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Nash G manifold structures of compact or compactifiable $C^{\infty}G$ manifolds

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1. Introduction.

Let G be a compact affine Nash group. We say that a $C^{\infty}G$ manifold X *admits* a (resp. an *affine*, a *nonaffine*) *Nash G manifold structure* if there exists a (resp. an affine, a nonaffine) Nash G manifold Y such that X is $C^{\infty}G$ diffeomorphic to Y. In the present paper we consider Nash G manifold structures of compact or compactifiable $C^{\infty}G$ manifolds.

We have the following when X is compact.

THEOREM 1. *Let G be a compact affine Nash group and let X be a compact $C^{\infty}G$ manifold with* $\dim X \geqq 1$.

(1) *X admits exactly one affine Nash G manifold structure up to Nash G diffeomorphism.*

(2) *If G acts on X transitively then a Nash G manifold structure of X is unique up to Nash G diffeomorphism.*

(3) *If X is connected and the action on X is not transitive, then X admits a continuum number of nonaffine Nash G manifold structures.*

In the non-equivariant category, M. Shiota in [**4**] proved that any compactifiable C^{∞} manifold X admits a continuum number of nonaffine Nash manifold structures. When X is not compact but compactifiable, an affine Nash compactification of X is not unique, and the number of affine ones can be investigated by the cardinality of the Whitehead torsion of X [**6**]. Here an affine Nash compactification of X means an affine Nash manifold Y with boundary so that X is C^{∞} diffeomorphic to the interior of Y.

We say that a $C^{\infty}G$ manifold X is *compactifiable as a $C^{\infty}G$ manifold* if there exists a compact $C^{\infty}G$ manifold Y with boundary so that X is $C^{\infty}G$ diffeomorphic to the interior of Y. We obtain the following.

THEOREM 2. *Let G be a compact affine Nash group and let X be a non-compact compactifiable $C^{\infty}G$ manifold with* $\dim X \geqq 1$.

(1) *X admits an affine Nash G manifold structure.*

(2) *X admits a continuum number of nonaffine Nash G manifold structures.*

This paper consists of two parts. The first half is to investigate Nash G manifold structures of compact $C^\infty G$ manifolds. We consider Nash G manifold structures of compactifiable (not compact) $C^\infty G$ manifolds in the latter half.

In this paper all Nash G manifolds and all Nash G maps are of class C^ω unless otherwise stated.

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2. Nash G manifolds.

First of all we recall the definition of Nash groups.

DEFINITION 2.1. A group is called a (resp. an *affine*) *Nash group* if it is a (resp. an affine) Nash manifold and that the multiplication $G\times G\to G$, the inversion $G\to G$ are Nash maps.

We remark that connected one-dimensional Nash groups are classified by J.J. Madden and C.M. Stanton [2].

Let G be an affine Nash group. In this paper, a *representation* of G means a Nash group homomorphism $G\to GL(\boldsymbol{R}^n)$ for some $\boldsymbol{R}^n$. Here a Nash group homomorphism means a group homomorphism which is a Nash map. We use a representation as a representation space.

DEFINITION 2.2. Let G be an affine Nash group.

(1) An affine Nash submanifold in some representation of G is called an *affine Nash G submanifold* if it is G invariant. A Nash manifold X with G action is said to be a *Nash G manifold* if the action map $G\times X\to X$ is a Nash map.

(2) Let X and Y be Nash G manifolds. A Nash map $f: X\to Y$ is called a *Nash G map* if it is a G map. We say that X is *Nash G diffeomorphic to Y* if there exist Nash G maps $f: X\to Y$, $h: Y\to X$ so that $f\circ h=id$, $h\circ f=id$.

(3) A Nash G manifold X is said to be *affine* if there exists an affine Nash G submanifold Y so that X is Nash G diffeomorphic to Y.

Tubular neighborhood theorem and collaring theorem are well known in the smooth equivariant category. They are proved in the Nash category by M. Shiota (Lemma 1.3.2 [7], Lemma 6.1.6 [7]). Since M. Shiota's proofs work in the equivariant Nash category, the following two propositions are obtained.

