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A remark on the exotic free actions in dimension 4

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Donaldson's polynomial invariants ([**2**]) are powerful tools for studying smooth 4-manifolds. For example the diffeomorphism types of elliptic surfaces with positive geometric genus were completely classified by them ([**4**], [**20**], [**21**]). The examples of smooth closed 1-connected noncomplex 4-manifolds with infinitely many smooth structures were first given by [**9**] and then were constructed by various methods ([**6**], [**13**], [**16**], [**26**]). In this paper we will give some examples of infinitely many exotic 4-manifolds whose universal coverings are mutually diffeomorphic. In fact we will show that the fundamental group of any spherical 3-manifold other than the 3-sphere acts freely on certain 4-manifolds in infinitely many different ways so that their orbit spaces are exotic (Main Theorem). Throughout this paper we denote the $K3$ surface by K, and for any finite group G we denote by $|G|$ the order of G. For any closed oriented 4-manifold X, we denote by $b_2^+=b_2^+(X)$ the rank of the maximal positive subspace $H^+=H^+(X)$ of the intersection form q_X of X, and nX denotes the connected sum of n copies of X.

MAIN THEOREM. *Let G be the fundamental group of any spherical 3-manifold other than the 3-sphere. Let $X=(2n-1)\boldsymbol{CP}^2\#(10n-1)\overline{\boldsymbol{CP}}^2$ or $X=nK\#(n-1)S^2\times S^2$. Then if n is divided by $|G|$ and $|G|<n$, there exist infinitely many smooth orientation-preserving free G-actions on X such that their orbit spaces are mutually homeomorphic but non-diffeomorphic to each other.*

In §1 we will construct the manifolds which will be the orbit spaces for the actions in Main Theorem. These manifolds are connected sums of rational homology 4-spheres and 1-connected 4-manifolds with $b_2^+>1$ which are derived from certain elliptic surfaces. In §2 and §4 we will describe the simple invariants for these manifolds to distinguish their diffeomorphism types. In §3 the list of $SO(3)$-representations for the above G will be given for the estimates of the simple invariants in §4. In §5 we will complete the proof of Main Theorem. Here we note that the argument in §4 is similar to that in [**15**] in which the connected sums of $\boldsymbol{Z}_2$ homology 4-spheres and some manifolds with

$b_2^+=1$ and their Kotschick invariants are discussed. The author would like to thank the referee for pointing out some errors in the preliminary draft of this paper.

§ 1. Constructions.

Let S_k be the relatively minimal elliptic surface over $\boldsymbol{CP}^1$ with euler number $12k$ and with a cross section Σ, and let $S_k(p, q)$ be the relatively minimal elliptic surface over $\boldsymbol{CP}^1$ with euler number $12k$ with two multiple fibers of multiplicity p and q for $k>0$, $p\geqq 1$ and $q\geqq 1$. Here the multiple fiber of multiplicity 1 means the general fiber and $S_k(1, 1)$ is identified with S_k. We denote by f the general fiber of any elliptic surface. We start with the manifolds S_k^σ and $S_k^\sigma(p, q)$ for $k\geqq 2$ which are the analogues of the examples in [**9**] and are constructed as follows. Let S_k^0 (resp. $S_k^0(p, q)$) be the manifold obtained from S_k (resp. $S_k(p, q)$) by removing the tubular neighborhood of f. Now we fix the diffeomorphism σ from ∂S_{k-1}^0 to $\partial S_1^0(p, q)=\partial S_1^0$ which does not preserve the general fibers of S_{k-1}^0 and $S_1^0(p, q)$ (or S_1^0), and consider the manifolds S_k^σ and $S_k^\sigma(p, q)$ defined by

$$S_k^\sigma = S_1^0 \cup_\sigma S_{k-1}^0, \quad S_k^\sigma(p, q) = S_1^0(p, q) \cup_\sigma S_{k-1}^0 .$$

We note that S_k^σ is diffeomorphic to S_k since any diffeomorphism of ∂S_1^0 extends to that of S_1^0 ([**7**], [**18**]) but we use this symbol for convenience. Next we construct some rational homology 4-spheres as follows.

DEFINITION 1-1. For any closed oriented smooth 3-manifold M, let $s(M)$ and $s'(M)$ be the 4-manifolds defined as follows.

$$s(M) = (M \setminus \operatorname{Int} D^3)\times S^1 \cup_{id} S^2\times D^2$$

$$s'(M) = (M \setminus \operatorname{Int} D^3)\times S^1 \cup_{\tau} S^2\times D^2 .$$

Here D^3 is a small 3-ball in M and $\tau: S^2\times\partial D^2 \to \partial(M\setminus \operatorname{Int} D^3)\times S^1$ is a self-diffeomorphism of $S^2\times S^1$ defined by $\tau(x, \theta)=(\rho_\theta(x), \theta)$ for $x\in S^2$, $\theta\in \boldsymbol{R}/2\pi \boldsymbol{Z}=S^1$, where ρ_θ is a rotation through angle θ in a fixed axis on S^2. The manifolds $s(M)$ and $s'(M)$ are called an untwisted spin and a twisted spin of M respectively (see [**24**] for example).

REMARK 1-2. If M is a lens space $L(p, q)$ then $s(M)$ is diffeomorphic to $s'(M)$. Moreover the diffeomorphism type of $s(M)$ in this case depends only on p (not depending on q) ([**23**]).

Hereafter we consider the spins for the spherical 3-manifold M_G with $\pi_1 M=G$.

PROPOSITION 1-3. (1) *Both* $s(M_G)$ *and* $s'(M_G)$ *are rational homology* 4-*spheres with* $\pi_1 s(M_G)=\pi_1 s'(M_G)=G$. *If* M_G *is a* $\boldsymbol{Z}_2$-*homology sphere then so is* $s(M_G)$ (*and so is* $s'(M_G)$). (2) *The universal covering* $s(\tilde{M}_G)$ *of* $s(M_G)$ *is diffeomorphic to* $(|G|-1)S^2\times S^2$. *The same claim holds for the universal covering* $s'(\tilde{M}_G)$ *of* $s'(M_G)$.

PROOF. The proof of (1) is straightforward. The second claim is proved in [**24**], § 2. The proof goes as follows. The universal covering of $M_G\setminus \mathrm{Int}\, D^3$ is a $|G|$-punctured 3-sphere $S_0^3=S^3\setminus\bigcup_{i=1}^{|G|}\mathrm{Int}\, D_i^3$ where D_i^3 is a copy of a small 3-ball in S^3. Therefore $s(\tilde{M}_G)$ (resp. $s'(\tilde{M}_G)$) is the union of $S_0^3\times S^1$ and $|G|$ copies of $S^2\times D^2$ each of which is attached along each boundary component of $S_0^3\times S^1$ via the identity (resp. τ). Then $s(\tilde{M}_G)$ is obtained from S^4 by untwisted surgery along $|G|-1$ (unknotted) circles in S^4. It follows that $s(\tilde{M}_G)$ is diffeomorphic to $(|G|-1)S^2\times S^2$. In case of $s'(\tilde{M}_G)$ the small balls D_i^3 in S^3 are located so that a rotation in a fixed circle of S^3 through angle θ leaves all D_i^3's invariant and induces the same rotation on each D_i^3. Therefore the copies of the above map τ on $\partial S_0^3\times S^1$ extends to the diffeomorphism of $S_0^3\times S^1$ and hence $s'(\tilde{M}_G)$ is diffeomorphic to $s(\tilde{M}_G)$. This proves the second claim.

