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Prinjective modules, propartite modules, representations of bocses and lattices over orders

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1. Introduction.

Throughout this paper we assume that K is a field. Given a K-algebra Λ we denote by $\mathrm{mod}(\Lambda)$ the category of finitely generated right Λ-modules and by $\mathrm{fin}(\Lambda)$ the full subcategory of $\mathrm{mod}(\Lambda)$ consisting of finite dimensional modules. Moreover we denote by $\mathrm{pr}(\Lambda)$ and $\mathrm{inj}(\Lambda)$ the full subcategories of $\mathrm{mod}(\Lambda)$ consisting of projective and injective modules, respectively.

By a *bipartite algebra* we mean a K-algebra (not necessarily finite dimensional) of the upper triangular form

$$R=\begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix} \tag{1.1}$$

where A, B are K-algebras and ${}_AM_B$ is an A-B-bimodule (see [22, Section 17.4]). Right modules X in $\mathrm{mod}(R)$ will be identified with the systems

$$X=(X'_A,\ X''_B,\ \varphi : X'\otimes_A M_B \to X''_B) \tag{1.2}$$

where X'_A is in $\mathrm{mod}(A)$, X''_B is in $\mathrm{mod}(B)$ and φ is a B-homomorphism. Note that φ is uniquely determined by the B-homomorphism

$$\bar{\varphi} : X'_A \longrightarrow \mathrm{Hom}_B({}_AM_B,\ X''_B) \tag{1.3}$$

adjoint to φ and defined by formula $\bar{\varphi}(x)(m)=\varphi(x\otimes m)$.

We recall from [16] that a module X is said to be *prinjective* if X'_A is A-projective and X''_B is B-injective. We denote by $\mathrm{prin}(R)^A_B$ the category of finitely generated prinjective right R-modules.

We define the module X in $\mathrm{mod}(R)$ to be *propartite* if X'_A is A-projective and X''_B is B-projective, or equivalently, if X viewed as a module over the subalgebra $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ of R is projective. Following [21] we call the propartite

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module X (1.2) *B-complete* if φ is surjective. We denote by $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ the category of finitely generated propartite modules and B-complete propartite modules over $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, respectively. We often write $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)$ instead of $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, respectively.

The aim of this paper is to establish basic representation theory properties of the categories $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, and their relations with prinjective modules, bimodule matrix problems (see [**5**], [**22**]), representations of bocses (see [**4**], [**19**], [**27**]) and representation theory of orders [**17**].

We show in the paper that $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is an additive Krull-Schmidt category which is closed under kernels of epimorphisms, under direct sums and summands and under extensions. It is shown that the Grothendieck group of the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is free abelian of finite rank. If M_B is a projective B-module or $\dim_K R$ is finite then $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ has enough relative projective objects and the indecomposable relative projective modules in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ are the modules (3.5). Moreover, $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is a hereditary subcategory of $\mathrm{mod}(R)$ in the sense that $\mathrm{Ext}_R^2(X, Y)=0$ for any pair of modules X and Y in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ (see Proposition 3.7). If M_B is projective, then the projective dimension $\mathrm{pd}_R X$ is $\leqq 1$ for any module X in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$.

It is shown in the paper that the propartite modules play an important role in the study of the representation types of matrix problems and orders. We prove in Theorem 2.8 that for any pair of Krull-Schmidt categories $\boldsymbol{K}$ and $\boldsymbol{L}$ having only a finite number of the isomorphism classes of indecomposable objects, and for any $\boldsymbol{L}$-$\boldsymbol{K}$-bimodule

$$\mathbf{M} : \boldsymbol{K}^{\mathrm{op}} \times \boldsymbol{L} \longrightarrow \mathcal{A}b$$

which is finitely generated projective viewed as a left $\boldsymbol{L}$-module there exist a bipartite semiperfect ring $\mathbf{R}'_{\mathbf{M}}$ and an equivalence of categories

$$\mathrm{Mat}({}_{\boldsymbol{L}}\mathbf{M}_{\boldsymbol{K}}) \cong \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathbf{R}'_{\mathbf{M}})$$

where $\mathrm{Mat}({}_{\boldsymbol{L}}\mathbf{M}_{\boldsymbol{K}})$ is the category of ${}_{\boldsymbol{L}}\mathbf{M}_{\boldsymbol{K}}$-matrices in the sense of Drozd (see [**22**]). We recall from [**16**] that, in case the bimodule ${}_{\boldsymbol{L}}\mathbf{M}_{\boldsymbol{K}}$ is finite dimensional, the study of $\mathrm{Mat}({}_{\boldsymbol{L}}\mathbf{M}_{\boldsymbol{K}})$ reduces to the study of prinjective modules. This is not the case in general if the bimodule is infinite dimensional. However, in this case the category of propartite modules applies.

A motivation for the study of $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ over infinite dimensional algebras R is the fact proved in Section 7 that for any D-order Λ in a semisimple D_0-algebra C the study of the category $\mathrm{latt}(\Lambda)$ of right Λ-lattices and its representation type reduces by the functors in the diagram (7.14) to the study of $\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}$, where Γ is a maximal order in C containing Λ. We apply this reduction to the study of the representation type of the category $\mathrm{latt}(\Lambda)$ by reducing the problem to the finite dimensional case (compare with [7], [18], [13]). Our main results of Section 7 are Theorem 7.19 and the adjustment functor (7.21)

$$G:\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}\longrightarrow \mathrm{latt}(\Lambda) \tag{1.4}$$

which is full, dense, preserves tameness and wildness, and vanishes only on finitely many indecomposable modules up to isomorphism. Every module in $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ has the projective dimension at most 1.

In a subsequent paper the interpretation of Λ-lattices in terms of Γ-complete propartite modules over the infinite dimensional algebra $\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ will be essentially used in defining a covering technique for $\mathrm{latt}(\Lambda)$ (compare with [12]).

In case the bipartite algebra $R=\begin{pmatrix}A & M\\ 0 & B\end{pmatrix}$ is finite dimensional our main results are the following statements proved in Sections 4-6.

(a) The category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ has Auslander-Reiten sequences, source maps and sink maps, and has enough relative projective and relative injective objects. Sink maps ending at indecomposable relative projective objects in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ are described in Corollary 4.8.

(b) The category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is equivalent to the category of prinjective modules over a bipartite algebra, and to the category $\mathrm{rep}(\mathfrak{B}_R)$ of K-linear representations of a free triangular bocs $\mathfrak{B}_R$ associated to R. The equivalences preserve tameness and wildness.

(c) Tame-wild dichotomy holds for the categories $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, and $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is an open subcategory of $\mathrm{mod}(R)$ in the sense of [5]. The representation properties of the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ are studied by means of affine varieties prop_v^R and an algebraic group action $\mathbf{G}_v^R\times\mathrm{prop}_v^R\to\mathrm{prop}_v^R$, where v is a vector in the Grothendieck group $\mathbf{K}_0\left(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A\right)\cong \boldsymbol{Z}^{n+m}$. In particular the tameness of $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and of $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is characterized by a dimension condition of $\mathbf{G}_v^R$-orbits of prop_v^R and of $\widehat{\mathrm{prop}}_v^R$.

One of the main results of this paper having important consequences in applications (see [16] and Section 7) is Theorem 6.10 which asserts that the adjustment functors in diagram (2.2) preserve and reflect tame representation type and the polynomial growth property. If R is finite dimensional, then they preserve also wildness.

Let us recall that the propartite modules over a class of finite dimensional algebras were considered by Green and Reiner [7, p. 61] in a connection with the study of lattices over orders, and by the author [22, Theorem 17.81] in a connection with a module-theoretical description of arbitrary bipartite matrix problems.

It was shown in [**22**, Theorem 17.81] that for any finite dimensional bipartite K-algebra $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ there exists a bipartite K-algebra $\bar{R}=\begin{pmatrix} A & \bar{M} \\ 0 & B \end{pmatrix}$, with $\bar{M}_B$ B-projective, and an equivalence of categories $\mathrm{prin}(R)^A_B \cong \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(\bar{R})^A_B$.

Let us mention that propartite modules appear implicitly in the Drozd's study of module categories by means of representations of bocses. We recall that for any finite dimensional K-algebra S the study of the category $\mathrm{mod}(S)$ was reduced by Drozd [**4**, Proposition 13] to the study of the category $\mathrm{rep}(\mathfrak{B}_S)$ of K-linear representations of a free triangular bocs $\mathfrak{B}_S$ associated to S in such a way that $\mathrm{rep}(\mathfrak{B}_S)$ is equivalent to the full subcategory $\widetilde{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$ of $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$ consisting of all modules (X'_S, X''_S, φ) such that $\mathrm{Im}\,\varphi \subseteqq \mathrm{rad}(X'')$. Moreover the equivalence respects representation types. The construction of $\mathfrak{B}_S$ involves the bipartite algebra $\begin{pmatrix} S & J^* \\ 0 & S \end{pmatrix}$, where $J=\mathrm{rad}(S)$ and $J^*=\mathrm{Hom}_K(J, K)$ (see [**2**, Proposition 6.1]). It was shown in [**25**, Lemma 2.7] that $\widetilde{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} S & S \\ 0 & S \end{pmatrix} \cong \mathrm{prin}\begin{pmatrix} S & J^* \\ 0 & S \end{pmatrix}$.

The main connection between $\mathrm{rep}(\mathfrak{B}_S)$ and $\mathrm{mod}(S)$ is given by the cokernel functor $\mathrm{rep}(\mathfrak{B}_S) \cong \widetilde{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} S & S \\ 0 & S \end{pmatrix} \to \mathrm{mod}(S)$, $(X'_S, X''_S, \varphi) \mapsto \mathrm{Coker}\,\varphi$. Our functor (1.4) is a counterpart of this construction for the category $\mathrm{latt}(\Lambda)$.

Main results of this paper were presented on the representation theory seminar in the University of Paderborn in May 1993.

2. Propartite modules, the adjustment functor and the category of matrices.

Throughout this paper we suppose that $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a bipartite K-algebra, where the algebras A and B are semi-perfect. We fix two complete sets

$$(2.1) \qquad \{e_1, \cdots, e_n\} \subseteqq A, \quad \{e_{n+1}, \cdots, e_{n+m}\} \subseteqq B$$

of primitive orthogonal idempotents in A and B, respectively.

A B-complete module $X=(X'_A, X''_B, \varphi)$ in $\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is defined to be *superfluous* if the kernel of the map $\bar{\varphi}$ adjoint to φ (1.3) is contained in the radical $\mathrm{rad}(X'_A)$ of X'_A. We denote by $\cdot\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the full subcategory of $\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ consisting of all superfluous modules.

Following [**21**] and [**16**] we call a module X in $\mathrm{mod}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *projectively adjusted* if X'_A is in $\mathrm{mod}(A)$, X''_B is in $\mathrm{pr}(B)$ and $\bar{\varphi}$ is injective. If, in addition, φ is surjective we call X *B-complete*. We denote by $\mathrm{mod}_{\mathrm{pr}}(R)^A_B$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}(R)^A_B$ the category of projectively adjusted modules and B-complete projectively adjusted modules over $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, respectively. Consider the commutative diagram

$$
(2.2)\qquad
\begin{array}{ccccc}
\bullet\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \hookrightarrow & \widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \hookrightarrow & \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \\
 & {\scriptstyle \bullet\hat{\Theta}^A}\searrow & \downarrow{\scriptstyle \hat{\Theta}^A} & & \downarrow{\scriptstyle \Theta^A} \\
 & & \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \hookrightarrow & \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}
\end{array}
$$

where Θ^A is the adjustment functor defined by the formula $\Theta^A(X)=(\mathrm{Im}\,\bar{\varphi},\, X''_B,\, \tilde{\varphi})$ and $\tilde{\varphi}:\mathrm{Im}\,\bar{\varphi}\otimes_A M_B \to X''_B$ is the map adjoint to the inclusion $\mathrm{Im}\,\bar{\varphi} \hookrightarrow \mathrm{Hom}_B({}_AM_B, X''_B)$ (see [**20**]). We denote by

$$\varepsilon_X : X \longrightarrow \Theta^A(X)$$

the canonical epimorphism. The functors $\bullet\hat{\Theta}^A$ and $\hat{\Theta}^A$ are the restrictions of Θ^A to $\bullet\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ and to $\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, respectively.

If $X=(X'_A, X''_B, \varphi)$ is a module in one of the categories $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ and $\mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, then X''_B is in $\mathrm{pr}(B)$ and there are decompositions

$$P_A(X'_A) \cong (e_1A)^{s_1}\oplus \cdots \oplus(e_nA)^{s_n}, \quad X''_B \cong (e_{n+1}B)^{s_{n+1}}\oplus \cdots\oplus(e_{n+m}B)^{s_{n+m}}$$

where $P_A(X'_A)$ is the A-projective cover of the A-module X'_A. The bipartite integral vector

$$(2.3)\qquad \mathrm{cdn}(X) = (s_1, \cdots, s_n\,;\, s_{n+1}, \cdots, s_{n+m}) \in \boldsymbol{N}^n\times\boldsymbol{N}^m$$

is called a *coordinate vector* of X (compare with [**22**, Section 17.9]).

PROPOSITION 2.4. (a) *The adjustment functors* Θ^A *and* $\hat{\Theta}^A$ *in* (2.2) *are full dense, and* $\mathrm{Ker}\,\Theta^A=[\mathrm{pr}(A)]$, $\mathrm{Ker}\,\hat{\Theta}^A=[\mathrm{pr}(A)]$, *that is,* $\mathrm{Ker}\,\Theta^A$ *and* $\mathrm{Ker}\,\hat{\Theta}^A$ *are the two-sided ideals in* $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R)^A_B$ *and in* $\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *respectively, consisting of all homomorphisms having a factorization through a module in* $\mathrm{pr}(A)$ *viewed as a full subcategory of* $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R)^A_B$ *and of* $\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *via the algebra surjection* $R\to A$.

(b) *For any* X *in* $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R)^A_B$ *and any* Y *in* $\mathrm{mod}_{\mathrm{pr}}(R)^A_B$ *there exists a natural isomorphism* $\mathrm{Hom}_R(\Theta^A(X), Y)\cong\mathrm{Hom}_R(X, Y)$. *For any* Z *in* $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R)^A_B$ *the map* $\varepsilon^*_X: \mathrm{Hom}_R(Z, X)\to\mathrm{Hom}_R(Z, \Theta^A(X))$ *induced by the canonical epimorphism* $\varepsilon_X: X\to\Theta^A(X)$ *is an epimorphism.*

(c) *Assume that* $X=(X'_A, X''_B, \varphi)$ *is an indecomposable module in* $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R)^A_B$. *Then* $\Theta^A(X)=0$ *if and only if* $X\cong e_jA$ *for some* $j=1, \cdots, n$. *If* $\Theta^A(X)\neq 0$ *then* $\Theta^A(X)$ *is indecomposable, the natural epimorphism* $X'_A\to\mathrm{Im}\,\varphi$ *is the* A*-projective cover and*

$$(2.5)\qquad \mathrm{cdn}\,\Theta^A(X) = \mathrm{cdn}\,X\,.$$

(d) *The functor* $\cdot\hat{\Theta}^A : \cdot\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \to \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a representation equivalence.*

PROOF. The proof of Lemma 11.33 in [22] extends to our situation. We leave it to the reader. □

Following an idea in [7] and [18] we shall show in Section 7 that for any order Λ in a semisimple algebra C there is an equivalence of categories $\mathrm{latt}(\Lambda) \cong \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}$, where Γ is a maximal order in C containing Λ. In the present paper we shall apply this equivalence to the study of the representation type of the category $\mathrm{latt}(\Lambda)$ of lattices over an order Λ.

Let us finish this section by describing a functorial connection between the category of propartite modules over a bipartite algebra and the category of matrices $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$ in the sense of Drozd defined as follows (see [4] or [22, Section 17.9]).

Let $\boldsymbol{K}$ and $\boldsymbol{L}$ be additive categories having the finite unique decomposition property (see [22, Chapter 1]). Suppose that ${}_{L}\mathbf{M}_{K}$ is an $\boldsymbol{L}$-$\boldsymbol{K}$-bimodule, that is,

$$\mathbf{M} : \boldsymbol{K}^{\mathrm{op}} \times \boldsymbol{L} \longrightarrow \mathcal{A}b \tag{2.6}$$

is an additive functor, where $\mathcal{A}b$ is the category of abelian groups. The objects of $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$ are triples (x, y, m), where $x \in \mathrm{ob}\,\boldsymbol{K}$, $y \in \mathrm{ob}\,\boldsymbol{L}$ and $m \in \mathbf{M}(x, y)$. A morphism from (x, y, m) to (x', y', m') in $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$ is a pair (φ, ϕ), where $\varphi \in \boldsymbol{K}(x, x')$, $\phi \in \boldsymbol{L}(y, y')$ are such that $\mathbf{M}(x, \phi)m = \mathbf{M}(\varphi, y')m'$.