PROPOSITION 2.3. *Let G be a compact affine Nash group and let X be an affine Nash G submanifold in a representation Ω of G. Then there exists a Nash G tubular neighborhood (U, p) of X in Ω, namely, U is an affine Nash G* *submanifold in Ω and the orthogonal projection $p: U \to X$ is a Nash G map.* □

PROPOSITION 2.4. *Let G be a compact affine Nash group. Any compact affine Nash G manifold X with boundary ∂X admits a Nash G collar, that is, there exists a Nash G imbedding $\phi: \partial X \times [0, 1] \to X$ so that $\phi|_{\partial X \times 0} = id_{\partial X}$, where the action on the closed unit interval [0, 1] is trivial.* □

3. Compact $C^{\infty}G$ manifolds.

Recall a theorem proved by K. H. Dovermann, M. Masuda, and T. Petrie [**1**], which is a partial solution of the equivariant Nash conjecture.

THEOREM **3.1** [**1**]. *Let G be a compact affine Nash group and let X be a compact $C^{\infty}G$ manifold so that X is G cobordant to a nonsingular algebraic G set. Then X is $C^{\infty}G$ diffeomorphic to a nonsingular algebraic G set. Here an algebraic G set means a G invariant algebraic subset of some representation of G.* □

PROOF OF THEOREM 1. The disjoint union $X \amalg X$ is null cobordant. By Theorem 3.1, $X \amalg X$ is $C^{\infty}G$ diffeomorphic to a nonsingular algebraic G set in some representation Ω of G. Since a G invariant collection of connected components of a nonsingular algebraic G set is an affine Nash G submanifold in Ω, X admits an affine Nash G manifold structure $Y \subset \Omega$. Let Z be another affine Nash G manifold structure of X in Ω'. We have to prove Y is Nash G diffeomorphic to Z. Let f be a $C^{\infty}G$ diffeomorphism from Y to Z. Let F denote the composition of f with the inclusion $Z \to \Omega'$. By [**1**] F can be approximated by a polynomial G map $q: Y \to \Omega'$. By Proposition 2.3, we have a Nash G tubular neighborhood (U, p) of Z in Ω'. Since Y is compact, if the approximation is close then the image of q lies in U. Thus $k := p \circ q$ is an approximation of f. If the approximation is close then a Nash G map $k: Y \to Z$ is a Nash G diffeomorphism. Therefore (1) is proved.

Next we prove (2). Let X_1, X_2 be two Nash G manifold structures (may not be affine) of X and let k be a $C^{\infty}G$ diffeomorphism from X_1 to X_2. Fix $x_1 \in X_1$, and let $x_2 = k(x_1)$. Then the map $f_i: G \to X_i: f_i(g) = gx_i$ $(i=1, 2)$ is a surjective Nash G map because G acts on X_i $(i=1, 2)$ transitively, and $f_2 = k \circ f_1$.

To prove k is a Nash map, it is enough to show k is a C^0 Nash map. By [**4**] we can find a C^0 Nash imbedding I_i from X_i to some Euclidean space $\boldsymbol{R}^s$ $(i=1, 2)$. Let $X'_i = I_i(X_i)$ $(i=1, 2)$, $f'_i = I_i \circ f_i$ $(i=1, 2)$ and $k' = I_2 \circ k \circ I_1^{-1}$. Then $f'_i: G \to X'_i$ $(i=1, 2)$ is a C^0 Nash map. Since G and X_i $(i=1, 2)$ are affine, there exists a finite semialgebraic open covering $\{O_t\}_t$ of G such that each $f'_i|O_t$ is semialgebraic. Therefore f'_i $(i=1, 2)$ is semialgebraic. Since k' is C^0 Nash if and only if k is C^0 Nash, we have only to show that k' is C^0 Nash.