For later use we choose one of $s(M_G)$ and $s'(M_G)$, and denote it by W_G. (The arguments below do not depend on the choice of W_G.) Let $X_G(p, q)=W_G\# S_k^q(p, q)$ where p and q are natural numbers with $\gcd(p, q)=1$. Note that the last condition on p and q implies that $S_k(p, q)$ and $S_k^q(p, q)$ are 1-connected. To choose the candidates for the orbit spaces of the actions in Main Theorem from the manifolds of the form $X_G(p, q)$ we need some further observations for them. Recall that the regular neighborhood N_k in S_k of the union of the cusp fiber and the cross section Σ, and the manifold $N_k(p, q)$ obtained from N_k by performing logarithmic transforms of multiplicity p and q at the general fibers in N_k are called nuclei in [7]. Here $N_k(p, q)$ is contained in $S_k(p, q)$. Then $H_2(N_k, \boldsymbol{Z})$ is generated by f and Σ with

$$f\cdot f=0, \quad f\cdot\Sigma=1, \quad \Sigma\cdot\Sigma=-k$$

and $H_2(N_k(p, q), \boldsymbol{Z})$ is generated by some 2-cycles κ and Δ with

$$\kappa\cdot\kappa=0, \quad \kappa\cdot\Delta=1, \quad \Delta\cdot\Delta=-(p+q)^2-k(pq)^2.$$

Here κ is a primitive element which is a positive rational multiple of f (where $f=pq\kappa$). Next note that S_k^q (resp. $S_k^q(p, q)$) contains N_2 (resp. $N_2(p, q)$) and N_k which are mutually disjoint. In fact in S_k^q we have smoothly embedded 2-spheres Σ^σ and Σ' where Σ^σ (resp. Σ') is a union of the cross sectional 2-disk of S_{k-1}^0 (resp. S_1^0) and a vanishing 2-disk in S_1^0 (resp. S_{k-1}^0) which are attached along their boundaries via σ. Then N_k (resp. N_2) in S_k^q is constructed as a regular neighborhood of the union of Σ^σ (resp. Σ') and the cusp fiber in S_{k-1}^0 (resp. S_1^0). In case of $S_k^q(p, q)$ we have only to replace N_2 by $N_2(p, q)$. Since $\partial N_k(p, q)$ is the Brieskorn homology 3-sphere ([7]), $H_2(S_k^q(p, q), \boldsymbol{Z})$ is the direct sum of $H_2(N_k, \boldsymbol{Z})$ (generated by the general fiber f^σ in S_{k-1}^0 and Σ^σ), $H_2(N_2(p, q), \boldsymbol{Z})$ (generated by κ' and Δ' corresponding to κ and Δ above), and their orthogonal complement $\langle f^\sigma, \Sigma^\sigma\rangle^\perp \cap \langle \kappa', \Delta'\rangle^\perp$ on which the intersection form is even. In case of S_k^q we can replace κ' and Δ' by the general fiber f' in S_1^0 and the above Σ' respectively. In the remainder of this paper the manifolds $S_k(1, 1)$, $S_k^q(1, 1)$, and $X_G(1, 1)$ are identified with S_k, S_k^q, and $W_G \# S_k^q$ respectively.

§ 2. Simple invariants for $S_k^q(p, q)$.

First we recall the definition of the simple invariants for simply connected 4-manifolds. Let X be an oriented smooth closed 1-connected 4-manifold with $b_2^+(X)$ odd and greater than 1. Put $l_X = -3(1+b_2^+(X))/2$ and define C_X to be the set of nonzero elements of $H^2(X, \boldsymbol{Z}_2)$ each of which is a mod 2 reduction of some $c \in H^2(X, \boldsymbol{Z})$ with $q_X(c) \equiv l_X \pmod 4$. For any $\eta \in C_X$ let P_η be a principal $SO(3)$-bundle over X with $w_2 = \eta$ and $p_1 = l_X$ (which exists uniquely up to equivalence). Note that there are no flat connections on P_η since X is 1-connected and w_2 is nonzero. Then the moduli space $\mathcal{M}_X(l_X, \eta, g)$ of g-ASD connections on P_η for a generic Riemann metric g of X consists of finitely many points with sign ± 1 where the signs are determined by the choice of the integral lift c of η and the orientation of $H^+(X)$.

DEFINITION 2-0 ([2], [9]). For any $\eta \in C_X$ the number of the points in $\mathcal{M}_X(l_X, \eta, g)$ counted with sign does not depend on g and is denoted by $\gamma_X(\eta)$. The values $\gamma_X(\eta)$'s are called simple invariants for X.

Thus γ_X is a well defined map on C_X up to sign and by its naturality with respect to the diffeomorphisms the value $\max\{|\gamma_X(\eta)| \mid \eta \in C_X\}$ is the diffeomorphism invariant of X. In [9] the set $\hat{C}_X$ of the integral lifts of $\eta \in C_X$ modulo some equivalence is used to define γ_X. But this point is not crucial for our purpose since we only need the absolute value of γ_X. In case $X = S_k(p, q)$ or $X = S_k^q(p, q)$ with $k \geqq 2$ and $\gcd(p, q) = 1$, we see that C_X consists of the elements $\eta \in H^2(X, \boldsymbol{Z}_2)$ with $\eta \neq 0$ and with $q_X(c) \equiv k \pmod 4$ for any integral lift c of η. Now the results in [6], [14] show the following.

THEOREM 2-1 (see [6], [14]). *The simple invariants for $X = S_k(p, q)$ with $k > 2$ and $\gcd(p, q) = 1$ are given as follows (up to sign).*

$$|\gamma_X(\eta)| = \begin{cases} 1 & \text{if } \eta \cdot PD_2\kappa = 1 \text{ and both } p \text{ and } q \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

Here PD_2 denotes the Poincaré dual mod 2.

REMARK 2-2. (1) Proposition 2-1 is not true for the cases with $k=2$ which were completely determined in [**9**]. (2) The above results are proved in [**11**], [**12**], [**26**] for some particular η and are proved for the general cases by mutually different methods in [**6**] and [**14**].

The above results show that γ_X is far from sufficient to determine the diffeomorphism types of $S_k(p, q)$ (see [**20**]). So we choose $S_k^\sigma(p, q)$ in place of $S_k(p, q)$ for our construction.

THEOREM 2-3. *The simple invariants for $X=S_k^\sigma(p, q)$ with $k>2$ and* $\gcd(p, q)=1$ *are given as follows (up to sign).*

$$|\gamma_X(\eta)| = \begin{cases} pq & \text{if } \eta\cdot PD_2 f^\sigma = 1 \text{ and } \eta\cdot PD_2\kappa' = 0 \\ 1 & \text{if } \eta\cdot PD_2 f^\sigma = \eta\cdot PD_2\kappa' = 1 \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 2-4. We can see from the above propositions that $S_k^\sigma(p, q)$ for $k>2$ is not diffeomorphic to any $S_k(p', q')$ (and in fact never diffeomorphic to a complex surface). But this does not hold if $k=2$ since $S_2^\sigma(p, q)$ is diffeomorphic to $S_2(p, q)$. So further logarithmic transforms are needed to get similar examples for $k=2$ ([**9**]).

PROOF OF THEOREM 2-3. First note that there is a diffeomorphism ϕ from S_k^σ to S_k which is the identity on the part S_{k-1}^0, and maps Σ^σ to Σ and f^σ to f ([**9**], [**18**]). Therefore the claim for the cases with $p=q=1$ is deduced from Theorem 2-1. For the general case note that $\eta\in C_X$ for $X=S_k^\sigma(p, q)$ is represented as

$$\eta = PD_2(a\Sigma^\sigma+bf^\sigma+a'\Delta'+b'\kappa'+w)$$

where a, a', b, b' are either 0 or 1, and w is either 0 or a primitive element in $\langle\Sigma^\sigma, f^\sigma\rangle^\perp\cap\langle\Delta', \kappa'\rangle^\perp$, which satisfies

$$-a^2k+2ab-a'^2((p+q)^2+2(pq)^2)+2a'b'+q_X(w)\equiv k \pmod 4.$$

Case I. $a=a'=1$. In this case the above equation shows that $(p+q)^2$ must be even since $q_X(w)$ is even. Since $\gcd(p, q)=1$ both p and q must be odd. Then we use Gompf-Mrowka's formula in [**9**], Proposition 4.4 twice to get

$$|\gamma_X(\eta)| = |\gamma_{S_k^\sigma}(\eta')|$$

where $\eta'\in C_{S_k^\sigma}$ is the Poincaré dual mod 2 of the element of the form $\Sigma^\sigma+bf^\sigma+\Sigma'+b'f'+w$ for $w\in\langle\Sigma^\sigma, f^\sigma\rangle^\perp\cap\langle\Sigma', f'\rangle^\perp$. Then via the above ϕ the left hand side of the above formula equals $|\gamma_{S_k}(\eta'')|$ for some $\eta''\in C_{S_k}$ with $\eta''\cdot PD_2 f=1$ and hence equals 1 by Theorem 2-1.

Case II. $a=0$ and $a'=1$. Again using [**9**], Proposition 4.4 and the above ϕ we see that $|\gamma_X(\eta)|$ equals 0 if one of p and q is even and otherwise equals $|\gamma_{S_k}(\eta'')|$ for some $\eta''\in C_{S_k}$ with $\eta''\cdot PD_2f=0$. By Theorem 2-1 the last value is also 0.