It is easy to check that $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$ is an additive category with the finite unique decomposition property. The direct sum of two objects (x, y, m) and (x', y', m') of $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$ is the object $(x \oplus x', y \oplus y', m \oplus m')$, where

$$m \oplus m' = \begin{pmatrix} m & 0 \\ 0 & m' \end{pmatrix} \in \begin{pmatrix} \mathbf{M}(x, y) & \mathbf{M}(x, y') \\ \mathbf{M}(x', y) & \mathbf{M}(x', y') \end{pmatrix} = \mathbf{M}(x \oplus x', y \oplus y')$$

under the obvious identifications. By a *bipartite bimodule matrix problem* we shall mean the classification of indecomposable objects in the category $\mathrm{Mat}({}_{L}\mathbf{M}_{K})$.

Assume that the sets of representatives of the isomorphism classes of the indecomposable objects

$$\mathrm{ind}\,\boldsymbol{K} = \{x_1, \cdots, x_n\}, \quad \mathrm{ind}\,\boldsymbol{L} = \{y_1, \cdots, y_m\}$$

of $\boldsymbol{K}$ and $\boldsymbol{L}$ are finite. We set

$$X_K = x_1 \oplus \cdots \oplus x_n \quad \text{and} \quad Y_L = y_1 \oplus \cdots \oplus y_m .$$

In this case we associate with the bimodule ${}_{L}\mathbf{M}_{K}$ the bipartite ring

$$\mathbf{R}'_{\mathrm{M}}=\begin{pmatrix} A & {}_AM'_B \\ 0 & B \end{pmatrix} \tag{2.7}$$

where $A=\boldsymbol{K}(X_K, X_K)$, $B=\boldsymbol{L}(Y_L, Y_L)$ and ${}_AM'_B=\mathrm{Hom}_B(\mathbf{M}(X_K, Y_L), B)$.

THEOREM 2.8. *Let ${}_L\mathbf{M}_K$ be an $\boldsymbol{K}$-$\boldsymbol{L}$-bimodule satisfying the conditions stated above and let $\mathbf{R}'_{\mathrm{M}}$ be the bipartite ring* (2.7) *associated with ${}_L\mathbf{M}_K$. If ${}_L\mathbf{M}$ is finitely generated projective viewed as a left $\boldsymbol{L}$-module then there exists an equivalence of categories (see* (2.10) *below) $\mu^*: \mathrm{Mat}({}_L\mathbf{M}_K)\to \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathbf{R}'_{\mathrm{M}})$.*

PROOF. Let $X=X_K$, $Y=Y_L$ and let $\omega: \boldsymbol{K}\to \mathrm{pr}(A)$, $\omega': \boldsymbol{L}\to\mathrm{pr}(B)$ be the Yoneda equivalences given by the formulas $\omega(-)=\boldsymbol{K}(X, -)$, $\omega'(-)=\boldsymbol{L}(Y, -)$, where $\mathrm{pr}(A)$ and $\mathrm{pr}(B)$ are the categories of finitely generated projective modules over A and B, respectively.

We recall that the correspondences $H\mapsto H(X)$ and $T\mapsto T(Y)$ define equivalences of categories

$$\mathrm{Add}(\boldsymbol{K}^{\mathrm{op}}, \mathcal{A}b)\cong \mathrm{Mod}(A), \quad \text{and} \quad \mathrm{Add}(\boldsymbol{L}, \mathcal{A}b)\cong \mathrm{Mod}(B^{op})$$

where $\mathrm{Add}(\boldsymbol{K}^{\mathrm{op}}, \mathcal{A}b)$ is the category of all additive contravariant functors $H: \boldsymbol{K}\to\mathcal{A}b$ and $\mathrm{Add}(\boldsymbol{L}, \mathcal{A}b)$ is the category of all additive covariant functors $T: \boldsymbol{L}\to\mathcal{A}b$. By our assumption, $M(X, Y)$ viewed as a left module over $B=\mathrm{End}(Y)$ is finitely generated projective. This together with Yoneda's Lemma yields natural isomorphisms

$$\begin{aligned} \mathbf{M}(x, y) &\cong \mathrm{Nat}(\boldsymbol{K}(-, x), \mathbf{M}(-, y)) \\ &\cong \mathrm{Hom}_A(\boldsymbol{K}(X, x), \mathbf{M}(X, y)) \\ &\cong \mathrm{Hom}_A(\boldsymbol{K}(X, x), \boldsymbol{L}(Y, y)\otimes_B \mathbf{M}(X, Y)) \\ &\cong \mathrm{Hom}_A(\omega(x), \mathrm{Hom}_B({}_AM'_B, \omega'(y))) \\ &\cong \mathrm{Hom}_B(\omega(x)\otimes_A M'_B, \omega'(y)) \end{aligned} \tag{2.9}$$

where $\mathrm{Nat}(\boldsymbol{K}(x, -), \mathbf{M}(-, y))$ is the abelian group consisting of all natural transformations of functors $\boldsymbol{K}(x, -)\to\mathbf{M}(-, y)$. The fourth isomorphism in (2.9) is a consequence of the fact that the map

$$P'\otimes_B \mathbf{M}(X, Y)\longrightarrow \mathrm{Hom}_B({}_AM'_B, P'), \quad p'\otimes m\longmapsto (\varphi\mapsto p'\cdot\varphi(m)),$$

is natural with respect to B-homomorphisms $P'\to P'_1$, and it is bijective if $P'=B$ or if P' is a finitely generated projective right B-module.

It follows that the functor

$$\mu^*: \mathrm{Mat}({}_L\mathbf{M}_K)\longrightarrow \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(\mathbf{R}'_{\mathrm{M}}) \tag{2.10}$$

$(x, y, m)\mapsto(\omega(x), \omega'(y), \mu(m))$, is an equivalence of categories, where μ is the composed isomorphism (2.9). □

3. The Grothendieck group and relatively projective propartite modules.

Throughout this section we suppose that R is a semiperfect bipartite K-algebra of the form (1.1).

DEFINITION 3.1. The Grothendieck group $\mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$ of $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is the abelian group generated by the isomorphism classes $[X]$ of modules X in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ subject to the set of relations $[X]=[X']+[X'']$ corresponding to all exact sequences

$$(*) \qquad 0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

in $\mathrm{mod}(R)$ with all terms in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$. □

LEMMA 3.2. *Let* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *be a basic semiperfect bipartite* K*-algebra.*

(a) *If* $0\to X\to Y\to Z\to 0$ *is an exact sequence in* $\mathrm{mod}(R)$ *consisting of propartite modules then* $\mathrm{cdn}(X)+\mathrm{cdn}(Z)=\mathrm{cdn}(Y)$.

(b) *The map* $X\mapsto \mathrm{cdn}(X)$ *induces a group isomorphism*

$$(**) \qquad \mathrm{cdn}: \mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A) \longrightarrow \boldsymbol{Z}^{n+m}$$

and the elements $[e_1A], \cdots, [e_nA], [e_{n+1}B], \cdots, [e_{n+m}B]$ *form a set of free generators of the group* $\mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$.

If, in addition, $\dim_K R$ *is finite and* $C(A)=[a'_{ij}]$, $C(B)=[b'_{st}]$ *are the Cartan matrices of* A *and* B *(see* [**16**]*) with respect to the fixed complete sets* $\{e_1, \cdots, e_n\}$, $\{e_{n+1}, \cdots, e_{n+m}\}$ (2.1) *of primitive orthogonal idempotents of* A *and* B, *where* $a'_{ij}=\dim_K(e_jAe_i)$, $b'_{st}=\dim_K(e_tBe_s)$, *then the following statements hold.*

(c) *The natural group homomorphism* $\beta: \mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)\to \mathbf{K}_0(\mathrm{mod}(R))$, $[X] \mapsto [X]$, *makes the diagram*

$$(3.3) \qquad \begin{array}{ccc} \mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A) & \xrightarrow{\mathrm{cdn}} & \boldsymbol{Z}^{n+m} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle d^{\bullet}} \\ \mathbf{K}_0(\mathrm{mod}(R)) & \xrightarrow{\mathrm{dim}} & \boldsymbol{Z}^{n+m} \end{array}$$

commutative, where $\dim$ *is the group isomorphism induced by attaching to any* R*-module* X *the dimension vector* $\dim(X)$ *of* X, *and* $d^{\bullet}$ *is the group homomorphism defined by the formula*

$$(3.4) \qquad d^v = v\cdot\begin{pmatrix} C(A) & 0 \\ 0 & C(B) \end{pmatrix}^{\mathrm{tr}}.$$

In particular, the equality

$$\dim(X) = \mathrm{cdn}(X)\cdot\begin{pmatrix} C(A) & 0 \\ 0 & C(B) \end{pmatrix}^{\mathrm{tr}} = d^{\mathrm{cdn}\, X}$$

holds for any module X in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$.

(d) *The homomorphism β is an isomorphism if and only if the matrix $C(A)C(B)$ is $\boldsymbol{Z}$-invertible. If this is the case then $v=d^v\begin{pmatrix} C(A)^{\mathrm{tr}} & 0 \\ 0 & C(B)^{\mathrm{tr}} \end{pmatrix}^{-1}$ for any vector $v\in\boldsymbol{Z}^{n+m}$.*

PROOF. The statement (a) follows immediately from the definition.

(b) It follows from (a) that the group homomorphism (**) is well defined. Given a module $X=(X'_A, X''_B, \varphi)$ in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ with $v=\mathrm{cdn}(X)$ there is an exact sequence $0\to X''_B\to X\to X'_A\to 0$, where

$$X'_A \cong (e_1A)^{s_1}\oplus\cdots\oplus(e_nA)^{s_n}, \quad X''_B \cong (e_{n+1}B)^{s_{n+1}}\oplus\cdots\oplus(e_{n+m}B)^{s_{n+m}}.$$

It follows that the equality

$$[X] = \sum_{j=1}^{n} s_j[e_jA]+\sum_{i=1}^{m} s_{n+i}[e_{n+i}B]$$

holds in the group $\mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$ and therefore the set

$$\mathcal{Z} = \{[e_1A], \cdots, [e_nA], [e_{n+1}B], \cdots, [e_{n+m}B]\}$$

generates the group $\mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$. Since $\mathrm{cdn}[e_{n+i}B]=\xi_{n+i}$ for $i=1, \cdots, m$, and $\mathrm{cdn}[e_jA]=\xi_j$, for $j\leq n$, where ξ_j is the j-th standard basis vector of $\boldsymbol{Z}^{n+m}$, then the set $\mathcal{Z}$ is a free basis of the group $\mathbf{K}_0(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$ and the map (**) is an isomorphism.

It follows from the definition that β is a well defined group homomorphism and a simple computation shows that the diagram (3.3) is commutative. Since the remaining parts of (c) and (d) are easy, the proof is complete. □

Suppose that R is a semiperfect bipartite K-algebra of the form (1.1). An important role in a study of the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ of propartite R-modules is played by the following family of indecomposable modules in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$:

$$\bar{P}_1, \cdots, \bar{P}_n, P_{n+1}, \cdots, P_{n+m} \tag{3.5}$$

where $P_j=e_jR$ for $j=1, \cdots, n+m$ and we set

$$\bar{P}_j = (e_jA, \overline{e_jM_B}, t_j) \quad \text{for } j\leq n,$$

where

$$\overline{e_jM_B} = \mathrm{Hom}_B(P_B(\mathrm{Hom}_B(e_jM_B, B)), B)$$

$P_B({}_BZ)$ is the B-projective cover of a B-module ${}_BZ$ and $t_j\colon e_jA\otimes_A M_B\to\overline{e_jM_B}$ is the composed B-homomorphism

$$e_jA\otimes_A M_B \cong e_jM_B \xrightarrow{\mathrm{ev}} \mathrm{Hom}_B(\mathrm{Hom}_B(e_jM_B, B), B) \xrightarrow{\varepsilon_j^*} \overline{e_jM_B}.$$

Here $\mathrm{ev}: e_jM_B \to \mathrm{Hom}_B(\mathrm{Hom}_B(e_jM_B, B), B)$, $m \mapsto (\varphi \mapsto \varphi(m))$, is the evaluation homomorphism and

$$\varepsilon_j^*: \mathrm{Hom}_B(\mathrm{Hom}_B(e_jM_B, B), B) \longrightarrow \mathrm{Hom}_B(P_B(\mathrm{Hom}_B(e_jM_B, B)), B) = \overline{e_jM_B}$$

is the map induced by the projective cover epimorphism

$$\varepsilon_j: P_B(\mathrm{Hom}_B(e_jM_B, B)) \longrightarrow \mathrm{Hom}_B(e_jM_B, B).$$

We note that if M_B is a finitely generated projective B-module then $\overline{e_jM_B}=e_jM_B$ for any j, and therefore $\bar{P}_1=P_1, \cdots, \bar{P}_n=P_n$.

For any $j=1, \cdots, n$ and $i=n+1, \cdots, n+m$, we consider the homomorphisms

$$(3.6)\qquad \begin{aligned} &(\xi_j, \mathrm{id}): (P_A(\mathrm{rad}\, e_jA), \overline{e_jM_B}, t_j') \longrightarrow \bar{P}_j \\ &(0, \hat{\varepsilon}_i): (\hat{P}_i, P_B(\mathrm{rad}\, P_i), \hat{\varepsilon}_i) \longrightarrow P_i \end{aligned}$$

where ξ_j is the composed map $P_A(\mathrm{rad}\, e_jA) \to \mathrm{rad}(e_jA) \hookrightarrow e_jA$, $t_j'=t_j\ (\xi_j \otimes \mathrm{id})$, ε_i is the composed map $P_B(\mathrm{rad}\, e_iB) \to \mathrm{rad}(e_iB) \hookrightarrow e_iB=P_i$, $\hat{P}_i$ is the A-projective cover of the kernel of the homomorphism

$$\mathrm{Hom}_B({}_AM_B, \varepsilon_i): \mathrm{Hom}_B({}_AM_B, P_B(\mathrm{rad}\, e_iB)) \longrightarrow \mathrm{Hom}_B({}_AM_B, e_iB)$$

and $\hat{\varepsilon}_i: \hat{P}_i \otimes_A M_B \to P_B(\mathrm{rad}\, P_i)$ is the adjoint map to the natural homomorphism $\hat{P}_i \to \mathrm{Hom}_B({}_AM_B, P_B(\mathrm{rad}\, P_i))$. □

A module N in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is said to be *relatively projective* (resp. *relative injective*) if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}(R)$, with X, Y, Z in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, the induced map $\mathrm{Hom}_R(N, Y) \to \mathrm{Hom}_R(N, Z)$ is surjective (resp. $\mathrm{Hom}_R(Y, N) \to \mathrm{Hom}_R(X, N)$ is injective).

We recall from [**24**] that a subcategory $\mathcal{A}$ of $\mathrm{Mod}(R)$ is defined to be a *representation subcategory* of $\mathrm{Mod}(R)$ if the following conditions are satisfied:

(a) $\mathcal{A}$ is a full subcategory of $\mathrm{mod}(R)$, which is closed under finite direct sums, summands, extensions and isomorphic images.

(b) $\mathcal{A}$ has the finite unique decomposition property (see [**22**, Chapter 1]).

The main homological properties of the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ are listed in the following proposition.

PROPOSITION 3.7. *Suppose that* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a bipartite semiperfect ring. If the right B-module M_B is finitely generated projective or the algebra R is finite dimensional over a field K, then the following statements hold.*

(a) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is a representation subcategory of* $\mathrm{mod}(R)$.

(b) *If X is an R-module in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *then there exists an exact sequence*

$$(3.8)\qquad 0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{mod}(R)$, *where* L_0 *is a direct sum of copies of modules* (3.5) *and* L_1 *is a projective* B-*module, which is a direct sum of copies of modules* $P_{n+1}, \cdots, P_{n+m}$. *If* M_B *is* B-*projective,* L_0 *is projective and* $\mathrm{pd}_R X \leqq 1$.

(c) *A module* L *in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is relatively projective if and only if* $\mathrm{Ext}_R^1(L, X)=0$ *for all* X *in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$. *The modules* (3.5) *form a complete set of pairwise non-isomorphic indecomposable relatively projective modules in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$.

(d) *If* M_B *is* B-*projective, then* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is a hereditary subcategory of* $\mathrm{mod}(R)$, *that is,* $\mathrm{Ext}_R^2(X, Y)=0$ *for all* X, Y *in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$.

PROOF. (a) Follows immediately from definitions.