Since f_i' $(i=1, 2)$ is a C^0 Nash map, there exist finite systems of coordinate neighborhoods $\{\phi_i : W_i \to \boldsymbol{R}^m\}$ of G, $\{\psi_j : U_j \to \boldsymbol{R}^n\}$ of X_1, and $\{\varphi_l : V_l \to \boldsymbol{R}^n\}$ of X_2 such that, for any i, j and l, $\phi_i((f_1')^{-1}(U_j)\cap W_i)$, $\phi_i((f_2')^{-1}(V_l)\cap W_i)$ are semialgebraic, and that $\psi_j \circ f_1' \circ \phi_i^{-1} : \phi_i((f_1')^{-1}(U_j)\cap W_i) \to \boldsymbol{R}^n$, $\varphi_l \circ f_2' \circ \phi_i^{-1} : \phi_i((f_2')^{-1}(V_l)\cap W_i) \to \boldsymbol{R}^n$ are C^0 Nash maps, where m (resp. n) denotes the dimension of G (resp. X_1). We have only to show that each $\varphi_l \circ k' \circ \psi_j^{-1}$ is semialgebraic. For a map h, let *graph*(h) denote the graph of h. For j and l, let

$$K = \bigcup_i graph(\psi_j \circ f_1' \circ \phi_i^{-1}) \times graph(\varphi_l \circ f_2' \circ \phi_i^{-1}).$$

Then K is semialgebraic in $(\boldsymbol{R}^m \times \boldsymbol{R}^n) \times (\boldsymbol{R}^m \times \boldsymbol{R}^n)$, hence the image K' of K by the projection $(\boldsymbol{R}^m \times \boldsymbol{R}^n) \times (\boldsymbol{R}^m \times \boldsymbol{R}^n) \to \boldsymbol{R}^n \times \boldsymbol{R}^n$ is semialgebraic in $\boldsymbol{R}^n \times \boldsymbol{R}^n$. Since f_i' $(i=1, 2)$ is surjective and $f_2' = k' \circ f_1'$, $graph(\varphi_l \circ k' \circ \psi_j^{-1}) = K'$. Thus each $\varphi_l \circ k' \circ \psi_j^{-1}$ is semialgebraic. Hence k' is a C^0 Nash. Therefore k is a Nash G diffeomorphism.

Now we prove (3). By (1) we can assume that X is an affine Nash G submanifold of a representation Ω of G. For any $x \in X$, the orbit $G(x)$ of x is a $C^\infty G$ submanifold of Ω because G is compact. Moreover $G(x)$ is a semialgebraic set. Hence $G(x)$ is an affine Nash G submanifold in Ω. Since the action on G is not transitive and by Proposition 2.3, there exists some Nash G tubular neighborhood (U', p) of some orbit $G(x)$ in Ω with $X \neq U := U' \cap X$.

For $0<c<1$, set

$$a = 2^{2 \cdot 5}(1+c)^2/(1-c)^2,$$

$$d = 2+2^{0 \cdot 5}3a+a^2-(a+\sqrt{2})\sqrt{a^2+2^{2 \cdot 5}a}\,.$$

Then $a>2^{2 \cdot 5}$, $1<d<2$. Suppose k is a Nash function satisfying

$$\sqrt{2}(x+k(x)) = (x-k(x))^2/a\,.$$

The graph of k comes to a rotation of the graph of $y=x^2/a$ with center at the origin. It follows from this and $a>2^{2 \cdot 5}$ that k and its Nash extension k' to

$$[1-2^{-0.25}\sqrt{a},\ 1+2^{-0.25}\sqrt{a}]\,(\supset(-1, 3))$$

is well-defined, and that k' satisfies

$$k'[1-2^{-0.25}\sqrt{a},\ 1+2^{-0.25}\sqrt{a}] = [1-2^{-0.25}\sqrt{a},\ 1+2^{-0.25}\sqrt{a}],$$

the derivative of k' is negative, $k' \circ k' = id$.

Let

$$N_1 = (-\infty, d), \quad N_2 = (0, \infty), \quad N_3 = (0, 1).$$

Define the Nash maps $h_1 : N_3 \to N_1$, $h_2 : N_3 \to N_2$ by

$$h_1(t) = t^2 + k(t)^2 \quad \text{and} \quad h_2(t) = 2t - t^2.$$

Then h_1 and h_2 are Nash imbeddings so that $h_1(N_3)=(0, d)$, $h_2(N_3)=(0, 1)$. We can extend h_1 to

$$h_1': [1-2^{-0.25}\sqrt{a},\ 1+2^{-0.25}\sqrt{a}] \to \boldsymbol{R}$$

as a Nash function such that the derivative vanishes at only 0 and that $h_1'=h_1'\circ k'$ because the derivative of k' is negative and $k'\circ k'=id$.

