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Sets of determination for harmonic functions in an NTA domain

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1. Introduction.

Let D be an NTA domain in $\boldsymbol{R}^N (N \geqq 2)$ with Green function $G(x, y)$. Without loss of generality we may assume that D contains the origin 0. It is proved that the Martin compactification of D is homeomorphic to the Euclidean closure of D and that every boundary point is minimal ([**15**]). Thus the ratio

$$K(x, y) = \frac{G(x, y)}{g(y)} \qquad \text{with} \quad g(y) = G(y, 0)$$

has a continuous extension on $D \times \bar{D}$. By the same symbol we denote the continuous extension. If $y \in \partial D$, then $K(\cdot, y)$ is a minimal harmonic function on D. Sometimes we write K_y for $K(\cdot, y)$. By definition $K_y(0)=1$. For every nonnegative superharmonic function u on D which is harmonic near the origin, there is a unique measure μ_u on D such that

$$u = K\mu_u = \int_{D \cup \partial D} K(\cdot, y) d\mu_u(y).$$

This measure μ_u is called the representing measure of u. For every nonnegative harmonic function h on D there is a unique measure μ_h concentrated on ∂D such that $h = K\mu_h$ and $\|\mu_h\| = h(0)$. Here, we say, in general, that a measure μ is concentrated on A if any Borel set outside A has μ measure zero.

Following Beurling [**4**] and Dahlberg [**7**], we introduce the notion of determination of a point measure.

DEFINITION A. A set $E \subset D$ is said to determine the point measure at $y \in \partial D$ if for all positive harmonic functions h with representing measure μ_h we have $\mu(\{y\}) = \inf\{h(x)/K(x, y) : x \in E\}$.

Let $B(x, r)$ be the open ball with center at x and radius r. We write $\delta(x)$ for the distance between x and ∂D. For $0 < \rho < 1$ let

$$E_\rho = \bigcup_{x \in E} B(x, \rho\delta(x)).$$

For a smooth domain the following characterization of sets determining a point measure was given by Dahlberg [7] (see also [1, 2], [4], [9], [18] and [20]).

THEOREM A. *Let D be a Liapunov-Dini domain. Suppose $E\subset D$ and $y\in\partial D$. Then the following are equivalent:*

(i) *E determines the point measure at y;*

(ii) *there is ρ, $0<\rho<1$, such that*

$$\int_{E_\rho}|x-y|^{-N}dx=\infty\,; \tag{1}$$

(iii) (1) *holds for all ρ, $0<\rho<1$.*

A set $E\subset D$ is said to be minimally thin at $y\in\partial D$ if the regularized reduced function $\hat{R}^E_{K_y}$ is a Green potential. The minimal thinness is closely related to the notion of determination of a point measure. In fact, the essential part of Theorem A is based on the following theorem.

THEOREM B. *Let D be a Liapunov-Dini domain. Suppose $y\in\partial D$. If E is a measurable subset of D such that*

$$\int_E|x-y|^{-N}dx=\infty\,, \tag{2}$$

then E is not minimally thin at y.

Using Theorems A and B, Gardiner [11] extended some results of [5, 6] and [13] to functions on balls. Essén [10] and Dudeley-Ward [9] also used Theorems A and B and gave generalizations. They dealt with smooth domains.

The aim of this paper is to consider a generalization to NTA domains. Hereafter we let D be an NTA domain. Since a general NTA domain may have wedges, Theorems A and B do not hold for an NTA domain. In [2, Section 4] we characterized minimal thinness and obtained integral characterizations corresponding to (1) and (2). For a Lipschitz domain, see Ancona [3, Theorem 7.4] and Zhang [21]. However, they are rather complicated (they depend on the boundary point y). In this paper, we generalize some results of [10] and [11] without using integral characterizations like (1) and (2).

It is not hard to see that $E\subset D$ is minimally thin at $y\in\partial D$ if and only if there is a finite measure μ on $\bar{D}$ such that $\mu(\{y\})=0$ and $K_y\leqq K\mu$ on E (see [5], [8] and [19]). From this observation, we introduce sets "minimally thin for harmonic functions" as follows.