(b) Assume first that M_B is finitely generated projective and let $X=(X'_A, X''_B, \varphi)$ be a propartite R-module. Since X'_A is projective and the right B-module M_B is projective, then the R-module $L=(X'_A, X'\oplus_A M_B, \mathrm{id})$ is projective and therefore L is a direct sum of copies of the projective modules $\bar{P}_1=P_1, \cdots, \bar{P}_n=P_n$. The maps $\mathrm{id}_{X'_A}$ and $\mathrm{id}_{X''_B}$ induce an exact sequence

$$0 \longrightarrow (0, Y_B, 0) \longrightarrow L\oplus(0, X''_B, 0) \longrightarrow X \longrightarrow 0$$

in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, where Y_B is a projective B-module. Since the modules $(0, Y_B, 0)$, $(0, X''_B, 0)$ are direct sum of copies of $P_{n+1}, \cdots, P_{n+m}$ then by taking $L_1=(0, Y_B, 0)$, $L_0=L\oplus(0, X''_B, 0)$ we get the exact sequence required in (b), which is a projective resolution of X. Then $\mathrm{pd}_R X\leqq 1$ as required.

If R is a finite dimensional algebra the statement (b) follows from Proposition 4.5 proved below.

(c) If M_B is a finitely generated projective B-module then $\overline{e_j M_B}=e_j M_B$, $\bar{P}_1=P_1, \cdots, \bar{P}_n=P_n$ and $P_{n+1}, \cdots, P_{n+m}$ are projective modules and therefore they are relatively projective in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. If R is a finite dimensional algebra then according to Proposition 4.5 below the modules $\bar{P}_1, \cdots, \bar{P}_n$ and $P_{n+1}, \cdots, P_{n+m}$ are relative projective. Then (c) follows immediately from (b).

(d) By our assumption and the final part of (b) the exact sequence (3.8) is a projective resolution of X and therefore (d) follows.

4. Propartite modules over artin algebras and representations of bocses.

The aim of this section is to describe a relation between propartite modules, prinjective modules and representations of bocses.

Assume that B is an artin algebra and let $D: \mathrm{mod}(B)\to\mathrm{mod}(B^{op})$ be the standard duality. Then the Nakayama functor (see [28])

$$\mathfrak{N}_B = D\,\mathrm{Hom}(-, B): \mathrm{mod}(B) \longrightarrow \mathrm{mod}(B) \tag{4.1}$$

induces the Nakayama equivalence

(4.2) $$\mathfrak{N}_B : \mathrm{pr}(B) \xrightarrow[\cong]{} \mathrm{inj}(B)$$

where $\mathrm{pr}(B)$ and $\mathrm{inj}(B)$ are the full subcategories of $\mathrm{mod}(B)$ consisting of projective and injective modules, respectively.

Assume that X_B is a module in $\mathrm{pr}(B)$. Then for any left module N in $\mathrm{mod}(B^{op})$ there is a natural isomorphism (see [**16**, (0.3)] or [**22**, 17.79])

(4.3) $$X \otimes N \xrightarrow[\cong]{} \mathrm{Hom}_B(D(N), \mathfrak{N}_B(X)).$$

Further, we observe that for any module M in $\mathrm{mod}(B)$ and X in $\mathrm{pr}(B)$ the Nakayama functor $\mathfrak{N}_B$ induces the natural isomorphism

(4.4) $$\mathfrak{N}_B^{\sharp} : \mathrm{Hom}_B(M, X) \xrightarrow[\cong]{} \mathrm{Hom}_B(\mathfrak{N}_B(M), \mathfrak{N}_B(X)), \quad \varphi \mapsto \mathfrak{N}_B(\varphi).$$

For this purpose we note that the map

$$X \otimes_B \mathrm{Hom}_B(M, B) \longrightarrow \mathrm{Hom}_B(M, X), \quad x \otimes \varphi \mapsto (m \mapsto x \cdot \varphi(m)),$$

is natural in the B-module X_B, and is bijective if X_B is finitely generated projective. Then, by applying (4.3) to $N = \mathrm{Hom}_B(M, B)$, we get (4.4).

The main connection between propartite modules and prinjective modules is given by the following result.

PROPOSITION 4.5. *Suppose that* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is an artin algebra.*

(a) *There exists a commutative diagram*

(4.6) $$\begin{array}{ccc} \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow[\cong]{T} & \mathrm{prin}\begin{pmatrix} A & \hat{M} \\ 0 & B \end{pmatrix} \\ \downarrow \Theta^A & & \downarrow \Theta^A \\ \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow[\cong]{T'} & \mathrm{mod}_{\mathrm{ic}}\begin{pmatrix} A & \hat{M} \\ 0 & B \end{pmatrix} \end{array}$$

where ${}_A\hat{M}_B = \mathfrak{N}_B({}_AM_B)$, *the right-hand functor* Θ^A *is defined in* [**16**, (3.1)], *the functors* T *and* T' *induced by* $\mathfrak{N}_B$ *and defined below (see* [**22**, 17.84]) *are equivalences of categories and the following conditions are satisfied.*

(b) *The functors* T *and* T' *carry exact sequences to exact ones, induce the group isomorphisms*

$$\mathrm{Ext}_R^1(X, Y) \cong \mathrm{Ext}_{\hat{R}}^1(T(X), T(Y)) \quad \textit{and} \quad \mathrm{Ext}_R^1(Z, U) \cong \mathrm{Ext}_{\hat{R}}^1(T'(Z), T'(U))$$

and the equalities $\mathrm{cdn}\, X = \mathrm{cdn}\, T(X)$ *and* $\mathrm{cdn}\, Z = \mathrm{cdn}\, T'(Z)$ *hold for all* X, Y *in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *and all* Z, U *in* $\mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $\mathrm{cdn}\, T(X)$ *and* $\mathrm{cdn}\, T'(Z)$ *are the coordinate vectors of the prinjective modules* $T(X)$ *and* $T'(Z)$ *defined in* [**16**].

(c) *The modules* $\bar{P}_1, \cdots, \bar{P}_n$ (3.5) *are relative projective in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

(d) *There exists an equivalence of categories*

$$\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \cong \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & \overline{M} \\ 0 & B \end{pmatrix}$$

where $\overline{M}$ is the bimodule

$$(4.7) \qquad {}_A\overline{M}_B = \mathrm{Hom}_B(P_B(\mathrm{Hom}_B({}_AM_B,\ B)),\ B)$$

which is a projective B-module.

PROOF. (a) Given a module $X=(X'_A,\ X''_B,\ \varphi)$ in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ or in $\mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ we set $T(X)=T'(X)=(X'_A,\ \mathfrak{N}_B(X''_B),\ \hat{\varphi})$, where $\hat{\varphi} : X'\otimes_A \hat{M}_B \to \mathfrak{N}_B(X''_B)$ is the B-module homomorphism adjoint to the composed A-module one

$$X'_A \xrightarrow{\bar{\varphi}} \mathrm{Hom}_B({}_AM_B,\ X''_B) \cong \mathrm{Hom}_B({}_A\hat{M}_B,\ \mathfrak{N}_B(X''_B)).$$

The isomorphism follows from (4.4). It is clear that T and T' are equivalences of categories making the diagram (4.6) commutative.

The proof of (b) follows immediately from definitions.

(c) It follows from (b) that a module X in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is relative projective if and only if the module TX is prin-projective. Moreover, there is an isomorphism $\overline{P}_1\oplus\cdots\oplus\overline{P}_n\cong(A,\ \overline{M}_B,\ t)$, where ${}_A\overline{M}_B$ is the bimodule (4.7) and $t=\varepsilon\cdot\mathrm{ev} : A\otimes_A M_B \to \overline{M}_B$ is defined in a similar way as the map t_j in the formula (3.5). Then it is sufficient to show that $T(A,\ \overline{M}_B,\ t)$ is a prin-projective module. For this purpose we observe that (4.4) applied to $X_B=\overline{M}_B$, together with the composed B-isomorphism

$$\mathfrak{N}_B(\overline{M}_B) \cong DP_B(\mathrm{Hom}_B(M_B,\ B)) \cong E_B(D\mathrm{Hom}_B(M_B,\ B)) \cong E_B(\hat{M}_B)$$

induces the composed A-module isomorphism

$$\begin{aligned}\mathrm{Hom}_B(A\otimes_A M_B,\ \overline{M}_B) &\cong \mathrm{Hom}_B(M_B,\ \overline{M}_B)\\ &\cong \mathrm{Hom}_B(\hat{M}_B,\ \mathfrak{N}_B(\overline{M}_B))\\ &\cong \mathrm{Hom}_B(\hat{M}_B,\ E_B(\hat{M}_B)),\end{aligned}$$

which carries the B-homomorphism $t : A\otimes_A M_B \to {}_A\overline{M}_B$ to the inclusion B-homomorphism $s' : \hat{M}_B \to E_B(\hat{M}_B)$ from $\hat{M}_B=\mathfrak{N}_B(M_B)$ to its injective envelope $E_B(\hat{M}_B)$. It follows that $T(A_A,\ \overline{M}_B,\ t)\cong(A_A,\ E_B(\hat{M}_B),\ s')$, and in view of [**16**, Proposition 2.4] the module $(A_A,\ E_B(\hat{M}_B),\ s')$ is prin-projective. This finishes the proof of (c).

(d) Let $T(\overline{P}_j)=\hat{P}_j$ for $j=1,\cdots,n$ and let $T(P_{n+i})={}^{\circ}Q_i$ for $i=1,\cdots,m$. It follows from Theorem 17.89 in [**22**] that $\mathrm{prin}\begin{pmatrix} A & \hat{M} \\ 0 & B \end{pmatrix}\cong\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M' \\ 0 & B \end{pmatrix}$, where

$$\begin{aligned}{}_AM'_B &= \mathrm{Hom}({}^{\circ}Q_1\oplus\cdots\oplus{}^{\circ}Q_m,\ \hat{P}_1\oplus\cdots\oplus\hat{P}_n)\\ &\underset{\cong}{\xrightarrow{T^{-1}}} \mathrm{Hom}(P_{n+1}\oplus\cdots\oplus P_{n+m},\ \overline{P}_1\oplus\cdots\oplus\overline{P}_n)\end{aligned}$$

$$\cong \operatorname{Hom}_R(B, \bar{P}_1 \oplus \cdots \oplus \bar{P}_n)$$
$$\cong \operatorname{Hom}_B(B, {}_A\bar{M}_B) \cong {}_A\bar{M}_B$$

(see (4.7)). This together with (b) yields (d) and finishes the proof. □

COROLLARY 4.8. *Suppose that* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a bipartite artin algebra.*

(a) *The category* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is functorially finite. The categories* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *and* $\operatorname{mod}_{\mathrm{pr}}(R)_B^A$ *have Auslander-Reiten sequences, source morphisms and sink morphisms. The left-hand term of any Auslander-Reiten sequence in* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A \cong \operatorname{prin}\begin{pmatrix} A & \hat{M} \\ 0 & B \end{pmatrix}$ *can be constructed from the right-hand term by applying two partial Coxeter operators defined in* [**16**, (3.8)] *for prinjective modules.*

(b) *The homomorphisms* (3.6) *are minimal right almost split morphisms in* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$.

(c) *If* R *is a directed* K*-algebra,* K *is an algebraically closed field and* $\mathbf{b}_R(x, y)=\sum_{i,j=1}^n x_j y_i \dim_K(e_i A e_j)-\sum_{i=1}^n \sum_{s=1}^m x_i y_{n+s} \dim_K \operatorname{Hom}_B(e_i M, e_{n+t}B)+\sum_{s,t=n+1}^{n+m} x_t y_s \dim_K(e_s A e_t)$ *is the* $\boldsymbol{Z}$*-bilinear form of* R*, then for any pair of modules* X *and* Z *in* $\operatorname{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$

$$\mathbf{b}_R(\operatorname{cdn} X, \operatorname{cdn} Z) = \dim_K \operatorname{Hom}_R(X, Z) - \dim_K \operatorname{Ext}_R^1(X, Z). \tag{4.8'}$$

PROOF. The statement (a) is a consequence of Proposition 4.5, [**26**] and [**16**, Theorem 3.4]. The statement (c) is a consequence of the exact sequence (4.3) in [**16**] (see also the proof of Proposition 11.93 in [**22**]).

(b) It follows from Proposition 4.5 that f is a right minimal almost split morphism if and only if $T(f)$ is right minimal almost split. Since one easily check that the functor T carries the morphisms (ξ_j, id) to the right minimal almost split morphisms described in [**16**, 2.5] then (ξ_j, id) is minimal right almost split. Since the fact that $(0, \hat{\varepsilon}_i)$ is minimal right almost split follows by a straightforward calculation then (b) follows. □

Now we shall relate the categories of propartite modules with categories of representations of bocses. We recall that a *bocs* is a pair

$$\mathfrak{B} = (B, {}_B V_B)$$

where B is a K-algebra and ${}_B V_B$ is a B coalgebra, that is, V is a B-B-bimodule equipped with two B-B-bimodule maps $\varepsilon: V \to B$ (the counit) and $\mu: V \to V \otimes_B V$ (the comultiplication), satisfying the usual counity and coassociativity laws $(1\otimes\varepsilon)\mu=(\varepsilon\otimes 1)\mu=\mathrm{id}_V$ and $\mu(1\otimes\mu)=\mu(\mu\otimes 1)$.

If S is a finitely generated K-algebra then the category $\operatorname{rep}(\mathfrak{B}, S)$ of *left* S*-module representations of the bocs* $\mathfrak{B}=(B, {}_B V_B)$ has as objects the S-B-bimodules ${}_S X_B$ in $\operatorname{mod}(B\otimes S^{\mathrm{op}})$, which are finitely generated projective when viewed as left S-modules. A morphism from ${}_S X_B$ to ${}_S Y_B$ is the S-B-bimodule homomorphisms $f: {}_SX\otimes_B V_B \to {}_SY_B$. The composition of $f: {}_SX_B \to {}_SY_B$ and $g: {}_SY_B \to {}_SZ_B$ in rep($\mathfrak{B}$, S) is defined to be the composed S-B-bimodule homomorphism

$$ {}_SX\otimes_B V_B \xrightarrow{1\otimes\mu} {}_SX\otimes_B V\otimes_B V_B \xrightarrow{f\otimes 1} {}_SY\otimes_B V_B \xrightarrow{g} {}_SZ_B $$

(see [2], [4], [5]). We set $\mathrm{rep}_K(\mathfrak{B})=\mathrm{rep}(\mathfrak{B}, K)$ and we call it the category of K-linear representations of $\mathfrak{B}$. For the concepts of a free triangular bocs and the representation type of a bocs we refer to [2], [4], [5].

PROPOSITION 4.9. *Suppose that* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a bipartite finite dimensional algebra over an algebraically closed field* K.

(a) *There exist a free triangular bocs* $\mathfrak{B}_R$ *and an equivalence of categories*

$$ F_S : \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A\otimes S^{\mathrm{op}} & M\otimes S^{\mathrm{op}} \\ 0 & B\otimes S^{\mathrm{op}} \end{pmatrix}_{B\otimes S^{\mathrm{op}}}^{A\otimes S^{\mathrm{op}}} \longrightarrow \mathrm{rep}(\mathfrak{B}_R, S) \tag{4.10} $$

for any finitely generated K-algebra S. In particular the functor

$$ F_K : \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A \longrightarrow \mathrm{rep}_K(\mathfrak{B}_R) \tag{4.11} $$

is an equivalence of categories.

(b) *For any module L in* $\mathrm{fin}(S^{\mathrm{op}})$ *and any S-R-bimodule ${}_SX_A$ in the category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A\otimes S^{\mathrm{op}} & M\otimes S^{\mathrm{op}} \\ 0 & B\otimes S^{\mathrm{op}} \end{pmatrix}_{B\otimes S^{\mathrm{op}}}^{A\otimes S^{\mathrm{op}}}$ *there exists an isomorphism*

$$ F_K(L\otimes_S X_R) \cong L\otimes_S F_S({}_SX_R) \tag{4.12} $$

which is functorial with respect to the S-homomorphisms $L \to L'$ and S-R-bimodule maps ${}_SX_R \to {}_SX'_R$.