Case III. $a=1$ and $a'=0$. In this case use [**9**], Corollary 4.3 twice and the above ϕ to get

$$|\gamma_X(\eta)| = pq|\gamma_{S_k}(\eta'')|$$

where η'' is some element of C_{S_k} with $\eta''\cdot PD_2f=1$. Therefore the left hand side of the above formula is pq.

Case IV. $a=a'=0$. If either $b=1$ or $w\neq 0$ the same procedure in Case III shows that $|\gamma_X(\eta)|=pq|\gamma_{S_k}(\eta'')|$ for some $\eta''\in C_{S_k}$ with $\eta\cdot PD_2f=0$ and hence this value is 0. But if $\eta=\kappa'$ (in this case $k\equiv 0 \pmod 4$) we cannot appeal to the method in [**9**]. For, if we put Y to be the manifold obtained from X by removing the tubular neighborhood of either one of the multiple fibers, then κ' may be zero in $H^2(Y, \boldsymbol{Z}_2)$. However since $k\geqq 4$ in this case we can decompose X as the torus sum of $S_2^\sigma(p, q)$ (which contains the support of κ') and S_{k-2} and apply the vanishing theorem of γ_X ([**14**]) to them. Then we see that $\gamma_X(\eta)=0$ also in this case. This proves Theorem 2-3.

COROLLARY 2-5. *Put* $\eta_0=PD_2(\Sigma^\sigma-kf^\sigma)$. *Then* η_0 *is contained simultaneously in* $C_{S_k^\sigma(p,q)}$ *and* $\max\{|\gamma_{S_k^\sigma(p,q)}(\eta)|\,|\,\eta\in C_{S_k^\sigma(p,q)}\}=|\gamma_{S_k^\sigma(p,q)}(\eta_0)|=pq$ *for any* k, p, q. *Hence* pq *is the diffeomorphism invariant for* $S_k^\sigma(p, q)$*'s.*

PROOF. The claim for $k=2$ comes from the results in [**9**]. The other cases are proved immediately by Theorem 2-3.

§3. $SO(3)$-representations for G.

In this section we consider the fundamental group of the spherical 3-manifold $G=\pi_1M_G$ and the set $R(G)$ of $SO(3)$-representations for G. Let $\chi(G)$ be the set of the conjugacy classes of the elements of $R(G)$ by $SO(3)$. In the next section any element in $\chi(G)$ will be considered as an equivalence class of a flat $SO(3)$-connection over the (twisted or untwisted) spin W_G of M_G. First of all if M_G is not a lens space then M_G has a Seifert fibration over the 2-orbifold of genus 0 with exactly 3 singular points $S^2(p_1, p_2, p_3)$ where the set of multiplicities of the singular points (p_1, p_2, p_3) is either (2, 3, 3), (2, 3, 4), (2, 3, 5), or (2, 2, n) for $n\geqq 2$ ([**22**]). Moreover M_G is represented by the Seifert invariants of the form $\{(p_1, a_1), (p_2, a_2), (p_3, a_3)\}$ with $\gcd(p_i, a_i)=1$ ($i=1, 2, 3$) and G has the following representation.

$$\{x, y, z, h\,|\,x^{p_1}h^{a_1}=y^{p_2}h^{a_2}=z^{p_3}h^{a_3}=xyz=1,\ h \text{ is central}\}.$$

Here h corresponds to the general fiber of M_G and x, y, z correspond to the lifts of the meridians for 3 singular points on the base orbifold. Note that x, y, z can be chosen so that they satisfy $xyz=1$ in G (therefore a_i may be negative in the above representation). Moreover $-M_G$ is represented by $\{(p_1, -a_1), (p_2, -a_2), (p_3, -a_3)\}$. It suffices to consider one of $\pm M_G$. Consequently replacing x, y, z by other lifts if necessary we have only to consider the following cases. (For example if $M_G=\{(2, 1), (3, 2), (n, b)\}$ then $-M_G=\{(2, -1), (3, -2), (n, -b)\}=\{(2, 1), (3, 1), (n, -b-2n)\}$.)

(1) $M_G = L(p, q)$.
(2) $M_G = \{(2, 1), (2, 1), (n, b)\}$ with $\gcd(n, b) = 1$, $n \geqq 2$.
(3) $M_G = \{(2, 1), (3, 1), (3, b)\}$ with $\gcd(3, b) = 1$.
(4) $M_G = \{(2, 1), (3, 1), (4, b)\}$ with $\gcd(4, b) = 1$.
(5) $M_G = \{(2, 1), (3, 1), (5, b)\}$ with $\gcd(5, b) = 1$.

For the cases (2)-(5) $G/[G, G]=H_1(M_G, \boldsymbol{Z})$ is given by the table below where x, z are the images in $G/[G, G]$ of the corresponding generators of G and $\boldsymbol{Z}_p[u]$ denotes the $\boldsymbol{Z}_p$-factor generated by u.

(3-0).

$$G/[G, G] = \begin{cases} \boldsymbol{Z}_p & \text{for Case (1)} \\ \boldsymbol{Z}_{|2(n+b)|}[x] \oplus \boldsymbol{Z}_2[z+2x] & \text{for Case (2) with } n \text{ even} \\ \boldsymbol{Z}_{|4(n+b)|}[x] & \text{for Case (2) with } n \text{ odd} \\ \boldsymbol{Z}_{|6b+15|}[z+2x] & \text{for Case (3)} \\ \boldsymbol{Z}_{|6b+20|}[z+2x] & \text{for Case (4)} \\ \boldsymbol{Z}_{|6b+25|}[z+2x] & \text{for Case (5).} \end{cases}$$

Note that M_G is a $\boldsymbol{Z}_2$-homology sphere if and only if M_G belongs to (3), (5), or (1) with p odd. Now we consider $R(G)$. For any element $\rho \in R(G)$ let $\mathrm{Stab}(\rho)$ be the stabilizer of ρ. Moreover for any $\gamma \in G$ we denote by $\widetilde{\rho(\gamma)}$ the lift of $\rho(\gamma)$ to the universal covering $SU(2)$ of $SO(3)$ (determined only up to sign). Hereafter $SU(2)$ is identified with the set of unit quaternions S^3. Note that $w_2(\rho)=0$ if and only if ρ can be lifted to an $SU(2)$-representation $\tilde{\rho}: \pi \to SU(2)$. (In this case we can put $\widetilde{\rho(\gamma)}=\tilde{\rho}(\gamma)$.) First let us consider the set of abelian representations $R_{ab}(G)$ (identified with $\mathrm{Hom}(G/[G, G], SO(3))$) and the set of their conjugacy classes $\chi_{ab}(G)$. Hereafter to determine $\rho \in R(G)$ only the lift $\widetilde{\rho(u)} \in S^3$ for each generator u of G (or of $G/[G, G]$ if $\rho \in R_{ab}(G)$) will be given. Note that $\pm\exp(i\theta) \in S^3$ projects to $\begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$, and $\pm j$ and $\pm k$ project to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ respectively.

(3-1). Abelian representations for G.

Case (I). $G/[G, G]$ is cyclic (say, of order r). M_G belongs to this case unless $M_G=\{(2, 1), (2, 1), (n, b)\}$ with n even. Fix the generator u of $G/[G, G]$. Then any $\rho\in R_{ab}(G)$ is conjugate in $SO(3)$ to one of ρ_l defined by

$$\widetilde{\rho_l(u)} = \pm\exp(\pi il/r) \quad (0 \leqq l \leqq [r/2]).$$

(3-1-1). The case when r is odd. In this case $\chi_{ab}(G)$ consists of the conjugacy classes of the trivial representation ρ_0 (with stabilizer $SO(3)$), and $(r-1)/2$ nontrivial abelian representations ρ_l $(1\leqq l\leqq (r-1)/2)$ (with stabilizer $SO(2)$). In either case ρ_l is lifted to the $SU(2)$-representation $\tilde{\rho}_l$ $(\tilde{\rho}_l(u)=(-1)^l\exp(\pi il/r))$, so $w_2(\rho_l)=0$.

(3-1-2). The case when r is even. In this case $H^2(G, \boldsymbol{Z}_2)=\boldsymbol{Z}_2$ and $w_2(\rho_l)=0$ if and only if l is even. If $r\equiv 0$ (4), then $\chi_{ab}(G)$ consists of the conjugacy classes of the trivial representation ρ_0, the representation $\rho_{r/2}$ with $\mathrm{Im}(\rho_{r/2})=\boldsymbol{Z}_2$ (with $\mathrm{Stab}(\rho_{r/2})=O(2)$, which is covered by $S^1\coprod S^1 j$ and with $w_2(\rho_{r/2})=0$), and $(r/2-1)$ representations ρ_l with stabilizer $SO(2)$ $(1\leqq l\leqq r/2-1)$. The number of ρ_l $(1\leqq l\leqq r/2-1)$ with $w_2(\rho_l)=0$ is just $r/4-1$.