DEFINITION 1. *A set $E\subset D$ is said to be minimally thin at $y\in\partial D$ for harmonic functions if there is a finite measure μ concentrated on ∂D such that $\mu(\{y\})=0$ and $K_y\leqq K\mu$ on E.*

REMARK. In view of Definitions 1 and A, it follows that E is minimally thin at y for harmonic functions if and only if E does not determine the point measure at y. Let us remark that Hayman [**12**, p. 481 and Theorem 7.37] defined sets "rarefied for harmonic functions", which correspond to rarefied sets given first by Lelong-Ferrand [**17**].

By definition if E is minimally thin at y for harmonic functions, then E is minimally thin at y. In view of the Harnack principle, more is true. If E is minimally thin at y for harmonic functions, then E_ρ, $0<\rho<1$, is minimally thin at y for harmonic functions, and hence minimally thin at y. Let us prove that the converse is true. This is the key theorem for the succeeding argument.

THEOREM 1. *Let $y\in\partial D$ and $E\subset D$. Then the following are equivalent:*
(i) *E is minimally thin at y for harmonic functions.*
(ii) *E_ρ is minimally thin at y for some ρ, $0<\rho<1$.*
(iii) *E_ρ is minimally thin at y for all ρ, $0<\rho<1$.*

The following theorem is well-known as the minimal fine limit theorem (see [**5**], [**8**] and [**19**]).

THEOREM C. *Let $h=K\mu_h$ and $H=K\mu_H$ be positive harmonic functions on D. Let u be a Green potential. Then, for μ_h almost every boundary point y, there are sets E and F which are minimally thin at y such that*

$$\lim_{\substack{x\to y\\ x\in D\setminus E}}\frac{H(x)}{h(x)}=\frac{d\mu_H}{d\mu_h}(y)\quad \textit{and}\quad \lim_{\substack{x\to y\\ x\in D\setminus E}}\frac{u(x)}{h(x)}=0.$$

Theorem 1 improves the first assertion of Theorem C.

COROLLARY 1. *Let $h=K\mu_h$ and $H=K\mu_H$ be positive harmonic functions on D. Then, for μ_h almost every boundary point y, there exists a set E minimally thin at y for harmonic functions such that*

$$\lim_{\substack{x\to y\\ x\in D\setminus E}}\frac{H(x)}{h(x)}=\frac{d\mu_H}{d\mu_h}(y).$$

For two measures μ and ν on ∂D we say $\nu\leqq\mu$ if $\nu(A)\leqq\mu(A)$ for every Borel subset A of ∂D. It is easy to see that $\nu\leqq\mu$ if and only if $d\mu/d\nu\geqq1$ for ν almost every point on ∂D. Since D is not minimally thin at any point $y\in\partial D$, it follows from Theorem C (or Corollary 1) that $\nu\leqq\mu$ if and only if $K\nu\leqq K\mu$ on D.

DEFINITION 2. Let ν be a finite measure on ∂D. A set $E\subset D$ is said to determine the measure ν if for every finite measure μ concentrated on ∂D, the inequality $K\nu\leqq K\mu$ on E implies that $\nu\leqq\mu$, or equivalently $K\nu\leqq K\mu$ on D.

The following theorem is a generalization of [**11**, Theorem 2].

THEOREM 2. *Let ν be a finite measure on ∂D and $E\subset D$. Then the following are equivalent:*

(i) *E determines the measure ν.*

(ii) *E determines the point measure at y for ν almost every boundary point y.*

(iii) *E is not minimally thin at y for harmonic functions for ν almost every boundary point y.*

(iv) *E_ρ is not minimally thin at y for ν almost every boundary point y and for some (or all) ρ, $0<\rho<1$.*

(v) $\inf_{x\in E}(H(x)/K\nu(x))=\inf_{x\in D}(H(x)/K\nu(x))$ *for all positive harmonic functions H.*

Let $\omega(x, A)$ be the harmonic measure at $x\in D$ of $A\subset\partial D$. We write simply ω for the harmonic measure at the origin 0. We observe that $K\omega\equiv1$ on D. In case ν is the harmonic measure ω we can give further equivalent condition to Theorem 2. Let $\alpha>0$. For each $y\in\partial D$ we associate the nontangential region $\Gamma_\alpha(y)=\{x\in D: |x-y|<(1+\alpha)\delta(x)\}$. We say that $\{x_j\}$ is a nontangential sequence converging to y if $x_j\to y$ and $x_j\in\Gamma_\alpha(y)$ for some $\alpha>0$.