PROOF. Let ${}_B\overline{M}_A=\mathrm{Hom}_B({}_AM_B, B)$ and we view it as a B-A-bimodule. For any finitely generated K-algebra S we consider the category $\mathrm{rep}({}_B\overline{M}_A\,;S)$ of left S-representations of the bimodule ${}_B\overline{M}_A$, whose objects are the triples $({}_SV'_B, {}_SV''_A, t: {}_SV''_A \to {}_SV'\otimes_B\overline{M}_A)$, where ${}_SV'_B$ and ${}_SV''_A$ are bimodules which are finitely generated projective over S, and t is an S-A-bimodule homomorphism (see [4]). The morphism from $({}_SV'_B, {}_SV''_A, t)$ to $({}_SU'_B, {}_SU''_A, s)$ is a pair (f', f''), where $f'': {}_SV''_B \to {}_SU''_B$ and $f': {}_SV'_A \to {}_SU'_A$ are such that $(f'\otimes\mathrm{id})t=sf''$. Let us define the equivalence of categories

$$ F'_S : \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A\otimes S^{\mathrm{op}} & M\otimes S^{\mathrm{op}} \\ 0 & B\otimes S^{\mathrm{op}} \end{pmatrix}_{B\otimes S^{\mathrm{op}}}^{A\otimes S^{\mathrm{op}}} \longrightarrow \mathrm{rep}({}_B\overline{M}_A\,; S) $$

by the formula $F'_S({}_SX'_A, {}_SX''_B, \varphi)=({}_SX''_B, {}_SX'_A, \bar{\varphi})$, where $\bar{\varphi}$ is the composed S-A-bimodule homomorphism

$$ {}_SX'_A \xrightarrow{\bar{\varphi}} \mathrm{Hom}_B({}_AM_B, X''_B) \cong {}_SX''\otimes_B\overline{M}_A. $$

Let $\mathfrak{B}_R$ be the bocs associated with the bimodule ${}_B\overline{M}_A$ in [4, Proposition 11] or in [5, Theorem 1.1] (see also the proof of Proposition 6.1 in [2]) and let $F''_S: \operatorname{rep}({}_B\overline{M}_A\,;S)\to\operatorname{rep}(\mathfrak{B}_R, S)$ be the equivalence of categories defined there. Applying Drozd [4, Propositions 11 and 13] and [5, Theorem 1.1] (see also the proof of Proposition 6.1 in [2]) it is easy to check that the composed functor $F_S=F''_S\circ F'_S$ satisfies the required conditions. □

5. Varieties of propartite modules.

Suppose that $R=\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}$ is a bipartite finite dimensional algebra over an algebraically closed field K and we fix primitive orthogonal idempotents (2.1). Throughout this section we use the notation introduced in [22, Section 14.5].

Given a dimension vector $d=(d_A, d_B)\in \boldsymbol{Z}^n\times\boldsymbol{Z}^m$ we denote by $\operatorname{mod}^R(d)$ the affine variety (in Zariski topology) of all R-modules X of the dimension vector d, that is $\dim(X)=d$ (see [14], [15]). Let

(5.1) $$*:\operatorname{Gl}(d)\times\operatorname{mod}^R(d)\longrightarrow\operatorname{mod}^R(d)$$

be the action of the algebraic group

$$\operatorname{Gl}(d)=\operatorname{Gl}(d_A)\times\operatorname{Gl}(d_B)=\prod_{i=1}^{n+m}\operatorname{Gl}(d_i, K)$$

on the variety $\operatorname{mod}^R(d)$. We denote by $\operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}(d, R)$ and $\operatorname{indmod}^{\mathrm{pr}}_{\mathrm{pr}}(d, R)$ the subvarieties defined by propartite modules and indecomposable propartite modules respectively. To any bipartite coordinate vector

$$v=(v_A, v_B)\in\boldsymbol{Z}^n\times\boldsymbol{Z}^m=\mathbf{K}_0(\operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}(R))$$

we associate the standard projective modules

(5.2) $$\begin{aligned} P'_A(v)&=(e_1A)^{v(1)}\oplus\cdots\oplus(e_nA)^{v(n)}\\ P''_B(v)&=(e_{n+1}B)^{v(n+1)}\oplus\cdots\oplus(e_{n+m}B)^{v(n+m)}\end{aligned}$$

over A and B, respectively. If $X=(X'_A, X''_B, \varphi)$ is a propartite R-module such that $X'_A=P'_A(v)$ and $X''_B=P''_B(v)$ then $\operatorname{cdn}X=v$ and Lemma 3.2 (c) yields $\dim(X)=d^v$, where $d^v=v\cdot\begin{pmatrix} C(A) & 0\\ 0 & C(B)\end{pmatrix}^{\mathrm{tr}}$ (see (3.4)).

Let us denote by $\underline{\operatorname{mod}}^{\mathrm{pr}}_{\mathrm{pr}}(d^v, R)$ the subvariety of $\operatorname{mod}^{\mathrm{pr}}_{\mathrm{pr}}(d^v, R)$ defined by all modules $X=(X'_A, X''_B, \varphi)$ with $X'_A=P'_A(v)$ and $X''_B=P''_B(v)$, and let

$$\underline{\operatorname{Gl}}(d^v)=\underline{\operatorname{Gl}}(d^v_A)\times\underline{\operatorname{Gl}}(d^v_B)$$

be the algebraic subgroup of $\operatorname{Gl}(d^v)$ consisting of all pairs $g=(g_A, g_B)$ such that $g_A*P'_A(v)=P'_A(v)$ and $g_B*P''_B(v)=P''_B(v)$.

Let $\mathcal{A}_R\subseteq\mathcal{B}_R$ be representation subcategories of $\operatorname{mod}(R)$. We denote by

$\mathcal{A}_{(d,R)}$ the subset of the variety $\mathrm{mod}^R(d)$ defined by the modules in $\mathcal{A}_R$ of the dimension vector $d \in \boldsymbol{N}^{n+m}$. Following [5] we say that $\mathcal{A}_R$ is an open subcategory of $\mathcal{B}_R$ if for any $d \in \boldsymbol{N}^{n+m}$ the set $\mathcal{B}_{(d,R)}$ is a subvariety of $\mathrm{mod}^R(d)$ and $\mathcal{A}_{(d,R)}$ is an open subset of $\mathcal{B}_{(d,R)}$.

LEMMA 5.3. (a) *Under the notation above* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(d, R)$ *is an open subset of* $\mathrm{mod}^R(d)$, $\mathrm{indmod}_{\mathrm{pr}}^{\mathrm{pr}}(d, R)$ *is a constructible subset of* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(d, R)$ *and* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)$ *is an open subcategory of* $\mathrm{mod}(R)$.

(a′) $\bullet\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is an open subcategory of* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.

(b) *For any vector* $v \in \boldsymbol{N}^n \times \boldsymbol{N}^m$ *the variety* $\underline{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R)$ *is a closed subset of* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R)$ *which intersects every* $\mathrm{Gl}(d^v)$*-orbit of* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R)$ *in at least one point. If* $X=(X'_A, X''_B, \varphi)$ *and* $Y=(Y'_A, Y''_B, \psi)$ *are modules in* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)$ *such that* $X'_A=Y'_A=P'_A(v)$ *and* $X''_B=Y''_B=P''_B(v)$, *then* $X \cong Y$ *if and only if* X *and* Y *belong to the same* $\underline{\mathrm{Gl}}(d^v)$*-orbit of* $\underline{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R)$.

PROOF. Apply the arguments used in [22, Section 15.2] and [8, Lemma 2.12]. □

For any $v \in \boldsymbol{N}^n \times \boldsymbol{N}^m$ we consider the affine algebraic varieties

$$\mathrm{prop}_v^R = \mathrm{Hom}_B(P'_A(v) \otimes_A M_B,\ P''_B(v)) \quad \text{and} \quad \mathbf{G}_v^R = \mathrm{Aut}_A P'_A(v) \times \mathrm{Aut}_B P''_B(v)$$

with respect to Zariski topology, and the natural algebraic group action

$$\bullet : \mathbf{G}_v^R \times \mathrm{prop}_v^R \longrightarrow \mathrm{prop}_v^R \tag{5.4}$$

where $\mathbf{G}_v^R$ is viewed as an algebraic group. We call prop_v^R the *matrix variety of propartite R-modules* of coordinate vector $v \in \boldsymbol{N}^{n+m}$.

LEMMA 5.5. *For any coordinate vector* $v \in \boldsymbol{N}^n \times \boldsymbol{N}^m$ *there exist a K-variety isomorphism* $\tau : \mathrm{prop}_v^R \to \underline{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R)$, *and an algebraic group isomorphism* $\tau :$ $\mathbf{G}_v^R \to \mathrm{Gl}(d^v)$ *making the following diagram commutative*

$$\begin{array}{ccc} \mathbf{G}_v^R \times \mathrm{prop}_v^R & \overset{\bullet}{\longrightarrow} & \mathrm{prop}_v^R \\ \big\downarrow{\scriptstyle \tau \times \tau} & & \big\downarrow{\scriptstyle \tau} \\ \underline{\mathrm{Gl}}(d^v) \times \underline{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R) & \overset{*}{\longrightarrow} & \underline{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(d^v, R). \end{array}$$

PROOF. Apply the arguments used in [22, Proposition 15.12] and in [23, Proposition 2.3]. □

6. Representation type.

In the definition of tame and wild representation type we shall need the following notation (see [22, Chapter 14], and [24]). Let $K[y]$ be the K-algebra of all polynomials in one indeterminate y. Given a non-zero polynomial $h \in K[y]$ we denote by $K[y]_h$ the localization of $K[y]$ with respect to the multiplicative system $\{h_j\}_{j \in N}$. For any $\lambda \in K$ we consider the simple $K[y]$-module

$$K_\lambda^1 = K[y]/(y-\lambda). \tag{6.1}$$

Let R and Λ be K-algebras. We recall from [**24**] that an additive functor $H: \mathrm{fin}(\Lambda) \to \mathrm{Mod}(R)$ *preserves the indecomposability* if H carries indecomposable modules to indecomposable ones. The functor H is said to *respect the isomorphism classes* if the existence of an isomorphism $H(X) \cong H(Y)$ for X, Y in $\mathrm{mod}(\Lambda)$ implies that $X \cong Y$ holds.

The functor $T: \mathrm{fin}(\Lambda) \to \mathrm{Mod}(R)$ is defined to be a *representation embedding* if T is exact, preserves the indecomposability and respects the isomorphism classes.

Let $\mathcal{A}_R$ be one of the categories of the diagram (2.2). A representation embedding $T: \mathrm{fin}(\Lambda) \to \mathcal{A}_R \subseteq \mathrm{Mod}(R)$ is defined to be *smooth* if T is of the form $T \cong (-) \otimes_\Lambda M_R$, where ${}_\Lambda M_R$ is a Λ-R-bimodule having the following properties:

(s1) ${}_\Lambda M_R$ is flat as a left Λ-module and is finitely generated as a Λ-R-bimodule (see [**5**, (M1)-(M4)]).

(s2) The Λ-R-bimodule ${}_\Lambda M_R$ viewed as a right $R \otimes \Lambda^{\mathrm{op}}$-module belongs to $\mathcal{A}_{R \otimes \Lambda^{\mathrm{op}}}$.

We define $\mathcal{A}_R$ to be of *wild representation type* (resp. *smooth wild*) if there exists a $\mathcal{W}$-R-bimodule ${}_\mathcal{W} M_R$ which is flat as a left module over the free algebra

$$\mathcal{W} = K\langle t_1, t_2\rangle$$

of polynomials in two non-commuting indeterminates t_1 and t_2, and ${}_\mathcal{W} M_R$ induces a representation embedding functor (resp. smooth representation embedding)

$$\hat{M} = (-) \otimes_\mathcal{W} M : \mathrm{fin}(\mathcal{W}) \longrightarrow \mathcal{A}_R \subseteq \mathrm{Mod}(R).$$

It follows from the wildness correction lemma [**24**, Lemma 2.6] that one can suppose without loss of generality that the bimodule M is free when viewed as a left $\mathcal{W}$-module. If, in addition, R is a finite dimensional algebra then one can suppose without loss of generality that the bimodule M is finitely generated free when viewed as a left $\mathcal{W}$-module.

Moreover, it follows from [**24**, Theorem 2.7] that the category $\mathcal{A}_R$ is of wild representation type if and only if there exists an exact functor $T: \mathrm{mod}\begin{pmatrix} K & K^3 \\ 0 & K \end{pmatrix} \to \mathcal{A}_R$, which preserves the indecomposability and respects the isomorphism classes.

It follows from Theorem 6.5 below that the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ is of wild representation type if for any finitely generated K-algebra S there exists a smooth representation embedding $\hat{M}=(-)\otimes_S M: \mathrm{fin}(S)\to \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$. For some technical reasons it is more convenient sometimes to work with the smooth representation embeddings than with representation embeddings.

DEFINITION 6.2. Let S be a finitely generated K-algebra, let $\mathcal{B}_S$ be one of the categories $\mathrm{fin}(S)$, $\mathrm{ind}(S)$ and $\mathrm{ind}_1(S)$ and let R be a bipartite K-algebra. Suppose that $\mathcal{A}_R$ is a representation subcategory of $\mathrm{Mod}(R)$, and

$$(6.2') \qquad \hat{M}^{(1)}, \cdots, \hat{M}^{(s)}: \mathcal{B}_S \longrightarrow \mathcal{A}_R$$

are additive functors of the form $\hat{M}^{(j)}=(-)\otimes_S M_R^{(j)}$, where ${}_SM_R^{(1)}, \cdots, {}_SM_R^{(s)}$ are S-R-bimodules. The family (6.2′) is defined to be an *almost S-parametrizing family* for the category $\mathrm{ind}_v(\mathcal{A}_R)$ if the following conditions are satisfied:

(P1) All but finitely many modules in $\mathrm{ind}_v(\mathcal{A}_R)$ are isomorphic to modules in $\mathrm{Im}\,\hat{M}^{(1)}\cup\cdots\cup\mathrm{Im}\,\hat{M}^{(s)}$.

(P2) ${}_SM_R^{(1)}, \cdots, {}_SM_R^{(s)}$ are finitely generated as S-modules.

If the bimodules ${}_SM_R^{(1)}, \cdots, {}_SM_R^{(s)}$ are finitely generated as S-R-bimodules and satisfy the conditions (s1) and (s2) we call the family *smooth.*

The almost S-parametrizing family (6.2′) is defined to be a *strict almost S-parametrizing family* for the category $\mathrm{ind}_v(\mathcal{A}_R)$ if the following condition is satisfied:

(P3) The functors (6.2′) preserve the indecomposability and respect the isomorphism classes.

Let $\mathcal{A}_R$ be one of the categories of the diagram (2.2). We define $\mathcal{A}_R$ to be of *tame representation type* (resp. of *smooth tame*) if for any bipartite coordinate vector $v\in \boldsymbol{N}^{n+m}$ there exists a polynomial $h\in K[y]$ and a family of additive functors

$$(6.3) \qquad \hat{M}^{(1)}, \cdots, \hat{M}^{(s)}: \mathrm{ind}_1(K[y]_h) \longrightarrow \mathcal{A}_R \subseteqq \mathrm{Mod}(R)$$

forming an almost $K[y]_h$-parametrizing family (resp. of smooth family) for the category $\mathrm{ind}_v(\mathcal{A}_R)$.

Given a number $v\geqq 1$ or a vector $v\in \boldsymbol{N}^{n+m}$ we define $\boldsymbol{\mu}^1_{\mathcal{A}_R}(v)$ to be the minimal number s of functors (6.3) forming an almost $K[y]_h$-parametrizing family for the category $\mathrm{ind}_v(\mathcal{A}_R)$. The tame category $\mathcal{A}_R$ is defined to be of *polynomial growth* if there exists an integer $G\geqq 1$ such that $\boldsymbol{\mu}^1_{\mathcal{A}_R}(v)\leqq \|v\|^G$ for all vectors $v\in \boldsymbol{N}^{n+m}$ such that $\|v\|=\sum_{j=1}^{n+m} v_j\geqq 2$ (compare with [**22**, Section 15.10]). □

It follows from Lemma 6.4 below that our definition of tameness is equivalent to that one given in [**24**]

LEMMA 6.4. *Suppose that* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a basic finite dimensional bipartite* K*-algebra.*

(a) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is of tame representation type if and only if for any integer* $t \in \boldsymbol{N}$ *there exist a polynomial* $h \in K[y]_h$ *and additive functors*

$$\hat{M}^{(1)}, \cdots, \hat{M}^{(s)} : \mathrm{ind}_1(\mathrm{mod}(K[y]_h)) \longrightarrow \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$$

where $\hat{M}^{(j)} = (-)\otimes_{K[y]_h} M_R^{(j)}$, *forming an almost parametrizing family for the category* $\mathrm{ind}_t(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$.

(b) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is tame of polynomial growth if and only if there exists an integer* $G \geqq 1$ *such that* $\boldsymbol{\mu}_{\mathcal{A}_R}^1(t) \leqq t^G$, *with* $\mathcal{A}_R = \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, *for all integers* $t \geqq 2$.

PROOF. Given $v \in \boldsymbol{N}^{n+m}$ we set $\|v\| = \sum_{j=1}^{n+m} v_j$ and we denote by $d^v \in \boldsymbol{N}^{n+m}$ the vector $v \cdot \begin{pmatrix} C(A) & 0 \\ 0 & C(B) \end{pmatrix}^{\mathrm{tr}}$ (see (3.4)). It follows from Lemma 3.2 (c) that if X is a module in $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $v = \mathrm{cdn}(X)$, then $\dim(X) = d^v$. Hence we conclude that for any $v \in \boldsymbol{N}^{n+m}$ the inclusion $\mathrm{ind}_v(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A) \subseteqq \mathrm{ind}_r(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$ holds up to isomorphism with $r = \|d^v\|$. Furthermore, for any $t \in \boldsymbol{N}$ the inclusion $\mathrm{ind}_t(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A) \subseteqq \bigcup_{v \in L(t)} \mathrm{ind}_v(\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A)$ holds up to isomorphism, where $L(t) = \{v \in \boldsymbol{N}^{n+m} ; \|d^v\| = t\}$.