If $r\equiv 2$ (4), then $\chi_{ab}(G)$ consists of the conjugacy classes of ρ_0, the representation $\rho_{r/2}$ with stabilizer $O(2)$ and with $w_2(\rho_{r/2})\neq 0$, and $(r/2-1)$ representations ρ_l $(1\leqq l\leqq r/2-1)$ with stabilizer $SO(2)$. The number of ρ_l $(1\leqq l\leqq r/2-1)$ with $w_2(\rho_l)=0$ is $(r-2)/4$.

Case (II). $G/[G, G]$ is non-cyclic. In this case $G/[G, G]=H_1(M_G, \boldsymbol{Z})=\boldsymbol{Z}_{2|n+b|}\oplus\boldsymbol{Z}_2$ where $M_G=\{(2, 1), (2, 1), (n, b)\}$ with n even, $n\geqq 2$, and $\gcd(n, b)=1$. Put $p=|n+b|$ and fix the generators u and v of $\boldsymbol{Z}_{2p}$ and $\boldsymbol{Z}_2$ respectively (note that p is odd and $p\geqq 1$ since b must be odd). For $\rho\in R_{ab}(G)$ put $U=\rho(u)$, $V=\rho(v)$ and let $\tilde{U}$, $\tilde{V}$ be the lifts of U, V respectively. Then we can assume (up to conjugacy) that

$$U=\begin{pmatrix} R(2\pi il/2p) & 0 \\ 0 & 1\end{pmatrix}, \quad \tilde{U} = \pm\exp(\pi il/2p) \quad (0 \leqq l \leqq p).$$

On the other hand $\tilde{V}$ must satisfy $\tilde{V}^2=\pm 1$ and $\tilde{V}\tilde{U}\tilde{V}^{-1}=\pm\tilde{U}$. Therefore if $l=0$ then by further conjugation we can assume that $\tilde{U}=\pm 1$, and $\tilde{V}=\pm 1$ or $\pm i$. Likewise if $l=p$ then we can assume that $\tilde{U}=\pm i$ and $\tilde{V}=\pm 1$, $\pm i$, or $\pm j$. If $1\leqq l\leqq p-1$ then $\tilde{V}=\pm 1$ or $\pm i$. Consequently $\chi_{ab}(G)$ consists of the conjugacy classes of the following representations.

(3-1-3).

(1) the trivial representation ρ_0.

(2) $\rho_{0,1}$: $\tilde{U}=\pm 1$, $\tilde{V}=\pm i$. In this case $w_2(\rho_{0,1})\neq 0$ and $\mathrm{Stab}(\rho_{0,1})=O(2)$.

(3) $\rho_{1,0}$: $\tilde{U}=\pm i$, $\tilde{V}=\pm 1$. In this case $\mathrm{Stab}(\rho_{1,0})=O(2)$ and $w_2(\rho_{1,0})\neq 0$ since p is odd.
(4) $\rho_{1,1}$: $\tilde{U}=\tilde{V}=\pm i$. In this case $\mathrm{Stab}(\rho_{1,1})=O(2)$ and $w_2(\rho_{1,1})\neq 0$.
(5) ρ_δ: $\tilde{U}=\pm i$, $\tilde{V}=\pm j$. In this case $\mathrm{Stab}(\rho_\delta)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ (covered by $\{\pm 1, \pm i, \pm j, \pm k\}$) and $w_2(\rho_\delta)\neq 0$.
(6) $\rho_{l,0}$ $(1\leqq l\leqq p-1)$: $\tilde{U}=\pm\exp(\pi il/2p)$, $\tilde{V}=\pm 1$. In this case $\mathrm{Stab}(\rho_{l,0})=SO(2)$, and $w_2(\rho_{l,0})=0$ if and only if l is even.
(7) $\rho_{l,1}$ $(1\leqq l\leqq p-1)$: $\tilde{U}=\pm\exp(\pi il/2p)$, $\tilde{V}=\pm i$. In this case $\mathrm{Stab}(\rho_{l,1})=SO(2)$ and $w_2(\rho_{l,1})\neq 0$.

In Case (II) $H^2(G, \boldsymbol{Z}_2)$ is identified with $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ so that $w_2(\rho)=(\varepsilon_1, \varepsilon_2)$ if and only if $\tilde{U}^{2p}=(-1)^{\varepsilon_1}$, $\tilde{V}^2=(-1)^{\varepsilon_2}$ where ε_1, ε_2 is 0 or 1 mod 2.

(3-2). Nonabelian representations.

Put $R^*(G)=R(G)\setminus R_{ab}(G)$ and $\chi^*(G)=\chi(G)\setminus\chi_{ab}(G)$. Note that $R^*(G)$ may be nonempty only when M_G belongs to the cases (2)-(5) where G has the representation of the form

$$G=\{x, y, z, h\,|\,x^2h=y^qh=z^nh^b=xyz=1,\ h \text{ is central}\}$$

for $q=2$ or 3. We look for the representatives for the elements of $\chi^*(G)$. For $\rho\in R^*(G)$ put $X=\rho(x)$, $Y=\rho(y)$, $Z=\rho(z)$, $H=\rho(h)$, and let $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{H}$ be the lifts in S^3 of X, Y, Z, H respectively. Up to conjugation we may assume that $\tilde{H}\in S^1$. Since $\tilde{U}\tilde{H}\tilde{U}^{-1}=\pm\tilde{H}$ for any lift $\tilde{U}$ of $\rho(u)$ for any $u\in G$ and ρ is non-abelian, either $\tilde{H}=\pm 1$, or $\tilde{H}=\pm i$ and $\tilde{X}, \tilde{Y}, \tilde{Z}\in S^1\amalg S^1j$. In the second case since $\tilde{X}\tilde{Y}\tilde{Z}=\pm 1$ and since ρ is non-abelian, two of $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ belong to S^1j and the rest belongs to S^1. On the other hand $u^2=-1$ for any $u\in S^1j$. Since $\tilde{X}^2=\pm i$ and $\tilde{Y}^q=\pm i$ for $q=2$ or 3 this is a contradiction. Therefore only the first case can occur. Then $\tilde{H}=\pm 1$ and none of $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ is ± 1 since otherwise ρ would be abelian. By the relation $\tilde{X}^2=\pm 1$ we may assume that $\tilde{X}=\pm i$. By further conjugation by some elements in S^1 we can assume that $\tilde{Y}=\pm(u+tj)$ with $u\in\boldsymbol{C}$, $t\in\boldsymbol{R}$, $|u|^2+t^2=1$. On the other hand since $\tilde{Y}^q=\pm 1$, $\tilde{Y}$ is conjugate to $\pm\exp(\pi il/q)$ for $1\leqq l\leqq[q/2]$. Therefore we must have $\tilde{Y}=\pm(si+tj)$ for $s, t\in\boldsymbol{R}$, $s^2+t^2=1$ if $q=2$, and $\tilde{Y}=\pm(1/2+si+tj)$ for $s, t\in\boldsymbol{R}$, $s^2+t^2=3/4$ if $q=3$. Since $\tilde{Z}=\pm\tilde{Y}^{-1}\tilde{X}^{-1}$ we have $\tilde{Z}=\pm(s+tk)$ if $q=2$, and $\tilde{Z}=\pm(s+i/2+tk)$ if $q=3$. By conjugation by i, j, or k we can replace (s, t) by any of $(s, -t)$, $(-s, t)$, and $(-s, -t)$. Moreover $\tilde{Z}$ is conjugate to $\pm\exp(\pi il/n)$ for some l $(1\leqq l\leqq[n/2])$. Hence we can assume that $s=\cos(\pi l/n)$, and $t=\sin(\pi l/n)$ if $q=2$ or $t=(3/4-\cos^2(\pi l/n))^{1/2}$ if $q=3$. Now we can give all the representatives for $\chi^*(G)$ for each case. In each representation below ε_i $(0\leqq i\leqq 3)$ is ± 1 and moreover

$$\tilde{H}=\varepsilon_0,\quad \tilde{X}=\varepsilon_1 i$$

(and so only the images of the remaining generators will be written).