COROLLARY 2. *Let $E\subset D$. Then each statement with $\nu=\omega$ in Theorem 2 is equivalent to*

(vi) *E includes a nontangential sequence converging to y for ω almost every boundary point $y\in\partial D$.*

The above corollary as well as further equivalent conditions are given by Gardiner [**11**, Corollary 2] for the unit ball. We observe that the harmonic measure ω can be replaced by the surface measure if D is a Lipschitz domain.

DEFINITION 3. Let A be a closed subset of ∂D. A set $E\subset D$ is said to determine the closed set A if E determines the measure ν for all measures ν concentrated on A.

Bonsall and Walsh defined the notion of positive Poisson basic (P.P.B.). In our situation their definition is generalized as follows.

DEFINITION 4. Let A be a closed subset of ∂D. A set $E\subset D$ is said to be a positive Martin basic (abbreviated to P.M.B.) set for A if, for every positive continuous function f on A, there exist sequences $\{\lambda_j\}$ and $\{x_j\}$ with λ_j positive and x_j in E such that $f(y)=\sum_j\lambda_jK(x_j, y)$ for $y\in A$.

Let $\mathcal{H}(D, A)$ be the class of functions $h=h_1-h_2$, where $h_j=K\mu_{h_j}$ is a positive harmonic function with measure μ_{h_j} concentrated on A. Then [**6**, Theorem 10] becomes the following form.

THEOREM D. *Let A be a closed subset of ∂D and $E\subset D$. Then the follow-*

ing are equivalent:

(i) $\sup_{x\in E} h(x)=\sup_{x\in D} h(x)$ *for all* $h\in\mathcal{H}(D, A)$.

(ii) *E is a P.M.B. set for A.*

(iii) *For each* $x\in D$ *there is a measure* λ_x *concentrated on E such that* $\|\lambda_x\|=1$ *and* $K(x, y)=\int_E K(\xi, y)d\lambda_x(\xi)$ *for* $y\in A$.

It is easy to see that if E determines A, then E is a P.M.B. set for A. The converse is not true in general. In fact, let A be a singleton $\{y\}$ with $y\in\partial D$. Then any nonempty set E (even a compact subset of D) is a P.M.B. set for A. The following theorem shows that this is rather an exceptional case. For the unit disk the theorem was proved by Essén [**10**, Theorem 2].

THEOREM 3. *Let A be a closed subset of* ∂D *and assume that A contains at least two points. Suppose* $E\subset\bigcup_{y\in A}\Gamma_\alpha(y)$ *for some* $\alpha>0$. *Then the following are equivalent*:

(i) $\sup_{x\in E} h(x)=\sup_{x\in D} h(x)$ *for all* $h\in\mathcal{H}(D, A)$.

(ii) *E is a P.M.B. set for A.*

(iii) *E determines A.*

(iv) *E determines the point measure at y for every* $y\in A$.

(v) *E is not minimally thin at any* $y\in A$ *for harmonic functions.*

(vi) E_ρ *is not minimally thin at any* $y\in A$ *for some (or all)* ρ, $0<\rho<1$.

Let us consider the case when $A=\partial D$. Since $\bigcup_{y\in\partial D}\Gamma_\alpha(y)$ includes a neighborhood of ∂D, we readily obtain a generalization of [**11**, Theorem 1].

COROLLARY 3. *The following are equivalent*:

(i) $\sup_{x\in E} h(x)=\sup_{x\in D} h(x)$ *for all* $h\in\mathcal{H}(D, \partial D)$.

(ii) *E is a P.M.B. set for* ∂D.

(iii) *E determines the boundary* ∂D.

(iv) *E determines the point measure at y for every* $y\in\partial D$.

(v) *E is not minimally thin at any* $y\in\partial D$ *for harmonic functions.*

(vi) E_ρ *is not minimally thin at any* $y\in\partial D$ *for some (or all)* ρ, $0<\rho<1$.

Finally we add below a further equivalent condition to Theorem 1.