Since the kernel of the group homomorphism $d^\bullet : \boldsymbol{Z}^{n+m} \to \boldsymbol{Z}^{n+m}$, $v \mapsto d^v$, restricted to $\boldsymbol{N}^{n+m}$ is zero then for any $d \in \boldsymbol{N}^{n+m}$ there are only finitely many vectors $v \in \boldsymbol{N}^{n+m}$ such that $d = d^v$ (see also [**9**]). Hence we conclude that the set $L(t)$ is finite for any t, and therefore the lemma follows from the definition of tame representation type and of polynomial growth. □

THEOREM 6.5. *Suppose that* $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a basic finite dimensional algebra over an algebraically closed field* K, *and we have fixed a set of primitive orthogonal idempotents* (2.1). *The following conditions are equivalent.*

(a) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is of tame representation type.*

(a′) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is of smooth tame representation type.*

(b) *For any* $v \in \boldsymbol{N}^n \times \boldsymbol{N}^m$ *there exists a constructible subset* $C(v)$ *of the subvariety* $\mathrm{indprop}_v^R$ *of* prop_v^R (*defined by the indecomposable propartite modules*) *such that* $\mathbf{G}_v^R * C(v) = \mathrm{indprop}_v^R$ *and* $\dim C(v) \leqq 1$.

(c) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is not of wild representation type.*

(c′) *The category* $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ *is not of smooth wild representation type.*

(d) *The category* $\mathrm{rep}_K(\mathfrak{B}_R)$ *is of tame representation type.*

(d′) *The category* $\mathrm{rep}_K(\mathfrak{B}_R)$ *is not of wild representation type.*

PROOF. The implications (a′)⇒(a) and (c)⇒(c′) are obvious. The implications (a)⇒(b)⇒(c) can be proved by a slight modification of the arguments in [4], [**14**], [**22**, Corollary 15.17, Theorem 15.13], or in [**10**, Section 3].

It follows from Proposition 4.9 (b), [**5**, Theorem 1] and the definition of smooth wild and smooth tame representation type that $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)$ is of smooth wild (resp. smooth tame) representation type if and only if the category $\mathrm{rep}_K(\mathfrak{B}_R)$ is of wild (resp. tame) representation type. Since (d)$\Leftrightarrow$(d′) follows from [**4**] or [**5**] then the implications (c′)$\Leftrightarrow$(d′)$\Leftrightarrow$(d)$\Rightarrow$(a′) follow and the proof is complete. $\square$

COROLLARY 6.6. *Under the assumption of Theorem* 6.5 *the following conditions are equivalent.*

(a) *The category* $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ *is of tame representation type.*

(a′) *The category* $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ *is of smooth tame representation type.*

(b) *For any* $v\in \boldsymbol{N}^n\times\boldsymbol{N}^m$ *there exists a constructible subset* $C(v)$ *of the variety*

$$\underline{\mathrm{ind}\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}}(d^v, R):=\underline{\mathrm{indmod}_{\mathrm{pr}}^{\mathrm{pr}}}(d^v, R)\cap\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(d^v, R)$$

such that $\underline{\mathrm{Gl}}(d^v)*C(v)=\underline{\mathrm{ind}\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}}(d^v, R)$ *and* $\dim C(v)\leqq 1$.

(c) *The category* $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ *is not of wild representation type.*

(c′) *The category* $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ *is not of smooth wild representation type.*

PROOF. The implications (a′)$\Rightarrow$(a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(c′) follow in a similar way as in the proof of Theorem 6.5.

(c′)$\Rightarrow$(a′) Since according to Lemma 5.3 $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ is an open subcategory of $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ then by Proposition 4.9 the equivalence $F_K: \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A \xrightarrow{\cong} \mathrm{rep}_K(\mathfrak{B}_R)$ carries $\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A$ to the open subcategory $F_K(\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A)$ of $\mathrm{rep}_K(\mathfrak{B}_R)$ and Theorem 2 in [**5**] applies to $F_K(\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A)$. If (c′) holds then from Proposition 4.9 it follows that the category $F_K(\widehat{\bullet\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}(R)_B^A)$ is not of wild representation type, and by [**5**, Theorem 2] it is of tame representation type. Applying Proposition 4.9 again we conclude that (a′) follows. $\square$

We say that a functor $T:\mathcal{A}\to\mathcal{B}$ *preserves tame* (resp. wild) *representation type* if the tameness (resp. wildness) of $\mathcal{A}$ implies the tameness (resp. wildness) of $\mathcal{B}$. The functor T *reflects tame* (resp. wild) *representation type* if the tameness (resp. wildness) of $\mathcal{B}$ implies the tameness (resp. wildness) of $\mathcal{A}$.

As a consequence of the proof of Theorem 6.5 we get the following.

COROLLARY 6.7. *The functor* $F_K: \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B\end{pmatrix}\to \mathrm{rep}_K(\mathfrak{B})$ (4.11) *preserves and reflects representation types.* $\square$

COROLLARY 6.8. (a) *The functors* T *and* T' *in diagram* (4.6) *preserve and reflect wild and tame representation type.*

(b) *The category* $\mathrm{prin}(R)_B^A$ *is either of tame representation type or of wild representation type, and these types are mutually exclusive.*

PROOF. (a) It follows from Proposition 4.5 that T and T' carry exact sequences to exact ones. Then the corollary is a consequence of [**24**, Theorem 2.7] for the wild representation type. The statement (a) for tame representation type follows from [**9**, Lemma 2.3]. The statement (b) follows from (a) and Theorem 6.5. □

We shall observe below that the functors Θ^{A} and $\bullet\hat{\Theta}^{A}$ lower and lift representation embedding functors with the domain $\mathrm{ind}_1(\mathrm{fin}(K[y]))$ in the following sense.

DEFINITION 6.9. Let S, R and Λ be K-algebras and assume that S is a commutative domain. Let $\mathcal{A}$ be a representation subcategory of $\mathrm{Mod}(\Lambda)$ and $\mathcal{B}_S \subseteqq \mathrm{Mod}(S)$ be one of the categories $\mathrm{mod}(S)$, $\mathrm{fin}(S)$, $\mathrm{ind}(\mathrm{fin}(S))$ and $\mathrm{ind}_1(\mathrm{fin}(S))$.

We say that an additive functor $H: \mathcal{A} \to \mathrm{Mod}(R)$ *lowers* the representation embedding functor $\hat{M}=(-)\otimes_S M_\Lambda : \mathcal{B}_S \to \mathcal{A} \subseteqq \mathrm{Mod}(\Lambda)$ defined by the S-Λ-bimodule ${}_SM_\Lambda$ if there exist a non-zero element $h \in S$ and a representation embedding $\hat{M}'=(-)\otimes_{S_h} M'_R : \mathcal{B}_{S_h} \to \mathrm{Mod}(R)$ defined by the S_h-R-bimodule ${}_{S_h}M'_R$ and the following conditions are satisfied:

(e1) The diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "H"] \\ \mathcal{B}_{S_h} \arrow[ur, "{}_h\hat{M}"] \arrow[r, "\hat{M}'"'] & \mathrm{Mod}(R) \end{tikzcd}$$

is commutative up to a natural equivalences of functors, where S_h is the localization of S at h and ${}_hM=S_h\otimes_S M$.

(e2) Given a module X in $\mathcal{B}_{S_h}$ the module ${}_h\hat{M}(X)$ is indecomposable if and only if $\hat{M}'(X)$ is indecomposable, and there is isomorphism $\hat{M}'(X)\cong\hat{M}'(Y)$ if and only if there is an isomorphism ${}_h\hat{M}(X)\cong{}_h\hat{M}(Y)$.

We say the functor $H: \mathcal{A} \to \mathrm{Mod}(R)$ *lifts* the representation embedding functor $\hat{M}: \mathcal{B}_S \to \mathrm{Mod}(R)$ if there exist a non-zero element $h \in S$ and a representation embedding $\hat{M}': \mathcal{B}_{S_h} \to \mathcal{A} \subseteqq \mathrm{Mod}(\Lambda)$ such that the diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "H"] \\ \mathcal{B}_{S_h} \arrow[ur, "\hat{M}'"] \arrow[r, "{}_h\hat{M}"'] & \mathrm{Mod}(R) \end{tikzcd}$$

is commutative up to a natural equivalences of functors and the condition (e2) is satisfied. □

We recall from [**23**, Proposition 2.4] that for any finite poset J the adjustment functor $\Theta: \mathrm{prin}(KJ) \to \mathrm{mod}_{\mathrm{sp}}(KJ) \cong J\text{-spr}$ lowers the representation embedding functors $\hat{M}: \mathrm{ind}_1(\mathrm{mod}(K[y]_h)) \to \mathrm{prin}(KJ)$ and lifts representation embedding functors $\hat{M}: \mathrm{ind}_1(\mathrm{mod}(K[y]_h)) \to \mathrm{mod}_{\mathrm{sp}}(KJ)$. The differential $\boldsymbol{\partial}_{(a,b)}: I\text{-sp} \to \delta_{(a,b)} I\text{-sp}$ with respect to a suitable pair (a, b) of elements of a poset I has also the lifting and the lowering property for representation embedding functors with the domain $\mathrm{ind}_1(\mathrm{mod}(K[y]_h))$ (see [**22**, Lemma 15.47]).

THEOREM 6.10. (a) *The adjustment functors Θ^A, $\hat{\Theta}^A$ and $\cdot\hat{\Theta}^A$ in the commutative diagram* (2.2) *lower and lift representation embedding functors with the domain* $\mathrm{ind}_1(\mathrm{mod}(K[y]_h))$.

(b) *The adjustment functors Θ^A, $\hat{\Theta}^A$ and $\cdot\hat{\Theta}^A$ preserve and reflect tame representation type and the polynomial growth property.*

(c) *If the algebra $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is finite dimensional then the functors*

$$\Theta^A: \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}, \quad \hat{\Theta}^A: \widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

preserve wild representation type.

PROOF. (a) If $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is finite dimensional then in view of Proposition 4.5 and Corollary 6.8 the statement (a) follows from [**9**, Section 2]. If $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is infinite dimensional we can prove (a) by applying the arguments used in [**9**, Section 2] (see also [**23**, Proposition 2.4]).

The statement (b) follows from (a) and Proposition 2.4 (c). The fact that the functors preserve tameness can be also proved by the arguments applied in the proof of (c) below.

(c) Assume that the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is of wild representation type. It follows from Theorem 6.5 and the wildness correction lemma [**24**, Lemma 2.6] that it is of smooth wild representation type and therefore there exists a representation embedding

$$\hat{U} = (-)\otimes_{\mathcal{W}} U_R : \mathrm{fin}(\mathcal{W}) \longrightarrow \mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$$

defined by an $\mathcal{W}$-R-bimodule ${}_{\mathcal{W}}U_R = ({}_{\mathcal{W}}U'_A, {}_{\mathcal{W}}U''_B, t)$ such that ${}_{\mathcal{W}}U$ is finitely generated free $\mathcal{W}$-module and the bimodules ${}_{\mathcal{W}}U'_A$, ${}_{\mathcal{W}}U''_B$ are finitely generated projective as bimodules. It follows from [**22**, Proposition 14.10] that there exists a full faithful representation embedding

$$\hat{N} = (-)\otimes_{\Lambda} N_{\mathcal{W}} : \mathrm{fin}(\Lambda) \longrightarrow \mathrm{fin}(\mathcal{W})$$

where $\Lambda = K[t_1, t_2]$ and ${}_{\Lambda}N_{\mathcal{W}}$ is a Λ-$\mathcal{W}$-bimodule which is finitely generated free as a left Λ-module. It follows that ${}_{\Lambda}N\otimes_{\mathcal{W}} U_R = ({}_{\Lambda}N\otimes_{\mathcal{W}} U'_A, {}_{\Lambda}N\otimes_{\mathcal{W}} U''_B, 1\otimes t)$. Since $L\otimes_{\Lambda} N\otimes_{\mathcal{W}} U''_B = \hat{U}\hat{N}(L)$ for any $L\in\mathrm{fin}(\Lambda)$, the right B-module $L\otimes_{\Lambda} N\otimes_{\mathcal{W}} U''_B$ is by definition finitely generated projective.

Since ${}_AM_B$ is finite dimensional then $P_B \otimes \mathrm{Hom}_B({}_AM_B, B) \cong \mathrm{Hom}_B({}_AM_B, P_B)$ for any projective right R-module P_B, and for any $L \in \mathrm{fin}(\Lambda)$ there are natural isomorphisms

$$(*) \qquad L \otimes_\Lambda \mathrm{Hom}_B({}_AM_B, {}_\Lambda N \otimes_{\mathcal{W}} U''_B) \cong L \otimes_\Lambda N \otimes_{\mathcal{W}} U'' \otimes_B \mathrm{Hom}_B({}_AM_B, B) \cong \mathrm{Hom}_B({}_AM_B, L \otimes_\Lambda N \otimes_{\mathcal{W}} U''_B).$$

Consider the commutative diagram of Λ-A-bimodules

$$(**) \qquad \begin{array}{ccccccc} {}_\Lambda N \otimes_{\mathcal{W}} U'_A & \xrightarrow{1 \otimes \bar{t}} & \mathrm{Hom}_B({}_AM_B, {}_\Lambda N \otimes_{\mathcal{W}} U''_B) & \longrightarrow & {}_\Lambda H_A & \longrightarrow & 0 \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\ \mathrm{Im}(1 \otimes \bar{t}) & \hookrightarrow & \mathrm{Hom}_B({}_AM_B, {}_\Lambda N \otimes_{\mathcal{W}} U''_B) & \longrightarrow & {}_\Lambda H_A & \longrightarrow & 0. \end{array}$$

It follows from our assumption that ${}_\Lambda H_A = \mathrm{Coker}(1 \otimes \bar{t})$ viewed as a Λ-A-bimodule is finitely generated and since $\dim_K A$ is finite then ${}_\Lambda H$ is a finitely generated Λ-module. By Lemma 6.11 below there exists a polynomial $h \in \Lambda$ such that the localization of ${}_\Lambda H_A$ with respect to h is zero or a finitely generated free module over the localization Λ_h of Λ. By a localization with respect to h we get the diagram (**), with $\Lambda = K[t_1, t_2]_h$ and ${}_\Lambda H_A$ a finitely generated free Λ-module.

In view of the isomorphism (*) the tensoring of the diagram (**) by a module L in $\mathrm{fin}(\Lambda)$ yields the commutative diagram

$$\begin{array}{ccccccc} L \otimes_\Lambda N \otimes_{\mathcal{W}} U'_A & \xrightarrow{\overline{1 \otimes 1 \otimes t}} & \mathrm{Hom}_B({}_AM_B, L \otimes_\Lambda N \otimes_{\mathcal{W}} U''_B) & \longrightarrow & L \otimes_\Lambda H_A & \longrightarrow & 0 \\ \downarrow & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\ L \otimes_\Lambda \mathrm{Im}(1 \otimes \bar{t}) & \hookrightarrow & \mathrm{Hom}_B({}_AM_B, L \otimes_\Lambda N \otimes_{\mathcal{W}} U''_B) & \longrightarrow & L \otimes_\Lambda H_A & \longrightarrow & 0 \end{array}$$

with exact rows. It follows that $\Theta^A(L \otimes_\Lambda N \otimes_{\mathcal{W}} U_R) \cong L \otimes_\Lambda \Theta^A({}_\Lambda N \otimes_{\mathcal{W}} U_R)$ and therefore the functor $(-) \otimes_\Lambda \Theta^A({}_\Lambda N \otimes_{\mathcal{W}} U_R) : \mathrm{fin}(K[t_1, t_2]_h) \to \mathrm{mod}_{\mathrm{pr}}(R)^A_B$ is a representation embedding. This proves that the category $\mathrm{mod}_{\mathrm{pr}}(R)^A_B$ is of wild representation type (see [**24**, Theorem 2.7]). The second part of (c) follows in a similar way. □

For a convenience of the reader we include the following well-known result (see [**1**]).