The list of representatives for $\chi^*(G)$.

(3-2-1). $M_G=\{(2, 1), (2, 1), (n, b)\}$.

$$\rho_l^* : \tilde{Y} = \varepsilon_2(\cos(\pi l/n)i+\sin(\pi l/n)j), \quad \tilde{Z} = \varepsilon_3(\cos(\pi l/n)+\sin(\pi l/n)k)$$

for $1\leqq l\leqq[(n-1)/2]$. Note that if $l=0$ or $l=n/2$ (in this case n is even) ρ_l^* would be abelian and so these cases must be removed. ρ_l^* can be lifted to the $SU(2)$-representation if and only if we can choose ε_i so that

$$\varepsilon_0 = -1, \quad \varepsilon_3^n = (-1)^{l+b}, \quad \varepsilon_1\varepsilon_2\varepsilon_3 = -1.$$

Thus $w_2(\rho_l^*)\neq 0$ if and only if both n and l is even (note that b is odd in this case). In either case $\mathrm{Stab}(\rho_l^*)=\boldsymbol{Z}_2$ (covered by $\{\pm 1, \pm k\}$).

(3-2-2). $M_G=\{(2, 1), (3, 1), (3, b)\}$.

$$\rho_{\mathrm{II}}^*: \quad \tilde{Y} = \varepsilon_2(1/2+i/2+j/\sqrt{2}), \quad \tilde{Z} = \varepsilon_3(1/2+i/2+k/\sqrt{2}).$$

Putting $\varepsilon_0=-1$, $\varepsilon_1=(-1)^b$, $\varepsilon_2=1$, $\varepsilon_3=(-1)^{b-1}$ we can lift ρ_{II}^* to the $SU(2)$-representation and hence $w_2(\rho_{\mathrm{II}}^*)=0$. Moreover $\mathrm{Stab}(\rho_{\mathrm{II}}^*)=1$.

(3-2-3). $M_G=\{(2, 1), (3, 1), (4, b)\}$.

$$\rho_{\mathrm{III}}^* : \quad \tilde{Y} = \varepsilon_2(1/2+i/\sqrt{2}+j/2), \quad \tilde{Z} = \varepsilon_3(1/\sqrt{2}+i/2+k/2).$$

$$\rho_{\mathrm{III}'}^* : \quad \tilde{Y} = \varepsilon_2(1/2+\sqrt{3}j/2), \quad \tilde{Z} = \varepsilon_3(i/2+\sqrt{3}k/2).$$

We have the $SU(2)$-lift of ρ_{III}^* by putting $\varepsilon_0=-1$, $\varepsilon_2=1$, $\varepsilon_1\varepsilon_3=-1$ and hence $w_2(\rho_{\mathrm{III}}^*)=0$. On the other hand to get the $SU(2)$-lift for $\rho_{\mathrm{III}'}^*$ we must have $(-1)^b=1$ from the relation $\tilde{Z}^4\tilde{H}^b=1$. But this contradicts the fact that b must be odd. Hence $w_2(\rho_{\mathrm{III}'}^*)\neq 0$. Easy computation shows that $\mathrm{Stab}(\rho_{\mathrm{III}}^*)=1$ and $\mathrm{Stab}(\rho_{\mathrm{III}'}^*)=\boldsymbol{Z}_2$ (covered by $\{\pm 1, \pm j\}$).

(3-2-4). $M_G=\{(2, 1), (3, 1), (5, b)\}$.

$$\rho_{\mathrm{IV}_n}^* : \quad \tilde{Y} = \varepsilon_2(1/2+s_n i+t_n j), \quad \tilde{Z} = \varepsilon_3(s_n+i/2+t_n k) \quad (n = 1, 2)$$

where $s_1=\cos\pi/5=(\sqrt{5}+1)/4$, $t_1=\sqrt{3/4-\cos^2\pi/5}=\sqrt{6-2\sqrt{5}}/4$ and $s_2=\cos 2\pi/5=(\sqrt{5}-1)/4$, $t_2=\sqrt{3/4-\cos^2 2\pi/5}=\sqrt{6+2\sqrt{5}}/4$. We have $\mathrm{Stab}(\rho_{\mathrm{IV}_n}^*)=1$ and $w_2(\rho_{\mathrm{IV}_n}^*)=0$ in either case (put $\varepsilon_0=-1$, $\varepsilon_2=1$, $\varepsilon_1=(-1)^b$, $\varepsilon_3=(-1)^{b-1}$ if $n=1$ and reverse the signs of ε_1 and ε_3 if $n=2$).

§4. Simple invariants for $X_G(p, q)$.

In this section we consider $X_G(p, q)=W_G\# S_k^q(p, q)$ with $\gcd(p, q)=1$. Put $X=X_G(p, q)$ and $S=S_k^q(p, q)$ for the moment. To get the well defined invariants for X we define C_X as the set of the elements $\eta\in H^2(X, \boldsymbol{Z}_2)$ satisfying

(1) η is a mod 2 reduction of some element $c \in H^2(X, \boldsymbol{Z})$ whose image in $H^2(X, \boldsymbol{Z})/\text{Torsion}$ is not divisible by 2,

(2) $q_X(c) \equiv l_X \pmod 4$

where $l_X = -3(1+b_2^+(X))/2$ as before. Then $\mathcal{C}_X$ is preserved by diffeomorphisms of X. Moreover $H^2(X, \boldsymbol{Z}) = H^2(W_G, \boldsymbol{Z}) \oplus H^2(S, \boldsymbol{Z})$ whose torsion part is $H^2(W_G, \boldsymbol{Z})$. Note that the class $\mathcal{C}_S$ for S is defined as in §2 and $l_S = l_X = -3k$. We denote the set of mod 2 reductions of the elements of $H^2(W_G, \boldsymbol{Z})$ in $H^2(W_G, \boldsymbol{Z}_2)$ by $\mathrm{Im}(H^2(W_G, \boldsymbol{Z}) \to H^2(W_G, \boldsymbol{Z}_2))$.

PROPOSITION 4-1. (1) *Let $c: W_G \to K(G, 1)$ be the classifying map for the universal covering $p: \widetilde{W}_G \to W_G$. Then c^* induces the isomorphisms between $H^2(G, \boldsymbol{Z}_2)$ and $\mathrm{Im}(H^2(W_G, \boldsymbol{Z}) \to H^2(W_G, \boldsymbol{Z}_2))$. Moreover $\eta = \eta_1 + \eta_2$ for $\eta_1 \in H^2(W_G, \boldsymbol{Z}_2)$ and $\eta_2 \in H^2(S, \boldsymbol{Z}_2)$ belongs to $\mathcal{C}_X$ if and only if $\eta_1 \in c^*(H^2(G, \boldsymbol{Z}_2))$ and $\eta_2 \in \mathcal{C}_S$.* (2) *For any $\eta = \eta_1 + \eta_2 \in \mathcal{C}_X$ with $\eta_1 \in H^2(W_G, \boldsymbol{Z}_2)$ and $\eta_2 \in H^2(S, \boldsymbol{Z}_2)$, there is a unique principal $SO(3)$-bundle P_η over X with $w_2 = \eta$ and $p_1 = l_X$ (up to equivalence) which is a fiber sum of the flat $SO(3)$ bundle $P_1 = \widetilde{W}_G \times_\rho SO(3)$ over W_G for some $SO(3)$-representation $\rho: G \to SO(3)$ with $c^* w_2(\rho) = \eta_1$, and a principal $SO(3)$ bundle P_2 over S with $w_2 = \eta_2$ and $p_1 = l_S = l_X$.*

PROOF. The spectral sequence for the universal covering $p: \widetilde{W}_G \to W_G$ yields the exact sequence of the form

$$0 \longrightarrow H^2(G, \boldsymbol{Z}) \xrightarrow{c^*} H^2(W_G, \boldsymbol{Z}) \xrightarrow{p^*} H^2(\widetilde{W}_G, \boldsymbol{Z}).$$

Since $H^2(W_G, \boldsymbol{Z})$ is torsion and $H^2(\widetilde{W}_G, \boldsymbol{Z})$ is torsion-free, c^* gives an isomorphism. Then mod 2 reduction yields the required isomorphism c^* in (1). It follows that for any $\eta_1 \in \mathrm{Im}(H^2(W_G, \boldsymbol{Z}) \to H^2(W_G, \boldsymbol{Z}_2))$ there is a principal flat $SO(3)$ bundle over W_G with $w_2 = \eta_1$. The other claims follows easily from these results.