THEOREM 4. *Let* $y\in\partial D$ *and* $E\subset D$. *Then E is minimally thin at y for harmonic functions if and only if there is a positive harmonic function H such that*

$$\liminf_{\substack{x\to y\\ x\in D}}\frac{H(x)}{g(x)}<\liminf_{\substack{x\to y\\ x\in E}}\frac{H(x)}{g(x)}.$$

If D is the unit ball $B(0, 1)$, then it is easy to see that $g(x)/(1-|x|)$ has a positive limit as $x\to y$ for every boundary point y. Hence, for the unit ball, Theorem 3 is the same result given by Gardiner [**11**, Theorem 3].

So far, we have observed that many results of [**5**, **6**, **10**, **11**, **13**] can be extended to an NTA domain without Theorems A and B. Therefore We raise

QUESTION. *Can one extend these result to a general Martin space?*

Our argument here depends on the estimates of the Martin kernel and the boundary Harnack principle, so it is not applicable to a general Martin space.

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2. Proof of Theorem 1 and Corollary 1.

By the symbol M we denote an absolute positive constant whose value is unimportant and may change from line to line. We shall say that two positive functions f_1 and f_2 are comparable, written $f_1 \approx f_2$, if and only if there exists a constant $M \geqq 1$ such that $M^{-1} f_1 \leqq f_2 \leqq M f_1$. The constant M will be called the constant of comparison. First we recall the well-known Harnack inequality.

LEMMA 1. *For $0<\rho<1$ there exists a positive constant $M(\rho)$ with the following properties:*

(i) $M(\rho) \downarrow 0$ *as* $\rho \downarrow 0$.

(ii) *Suppose $x \in D$ and $x' \in B(x, \rho\delta(x))$. Then for any positive harmonic functions h and H on D*

$$(1-M(\rho))\frac{H(x')}{h(x')} \leqq \inf_{B(x,\rho\delta(x))} \frac{H}{h} \leqq \sup_{B(x,\rho\delta(x))} \frac{H}{h} \leqq (1+M(\rho))\frac{H(x')}{h(x')}\,.$$

For the proof of Theorem 1 we use the following estimate of the Martin kernel, whose proof will be given in Section 5.

LEMMA 2. *Let $0<\rho<1<\gamma$. Suppose $x, y \in D$. If $|x-y| \geqq \rho\delta(x)$, then*

$$K(x, y) \leqq MK(x, z) \qquad \textit{for all} \quad z \in B(y, \gamma\delta(y)) \cap \partial D$$

with M depending only on D, ρ and γ.

The following lemma includes the crucial part of the proof of Theorem 1. For the lemma we are motivated by the argument of Sjögren [**20**, Proof of Theorem 2].

LEMMA 3. *Let h be a positive harmonic function on D. Suppose $E \subset D$, $0<\rho<1$ and there is a Green potential u such that $u \geqq h$ on E_ρ. Suppose F is a subset of ∂D and there is $\eta>1$ such that*

$$B(x, \eta\delta(x)) \cap F \neq \emptyset \qquad \textit{for} \quad x \in E\,. \tag{3}$$

Then there exists a finite measure μ concentrated on a countable subset of F such that $K\mu \geqq h$ on E_ρ.

REMARK. Let F be a dense subset of ∂D. Then (3) holds for any $E \subset D$ and $\eta > 1$.

PROOF OF LEMMA 3. By the covering lemma ([**16**, Lemma 3.2] and [**22**, Theorem 1.3.5]) we can find sequences $\{x_j^i\}_j \subset E$, $i=1, \cdots, N$, such that N depends only on the dimension, $\{B(x_j^i, \rho\delta(x_j^i))\}_j$ is mutually disjoint and

$$E \subset \bigcup_{i=1}^{N} \bigcup_j B(x_j^i, \rho\delta(x_j^i)). \tag{4}$$

Obviously, the right hand side is included in E_ρ and hence the Green potential u majorizes h on $E_\rho^i = \bigcup_j B(x_j^i, \rho\delta(x_j^i))$. Therefore $\hat{R}_h^{E_\rho^i}$ is a Green potential, which is represented as $K\nu_i$ with finite measure ν_i concentrated on $\partial E_\rho^i \cap D$. Note that

$$K\nu_i \geqq h \quad \text{on} \quad E_\rho^i. \tag{5}$$

Since $\{B(x_j^i, \rho\delta(x_j^i))\}_j$ do not accumulate in D for each $i=1, \cdots, n$, it follows that

$$\partial E_\rho^i \cap D = \bigcup_j \partial B(x_j^i, \rho\delta(x_j^i)).$$

Thus the measure ν_i is concentrated on the union of spheres $\partial B(x_j^i, \rho\delta(x_j^i))$.