LEMMA 6.11. *Let Λ be a noetherian domain and let N be a finitely generated Λ-module. Then there exists a non-zero element $h \in \Lambda$ such that the localization $N_h = N \otimes_\Lambda \Lambda_h$ of N with respect to the multiplicative system $\{h^n\}_{n \in \mathbf{N}}$ is zero or a free module over the localization Λ_h of Λ.*

PROOF. Choose an exact sequence $\Lambda^m \xrightarrow{\alpha} \Lambda^m \to N \to 0$. Assume that $A \in \boldsymbol{M}_m(\Lambda)$ is the $m \times m$ matrix of α in the standard basis of Λ^m. If Q is the field of fractions of Λ then there exist matrices B, $C \in \mathrm{Gl}(m, Q)$ such that $C^{-1}AB = \mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ is the diagonal matrix with u identities and $m-u$ zeros on the main diagonal. Let $h \in \Lambda$ be a non-zero element such that B, B^{-1}, C, $C^{-1} \in \mathrm{Gl}(m, \Lambda_h)$. We get the commutative diagram of Λ_h-modules

$$\begin{array}{ccccccc} \Lambda_h^m & \xrightarrow{\alpha_h} & \Lambda_h^m & \longrightarrow & N_h & \longrightarrow & 0 \\ \downarrow \gamma & & \downarrow \beta & & & & \\ \Lambda_h^m & \xrightarrow{\delta} & \Lambda_h^m & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

with exact rows, where β and γ are the isomorphisms defined by B and A, and $\delta(x_1, \cdots, x_m) = (x_1, \cdots, x_u, 0, \cdots, 0)$. It follows that N' is a free Λ_h-module of rank $m-u$ and there is an isomorphism $N_h \cong N'$. $\square$

COROLLARY 6.12. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a finite dimensional algebra and let $\mathcal{A}_R \subseteq \mathrm{mod}(R)$ be one of the categories $\mathrm{mod}_{\mathrm{pr}}(R)_B^A$ or $\widehat{\mathrm{mod}}_{\mathrm{pr}}(R)_B^A$. Then the following conditions are equivalent.*

(a) *$\mathcal{A}_R$ is of tame representation type.*

(b) *For any coordinate vector $v \in \boldsymbol{N}^{n+m}$ there exists a constructible subset $C(v)$ of the variety $\mathrm{ind}\, \mathcal{A}_{(d^v, R)} \subset \mathcal{A}_{(d^v, R)}$ (see Section 5) such that $\dim C(v) \leq 1$ and $\mathrm{Gl}(d^v) * C(v) = \mathrm{ind}\, \mathcal{A}_{(d^v, R)}$.*

(c) *$\mathcal{A}_R$ is not of wild representation type.*

PROOF. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) can be proved by standard algebraic geometry arguments as in Theorem 15.13 and Corollary 15.17 of [**22**] (see also [**10**, Section 3]). For the proof of (c)$\Rightarrow$(a) we denote by $\mathcal{A}'_R$ one of the categories $\mathrm{mod}_{\mathrm{pi}}^{\mathrm{p}}(R)_B^A$ or $\widehat{\mathrm{mod}}_{\mathrm{pi}}^{\mathrm{p}}(R)_B^A$ respectively. Let $\Theta : \mathcal{A}'_R \to \mathcal{A}_R$ be one of the adjustment functors Θ^A or $\hat{\Theta}^A$, respectively (see (2.2)).

If (c) holds then in view of Theorem 6.10 (c) the category $\mathcal{A}'_R$ is not of wild representation type and by Theorem 6.5 and Corollary 6.6 it is of tame representation type. Since according to Theorem 6.10 (b) the functor Θ preserves tameness then $\mathcal{A}_R$ is of tame representation type. This finishes the proof. $\square$

7. Lattices over orders and propartite modules over bipartite algebras.

In this section we suppose that K is an algebraically closed field, D is a K-algebra which is a complete discrete valuation domain, F is a field of fractions of D, $\mathfrak{p}$ is the unique maximal ideal of D and $D/\mathfrak{p} \cong K$.

We assume that Λ is a *D-order* in a semisimple F-algebra C, that is,

(0_1) Λ viewed as a D-module is finitely generated free, and

(0_2) Λ is a D-subalgebra of C containing an F-basis of C, i.e., $\Lambda F = F\Lambda = C$.

We recall that a right Λ-module X is said to be a right *Λ-lattice* if X is finitely generated and X is free when viewed as a D-module [**17**]. We denote by latt(Λ) the category of right Λ-lattices.

Throughout we fix the following notation. Given X in latt(Λ) we view the module

(7.1) $$XF := X\otimes_D F$$

as a right Λ-module in a natural way. Since $C=F\Lambda$ we can view XF as a right C-module. Moreover we view X as a Λ-submodule of XF along the Λ-monomorphism $X\to X\otimes_D F=XF$.

Given X in latt(Λ) we denote by $\mathrm{rk}_D X$ the D-rank of X. For any number $r\in\boldsymbol{N}$ we denote by $\mathrm{ind}_r(\mathrm{latt}(\Lambda))$ the full subcategory of latt(Λ) consisting of pairwise non-isomorphic representatives of the isomorphism classes of indecomposable Λ-lattices of D-rank r.

Following an idea in [**21**, Example 3.8] and in [**22**, 13.0] we shall study the category latt(Λ) and its representation type by means of propartite modules. For this purpose we fix a D-order Γ such that

$$\Lambda\subseteqq\Gamma\subseteqq C$$

and Γ is maximal in C. It follows that Γ is a hereditary D-order and since Γ/Λ is a finitely generated torsion D-module then there exists a two-sided Γ-ideal I such that

(7.2) $$I\subseteqq\mathrm{rad}(\Lambda),\quad I\subseteqq\mathrm{rad}(\Gamma),\quad \dim_K\Gamma/I<\infty \quad\text{and}\quad I\cap D=(p^c),$$

for some $c\geqq 1$, where p is a generator of the maximal ideal $\mathfrak{p}\subseteqq D$.

Throughout this paper we fix a maximal D-order Γ and a two-sided ideal I satisfying the condition (7.2), and we consider the surjection $\varepsilon: \Lambda_\Gamma\to\bar{\Lambda}_\Gamma$ of bipartite K-algebras, where

(7.3) $$\Lambda_\Gamma=\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix},\quad \bar{\Lambda}_\Gamma=\begin{pmatrix}A & B\\ 0 & B\end{pmatrix}\quad\text{and}\quad A=\Lambda/I\hookrightarrow B=\Gamma/I$$

is a K-algebra injection induced by the inclusion $\Lambda\subseteqq\Gamma$. It follows from our assumption that $\bar{\Lambda}_\Gamma$ is a finite dimensional K-algebra. Throughout we shall view $\mathrm{mod}(\bar{\Lambda}_\Gamma)$ as a full subcategory of $\mathrm{mod}(\Lambda_\Gamma)$.

We recall from [**17**] that Λ and Γ are semiperfect K-algebras in the sense that their finitely generated modules admits projective covers. It follows that Λ_Γ is a semiperfect noetherian D-algebra.

Throughout we suppose that Λ is a basic K-algebra and we fix complete sets

(7.4) $$\{\eta_1,\cdots,\eta_n\}\subseteqq\Lambda \quad\text{and}\quad \{\eta_{n+1},\cdots,\eta_{n+m}\}\subseteqq\Gamma$$

of primitive orthogonal idempotents of Λ and of Γ, respectively. It follows that $\{\eta_1, \cdots, \eta_n, \eta_{n+1}, \cdots, \eta_{n+m}\}$ is a complete set of primitive orthogonal idempotents of the bipartite K-algebra Λ_Γ. Since Λ is basic then $\Lambda/\mathrm{rad}(\Lambda)\cong K\times\cdots\times K$ (n copies). Let

$$e_j = \overline{\eta_j} = \eta_j + I$$

be the coset of η_j modulo I for $j=1, \cdots, n+m$. Then the sets

$$\{e_1, \cdots, e_n\} \subseteq A \quad \text{and} \quad \{e_{n+1}, \cdots, e_{n+m}\} \subseteq B$$

are complete sets of primitive orthogonal idempotents of A and of B, respectively. Moreover the algebra A is basic.

Given a Λ-lattice X and a Γ-lattice Y the projective covers $P_\Lambda(X)$ and $P_\Gamma(Y)$ of X and Y have the forms

$$P_\Lambda(X) \cong (\eta_1\Lambda)^{w_1}\oplus\cdots\oplus(\eta_n\Lambda)^{w_n}$$

and

$$P_\Gamma(Y) \cong (\eta_{n+1}\Gamma)^{w_{n+1}}\oplus\cdots\oplus(\eta_{n+m}\Gamma)^{w_{n+m}}.$$

Following [**20**] and [**24**] the *coordinate vectors* of the Λ-lattice X and of the Γ-lattice Y are defined to be the vectors

$$\text{(7.5)} \qquad \mathrm{cdn}(X) = (w_1, \cdots, w_n) \in \boldsymbol{N}^n, \quad \mathrm{cdn}(Y) = (w_{n+1}, \cdots, w_{n+m}) \in \boldsymbol{N}^m.$$

Moreover given a vector $v\in\boldsymbol{N}^n$ we set $\|v\|=v_1+\cdots+v_n$, and we denote by $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ the full subcategory of $\mathrm{latt}(\Lambda)$ consisting of representatives of the isomorphism classes of indecomposable Λ-lattices X with $\mathrm{cdn}(X)=v$.

Let Λ be a D-order in a semisimple algebra C. We recall from [**24**] that a D-order Λ is said to be of *wild lattice type* (or the category $\mathrm{latt}(\Lambda)$ to be of wild representation type) if there exists a representation embedding functor $T: \mathrm{fin}(\mathcal{W})\to\mathrm{latt}(\Lambda)$, where $\mathcal{W}=K\langle t_1, t_2\rangle$ is the free algebra of polynomials in two non-commuting indeterminates t_1 and t_2. It is well-known that every such a functor T is of the form $T\cong(-)\otimes_{\mathcal{W}} M_\Lambda$, where ${}_{\mathcal{W}}M_\Lambda$ is a $\mathcal{W}$-Λ-bimodule. In view of [**24**, Lemma 2.6 and 3.6] one can suppose without loss of generality that the bimodule M is finitely generated free when viewed as a left $\mathcal{W}$-module. Moreover, it follows from [**24**, Theorem 3.5] that the category $\mathrm{latt}(\Lambda)$ is of wild representation type if and only if there exists an additive exact functor $T: \mathrm{mod}\begin{pmatrix} K & K^3 \\ 0 & K\end{pmatrix}\to\mathrm{latt}(\Lambda)$ which preserves the indecomposability and respects the isomorphism classes.

DEFINITION 7.6. Let S be a finitely generated K-algebra, let $\mathcal{B}_S$ be one of the categories $\mathrm{fin}(S)$, $\mathrm{ind}(S)$ and $\mathrm{ind}_1(S)$ and let Λ be a D-order in a semisimple F-algebra C. Suppose that

$$\text{(7.6')} \qquad \hat{M}^{(1)}, \cdots, \hat{M}^{(s)} : \mathcal{B}_S \longrightarrow \mathrm{latt}(\Lambda)$$

are additive functors of the form $\hat{M}^{(j)}=(-)\otimes_S M_\Lambda^{(j)}$, where ${}_SM_\Lambda^{(1)}, \cdots, {}_SM_\Lambda^{(s)}$ are S-Λ-bimodules. The family (7.6′) is defined to be an *almost S-parametrizing family* for the category $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ if the following conditions are satisfied:

(P0) The left S-modules ${}_SM^{(1)}, \cdots, {}_SM^{(s)}$ are flat.

(P1) All but finitely many lattices in $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ are isomorphic to lattices in $\mathrm{Im}\,\hat{M}^{(1)}\cup\cdots\cup\mathrm{Im}\,\hat{M}^{(s)}$.

(P2) ${}_SM_\Lambda^{(1)}, \cdots, {}_SM_\Lambda^{(s)}$ are finitely generated as S-Λ-bimodules.

(P3) $M_\Lambda^{(1)}, \cdots, M_\Lambda^{(s)}$ viewed as D-modules are torsion-free.

The family (7.6′) is defined to be a *weak almost parametrizing family* for the category $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ if it has the properties (P1) and (P2). The almost S-parametrizing family (7.6′) is defined to be a *strict almost S-parametrizing family* for the category $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ if the following condition is satisfied:

(P4) The functors (7.6′) preserve the indecomposability and respect the isomorphism classes. □

We define Λ to be of *tame lattice type* if for any vector $v\in\boldsymbol{N}^n$ there exists a polynomial $h\in K[y]$ and a family of additive functors

$$\hat{M}^{(1)}, \cdots, \hat{M}^{(s)} : \mathrm{ind}_1(K[y]_h) \longrightarrow \mathrm{latt}(\Lambda) \tag{7.7}$$

forming a weak almost $K[y]_h$-parametrizing family for the category $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$.

Given an integer $v\geqq 1$ or a vector $v\in\boldsymbol{N}^n$ we define $\boldsymbol{\mu}^1_{\mathrm{latt}(\Lambda)}(v)$ to be the minimal number s of functors (7.7) forming an almost $K[y]_h$-parametrizing family for the category $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$. The D-order Λ of tame lattice type is defined to be of *polynomial growth* if there exists an integer $G\geqq 1$ such that $\boldsymbol{\mu}^1_{\mathrm{latt}(\Lambda)}(v)\leqq\|v\|^G$ for all vectors $v\in\boldsymbol{N}^n$ such that $\|v\|\geqq 2$ (compare with [22, Section 15.10]). □

It follows from Theorems 3.6 and 3.8 in [24] that our definition of D-orders of tame lattice type and of wild lattice type is equivalent to that one given by Drozd and Greuel in [5]. Therefore the results of Drozd-Greuel [5] yield that any D-order Λ is of tame or of wild lattice type and these types are mutually exclusive. Note that if Λ is of finite lattice type then it is of tame lattice type.

Now we are going to reduce the study of the category $\mathrm{latt}(\Lambda)$ to the study of Γ-complete propartite $\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$-modules and to B-complete propartite $\begin{pmatrix}A & B\\ 0 & B\end{pmatrix}$-modules by means of the commutative diagram (7.14) below. The main idea for that is to view Λ-lattices as modules in $\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$. We do it by defining the pair of adjoint functors

$$\mathrm{latt}(\Lambda) \underset{\mathrm{res}_\Lambda}{\overset{\mathcal{I}_\Gamma}{\rightleftarrows}} \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \tag{7.8}$$

where res_Λ is the *restriction functor* defined by the formula $\mathrm{res}_\Lambda(Z'_\Lambda, Z''_\Gamma, \varphi)=Z'_\Lambda$. The *induction functor* $\mathcal{I}_\Gamma$ is defined as follows. Given a Λ-lattice X we set

$$\mathcal{J}_\Gamma(X) = (X, X\Gamma, \mu) \tag{7.9}$$

where $X\Gamma \subseteq XF = X\otimes_D F$ is the Γ-submodule of the C-module XF generated by X, and

$$\mu : X\otimes_\Lambda \Gamma_\Gamma \longrightarrow X\Gamma \tag{7.10}$$

is the Γ-homomorphism such that the Λ-homomorphism $\bar{\mu} : X \to \mathrm{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, X\Gamma) \cong X\Gamma$ adjoint to μ is the embedding $X \hookrightarrow X\Gamma$. Note that $\mathrm{Ker}\,\mu$ is the Γ-submodule of $X\otimes_\Lambda \Gamma$ consisting of all D-torsion elements and

$$X\Gamma \cong (X\otimes_\Lambda \Gamma)/\mathrm{Ker}\,\mu\,.$$

If $f : X \to Y$ is a homomorphism of Λ-lattices we set $\mathcal{J}_\Gamma(f) = (f, \hat{f}) : \mathcal{J}(X) \to \mathcal{J}(Y)$, where $\hat{f}$ is a unique Γ-homomorphism making the diagram

$$\begin{array}{ccc} X & \hookrightarrow & X\Gamma \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \hat{f}} \\ Y & \hookrightarrow & Y\Gamma \end{array}$$

commutative. It is easy to see that $\mathcal{J}$ is a covariant K-linear functor.

PROPOSITION 7.11. (a) *For any Λ-lattice X and any module $Z = (Z'_\Lambda, Z''_\Gamma, \varphi)$ in $\mathrm{mod}_{\mathrm{pi}}\begin{pmatrix}\Lambda & \Gamma \\ 0 & \Gamma\end{pmatrix}$ the restriction map*

$$\mathrm{res}_\Lambda : \mathrm{Hom}_{\Lambda\Gamma}(\mathcal{J}_\Gamma(X), Z) \longrightarrow \mathrm{Hom}_\Lambda(X, \mathrm{res}_\Lambda(Z))$$

$(f', f'') \mapsto f'$, is an isomorphism and it is natural with respect to the homomorphisms $X \to Y$ and $Z \to U$. In other words, the functor $\mathcal{J}_\Gamma$ is left adjoint to the exact functor res_Λ. Moreover (see (7.4))

$$\|\mathrm{cdn}(X\Gamma)\| \leqq m\|\mathrm{cdn}(X)\|(\mathrm{rk}_D \Lambda)\,. \tag{7.12}$$

(b) *The functor res_Λ is full faithful and establishes an equivalence of categories*

$$\mathrm{res}_\Lambda : \widehat{\mathrm{mod}}_{\mathrm{pi}}\begin{pmatrix}\Lambda & \Gamma \\ 0 & \Gamma\end{pmatrix} \longrightarrow \mathrm{latt}(\Lambda)\,. \tag{7.13}$$

The quasi-inverse of res_Λ is the functor $\mathcal{J}_\Gamma$.