Applying Proposition 2.3 to the boundary $\partial\bar{U}$ of the closure $\bar{U}$ of U in X, there exists a Nash G collar $\phi: \partial\bar{U}\times[0, 1]\to\bar{U}$. Let $D(\varepsilon)$ $(0<\varepsilon<1)$ denote $\phi(\partial\bar{U}\times(0, \varepsilon))$. Take a Nash diffeomorphism $f: \boldsymbol{R}\to(0, 1)$ (e.g., the inverse map of the composition of $f: (0, 1)\to(-1, 1): f(x)=2x-1$ with $h: (-1, 1)\to\boldsymbol{R}: h(x) =x/(1-x^2)$). Set

$$U_1 = D(f(d)),\quad U_2 = X-\overline{D(f(0))},\quad U_3 = D(f(1))-\overline{D(f(0))}.$$

Then each U_i is an open affine Nash G submanifold of X. Let

$$H_1 = \phi\circ(id\times(f\circ h_1\circ f^{-1}))\circ\phi^{-1}: U_3 \to U_1,$$
$$H_2 = \phi\circ(id\times(f\circ h_2\circ f^{-1}))\circ\phi^{-1}: U_3 \to U_2.$$

We define X_c by the quotient topological space of the disjoint union $\coprod_{i=1}^3 U_i$, and the equivalence relation $x\sim H_1(x)\sim H_2(x)$ for $x\in U_3$ on the union. Then one can check that X_c is a Nash G manifold which is $C^\infty G$ diffeomorphic to X. Next we prove X_c is nonaffine. To prove this, we use the following lemma.

LEMMA 3.2 (cf. REMARK 1.2.2.15 [7]). *Let f be a locally semialgebraic C^∞ map from a Nash manifold M to a Nash manifold N. If N is affine then f is a Nash map.* □

Fix $0<c<1$ and $z\in S(f(1))$, where $S(f(1))$ denotes $\phi(\partial\bar{U}\times\{f(1)\})$. Let ψ_c: $(f(0), f(1))\to X_c$ be the composition

$$(f(0), f(1)) \longrightarrow S(f(1))\times(f(0), f(1)) \longrightarrow U_3 \longrightarrow X_c,$$

where the first map is $x\to(z, x)$, the second is the natural Nash G diffeomorphism from $S(f(1))\times(f(0), f(1))$ to U_3, and the third is the natural imbedding from U_3 into X_c. Then ψ_c is an imbedding. We extend ψ_c as follows. Let l_{ci} $(i=1, 2, 3)$ be the natural imbedding $U_i\to X_c$ and let V_{ci} $(i=1, 2, 3)$ denote its image. Then

$$p\circ k_1^{-1}\circ l_1^{-1}\circ\psi_c = f\circ h_1\circ f^{-1},\quad p\circ k_2^{-1}\circ l_2^{-1}\circ\psi_c = f\circ h_2\circ f^{-1} \quad \text{on } (f(0), f(1)),$$

where p denotes the projection $\partial\bar{U}_3\times(f(0), f(d))\to(f(0), f(d))$ and k_i $(i=1, 2)$ stands for the natural imbedding $\partial\bar{U}_3\times(f(0), f(d))\to U_i$. We extend ψ_c to $(f(0), f(1+\varepsilon))$ for small positive ε. It suffices to consider $p\circ k_2^{-1}\circ l_2^{-1}\circ\psi_c$ because the image of ψ_c lies in V_{c2} and $\lim_{t\to f(1)}\psi_c(t)\in V_{c2}$. Now $p\circ k_2^{-1}\circ l_2^{-1}\circ\psi_c= f(2f^{-1}(t)-(f^{-1}(t))^2)$ on $(f(0), f(1))$. Thus $p\circ k_2^{-1}\circ l_2^{-1}\circ\psi_c$ and ψ_c are extensible to $(f(0), f(2))$ and

$$p\circ k_2^{-1}\circ l_2^{-1}\circ\psi_c(t)=f(2f^{-1}(t)-(f^{-1}(t))^2)\quad\text{on } [f(1),\ f(2)).$$