By (3) we can take a point $z_j^i \in F \cap B(x_j^i, \eta\delta(x_j^i))$. Define the measure μ_i by the summation of point masses at z_j^i of magnitude $\nu_i(\partial B(x_j^i, \rho\delta(x_j^i)))$. Then $\|\mu_i\| = \|\nu_i\| < \infty$, and hence $K\mu_i$ is a positive harmonic function. Let x be one of $\{x_j^i\}_j$, say x_j^i. If $y \in \partial B(x_j^i, \rho\delta(x_j^i))$, then $|x-y| \geqq \rho\delta(x)$ and

$$|y-z_j^i| \leqq |y-x_j^i| + |x_j^i - z_j^i| \leqq (\rho+\eta)\delta(x_j^i) \leqq \frac{\rho+\eta}{1-\rho}\delta(y).$$

Hence Lemma 2 with $\gamma = (\rho+\eta)/(1-\rho) > 1$ yields

$$K(x, y) \leqq MK(x, z_j^i)$$

with $M > 0$ independent of x, y and z_j^i. By the definition of the integration we have $K\nu_i(x) \leqq MK\mu_i(x)$. Therefore (5) and Lemma 1 show that

$$K\mu_i \geqq Mh \quad \text{on} \quad E_\rho^i.$$

Hence, by (4), the finite measure $\mu = M^{-1}\sum_{i=1}^N \mu_i$ satisfies

$$K\mu \geqq h \quad \text{on} \quad E.$$

Moreover μ is concentrated on $\bigcup_{i,j}\{z_i^j\}$, which is a countable subset of F. The lemma is proved.

PROOF OF THEOREM 1. We have only to prove (ii)$\Rightarrow$(i). Suppose E_ρ is minimally thin at y, in other words there is a Green potential which majorizes the harmonic function K_y on E_ρ. Observe that $F=\partial D\setminus\{y\}$ is a dense subset of ∂D. Hence, it follows from Lemma 3 and its remark that there is a finite measure μ concentrated on F such that $K\mu\geqq K_y$ on E_ρ. This implies that E_ρ is minimally thin at y for harmonic functions. Thus (i) follows.

The following lemma follows easily from Theorem 1 (cf. [5, Theorem II, 9 and Theorem XV, 9]).

LEMMA 4. *Let $y\in\partial D$. Suppose $E_j\subset D$ are minimally thin at y for harmonic functions for $j=1, \cdots$. Then there is a sequence of positive numbers r_j such that $E=\bigcup_j E_j\cap B(y, r_j)$ is minimally thin at y for harmonic functions.*

PROOF OF COROLLARY 1. By Theorem C, it is sufficient to show that if $\lim_{\substack{x\to y\\ x\in D\setminus F}} H(x)/h(x)=\alpha$ with a set F minimally thin at y, then there is a set E minimally thin at y for harmonic functions such that $\lim_{\substack{x\to y\\ x\in D\setminus E}} H(x)/h(x)=\alpha$.

For $0<\rho<1$ we let $F(\rho)=\{x\in F: B(x, \rho\delta(x))\subset F\}$. We observe that if $x\in D\setminus F(\rho)$, then there is a point $x'\in B(x, \rho\delta(x))\setminus F$. Hence Lemma 1 yields

$$\alpha(1-M(\rho))\leqq \liminf_{\substack{x\to y\\ x\in D\setminus F(\rho)}} \frac{H(x)}{h(x)}\leqq \limsup_{\substack{x\to y\\ x\in D\setminus F(\rho)}} \frac{H(x)}{h(x)}\leqq \alpha(1+M(\rho)).$$

Since $\bigcup_{x\in F(\rho)} B(x, \rho\delta(x))\subset F$, it follows from Theorem 1 that $F(\rho)$ is minimally thin at y for harmonic functions. From Lemma 4 we can find $r_j\to 0$ such that

$$E=\bigcup_j F\left(\frac{1}{j}\right)\cap B(y, r_j)$$

is minimally thin at y for harmonic functions. By the construction of E we see that $\lim_{\substack{x\to y\\ x\in D\setminus E}} H(x)/h(x)=\alpha$. The corollary is proved.