(c) *The functor res_Λ preserves representation types and the polynomial growth property.*

(d) *If the functor $\mathcal{J}_\Gamma$ is exact then $\mathcal{J}_\Gamma$ preserves and reflects representation types and the polynomial growth property.*

PROOF. The proof of (b) and the first part of (a) easily follows from definitions and we leave it to the reader. For the second part of (a) we note that $\mathrm{cdn}(\Gamma)=(1, \cdots, 1)\in \boldsymbol{N}^m$, $\mathrm{cdn}(X\otimes_D \Gamma)=(\mathrm{rk}_D X, \cdots, \mathrm{rk}_D X)\in \boldsymbol{N}^m$, the Γ-module $X\otimes_D \Gamma$ is free and there are Γ-module epimorphisms $X\otimes_D \Gamma \to X\otimes_\Lambda \Gamma \to X\Gamma$. It follows that the projective cover $P_\Gamma(X\Gamma)$ of $X\Gamma$ is a direct Γ-summand of $X\otimes_D \Gamma$ and therefore $\mathrm{cdn}(X\otimes_D \Gamma)-\mathrm{cdn}(X\Gamma)\in \boldsymbol{N}^m$. This together with the inequality $\mathrm{rk}_D X \leqq \|\mathrm{cdn}(X)\| \cdot \mathrm{rk}_D \Lambda$ (see Lemma 3.8 (a) in [**24**]) yields $\|\mathrm{cdn}(X\otimes_D \Gamma)\| = m(\mathrm{rk}_D X) \leqq m\|\mathrm{cdn}(X)\|(\mathrm{rk}_D \Lambda)$ and (a) follows.

(c) We note that given a K-algebra S any S-Λ_Γ-bimodule ${}_S N_{\Lambda_\Gamma}$ may be identified with the triple

$$ {}_S N_{\Lambda_\Gamma} = ({}_S N'_\Lambda,\ {}_S N''_\Gamma,\ \bar{\varphi}) $$

where ${}_S N'_\Lambda$ and ${}_S N''_\Gamma$ are bimodules and $\bar{\varphi} : {}_S N'_\Lambda \to {}_S N''_\Gamma$ is a homomorphism of S-Λ-bimodules. Moreover, for any right S-module U_S there exists a Λ_Γ-isomorphism

$$ U\otimes_S N_{\Lambda_\Gamma} \cong (U\otimes_S N'_\Lambda,\ U\otimes_S N''_\Gamma,\ \mathrm{id}\otimes\bar{\varphi}) $$

which is natural with respect to the S-homomorphisms $U\to V$. It is easy to see that for any right S-module U there is an isomorphism

$$ \mathrm{res}_\Lambda(U\otimes_S N_{\Lambda_\Gamma}) \cong U\otimes_S \mathrm{res}_\Lambda({}_S N_{\Lambda_\Gamma}) $$

which is natural with respect to S-homomorphisms $U\to V$. It follows that the functor res_Λ preserves wild representation type and carries $K[y]_h$-parametrizing families of functors to $K[y]_h$-parametrizing families. Moreover, since the set $H(v)=\{w\ ;\ \|w\|\leqq m\|v\|(\mathrm{rk}_D \Lambda)\}$ is finite and the inclusion

$$ \mathcal{J}_\Gamma(\mathrm{ind}_v(\mathrm{latt}(\Lambda))) \subseteq \bigcup_{w\in H(v)} \mathrm{ind}_{(v,\,w)}\left(\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}\right) $$

holds for any $v\in \boldsymbol{N}^n$, then the functor res_Λ preserves tame representation type and the polynomial growth property. To see this we shall show that if $v\in \boldsymbol{N}^n$ and

$$ \hat{N}^{(1)}, \cdots, \hat{N}^{(s)} : \mathrm{ind}_1(\mathrm{mod}(K[y]_h)) \longrightarrow \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix} $$

is an almost parametrizing family for $\bigcup_{w\in H(v)} \mathrm{ind}_{(v,\,w)}\left(\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}\right)$ then the functors $\mathrm{res}_\Lambda\circ\hat{N}^{(1)}, \cdots, \mathrm{res}_\Lambda\circ\hat{N}^{(s)}$ form an almost parametrizing family for $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$. This follows from the fact that if X is in $\mathrm{ind}_v(\mathrm{latt}(\Lambda))$ then the module $\mathcal{J}_\Gamma(X)$ belongs to $\bigcup_{w\in H(v)} \mathrm{ind}_{(v,\,w)}\left(\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}\right)$ and therefore $\mathcal{J}_\Gamma(X)\cong \hat{N}^{(j)}(K_\lambda^1)$ for some j and $\lambda\in K$ (see (6.1)). Hence we get $X\cong \mathrm{res}_\Lambda(\mathcal{J}_\Gamma(X))\cong K_\lambda^1\otimes_{K[y]_h} N_\Lambda'^{(j)}$, and we are done. Further, by applying the same type of arguments we show that the functor res_Λ preserves the polynomial growth property (see also the proof of Corollary 3.9 in [**24**]).

(d) It is easy to see that given an S-Λ-bimodule ${}_S M_\Lambda$ the right Λ_Γ-module

$$\mathcal{J}_\Gamma({}_S M_\Lambda) = ({}_S M_\Lambda,\ {}_S M\Gamma_\Gamma,\ \mu)$$

(see (7.9)) has a natural S-Λ_Γ-bimodule structure. If $\mathcal{J}_\Gamma$ is exact then there is an isomorphism

$$\mathcal{J}_\Gamma(U\otimes_S M_\Lambda) \cong U\otimes_S \mathcal{J}_\Gamma({}_S M_\Lambda)$$

which is natural with respect to S-homomorphisms $U\to V$. It follows that the functor $\mathcal{J}_\Lambda$ preserves wild representation type and carries $K[y]_h$-parametrizing families of functors to $K[y]_h$-parametrizing families. Further, it follows from (7.12) that

$$\mathrm{ind}_v(\mathrm{latt}(\Lambda)) \supseteqq \mathrm{res}_\Lambda\left(\mathrm{ind}_{(v,\,w)}\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}\right)$$

for any $w\in \boldsymbol{N}^m$ and any $v\in \boldsymbol{N}^n$. Hence (d) easily follows by applying the arguments used in the proof of (c). This finishes the proof.

CONVENTION. Following our discussion in Section 2 the right modules over the algebra $\Lambda_\Gamma=\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ will be identified with triples $X=(X'_\Lambda, X''_\Gamma, t: X'\to X'')$, where X' is a right Λ-module, X'' is a right Γ-module and t is a Λ-homomorphism. Analogously, right modules over $\bar{\Lambda}_\Gamma=\begin{pmatrix}A & B\\ 0 & B\end{pmatrix}$ will be identified with triples $Y=(Y'_A, Y''_B, t: Y'\to Y'')$, where Y' is a right A-module, Y'' is a right B-module and t is an A-homomorphism.

A fundamental role in applications is played by the functorial connections between lattices over orders, propartite modules and representations of bocses given in the commutative diagram induced by a maximal D-order $\Gamma\subseteq C$ containing Λ, and a two-sided ideal $I\subseteq \mathrm{rad}(\Lambda)\cap\mathrm{rad}(\Gamma)$ (see (7.2))

(7.14)

$$\begin{tikzcd}
\bullet\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \arrow[rrr, "F_4"] \arrow[dr] \arrow[ddd, "\bullet\hat{\Theta}^\Lambda"] & & & \bullet\widehat{\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix}A & B\\ 0 & B\end{pmatrix} \arrow[dl] \arrow[ddd, "\bullet\hat{\Theta}^A"] \\
& \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \arrow[r, "F_3"] \arrow[d, "\Theta^\Lambda"] & \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix}A & B\\ 0 & B\end{pmatrix} \arrow[d, "\Theta^A"] & \\
& \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \arrow[r, "F_2"] & \mathrm{mod}_{\mathrm{pr}}\begin{pmatrix}A & B\\ 0 & B\end{pmatrix} & \\
\widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \arrow[ur] \arrow[rrr, "F_1"] \arrow[d, "\mathrm{res}_\Lambda", "\cong"'] & & & \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix}A & B\\ 0 & B\end{pmatrix} \arrow[ul] \\
\mathrm{latt}(\Lambda) \arrow[urrr, "F"'] & & &
\end{tikzcd}$$

where $A=\Lambda/I \hookrightarrow B=\Gamma/I$, Θ^{A}, Θ^{A}, $\cdot\hat{\Theta}^{A}$, $\cdot\hat{\Theta}^{A}$ are the adjustment functors (see (2.2)) and res_{Λ} is the functor defined in (7.8). The functor $\boldsymbol{F}$ is defined by the formula

$$(7.15) \qquad \boldsymbol{F}(X)=(X/XI,\ X\Gamma/XI,\ u: X/XI \to X\Gamma/XI)$$

where u is the A-module embedding induced by the monomorphism $X \hookrightarrow X\Gamma$ (see [7], [18]). By Proposition 3.7 (d), the categories $\widehat{\cdot\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix} \subseteq \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}$, $\widehat{\cdot\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \subseteq \mathrm{mod}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$ consist of modules of projective dimension at most 1. Keeping the notation of (7.3) we set

$$\Lambda_{\Gamma}=\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}, \quad \bar{\Lambda}_{\Gamma}=\begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \quad \text{and} \quad A=\Lambda/I \to B=\Gamma/I.$$

In order to define the functors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ we consider the two-sided ideal

$$(7.16) \qquad \hat{I}=\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \subseteq \mathrm{rad}\begin{pmatrix} \Lambda & \Gamma \\ 0 & \Gamma \end{pmatrix}$$

of Λ_{Γ}. For any $j=1, 2, 3, 4$ and a Λ_{Γ}-module $X=(X'_{\Lambda}, X''_{\Gamma}, t: X' \to X'')$ we set

$$(7.17) \qquad \boldsymbol{F}_j(X)=X/X\hat{I}=(X'_{\Lambda}/X'_{\Lambda}I,\ X''_{\Gamma}/X''_{\Gamma}I,\ \bar{t})$$

where $\bar{t}: X'_{\Lambda}/X'_{\Lambda}I \to X''_{\Gamma}/X''_{\Gamma}I$ is the residue class A-homomorphism. If $f=(f', f''):$ $(X'_{\Lambda}, X''_{\Gamma}, t) \to (Y'_{\Lambda}, Y''_{\Gamma}, u)$ is a Λ_{Γ}-homomorphism we set $\boldsymbol{F}_j(f)=\bar{f}=(\bar{f}', \bar{f}'')$, where $\bar{f}: X/X\hat{I} \to Y/Y\hat{I}$, $\bar{f}': X'_{\Lambda}/X'_{\Lambda}I \to Y'_{\Lambda}/Y'_{\Lambda}I$ and $\bar{f}'': X''_{\Gamma}/X''_{\Gamma}I \to Y''_{\Gamma}/Y''_{\Gamma}I$ are the residue class homomorphisms induced by f, f' and f'', respectively. It is easy to check that $\boldsymbol{F}$, $\boldsymbol{F}_1, \cdots, \boldsymbol{F}_4$ are covariant additive K-linear functors making the diagram (7.14) commutative.

The proof of our main theorem of this section depends essentially on the following result on lowering and lifting bimodules.

LEMMA 7.18. *Assume that S is a commutative noetherian K-domain.*

(a) *Let ${}_SM_{\Lambda_\Gamma}=({}_SM'_{\Lambda}, {}_SM''_{\Gamma}, \varphi: {}_SM'_{\Lambda} \to {}_SM''_{\Gamma})$ be an S-Λ_{Γ}-bimodule and let*

$$\boldsymbol{F}_j({}_SM_{\Lambda_\Gamma})=({}_SN'_{A}, {}_SN''_{B}, \bar{\varphi}), \quad \Theta^{A}({}_SM_{\Lambda_\Gamma})=(\mathrm{Im}\,\varphi, {}_{\Lambda}M''_{\Gamma}, u).$$

where ${}_SN'_{A}={}_SM'_{\Lambda}/{}_SM'I$, ${}_SN''_{A}={}_SM''_{\Gamma}/{}_SM''I$ and u is the embedding $\mathrm{Im}\,\varphi \hookrightarrow {}_{\Lambda}M''_{\Gamma}$. *Then $\boldsymbol{F}_j({}_SM_{\Lambda_\Gamma})$ are S-$\bar{\Lambda}_{\Gamma}$-bimodules and $\Theta^{A}({}_SM_{\Lambda_\Gamma})$ is an S-Λ_{Γ}-bimodule. If ${}_SM''_{\Gamma}$ is a finitely generated projective S-Γ-bimodule then there exists an element $h \in S$ such that*

(i) *The left S_h-module $\Theta^{A}({}_hM_{\Lambda_\Gamma})$ is projective.*

(ii) *For any U in* $\mathrm{fin}(S_h)$ *there are isomorphisms*

$$\boldsymbol{F}_j(U \otimes_{S_h} {}_hM_{\Lambda_\Gamma}) \cong U \otimes_{S_h} \boldsymbol{F}_j({}_hM_{\Lambda_\Gamma}), \quad \Theta^{A}(U \otimes_{S_h} {}_hM_{\Lambda_\Gamma}) \cong U \otimes_{S_h} \Theta^{A}({}_hM_{\Lambda_\Gamma})$$

for $j=1, 2, 3, 4$, which are functorial with respect to the homomorphisms $U \to U'$.

(b) *For any S-$\bar{\Lambda}_{\Gamma}$-bimodule*

$$ {}_SN_{\bar{A}_\Gamma} = ({}_SN'_A,\ {}_SN''_B,\ \bar{\varphi}) $$

such that ${}_SN''_B$ is a finitely generated projective S-B-bimodule there exist an element $h \in S$ and an S-Λ_Γ-bimodule ${}_SM_{\Lambda_\Gamma}=({}_SM'_\Lambda, {}_SM''_\Gamma, \varphi)$ such that the left S-module ${}_SM$ is projective, ${}_SM''_\Gamma$ is a finitely generated projective S-Γ-bimodule and for any U in $\mathrm{fin}(S_h)$ there are isomorphisms

$$ \boldsymbol{F}_j(U \otimes_{S_h} {}_hM_{\Lambda_\Gamma}) \cong U \otimes_{S_h} \boldsymbol{F}_j({}_hM_{\Lambda_\Gamma}), \quad \Theta^A(U \otimes_{S_h} {}_hM_{\Lambda_\Gamma}) \cong U \otimes_{S_h} \Theta^A({}_hM_{\Lambda_\Gamma}) $$

for $j=1, 2, 3, 4$, which are functorial with respect to the homomorphisms $U \to U'$.

PROOF. (a) Assume that ${}_SM''_\Gamma$ is a finitely generated projective S-Γ-bimodule and consider the commutative diagram of bimodules

$$ \begin{tikzcd} {}_SM'I \arrow[r, hook] \arrow[d, "\varphi"] & {}_SM'_\Lambda \arrow[r, "\varepsilon'"] \arrow[d] & {}_SN'_A \arrow[r] \arrow[d, "\bar{\varphi}"] & 0 \\ {}_SM''I \arrow[r, hook] \arrow[d] & {}_SM''_\Gamma \arrow[r, "\varepsilon''"] \arrow[d] & {}_SN''_B \arrow[r] \arrow[d] & 0 \\ 0 & \mathrm{Coker}\,\varphi \arrow[r, "\bar{\varepsilon}''", "\cong"'] & \mathrm{Coker}\,\bar{\varphi} \arrow[r] & 0 \end{tikzcd} $$

with exact rows and columns. It follows that $\bar{\varepsilon}''$ is bijective. By our assumption A and B are finite dimensional algebras, and ${}_SN'_A$ and ${}_SN''_B$ are finitely generated bimodules. It follows that the bimodule $\mathrm{Coker}\,\bar{\varphi}$ is finitely generated and therefore the left S-modules ${}_SN'_A$, ${}_SN''_B$ and $\mathrm{Coker}\,\bar{\varphi}$ are finitely generated. By Lemma 6.11, there exists an element $h \in S$ such that the localizations of ${}_SN'_A$, ${}_SN''_B$ and $\mathrm{Coker}\,\bar{\varphi}$ with respect to h are projective S_h-modules. It follows that $S_h \otimes_S \mathrm{Coker}\,\varphi$ is a projective S_h-module and therefore $\mathrm{Im}(S_h \otimes \varphi) \cong S_h \otimes_S \mathrm{Im}\,\varphi$ is projective, because by the assumption the bimodule ${}_SM''_\Gamma$ is finitely generated projective and therefore the left S_h-module $S_h \otimes_S M''$ is projective. Hence (a) follows.