Clearly we can extend ψ_c to $[f(0),\ f(1)]$, and $\psi_c((f(0),\ f(2)))\subset\psi_c([f(0),\ f(1)])$. Hence

$$\psi_{c0}^{-1}\circ\psi_c(t)=f(2-f^{-1}(t))\quad\text{on } [f(1),\ f(2)),$$

$f(1)$ is the only and nondegenerate critical point, where ψ_{c0} denotes the homeomorphism $\psi_c:[f(0),\ f(1)]\to\psi_c([f(0),\ f(1)])$. In the same way, ψ_c can be defined on $(f(k'(1)),\ f(0)]$ satisfying

$$\psi_{c0}^{-1}\circ\psi_c(t)=f(k'(f^{-1}(t)))\quad\text{for } t\in(f(k'(1)),\ f(0)],$$

and the critical point is only $f(0)$ and nondegenerated. Repeating this argument, ψ_c is extensible on

$$(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a})),$$

and ψ_c is locally semialgebraic, the image of ψ_c is $\psi_c([f(0),\ f(1)])$, and that for any $e\in(f(0),\ f(1))$, $(\psi_{c0}^{-1}\circ\psi_c)^{-1}(e)$ is discrete and consists of infinitely many elements. The set of critical points of ψ_c is $(\psi_{c0}^{-1}\circ\psi_c)^{-1}(f(0))\cup(\psi_{c0}^{-1}\circ\psi_c)^{-1}(f(1))$, and they are nondegenerate ones. Since ψ_c is locally semialgebraic and not semialgebraic and by Lemma 3.2, X_c is not affine.

Finally we prove that X_c is not Nash G diffeomorphism to $X_{c'}$ if $0<c,\ c'<1$, $\alpha=\log f(c')/\log f(c)$ is irrational. Assume that there exists a Nash G diffeomorphism $u:X_c\to X_{c'}$. Then we have to prove $\log f(c')/\log f(c)$ is rational. Set

$$a=2^{2.5}(1+c)^2/(1-c)^2,\quad a'=2^{2.5}(1+c')^2/(1-c')^2,$$

$$\psi_c:(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a}))\to X_c,$$

$$\psi_{c'}:(f(1-2^{-0.25}\sqrt{a'}),\ f(1+2^{-0.25}\sqrt{a'}))\to X_{c'}.$$

We also write

$$\psi_{c0}=\psi_c|[f(0),\ f(1)]:[f(0),\ f(1)]\longrightarrow\psi_c([f(0),\ f(1)]),$$

$$\psi_{c'0}=\psi_{c'}|[f(0),\ f(1)]:[f(0),\ f(1)]\longrightarrow\psi_{c'}([f(0),\ f(1)]).$$

Let L_1 be the composition of the diffeomorphism $S(f(1))\times(f(-10d),\ f(10d))\to D(f(10d))-\overline{D(f(-10d))}$ with the projection $D(f(10d))\to X_{c'}$, and let L_2 be the projection $S(f(1))\times(f(-10d),\ f(-10d))\to S(f(1))$. By Lemma 3.2 and the infinite vibration of ψ_c, $L_2\circ L_1^{-1}\circ u\circ\psi_c$ is constant. Let z' denote this constant. Clearly the images of $\phi(z'\times f(N_2))$ and $\phi(z'\times(f(1),\ f(d)))$ via $\pi_{c'}$ in $X_{c'}$ are affine Nash G submanifolds. Let k_c be the natural homeomorphism from $\bar{U}_3$ into X_c. Thus $u\circ k_c\circ\phi(z\times[f(0),\ f(1)])$ is not contained in these affine Nash G submanifolds because the image of ψ_c is not affine. This implies that

$$u\circ k_c\circ\phi(z\times[f(0),\ f(1)])\subset k_{c'}\circ\phi(z'\times[f(0),\ f(1)]).$$