3. Proof of Theorem 2 and Corollary 2.

We shall show Theorem 2. The essential part will be (i)$\Rightarrow$(iii). For the proof we shall invoke Lemma 3.

PROOF OF THEOREM 2. The equivalence (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) follows from Theorem 1. Let us prove (iii)$\Rightarrow$(i), (i)$\Leftrightarrow$(v) and (i)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i): Suppose (iii) holds, i.e., E is not minimally thin at y for harmonic functions for ν almost every $y\in\partial D$. Let μ be a finite measure on ∂D and suppose $K\nu\leqq K\mu$ on E. Then Corollary 1 yields

$$\frac{d\mu}{d\nu}\geqq 1 \quad \nu\text{-a.e. on } \partial D.$$

Thus $\nu\leqq\mu$ and (i) follows.

(i)$\Leftrightarrow$(v): Suppose (i) holds. Take a positive harmonic function $H=K\mu_H$ on D and let $\alpha=\inf_E H/K\nu$. Then $\alpha K\nu\leqq K\mu_H$ on E. Since E determines the measure ν, it follows that $\alpha K\nu\leqq K\mu_H$ on D, which implies that $\alpha=\inf_D H/K\nu$. Thus (v) follows.

Next suppose (v) holds. Take a finite measure μ on ∂D such that $K\nu\leqq K\mu$ on E. Theorem C says that the ratio $K\mu/K\nu$ has minimal fine limit $d\mu/d\nu$ for ν-almost every boundary point. This limit is greater than or equal to 1 since

$$1\leqq\inf_E\frac{K\mu}{K\nu}=\inf_D\frac{K\mu}{K\nu}.$$

Hence $\nu\leqq\mu$ and (i) follows.

(i)$\Rightarrow$(iii): Suppose (iii) does not hold, i.e., there is $A\subset\partial D$ with $\nu(A)>0$ such that E is minimally thin at any $y\in A$ for harmonic functions, or equivalently E_ρ is minimally thin at any $y\in A$.

Suppose first that there is $y\in A$ with $\nu(\{y\})>0$. Since E is not minimally thin at y for harmonic functions, we can find a measure μ_y on ∂D such that $\mu_y(\{y\})=0$ and $K\mu_y\geqq K_y$ on E. Observe that the measure

$$\mu=\nu|_{\partial D\setminus\{y\}}+\nu(\{y\})\mu_y$$

satisfies $\nu\nleqq\mu$ and $K\mu\geqq K\nu$ on E. Thus E does not determine the measure ν.

Suppose next that $\nu(\{y\})=0$ for any $y\in A$. Let $\nu'=\nu|_A$. Then $\hat{R}^{E_\rho}_{K\nu'}$ is a Green potential ([**5**, Corollary to Theorem XV, 11]); in other words there is a Green potential which majorizes the harmonic function $K\nu'$ on E_ρ. By Lemma 3 we can find a finite measure ν^* concentrated on a countable subset A^* of ∂D such that $K\nu^*\geqq K\nu'$ on E. Then the measure $\mu=\nu^*+\nu|_{\partial D\setminus A}$ satisfies $K\mu\geqq K\nu$ on E. Observe that

$$\nu(A\setminus A^*)=\nu(A)>0,$$

$$\mu(A\setminus A^*)=\nu^*(A\setminus A^*)=0.$$

This implies $\nu\nleqq\mu$. Thus E does not determine the measure ν. Therefore (i) does not hold. Thus (i)$\Rightarrow$(iii) follows.

In view of [**2**, Theorem 4], we obtain the following lemma. Actually, this lemma is not so difficult. For a Lipschitz domain, Hunt and Wheeden [**14**] used this fact as the property of *n.t.* $\mathscr{B}$ sets.