(b) It follows from Lemma 4 in [**5**] and its proof that for any S-$\bar{A}_\Gamma$-bimodule ${}_SN_{\bar{A}_\Gamma}=({}_SN'_A, {}_SN''_B, \bar{\varphi})$ as in (a) there exists a commutative diagram above, where the bimodule ${}_SM''_\Gamma$ is finitely generated projective and the localization of $\mathrm{Coker}\,\varphi$ with respect to some $h \in S$ is a flat S_h-module. Hence we conclude as in the proof of (a) that the conditions required in (b) are satisfied. □

Now we are able to prove the main result of this section.

THEOREM 7.19. *In the notation and assumption made above the following statements hold.*

(a) *The functors $\boldsymbol{F}$, $\boldsymbol{F}_1$, $\boldsymbol{F}_4$ are K-linear representation equivalences, that is, they are full dense and reflect isomorphisms. The diagram* (7.14) *is commutative and*

$$(7.20)\qquad \mathrm{cdn}\, X = \mathrm{cdn}\, \boldsymbol{F}_j(X), \quad \mathrm{cdn}\, Y = \mathrm{cdn}\, \bullet\hat{\Theta}^A(Y), \quad \mathrm{cdn}\, Z = \mathrm{cdn}\, \bullet\hat{\Theta}^A(Z)$$

for $j=1, 2, 3, 4$, *for any indecomposable module* X *such that* $\boldsymbol{F}_j(X)\neq 0$, *and for all indecomposable modules* Y *in* $\widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma \end{pmatrix}$ *and* Z *in* $\widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$.

(b) *The functors* $\boldsymbol{F}$, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ *and the adjustment functors* $\bullet\hat{\Theta}^A$, Θ^A, Θ^A, $\bullet\hat{\Theta}^A$ *lower and lift smooth almost* $K[y]_h$*-parametrizing families. Moreover they preserve and reflect tame representation type as well as the polynomial growth property.*

(c) *The functors* $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ *preserve and reflect smooth wildness. The adjustment functors* $\bullet\hat{\Theta}^A$, $\bullet\hat{\Theta}^A$ *preserve smooth wildness.*

PROOF. (a) It was proved in [7] and [18] that the functor $\boldsymbol{F}$ is a representation equivalence. The arguments applied there also shows that the functors $\boldsymbol{F}_1$ and $\boldsymbol{F}_4$ are representation equivalences. The commutativity of the diagram (7.14) follows immediately from definition.

In order to prove (7.20) we note that since $I\subseteq \mathrm{rad}(A)$ and $I\subseteq \mathrm{rad}(\Gamma)$ then given a A_Γ-module $X=(X'_A, X''_\Gamma, t)$ the canonical epimorphisms $\varepsilon' : X'_A \to X'_A/X'I$ and $\varepsilon'' : X''_\Gamma \to X''_\Gamma/X''I$ are minimal epimorphisms and therefore they induce the isomorphisms $\mathrm{top}_A(X'_A)\cong \mathrm{top}_A(X'_A/X'I)$ and $\mathrm{top}_\Gamma(X''_\Gamma)\cong \mathrm{top}_B(X''_\Gamma/X''I)$. Hence (7.20) follows for the functors $\boldsymbol{F}_1, \cdots, \boldsymbol{F}_4$. The remaining part of (7.20) follows in a similar way.

(b) We shall show that $\boldsymbol{F}_j$ lowers smooth almost parametrizing families, preserves tame representation type and the polynomial growth property for $j=1, 2, 3, 4$. For, assume that $j=4$ and that

$$\hat{N}^{(1)}, \cdots, \hat{N}^{(s)} : \mathrm{ind}_1(\mathrm{mod}(K[y]_h)) \longrightarrow \widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma \end{pmatrix}$$

is a smooth almost parametrizing family for the category $\mathrm{ind}_v\left(\widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma \end{pmatrix}\right)$. It follows from (a) that

$$\boldsymbol{F}_4\left(\mathrm{ind}_v \widehat{\mathrm{mod}}_{\mathrm{pr}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma \end{pmatrix}\right) = \mathrm{ind}_v\left(\widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}\right).$$

We shall show that the functors $\boldsymbol{F}_4\circ\hat{N}^{(1)}, \cdots, \boldsymbol{F}_4\circ\hat{N}^{(s)}$ form an almost parametrizing family for $\mathrm{ind}_v\left(\widehat{\bullet\mathrm{mod}}^{\mathrm{pr}}_{\mathrm{pr}}\begin{pmatrix} A & B \\ 0 & B \end{pmatrix}\right)$.

It follows from Lemma 7.18 (a) that the polynomial $h\in K[t]$ can be chosen in such a way that the equivalence $\boldsymbol{F}_4\circ\hat{N}^{(i)}\cong(-)\otimes_{K[y]_h}\bar{N}^{(i)}$ holds for $i=1, \cdots, s$, where $\bar{N}^{(i)}=N^{(i)}/N^{(i)}\hat{I}$. Since $N^{(i)}$ is a finitely generated $K[y]_h$-A_Γ-bimodule then according to Lemma 3.4 (ii) in [24] $N^{(i)}$ is finitely generated over $D[y]_h$

and therefore $\bar{N}^{(i)}$ is a finitely generated module over the $K[y]_h$-algebra $(D/D\cap I)[y]_h$. We recall that $(D/D\cap I)\cong D/(p^c)$, $c\geq 1$, is a finite dimensional K-algebra. Hence $(D/D\cap I)[y]_h$ is a finitely generated module over $K[y]_h$ and therefore $\bar{N}^{(i)}$ is a finitely generated $K[y]_h$-module for $i=1, \cdots, s$.

Now if Y is a module in $\mathrm{ind}_v\left(\widehat{\mathrm{mod}_{\mathrm{pr}}}\begin{pmatrix} A & B \\ 0 & B\end{pmatrix}\right)$ then according to (a) there exists a module X in $\mathrm{ind}_v\left(\widehat{\mathrm{mod}_{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}\right)$ such that $\boldsymbol{F}_4(X)\cong Y$. By our assumption all but finitely many such modules X have the form $X\cong K_\lambda^1\otimes_{K[y]_h}N^{(i)}$ for some index i and $\lambda\in K$ (see (6.1)). Hence $Y\cong\boldsymbol{F}_4(X)\cong K_\lambda^1\otimes_{K[y]_h}\bar{N}^{(i)}$. This shows that the functors $\boldsymbol{F}_4\circ\hat{N}^{(1)}, \cdots, \boldsymbol{F}_4\circ\hat{N}^{(s)}$ form an almost parametrizing family for $\mathrm{ind}_v\left(\widehat{\mathrm{mod}_{\mathrm{pr}}}\begin{pmatrix} A & B \\ 0 & B\end{pmatrix}\right)$. It follows that the functor $\boldsymbol{F}_4$ lowers smooth almost $K[y]_h$-parametrizing families, preserves tame representation type and the polynomial growth property. The fact that $\boldsymbol{F}_4$ lifts smooth almost $K[y]_h$-parametrizing families, reflects tame representation type and the polynomial growth property can be proved in a similar way by applying Lemma 7.18 (b). The proof for the functors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ is analogous.

(c) Assume that the category $\bullet\widehat{\mathrm{mod}_{\mathrm{pi}}^{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}$ is of smooth wild representation type. It follows that for $S=K[t_1, t_2]$ there exists an S-A_Γ-bimodule ${}_SM_{A_\Gamma}$, which is finitely generated projective as a bimodule and induces an exact representation embedding $\hat{M}\colon \mathrm{fin}(S)\to\bullet\widehat{\mathrm{mod}_{\mathrm{pi}}^{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}$. According to Lemma 7.18 (a) there exists $h\in S$ such that the composed functor

$$\mathrm{fin}(S_h)\xrightarrow{{}_h\hat{M}}\bullet\widehat{\mathrm{mod}_{\mathrm{pi}}^{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}\xrightarrow{\boldsymbol{F}_4}\bullet\widehat{\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pi}}}\begin{pmatrix} A & B \\ 0 & B\end{pmatrix}$$

is exact and of the form $(-)\otimes_{S_h}N_{\bar{A}_\Gamma}$. Since S_h is representation-wild then the category $\bullet\widehat{\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}\begin{pmatrix} A & B \\ 0 & B\end{pmatrix}$ is of wild representation type (see [**24**]). This shows that $\boldsymbol{F}_4$ preserves wild representation type. The fact that $\boldsymbol{F}_4$ reflects wild representation type can be proved in a similar way by applying Lemma 7.18 (b). The proof for the functors $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ is analogous.

The fact that the adjustment functor $\bullet\hat{\Theta}^A$ preserves smooth wildness is analogous to that one for the functor $\boldsymbol{F}_4$. Then in view of Theorem 6.10 (c) the statement (c) is proved. □

The functors $\boldsymbol{F}$, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, $\boldsymbol{F}_4$ are reduction functors in the sense of Dieterich [**3**].

Under the assumption and notation above let us define the *lattice adjustment functor*

$$\boldsymbol{G}\colon\widehat{\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}\longrightarrow\mathrm{latt}(A) \tag{7.21}$$

to be the composed functor $\widehat{\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}\xrightarrow{\hat{\Theta}^A}\widehat{\mathrm{mod}_{\mathrm{pr}}}\begin{pmatrix} A & \Gamma \\ 0 & \Gamma\end{pmatrix}\xrightarrow{\mathrm{res}_A}\mathrm{latt}(A)$. It will play an important role in developing the covering technique for $\mathrm{latt}(\Lambda)$ (compare with [**12**]).

Note that according to Proposition 3.7 (b), the category $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ consists of modules of projective dimension at most 1.

Main properties of $\boldsymbol{G}$ are collected in the following corollary, which immediately follows from Proposition 2.4, Proposition 7.11 and Theorem 7.19 (b).

COROLLARY 7.22. *Let $\Lambda\subset\Gamma\subseteqq C$ be as above and let $\boldsymbol{G}$ be the functor* (7.21).

(a) *$\boldsymbol{G}$ is full dense, and $\boldsymbol{G}$ carries a homomorphism f to zero if and only if f has a factorization through a projective Λ-module viewed as a $\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$-module.*

(b) *Let $X=(X'_\Lambda, X''_\Gamma, \varphi)$ be an indecomposable module in $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$. Then $\boldsymbol{G}(X)=0$ if and only if $X''_\Gamma=0$. If $\boldsymbol{G}(X)\neq 0$ then* $\mathrm{cdn}(\boldsymbol{G}(X))=\mathrm{cdn}(X'_\Lambda)$.

(c) *The functor $\boldsymbol{G}$ preserves and reflects representation types, and the restriction*

$$\boldsymbol{G}\bullet : \bullet\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix} \longrightarrow \mathrm{latt}(\Lambda) \tag{7.23}$$

of $\boldsymbol{G}$ to $\bullet\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ is a representation equivalence. □

Let us finish the paper by a list of open problems on propartite modules.

PROBLEMS 7.24. (a) Is the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ a hereditary subcategory of $\mathrm{mod}(R)$ for any bipartite semiperfect ring $R=\begin{pmatrix}A & M\\ 0 & B\end{pmatrix}$? (Compare with Proposition 3.7 (d).)

(b) Complete Corollary 4.8 by constructing minimal left almost split morphisms in the category $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ starting with any indecomposable relative injective module.

(c) It would be interesting to know if the categories $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}\begin{pmatrix}\Lambda & \Gamma\\ 0 & \Gamma\end{pmatrix}$ have almost split sequences for any pair of D-orders $\Lambda\subseteqq\Gamma\subseteqq C$, where $\Gamma\subseteqq C$ is a hereditary order. It seems to us that this should be done by using the functor (7.21), the fact that $\mathrm{latt}(\Lambda)$ has almost split sequences and by applying the method used in [**22**, Theorem 11.68] and [**16**, Theorem 3.4] (compare also with [**26**]).

(d) It would be interesting to develop a tilting theory for the categories of the form $\mathrm{mod}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$ and $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(R)_B^A$, where $R=\begin{pmatrix}A & M\\ 0 & B\end{pmatrix}$ is a bipartite algebra. In this case a tilting module should be bipartite, that is, of the form $T=T'\oplus T''$, with $\mathrm{Hom}_R(T', T'')=0$. Then the tilted algebra is a bipartite one of the form $R'=\mathrm{End}(T)=\begin{pmatrix}A' & M'\\ 0 & B'\end{pmatrix}$, where $A'=\mathrm{End}(T')$, $B'=\mathrm{End}(T'')$ and $M'=\mathrm{Hom}_R(T'', T')$. This applied to $\widehat{\mathrm{mod}}_{\mathrm{pr}}^{\mathrm{pr}}(\Lambda_\Gamma)$ should provide with a tilting procedure for $\mathrm{latt}(\Lambda)$ (compare with [**6**]), because of the functor (7.21). An example of such a bipartite tilting is provided by Theorem 17.81 in [**22**].

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References

[1] N. Bourbaki, Algébre Commutative, Paris, Hermann, 1965, 1968, 1969.
[2] W. W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc., **56** (1988), 451-483.
[3] E. Dieterich, Reduction of isolated singularities, Comment. Math. Helv., **62** (1987), 654-676.
[4] Ju. A. Drozd, Tame and wild matrix problems (in Russian), in Representations and Quadratic Forms, Kiev 1979, pp. 39-74.
[5] J. A. Drozd and M. G. Greuel, Tame-wild dichotomy for Cohen-Macaulay modules, Math. Ann., **294** (1992), 387-394.
[6] H. Fujita and K. Nishida, Tilting lattices over orders associated with simple modules, Tsukuba J. Math., **15** (1991), 529-546.
[7] E. L. Green and I. Reiner, Integral representations and diagrams, Michigan Math. J., **25** (1978), 53-84, (1988), 312-336.
[8] S. Kasjan, On prinjective modules, tameness and the Tits form, Bull. Polish Acad. Sci. Math., **41** (1993), 327-341.
[9] S. Kasjan, Adjustment functors and tame representation type, Comm. Algebra, **22** (1994), 5587-5597.
[10] S. Kasjan and D. Simson, Varieties of poset representations, tameness and minimal posets of wild prinjective type, in Proceedings of the Sixth International Conference on Representations of Algebras, Canadian Mathematical Society Conference Proceedings, Vol. **14**, 1993, pp. 245-284.
[11] H. Kraft, Geometric methods in representation theory, in Representations of Algebras, in Lecture Notes in Math., No. **944**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1982, pp. 180-258.
[12] H. Lenzing and D. Simson, Cohen-Macaulay modules, representations of posets and tiled orders, in preparation.
[13] K. Nishida, Representations of orders and vector space categories, J. Pure Appl. Algebra, **33** (1984), 209-217.
[14] J. A. de la Peña, Functors preserving tameness, Fund. Math., **137** (1991), 177-185.
[15] J. A. de la Peña, On the dimension of the module varieties of tame and wild algebras, Comm. Algebra, **19** (1991), 1795-1807.
[16] J. A. de la Peña and D. Simson, Prinjective modules, reflection functors, quadratic forms and Auslander-Reiten sequences, Trans. Amer. Math. Soc., **329** (1992), 733-753.
[17] I. Reiner, Maximal Orders, Academic Press, London-New York-San Francisco, 1975.
[18] C. M. Ringel and K. W. Roggenkamp, Diagrammatic methods in the representation theory of orders, J. Algebra, **60** (1979), 11-42.
[19] A. V. Roiter, Matrix problems and representations of bocses (in Russian), in Representations and Quadratic Forms, Kiev 1979, 3-38.
[20] D. Simson, Vector space categories, right peak rings and their socle projective modules, J. Algebra, **92** (1985), 532-571.
[21] D. Simson, A narrow over-ring adjustment functor, J. Algebra, **136** (1991), 463-496.

[22] D. Simson, Linear Representations of Partially Ordered Sets and Vector Space Categories, Algebra, Logic and Applications, Vol. **4**, Gordon & Breach Science Publishers, 1992.

[23] D. Simson, Posets of finite prinjective type and a class of orders, J. Pure Appl. Algebra, **90** (1993), 77-103.

[24] D. Simson, On representation types of module subcategories and orders, Bull. Polish Acad. Sci. Math., **41** (1993), 77-93.

[25] D. Simson, Representation embedding problems, categories of extensions and prinjective modules, Canad. Math. Soc. Conf. Proc., **18** (1996), 601-639.

[26] S.O. Smalø, Functorial finite subcategories over triangular matrix rings, Proc. Amer. Math. Soc., **111** (1991), 651-656.

[27] K. Yamagata, On the representation theory of BOCS's, in Proceedings of Representation Theory Conference, Shimoda (Japan), 2.-5. November 1982, pp. 284-296, (in Japanese).

[28] K. Yamagata, Frobenius algebras, in Handbook of Algebra, (ed. M. Hazewinkel), Vol. 1, North-Holland Elsevier, 1996, pp. 841-887.

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