Applying the same argument to u^{-1}, we have

$$u^{-1}\circ k_{c'}\circ\phi(z'\times[f(0),\ f(1)])\subset k_c\circ\phi(z\times[f(0),\ f(1)]).$$

Therefore

$$u\circ k_c\circ\phi(z\times[f(0),\ f(1)])=k_{c'}\circ\phi(z'\times[f(0),\ f(1)]).$$

For any $e\in(f(0),\ f(1))$, let $(\psi_{c0}^{-1}\circ\psi_c)^{-1}(e)=\{e_i\}_{i\in\boldsymbol{Z}}$, $(\psi_{c'0}^{-1}\circ\psi_{c'})^{-1}(e)=\{e_i'\}_{i\in\boldsymbol{Z}}$. Then

$$\lim_{i\to\infty}(f(1+2^{-0.25}\sqrt{a})-e_{-i-2})/(f(1+2^{-0.25}\sqrt{a})-e_{-i})=f(c), \tag{3.1}$$

$$\lim_{i\to\infty}(f(1+2^{-0.25}\sqrt{a'})-e'_{-i-2})/(f(1+2^{-0.25}\sqrt{a'})-e'_{-i})=f(c'), \tag{3.2}$$

are obtained as follows. The map $t\rightarrow f(k'(2-f^{-1}(t)))$ has fixed points only at the end of the interval, it repels from $f(1+2^{-0.25}\sqrt{a})$, attracts to $f(1-2^{-0.25}\sqrt{a})$ and its derivatives at the latter point is $f(c)$. Thus $(f(1+2^{-0.25}\sqrt{a})-e_{-i-2})/(f(1+2^{-0.25}\sqrt{a})-e_{-i})$ converges $f(c)$ because $e_{-i-2}=f(k'(2-f^{-1}(e_{-i})))$. Hence we have (3.1). A similar argument shows (3.2). Since

$$u\circ k_c\circ\phi(z\times[f(0),\ f(1)])=k_{c'}\circ\phi(z'\times[f(0),\ f(1)]),$$

for a pair

$$e_0\in(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a}))$$

and

$$e_0'\in(f(1-2^{-0.25}\sqrt{a'}),\ f(1+2^{-0.25}\sqrt{a'}))$$

with

$$\psi_{c'}(e_0')=u\circ\psi_c(e_0)$$

there exists a homeomorphism

$$\tau:(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a}))\longrightarrow(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a}))$$

so that $\tau(e_0)=e_0'$ and $\psi_{c'}\circ\tau=u\circ\psi_c$ on $(f(1-2^{-0.25}\sqrt{a}),\ f(1+2^{-0.25}\sqrt{a}))$. Remember that all critical points of ψ_c, $\psi_{c'}$ are nondegenerate. This shows that τ is of class C^ω. Therefore, by Lemma 3.2, τ is a Nash diffeomorphism. Set

$$\psi_{c0}^{-1}\circ\psi_c(e_0)=e,\ \psi_{c'0}^{-1}\circ\psi_{c'}(e_0)=e',$$

$$(\psi_{c0}^{-1}\circ\psi_c)^{-1}(e)=\{e_i\}_{i\in\boldsymbol{Z}},$$

$$(\psi_{c'0}^{-1}\circ\psi_{c'})^{-1}(e')=\{e_i'\}_{i\in\boldsymbol{Z}}.$$

Then τ satisfies

$$\tau(e_i)=e_i' \quad \text{for any } i\in\boldsymbol{Z} \text{ or,}$$

$$\tau(e_i)=e'_{-i} \quad \text{for any } i\in\boldsymbol{Z}.$$

A map $f\circ(\text{translation})\circ f^{-1}$ takes $(f(1-2^{-0.25}\sqrt{a}), f(1+2^{-0.25}\sqrt{a}))$ to $(f(0), f(2^{0.75}\sqrt{a}))$, and a similar map $f\circ(\text{translation})\circ f^{-1}$ takes $(f(1-2^{-0.25}\sqrt{a'}), f(1+2^{-0.25}\sqrt{a'}))$ to $(f(0), f(2^{0.75}\sqrt{a'}))$, we may suppose that e_i and e_i' are convergent to 0 as $i\to\infty$. Assume that e and e' lie in $(f(0), f(1))$. Then it follows from (3.1) and (3.2) that