PROPOSITION 4-2. *For any $\eta \in \mathcal{C}_X$ for above X there are no flat connections on any principal $SO(3)$ bundle P over X with $w_2(P) = \eta$.*

PROOF. For any $\eta = \eta_1 + \eta_2 \in \mathcal{C}_X$ represented as above we can assume that η_2 is the Poincaré dual mod 2 of some primitive element $\xi \in H_2(S_k^q(p, q), \boldsymbol{Z})$ (since $\eta_2 \in \mathcal{C}_S$). Therefore there is $\delta \in H_2(S_k^q(p, q), \boldsymbol{Z})$ (which is a spherical element since $S_k^q(p, q)$ is 1-connected) with $\xi \cdot \delta = 1$ and hence $\langle \eta, \delta \pmod 2 \rangle = 1$. It follows that η does not come from $H^2(\pi_1 X, \boldsymbol{Z}_2)$ and hence there are no flat connections on P.

Hence for any $\eta \in \mathcal{C}_X$ we can define the set $\mathcal{M}_X(l_X, \eta, g)$ of g-ASD connections on P_η (defined in Proposition 4-1) modulo $\mathrm{Aut}\, P_\eta$ for a generic metric g on X. Moreover $\mathcal{M}_X(l_X, \eta, g)$ consists of finitely many points with sign ± 1

as in § 2 (which is fixed once the integral lift of η and the orientation of $H^+(X)$ is fixed, and does not depend on g since $b_X^+=b_S^+>1$). So we can define $\gamma_X(\eta)$ for $\eta\in C_X$ as the number of points in $\mathcal{M}_X(l_X, \eta, g)$ counted with sign as in Definition 2-0 so that $\max\{|\gamma_X(\eta)|\,|\,\eta\in C_X\}$ is a diffeomorphism invariant for X. In particular the element $\eta_0\in C_{S_k^q(p,q)}$ in Corollary 2-5 is also contained simultaneously in C_X for any $X=X_G(p, q)$. We also note that $\mathrm{Im}(H^2(W_G, \boldsymbol{Z})\to H^2(W_G, \boldsymbol{Z}_2))\cong c^*(H^2(G, \boldsymbol{Z}_2))$ is given by

$$c^*(H^2(G, \boldsymbol{Z}_2)) = \begin{cases} 0 & \text{if } G/[G, G] = \boldsymbol{Z}_r \text{ with } r \text{ odd} \\ \boldsymbol{Z}_2 & \text{if } G/[G, G] = \boldsymbol{Z}_r \text{ with } r \text{ even} \\ \boldsymbol{Z}_2\oplus\boldsymbol{Z}_2 & \text{if } G/[G, G] = \boldsymbol{Z}_{2p}\oplus\boldsymbol{Z}_2 \text{ with } p \text{ odd.} \end{cases}$$

According to the result in § 3 the above list covers all the possible cases.

PROPOSITION 4-3. *There is a constant c_G such that $|\gamma_{X_G(p,q)}(\eta)|=c_G|\gamma_{S_k^q(p,q)}(\eta_2)|$ for any $\eta=\eta_1+\eta_2\in C_{X_G(p,q)}$ with $\eta_1\in\mathrm{Im}(H^2(W_G, \boldsymbol{Z})\to H^2(W_G, \boldsymbol{Z}_2))$ and $\eta_2\in C_{S_k^q(p,q)}$. In particular for η_0 in Corollary 2-5 we have $|\gamma_{X_G(p,q)}(\eta_0)|=pqc_G$. The constant c_G depends only on G and is defined as follows.*

$$c_G = \begin{cases} r & \text{if } G/[G, G] = \boldsymbol{Z}_r \text{ with } r \text{ odd} \\ r/2 & \text{if } G/[G, G] = \boldsymbol{Z}_r \text{ with } r \text{ even} \\ p & \text{if } G/[G, G] = \boldsymbol{Z}_{2p}\oplus\boldsymbol{Z}_2 \text{ with } p \text{ odd.} \end{cases}$$

PROOF. Let $X_1=W_G$, $X_2=S_k^q(p, q)$, and $X=X_G(p, q)$. Let $D_j(r)$ be the geodesic ball in X_j with respect to the fixed metric g_j on X_j of radius r centered at the base point x_j ($j=1, 2$). Fix a large $N>0$ and choose $\lambda>0$ so that $N\sqrt{\lambda}$ is small. Then identifying the annuli $\Omega_j=D_j(N\sqrt{\lambda})\setminus D_j(N^{-1}\sqrt{\lambda})$ ($j=1, 2$) by some map f_λ we can construct a connected sum $X=X_1\# X_2$ as $X=(X_1\setminus D_1(N^{-1}\sqrt{\lambda}))\cup_{f_\lambda}(X_2\setminus D_2(N^{-1}\sqrt{\lambda}))$ with a metric g_λ on X such that g_λ is conformally equivalent to g_j over $X_j\setminus D_j(4N\sqrt{\lambda})$ for $j=1, 2$ ([**2**], § 7.2.1). We can choose g_2 and g_λ so that they are generic if λ is sufficiently small. For any element $\eta=\eta_1+\eta_2\in C_X$ with $\eta_1\in\mathrm{Im}(H^2(X_1, \boldsymbol{Z})\to H^2(X_1, \boldsymbol{Z}_2))$ and $\eta_2\in C_{X_2}$ we have an $SO(3)$-bundle P_η over X with $w_2(P_\eta)=\eta$ and $p_1(P_\eta)=-3k$. Note that there are no flat connections on any bundle over X_2 with $w_2=\eta_2$ since $\eta_2\neq 0$ and X_2 is 1-connected. Consider a sequence λ_i with $\lambda_i\to 0$ as $i\to\infty$ and $A_{\lambda_i}\in\mathcal{M}_X(-3k, \eta, g_{\lambda_i})$. Then by Uhlenbeck criterion and dimension counting argument we have a flat connection A_1 on the bundle P_1 over X_1 with $w_2=\eta_1$ and $p_1=0$, and a g_2-ASD connection A_2 on the bundle P_2 over X_2 with $w_2(P_2)=\eta_2$ and $p_1(P_2)=-3k$, such that (by passing to subsequences) $[A_{\lambda_i}]$ converges smoothly to $[A_j]$ over $X_j\setminus D_j(\sqrt{\lambda}/2)$ ($j=1, 2$). Here $[A]$ means the gauge equivalence class of A. If λ_i is small enough, $[A_{\lambda_i}]$ has the neighborhood $\mathcal{N}$ of the following form ([**2**], Theorem 7.2.62 and Theorem 7.3.2). For any connection A_i on P_i, let Γ_{A_i} be the stabilizer of A_i in the associated gauge group and $H^k_{A_i}$ be the k-th cohomology group of the deformation complex

$$\Omega^0_{X_i}(AdP_i) \xrightarrow{d_{A_i}} \Omega^1_{X_i}(AdP_i) \xrightarrow{d^+_{A_i}} \Omega^2_{X_i,+}(AdP_i).$$

Then there is a $\Gamma_{A_1}\times\Gamma_{A_2}$ equivariant map Ψ of the form

$$\Psi : SO(3)\times H^1_{A_1}\times H^1_{A_2} \longrightarrow H^2_{A_1}\times H^2_{A_2}$$

such that $\mathcal{N}$ is homeomorphic to $\Psi^{-1}(0)/\Gamma_{A_1}\times\Gamma_{A_2}$. Since $\mathcal{M}_X(-3k, \eta, g_{\lambda_i})$ consists of finitely many points we can assume that every point in $\mathcal{M}_X(-3k, \eta, g_{\lambda_i})$ for small λ_i belongs to one of $\mathcal{N}$ of the above form. Conversely every element in such an $\mathcal{N}$ corresponds to the unique element in $\mathcal{M}_X(-3k, \eta, g_{\lambda_i})$ ([**2**], Theorem 7.2.62. The construction of the ASD connections in [**2**], §7.2.2 is valid even when $\pi_1 X\neq 1$). So we will check the contribution of $\mathcal{N}$ to the moduli space over X for fixed $[A_2]$ and $[A_1]$ separately according to the type of A_1. Note that $\mathcal{M}_{X_2}(-3k, \eta_2, g_2)$ consists of finitely many points and $H^1_{A_2}=0$, $H^2_{A_2}=0$, and $\Gamma_{A_2}=1$ for any $[A_2]\in\mathcal{M}_{X_2}(-3k, \eta_2, g_2)$ since g_2 is generic (and $b^+(X_2)>1$). On the other hand since A_1 is a flat connection over P_1, it corresponds to a representation $\rho : G\to SO(3)$ with $w_2(\rho)=\eta_1$ (via the isomorphism c^* in Proposition 4-1). We denote by $\chi^{\eta_1}(G)$ the conjugacy classes of such representations. As is listed in §3, the stabilizer Stab(ρ) of ρ (which corresponds to $\Gamma(A_1)$) is either 1, $\boldsymbol{Z}_2$, $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, $SO(2)$, $O(2)$, or $SO(3)$ (up to conjugacy). For any such ρ we can see that $H^1_\rho=H^1(G, ad\rho)=0$ by direct computation using the list in §3 or by the method as in [**5**]. Also we can see that the dimension of $H^0_\rho=H^0(G, ad\rho)$ is 3 if Stab(ρ)$=SO(3)$, 1 if Stab(ρ)$=SO(2)$ or $O(2)$ (up to conjugacy), and is 0 otherwise. On the other hand index computation shows that $\dim H^0_\rho-\dim H^1_\rho+\dim H^2_\rho=3$ for any ρ. Hence we have