LEMMA 5. *Let $y\in\partial D$. Then a nontangential sequence converging to y is not minimally thin at y for harmonic functions.*

PROOF OF COROLLARY 2. By Lemma 5 we readily have (vi)$\Rightarrow$(iii). We prove the corollary by showing (v)$\Rightarrow$(vi). Suppose (vi) does not hold. Then there is $\alpha>0$ and $r>0$ such that $\omega(A)>0$ with $A=\{y\in\partial D: \Gamma_\alpha(y)\cap B(y, r)\cap E$

$=\emptyset\}$. Let

$$H(x)=\omega(x,\ \partial D\setminus A)=\int_{\partial D\setminus A} K(x,\ y)d\omega(y).$$

We observe from the minimal fine limit theorem (Theorem C) that for ω a.e. $y\in A$ there is a set F minimally thin at y such that $\lim_{\substack{x\to y\\ x\in D\setminus F}} H(x)=0$. Since $\omega(A)>0$, it follows, in particular, that

$$\inf_{x\in D} H(x)=0.$$

Hence, it is sufficient to show that $\inf_{x\in E}H(x)>0$.

For $x\in D$ we let $P_\alpha(x)=\{y\in\partial D: x\in\Gamma_\alpha(y)\}$. Let $D_0=\{x\in D: \delta(x)<r/(1+\alpha)\}$. By definition if $x\in E\cap D_0$, then $P_\alpha(x)\subset\partial D\setminus A$. Hence $H(x)\geqq\omega(x,\ P_\alpha(x))$. On the other hand, in view of the definition of $\Gamma_\alpha(y)$, $P_\alpha(x)=B(x,\ (1+\alpha)\delta(x))\cap\partial D$, and hence one of the Carleson estimates reads

$$\omega(x,\ P_\alpha(x))\geqq M \qquad \text{for} \quad x\in D_0,$$

where M is a positive constant independent of x (cf. [**15**, Lemma 4.2]). Therefore,

$$\inf_{x\in E} H(x)\geqq \min\{\inf_{x\in E\cap D_0} H(x),\ \inf_{x\in D\setminus D_0} H(x)\}\geqq \min\{M,\ \inf_{x\in D\setminus D_0} H(x)\}>0.$$

Thus the corollary is proved.

REMARK. Let $E\subset D$ and $A\subset\partial D$. Then E determines the point measure at y for ω a.e. $y\in A$ if and only if E includes a nontangential sequence converging to y for ω a.e. $y\in A$.

4. Proof of Theorem 3.

Let us prove Theorem 3. The essential part will be (i)$\Rightarrow$(v). For the proof we shall invoke Lemma 3 and the assumption that $E\subset\bigcup_{y\in A}\Gamma_\alpha(y)$.

PROOF OF THEOREM 3. We have observed in Theorem D that (i)$\Leftrightarrow$(ii). By definition (iii)$\Rightarrow$(iv) and by Theorem 2 (iv)$\Rightarrow$(iii). In view of Theorem 1 and the remark after Definition 1 we have (iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi). Let us prove (iii)$\Rightarrow$(i) and (i)$\Rightarrow$(v).

(iii)$\Rightarrow$(i): Let ω be the harmonic measure at the origin as before Corollary 2. We observe that $K\omega\equiv 1$ on D. Take $h=K\mu_1-K\mu_2\in\mathcal{H}(D,\ A)$ with measures μ_1 and μ_2 concentrated on A. Put $\alpha=\sup_E h$. Then

$$K\mu_1\leqq K(\mu_2+\alpha\omega) \text{ on } E.$$

Since E determines A, it follows from the discussion before Definition 2 that the same inequality holds on D; in other words

$$h=K\mu_1-K\mu_2\leqq \alpha K\omega=\alpha \text{ on } D.$$

Hence $\sup_D h\leqq\alpha$. Thus (i) follows.

(i)$\Rightarrow$(v): Let us prove the implication by contradiction. Suppose (v) does not holds, i.e., E is minimally thin at some point $y\in A$ for harmonic functions. It is sufficient to show that there is a measure μ concentrated on $A\setminus\{y\}$ such that $K_y\leqq K\mu$ on E since $h=K_y-K\mu\in\mathcal{H}(D, A)$ satisfies $\sup_E h\leqq 0$ and yet, by Theorem C,

$$\lim_{\substack{x\to y\\ x\in D\setminus F}}\frac{h(x)}{K_y(x)}=1 \qquad \text{with } F \text{ being minimally thin at } y,$$

which in particular implies that $\sup_D h>0$.