$$\lim_{i\to\infty} e_{-i-2}/e_{-i} = f(c), \tag{3.3}$$

$$\lim_{i\to\infty} e'_{-i-2}/e'_{-i} = f(c'). \tag{3.4}$$

Let Z denote the Zariski closure of $graph(\tau)$. This is of dimension 1 because τ is semialgebraic. It is clear that Z contains all (e_i, e_i'). Let $P(x, y) =\sum_{j=1}^{s}\delta_j x^{\beta_j}y^{\gamma_j}$ ($\delta_j\in\boldsymbol{R}$, β_j, $\gamma_j\in\boldsymbol{N}$) be a defining polynomial of Z. Then

$$P(e_i, e_i') = 0 \quad \text{for any } i\in\boldsymbol{Z}.$$

Since α is irrational,

$$\beta_i+\alpha\gamma_i \neq \beta_j+\alpha\gamma_j \quad \text{for } i\neq j. \tag{3.5}$$

Set

$$P_i(x, y) = x^{\beta_i}y^{\gamma_i}.$$

For each $n\in\boldsymbol{Z}$, let $E(n)$ denote the $s\times s$-matrix whose (i, j) entry is

$$P_i(e_{-n-2j+1}, e'_{-n-2j+1}).$$

Then

$$(\delta_1, \cdots, \delta_s)E(n) = (P(e_{-n-1}, e'_{-n-1}), \cdots, P(e_{-n-2s+1}, e'_{-n-2s+1})) = 0.$$

In particular $\det E(n)=0$. On the other hand, we have

$$\det E(n) = \Big(\prod_{i=1}^{s} P_i(e_{-n-1}, e'_{-n-1})\Big)\det F(n),$$

where $F(n)$ is the $s\times s$-matrix whose (i, j) entry is

$$P_i(e_{-n-2j+1}, e'_{-n-2j+1})/P_i(e_{-n-1}, e'_{-n-1}).$$

Now (3.3) and (3.4) mean that each entry of $F(n)$ converges to

$$(c^{\beta_i}(c')^{\gamma_i})^{j-1} = c^{(\beta_i+\alpha\gamma_i)(j-1)}$$

as $n\to\infty$. Thus $\det F(n)$ converges to a Vandermonde's determinant equals

$$\prod_{i<j}(c^{\beta_j+\alpha\gamma_j}-c^{\beta_i+\alpha\gamma_i}) \neq 0,$$

by (3.5). Therefore $\det E(n)\neq 0$ for large n. This proves the result. □

4. Compactifiable $C^\infty G$ manifolds.

The same argument of the proof of Theorem 1 (3) proves Theorem 2 (2). To prove Theorem 2 (1), we show a relative version of Theorem 3.1. After proving Theorem 4.2, we give a proof of Theorem 2 (1).

DEFINITION 4.1. (1) An algebraic subset of a representation of G is said to be an *algebraic G set* if it is G invariant. Moreover we call it a *nonsingular algebraic G set* if it is nonsingular.

(2) Let X be a $C^\infty G$ manifold and let X' be a $C^\infty G$ submanifold of X. A pair (X, X') is called *algebraically G cobordant* if there exist a nonsingular algebraic G set Y, a nonsingular algebraic G subset Y' of Y, a G cobordism N between X and Y, and a G cobordism N' between X' and Y' such that N' is a $C^\infty G$ submanifold of N.

THEOREM 4.2. *Let G be a compact affine Nash group, X a compact $C^\infty G$ manifold, and X' a compact $C^\infty G$ submanifold of X. If the pair (X, X') is algebraically G cobordant then there exist a nonsingular algebraic G set Z in $X\times\Omega$ for some representation Ω of G, a nonsingular algebraic subset Z' of Z, and a $C^\infty G$ diffeomorphism $\phi: X\to Z$ with $\phi(X')=Z'$.*

For any $C^\infty G$ manifold X and $C^\infty G$ submanifold X' of X, the pair $(X\amalg X, X'\amalg X')$ is algebraically G cobordant. Therefore we have the next corollary because a G invariant collection of connected components of a nonsingular algebraic G set is an affine Nash G submanifold in some representation of G.