$$\mathcal{N} = \Psi^{-1}(0)/\mathrm{Stab}(\rho) \quad \text{for } \Psi : SO(3) \to H^2_\rho$$

where $\dim H^2_\rho$ is 0 if Stab(ρ)$=SO(3)$, 2 if Stab(ρ)$=SO(2)$ or $O(2)$, and 3 otherwise. If ρ is the trivial connection then Stab(ρ)$=SO(3)$ and so the contribution of $\mathcal{N}$ to the moduli space over X is 1. For any ρ with Stab(ρ)$=SO(2)$ or $O(2)$ we can assume that Im $\rho\subset SO(2)$ (see the list of the $SO(3)$-representations in §3) and AdP_1 is a sum $L\oplus\varepsilon$ of a complex line bundle L (on which $SO(2)$ acts as rotations) and the trivial bundle ε of dimension 1. If Stab(ρ)$=SO(2)$ we have $H^2_\rho=H^2_+(X_1, L)=\boldsymbol{C}$ on which Stab(ρ) acts as rotations. Therefore $\mathcal{N}$ coincides with the zero of some section of the bundle $SO(3)\times_{SO(2)}\boldsymbol{C}$ over S^2 associated with the natural bundle $SO(3)\to SO(3)/SO(2)$ for each $[A_2]$ (cf. [**1**], Proposition 2.13). Therefore the contribution of such $\mathcal{N}$ to the moduli space over X is 2. If Stab(ρ)$=O(2)$ then also we have $H^2_\rho=H^2_+(X_1, L)=\boldsymbol{C}$. In this case $SO(2)$ acts on the fiber of L (and on H^2_ρ) as rotations as before, whereas $J=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$ (which is a representative of $O(2)/SO(2)$) acts as the complex conjugation on them. Hence the contribution of $\mathfrak{N}$ in this case is the same as half of the euler number of the above bundle $SO(3)\times_{SO(2)}\boldsymbol{C}\to S^2$ (which is the same as the euler number of the twisted bundle over $SO(3)/O(2)=\boldsymbol{RP}^2$) and hence equals 1. If $\mathrm{Stab}(\rho)=1$, $\boldsymbol{Z}_2$, or $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ then the contribution of $\mathfrak{N}$ is the same as the euler class of the bundle over $SO(3)/\mathrm{Stab}(\rho)$ with fiber $H^2_\rho=\boldsymbol{R}^3$, since $\mathrm{Stab}(\rho)$ acts freely on $SO(3)$. (In fact the lift $\widetilde{\mathrm{Stab}(\rho)}$ in S^3 of $\mathrm{Stab}(\rho)$ is generated by $k\in S^3$ if $\mathrm{Stab}(\rho)=\boldsymbol{Z}_2$, and is generated by i, j, $k\in S^3$ if $\mathrm{Stab}(\rho)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. So $SO(3)/\mathrm{Stab}(\rho)=S^3/\widetilde{\mathrm{Stab}(\rho)}$ is $SO(3)$ if $\mathrm{Stab}(\rho)=1$, the lens space $L(4, 1)$ if $\mathrm{Stab}(\rho)=\boldsymbol{Z}_2$, and the quaternionic space if $\mathrm{Stab}(\rho)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$.) Since the euler number of the $\boldsymbol{R}^3$-bundle over the 3-manifold is zero, the contribution for this case is zero ([**2**], [**15**]). Note that the orientation of the total moduli space is determined once the integral lift of η and the orientation of $H^+(X)$ are fixed (in fact, only $H^+(X)=H^+(X_2)$ and the integral lift of η_2 is essential since η_1 is a torsion [**1**]). Moreover the lists (3-1-1)-(3-1-3) in § 3 show that for any η_1 we can choose a common decomposition $AdP_1=L\oplus\varepsilon$ above such that each element of $\chi^{\eta_1}(G)$ with stabilizer $SO(2)$ or $O(2)$ has a representative ρ with $\mathrm{Im}(\rho)\in SO(2)$ and with $\mathrm{Stab}(\rho)=SO(2)$ or $O(2)$ which acts on the fiber of the common L (or $H^2_+(X_1, L)$) as defined above. Hence the contributions of $\chi^{\eta_1}(G)$ are of the same sign, and their total amount for each $[A_2]$ is the sum (which we denote by $c^{\eta_1}_G$) of the number of the elements in $\chi^{\eta_1}(G)$ with stabilizer $SO(3)$ or $O(2)$ and twice the number of the elements in $\chi^{\eta_1}(G)$ with stabilizer $SO(2)$. It follows that the number of points in $\mathcal{M}_X(-3k, \eta, g_{\lambda_i})$ counted with sign is $c^{\eta_1}_G$-times that of $\mathcal{M}_{X_2}(-3k, \eta_2, g_2)$. According to the lists (3-1-1) and (3-1-2) in § 3 we can see that $c^{\eta_1}_G=r$ if $G/[G, G]=\boldsymbol{Z}_r$ with r odd and $c^{\eta_1}_G=r/2$ if $G/[G, G]=\boldsymbol{Z}_r$ with r even for any η_1. If $G/[G, G]=\boldsymbol{Z}_{2p}\oplus\boldsymbol{Z}_2$ then the elements of $\chi^{\eta_1}(G)$'s except for ρ_δ with stabilizer $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ (whose contribution is zero) are divided into the following groups according to the 4 choices of η_1 (we use the notation in § 3. (3-1-3)).

(1) ρ_0 and $\rho_{l,0}$ with l even, $1\leqq l\leqq p-1$,
(2) $\rho_{1,0}$ and $\rho_{l,0}$ with l odd, $1\leqq l\leqq p-1$,
(3) $\rho_{0,1}$ and $\rho_{l,1}$ with l even, $1\leqq l\leqq p-1$,
(4) $\rho_{1,1}$ and $\rho_{l,1}$ with l odd, $1\leqq l\leqq p-1$.

These facts show that $c^{\eta_1}_G=p$ in any case. Consequently in either case $c^{\eta_1}_G$ depends only on G and we can denote it by c_G. We also deduce that $|\gamma_X(\eta_0)|=c_G|\gamma_{S^q_k(p,q)}(\eta_0)|=c_Gpq$ by Corollary 2-5. This proves Proposition 4-3.

COROLLARY 4-4. *For any G we have infinitely many pairs $\{p_i, q_i\}$ $(i \in \boldsymbol{N})$ such that the above $X_G(p_i, q_i)$'s are non-diffeomorphic to each other. In fact $X_G(p, q)$ and $X_G(p', q')$ can be diffeomorphic only when $pq = p'q'$.*

PROOF. Proposition 4-3 and Corollary 2-5 show that $\max\{|\gamma_{X_G(p,q)}(\eta)| \mid \eta \in C_{X_G(p,q)}\} = c_G \max\{|\gamma_{S_k^q(p,q)}(\eta_2)| \mid \eta_2 \in C_{S_k^q(p,q)}\} = c_G pq$. Since c_G is nonzero we obtain the desired result.

§ 5. Proof of Main Theorem.

First we discuss the homeomorphism types of $X_G(p, q) = W_G \# S_k^q(p, q)$ for $k \geqq 2$ and $\gcd(p, q) = 1$.

