus \bigoplus_{i=1}^{l} G^{(i)}(\phi)$. We have also $span_{S^2}\, Im\, \pi_{\underline{g}} \subset \underline{g} \oplus \bigoplus_{i=1}^{k} G^{(-i)}(\phi)$ similarly. On the other hand $\underline{f} \oplus \bigoplus_{i=1}^{l} G^{(i)}(\phi) \perp \underline{g} \oplus \bigoplus_{i=1}^{k} G^{(-i)}(\phi)$, we obtain $\mathrm{span}_{S^2}\, Im\, \pi_{\underline{f}} \perp \mathrm{span}_{S^2}\, Im\, \pi_{\underline{g}}$.

Conversely we assume $\mathrm{span}_{S^2}\, Im\, \pi_{\underline{f}} \perp \mathrm{span}_{S^2}\, Im\, \pi_{\underline{g}}$. We have

$$\bigoplus_{i=0}^{\infty} G^{(i)}(f) \subset \mathop{\mathrm{span}}_{S^2} Im\, \pi_{\underline{f}}$$
$$\bigoplus_{i=0}^{\infty} G^{(-i)}(g) \subset \mathop{\mathrm{span}}_{S^2} Im\, \pi_{\underline{g}},$$

where those left sides are finite sums. By the assumption, we have

$$\bigoplus_{i=0}^{\infty} G^{(i)}(f) \perp \bigoplus_{i=0}^{\infty} G^{(-i)}(g).$$

On the other hand, for $i \geqq 1$,

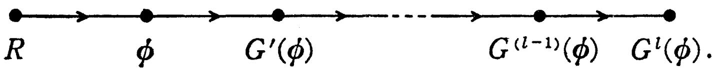
$$\begin{cases} G^{(i)}(\phi) = G^{(i)}(f) \\ G^{(-i)}(\phi) = G^{(-i)}(g) \end{cases}$$

so we have $G^{(i)}(\phi) \perp G^{(j)}(\phi)$ for $i \neq j$. □

§ 9. Conclusion.

PROPOSITION 9.1. *Let $\phi : \Sigma \to Gr_l(\boldsymbol{C}^n)$ be a harmonic map from a Riemann surface to a Grassmann manifold. Suppose that ϕ is strongly isotropic. Then a harmonic map $\mathcal{F}(\phi, \alpha)$ is also strongly isotropic.*

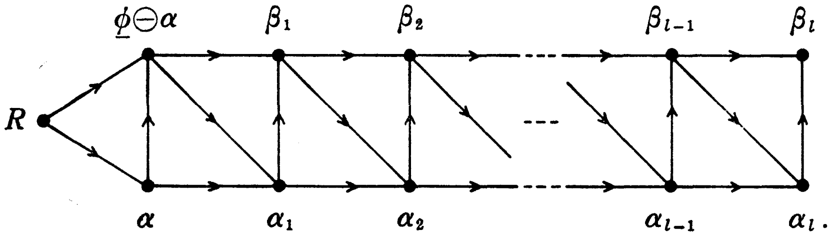
PROOF. Since ϕ is strongly isotropic, we have $\underline{\phi} \perp G^{(k)}(\phi)$ for any $k \geqq 1$. Now we have the following diagram



We set

$$\begin{cases} \alpha_i := Im\, A'_{G^{(i-1)}(\phi), G^{(i)}(\phi)} | \alpha_{i-1} \\ \beta_i := G^{(i)}(\phi) \ominus \alpha_i . \end{cases}$$

Then we get the diagram ([**Bu-Woo**, p. 275])



So we have

$$G'(\mathcal{F}(\phi, \alpha)) = \alpha_2 \oplus \beta_1$$
$$G^{(2)}(\mathcal{F}(\phi, \alpha)) = \alpha_3 \oplus \beta_2$$
$$\vdots$$
$$G^{(k)}(\mathcal{F}(\phi, \alpha)) = \begin{cases} \alpha_{k+1} \oplus \beta_k & \text{for any } 1 \leqq k \leqq l-1, \\ 0 & \text{for } k \geqq l \end{cases}$$

which imply $G^{(k)}(\mathcal{F}(\phi, \alpha)) \perp \underline{\mathcal{F}(\phi, \alpha)}$ for any $k \geqq 1$. That is to say, $\mathcal{F}(\phi, \alpha)$ is strongly isotropic. □

By Proposition 6.4, 6.5 and Proposition 9.1, we obtain the following corollary:

COROLLARY 9.2. *The harmonic map from a Riemann surface to $Gr_2(\boldsymbol{C}^n)$ obtained by finite times of forward replacements from a Frenet pair is isotropic.*

Now we are in position to state our main theorem. Putting together Lemma 3.2, Proposition 7.3, 7.9, 7.11, 8.9, 8.10 and 9.1, what we obtain is the following:

THEOREM 9.3. *The set of the full harmonic maps from S^2 to $Gr_2(\boldsymbol{C}^4)$ is classified with respect to the isotropy and the uniton number as follows:*

i) *isotropic case*;

1) *A constant map is a* 0-*uniton.*

2) *A (anti-)holomorphic map is a* 1-*uniton.*

3) *A* 2-*uniton is one of the following*:

- *A map* ϕ *which decomposes to the form* $\underline{\phi}=\underline{f}\oplus c$, *where* f *is a non (anti-)holomorphic harmonic map to* CP^2 *and* c *is a trivial line bundle or a map* ϕ *that* $\phi^{\perp}$ *decomposes to the form* $\underline{\phi}^{\perp}=\underline{f}\oplus c$ *as above.*
- *A Frenet pair.*
- *A non (anti-)holomorphic, harmonic map obtained by a forward replacement of a Frenet pair.*
- *A mixed pair* ϕ *defined by* $\underline{\phi}=\underline{f}\oplus\underline{g}$ *where* f *is holomorphic and* g *is anti-holomorphic with* $span_{S^2} Im\, \pi_{\underline{f}} \perp span_{S^2} Im\, \pi_{\underline{g}}$, *namely a strongly mixed pair.*
- *A non (anti-)holomorphic harmonic map obtained by a forward replacement of a strongly isotropic mixed pair.*

ii) *non-isotropic case*;

1) *A* 2-*uniton is a mixed pair* ϕ *defined by* $\underline{\phi}=\underline{f}\oplus\underline{g}$ *where* f *is holomorphic and* g *is anti-holomorphic with* $span_{S^2} Im\, \pi_{\underline{f}} \not\perp span_{S^2} Im\, \pi_{\underline{g}}$, *namely a non-strongly isotropic mixed pair.*

2) *The map obtained by a forward replacement from a non-strongly isotropic mixed pair is a* 3-*uniton.*

REMARK 9.4. We will observe that we can always do a forward replacement from every mixed pairs. Let $\phi:\Sigma\to Gr_2(\boldsymbol{C}^n)$ be a mixed pair from a Riemann surface defined by $\underline{\phi}=\underline{f}\oplus\underline{g}$ where f is holomorphic and g is antiholomorphic. Then $\underline{f}$ is a holomorphic subbundle of $\underline{\phi}$ so that we can do a forward replacement of $\underline{f}$ from ϕ:

$$\underline{\mathcal{F}(\phi,\alpha)} = (\underline{\phi}\ominus\underline{f})\oplus\underline{Im}(A'_{\phi}|\underline{f})$$
$$= \underline{g}\oplus G'(f).$$

Hence there exist 3-unitons into $Gr_2(\boldsymbol{C}^4)$ actually.

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