COROLLARY 4.3. *Let G be a compact affine Nash group, X a compact $C^\infty G$ manifold, and X' a compact $C^\infty G$ submanifold of X. Then there exist an affine Nash G manifold Y, an affine Nash G submanifold Y' of Y, and a $C^\infty G$ diffeomorphism $\phi: X\to Y$ so that $\phi(X')=Y'$.* □

PROOF OF THEOREM 4.2. By the proof of Theorem 1.3 [**1**], X' is G isotopic to a nonsingular algebraic G subset Z' of $X\times\Omega$ by an arbitrarily small isotopy, for some representation of G. Extending this isotopy, we may assume that it maps $X\times 0$ to some $C^\infty G$ manifold M in $X\times\Omega$ so that $M-X\times 0$ has compact closure and that the composition of the inclusion $M\to X\times\Omega$ with the projection $X\times\Omega\to X$ is a $C^\infty G$ diffeomorphism. In particular $Z'\subset M$. Since Z' is compact and by Lemma 4.7 [**1**], one can find a proper G invariant polynomial ρ such that $\rho^{-1}(0)=Z'$. Let $\alpha: X\to\Omega$ be a $C^\infty G$ map with compact support so that

$$M=\{(x, y)\in X\times\Omega \mid y=\alpha(x)\}.$$

Take a G invariant C^∞ function $\beta: X\times\Omega\to[0, 1]$ with compact support with $\beta(x, y)=1$ when $|y|<2|\alpha(x)|$. Let $\gamma: X\times\Omega\to\Omega$ be

$$\gamma(x, y) = \beta(x, y)(y-\alpha(x))+(1+\beta(x, y))\rho^2(x, y)y\,.$$

Then 0 is a regular value of γ, $\gamma^{-1}(0)=M$, and γ is equal to the polynomial $\rho^2(x, y)y$ outside of a G invariant compact set. By Lemma 5.1 [**1**], one can C^1 approximates $\gamma(x, y)-\rho^2(x, y)y$ by an equivariant entire rational map u: $(X\times\Omega, Z')\to(\Omega, 0)$. Here an entire rational map means a fraction of polynomial maps with nowhere vanishing denominator. This approximation is close on all $X\times\Omega$. Thus

$$w(x, y) = u(x, y)+\rho^2(x, y)y$$

is C^1 approximation of γ on $X\times\Omega$. Since ρ is proper and by equivariant Morse theory, there exists a $C^\infty G$ diffeomorphism from $Z:=w^{-1}(0)$ to $M=\gamma^{-1}(0)$ fixing Z'. □

PROOF OF THEOREM 2(1). Since X is compactifiable, there exists a $C^\infty G$ manifold X' with boundary ∂X so that X is $C^\infty G$ diffeomorphic to the interior of X'. Let Y be the double of X'. Applying Corollary 4.3 to the pair $(Y, \partial X')$, one can find a representation Ω of G and a $C^\infty G$ imbedding $F: Y\to\Omega$ such that $F(Y)$ and $F(\partial X')$ are affine Nash G manifolds. Hence $F(X)$ is an affine Nash G manifold. Therefore X admits an affine Nash G manifold structure. □

On the other hand, T. Petrie [**3**] proved that any nonsingular algebraic G set is compactifiable as a $C^\infty G$ manifold when G is an algebraic group. A similar proof shows the next theorem, because the number of connected components of the zeros of a Nash map is finite.

THEOREM 4.4. *Let G be a compact affine Nash group. Then every affine Nash G manifold is compactifiable as a $C^\infty G$ manifold.* □

M. Shiota studied compactifications of Nash manifolds as either C^∞ manifolds [**4**] or Nash manifolds [**5**].

By Theorem 2(1) and Theorem 4.4, we have the following.

THEOREM 4.5. *Let G be a compact affine Nash group. Then a $C^\infty G$ manifold is compactifiable if and only if it admits an affine Nash G manifold structure.* □

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