PROPOSITION 5-1. *$X_G(p, q)$ is homeomorphic to $W_G \# (k/2)K \# (k/2-1)S^2 \times S^2$ if k is even and both p and q are odd, and is homeomorphic to $W_G \# (2k-1)\boldsymbol{CP}^2 \# (10k-1)\overline{\boldsymbol{CP}}^2$ otherwise.*

PROOF. First note that there is a homeomorphism from $N_2(p, q)$ to N_2 (resp. to $N_2(2, 1)$) which is the identity on the boundaries if both p and q are odd (resp. one of p and q is even) ([**7**]). Since $N_2(p, q)$ is contained in $S_k^q(p, q)$ (and hence also in $X_G(p, q)$) such a homeomorphism extends to that from $X_G(p, q)$ to either $W_G \# S_k^q$ (if p and q are odd) or $W_G \# S_k^q(2, 1)$ (otherwise) which is the identity on the complement of $N_2(p, q)$ in $X_G(p, q)$. On the other hand $S_k^q(p, q)$ is a 1-connected manifold with euler number $12k$, with signature $-8k$, and it is spin if and only if k is even and both p and q are odd. Then applying Freedman's theorem [**3**] to $S_k^q(p, q)$ and using the smoothability of 0-handles in dimension 4 ([**25**]) we obtain the desired results.

Next we consider the universal covering $\tilde{X}_G(p, q)$ of $X_G(p, q)$. By Proposition 1-2 $\tilde{X}_G(p, q)$ is diffeomorphic to $(|G|-1)S^2 \times S^2 \# |G| S_k^q(p, q)$. The following propositions are essentially contained in [**7**], [**8**], [**9**].

PROPOSITION 5-2 ([**7**], [**8**], [**9**]). (1) *$N_2(p, q) \# S^2 \times S^2$ is diffeomorphic to $N_2 \# S^2 \times S^2$ if both p and q are odd, and is diffeomorphic to $N_2 \# \boldsymbol{CP}^2 \# \overline{\boldsymbol{CP}}^2$ otherwise by a diffeomorphism which induces the identity on the boundary.* (2) *$S_k^q(p, q) \# S^2 \times S^2$ is diffeomorphic to $k/2K \# k/2(S^2 \times S^2)$ if k is even and both p and q are odd, and is diffeomorphic to $2k\boldsymbol{CP}^2 \# 10k\overline{\boldsymbol{CP}}^2$ otherwise.*

PROOF. (1) is proved in [**9**], § 23 by applying Mandelbaum's lemma ([**17**]) to a fiber sum of N_2 and a manifold obtained from $T^2 \times S^2$ by performing logarithmic transforms of multiplicity p and q along two fibers and by the fact that $N_2(p, q)$ is spin if and only if both p and q are odd ([**7**]). Using the natural extension of the diffeomorphism in (1) and the diffeomorphism between

S_k and S_k^q we see that $S_k^q(p, q)\# S^2\times S^2$ is diffeomorphic to either $S_k\# S^2\times S^2$ or $S_k\#\boldsymbol{CP}^2\#\overline{\boldsymbol{CP}^2}$. On the other hand the results in [**17**] and [**19**] show that $S_k\# S^2\times S^2$ and $S_k\#\boldsymbol{CP}^2$ are diffeomorphic to either connected sums of copies of K's and $S^2\times S^2$, or connected sums of copies of $\boldsymbol{CP}^2$'s and $\overline{\boldsymbol{CP}}^2$'s. Then the computation of the euler number, the signature, and the type of the intersection form of $S_k^q(p, q)$ show (2).

PROPOSITION 5-3. *$\tilde{X}_G(p, q)$ is diffeomorphic to $(|G|k/2)K\#(|G|k/2-1)S^2\times S^2$ if k is even and p and q are odd, and is diffeomorphic to $(2k|G|-1)\boldsymbol{CP}^2\#(10k|G|-1)\overline{\boldsymbol{CP}^2}$ otherwise.*

PROOF. By Proposition 5-2 we can replace one copy of $S_k^q(p, q)\# S^2\times S^2$ in $\tilde{X}_G(p, q)=(|G|-1)S^2\times S^2\#|G|S_k^q(p, q)$ by a connected sum of K's and $S^2\times S^2$'s, or a connected sum of $\boldsymbol{CP}^2$'s and $\overline{\boldsymbol{CP}}^2$'s to reduce the number of the copies of $S_k^q(p, q)$ by 1. Note that if $S_k^q(p, q)$ is nonspin then $S_k^q(p, q)\#\boldsymbol{CP}^2\#\overline{\boldsymbol{CP}}^2$ can be replaced by $S_k^q(p, q)\# S^2\times S^2$ since these two manifolds are diffeomorphic. Thus we can repeat this process on $\tilde{X}_G(p, q)$ $|G|$ times to obtain the desired result.

PROOF OF MAIN THEOREM. Let $X=nK\#(n-1)S^2\times S^2$ where n is divided by $|G|$ with $1<|G|<n$. Put $k=2n/|G|$. Then by Proposition 4-3, Corollary 4-4 we can choose the pair of integers (p_i, q_i) for each $i\in\boldsymbol{N}$ satisfying

(1) $\gcd(p_i, q_i)=1$,
(2) both p_i and q_i are odd,
(3) $\max\{|\gamma_{X_i}(\eta)|\,|\,\eta\in C_{X_i}\}$ for $X_i=W_G\# S_k^q(p_i, q_i)$ are strictly increasing as i tends to ∞.

Then X_i's are mutually homeomorphic by Proposition 5-1, their universal coverings $\tilde{X}_i$ are diffeomorphic to X by Proposition 5-3, and X_i's are not diffeomorphic to each other by (3). Therefore the covering translations for such $\tilde{X}_i$'s give the desired G actions. Suppose that $X=(2n-1)\boldsymbol{CP}^2\#(10n-1)\overline{\boldsymbol{CP}^2}$ where n is divided by $|G|$ with $1<|G|<n$. Then put $k=n/|G|$. If k is odd choose p_i, q_i as in the first case. If k is even we can also choose $\{p_i, q_i\}$ satisfying the above conditions (1) and (3) so that p_i is even and q_i is odd by Proposition 4-3. Then in either case Proposition 5-1, Proposition 5-3, and Proposition 5-2 show that the covering translations for $\tilde{X}_i$ give the desired action as in the first case. This completes the proof.

REMARK 5-4. Consider $X_i=W_G\# S_k^q(p_i, q_i)$ satisfying (1), (2), (3) in the proof of Main Theorem equipped with a generic metric g_i. Let P_i be the $SO(3)$ bundle over X_i with $p_1=-3k$ and $w_2=\eta_0$ where η_0 is the element in Corollary 2-5. Next consider the pullback $\tilde{P}_i$ of P_i over the universal covering $\tilde{X}_i=\tilde{W}_G\#$

$|G|S_k^q(p_i, q_i)$ of X_i with the metric $\tilde{g}_i$ induced by g_i. Then $w_2(\tilde{P}_i)$ is the Poincaré dual mod 2 of the union of the lifts of $\Sigma^\sigma - kf^\sigma$ contained in $|G|$ copies of the complement S^0 of $N_2(p, q)$ in $S_k^q(p, q)$ and the virtual dimension of the moduli space of ASD connections on $\tilde{P}_i$ is also 0. Moreover applying Mandelbaum's lemma on $\tilde{X}_i$ as in the proof of Proposition 5-2 we have a diffeomorphism between $\tilde{X}_i$ and $\tilde{X}_j$ which is the identity on the common $|G|$ copies of S^0 for any i and j. Hence $\tilde{P}_i$'s are mutually isomorphic. The number of $\tilde{g}_i$-ASD connections on $\tilde{P}_i$ which are the pullbacks of g_i-ASD connections tends to ∞ as $i \to \infty$ by Proposition 4-3. On the other hand $\tilde{X}_i$ is completely decomposable (Proposition 5-3) and the support of $w_2(\tilde{P}_i)$ lies in different $|G|$ copies of S^0. So if we choose a metric g on $\tilde{X}_i$ so that $\tilde{X}_i$ has one thin neck which separates one copy of $S_k^q(p_i, q_i)$ from the other summands of the connected sum decompositions of $\tilde{X}_i$, and so that g is generic on both of the separated regions, then the moduli space of g-ASD connections on $\tilde{P}_i$ is empty ([**2**], Proposition 9.3.7). This implies that such g cannot be G-invariant. (In fact $\tilde{X}_i$ with G-invariant metric must have at least two neck regions invariant under the G-action and equivariant transversality theorem for the moduli spaces with G-actions fails in naive sense [**10**].)

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