Suppose first y is an isolated point of A. Since E is minimally thin at y for harmonic functions, it follows from Lemma 5 that E does not contain a nontangential sequence converging to y. From the assumption that $E\subset\bigcup_{y\in A}\Gamma_\alpha(y)$ it follows that dist$(E, y)>0$. Therefore there is $r>0$ such that $B(y, r)\cap(A\setminus\{y\})=\emptyset$ and $B(y, r)\cap E=\emptyset$. Let $y'\in A\setminus\{y\}$. By the boundary Harnack principle (see Lemma 6 below) we can show

$$K_y(x)\leqq MK_{y'}(x) \qquad \text{for} \quad x\in D\cap\partial B\left(y, \frac{r}{2}\right)$$

with M depending on y, y' and r but not on x. Hence the maximum principle yields the same inequality for $x\in D\setminus B(y, r/2)$. Thus $K_y\leqq K\mu$ **on** E with the point mass μ at y' of magnitude M.

Suppose next y is not an isolated point of A. Then we can find a sequence y'_j in A converging to y. We observe that

$$\Gamma_\alpha(y)\subset\bigcup_j\Gamma_{2\alpha}(y'_j).$$

Hence

$$E\subset\bigcup_{y'\in A\setminus\{y\}}\Gamma_{2\alpha}(y'),$$

which implies (3) with $F=A\setminus\{y\}$ and $\eta=1+2\alpha$. Let $0<\rho<1$. By Theorem 1 E_ρ is minimally thin at y, and hence we find a Green potential which majorizes K_y on E_ρ. Therefore Lemma 3 yields a measure μ on $A\setminus\{y\}$ such that $K\mu\geqq K_y$ on $E_\rho\supset E$. The theorem is proved.

5. Proof of Lemma 2.

In this section we prove Lemma 2. Let us recall the definition of an NTA domain. A bounded domain is called NTA when there exist positive constants M and r_0 such that

(a) Corkscrew condition. For any $z\in\partial D$, $r<r_0$ there exists a point $A_r(z)\in$

D such that $M^{-1}r<|A_r(z)-z|<r$ and $\delta(A_r(z))>M^{-1}r$.

(b) The complement of D satisfies the corkscrew condition.

(c) Harnack chain condition. If $\varepsilon>0$ and x_1 and x_2 belong to D, $\delta(x_j)>\varepsilon$ and $|x_1-x_2|<C\varepsilon$, then there exists a Harnack chain from x_1 and x_2 whose length depends on C, but not ε.

Without loss of generality we may assume that $r_0=1$ and $B(0, 2)\subset D$. The boundary Harnack principle ([**15**, Lemma 4.10]) is crucial.

LEMMA 6. *Let $z\in\partial D$ and let $0<r<1$. Suppose u and v are positive harmonic functions on $B(z, 2r)\cap D$ and that u and v vanish continuously on $B(z, 2r)\cap\partial D$. Then*

$$\frac{u}{u(A_r(z))}\approx\frac{v}{v(A_r(z))}\qquad \text{on}\quad B(z, r)\cap D$$

with constant of comparison independent of z, r, u and v.

The boundary Harnack principle is a powerful tool and produces many results. The following is an easy corollary.

LEMMA 7. *Let $z\in\partial D$ and let $0<r<1$. Suppose u is a positive harmonic function on $B(z, 2r)\cap D$ and that u vanishes continuously on $B(z, 2r)\cap\partial D$. Then*

$$\sup_{B(z,r)\cap D} u\approx u(A_r(z))$$

with constant of comparison independent of z, r and u.

Applying Lemma 7 to $g=G(\cdot, 0)$, we obtain

LEMMA 8. *Let $z\in\partial D$ and let $0<r<R<1$. Then $g(A_r(z))\leqq Mg(A_R(z))$.*

Let $\Theta(x, y)=K(x, y)/g(x)$. It is known that $\Theta(x, y)$ has a continuous extension on $\bar{D}\times\bar{D}$. By the same symbol we denote the continuous extension. The kernel Θ is referred to as the Naïm's Θ kernel for D. By definition Θ is symmetric.

LEMMA 9. *For $z\in\partial D$ we let $\theta_z(r)=\Theta(A_r(z), z)$. Then*

$$\theta_z(R)\leqq M\theta_z(r)\qquad \text{for}\quad 0<r<R<1,$$

where M depends only on D. Moreover, if x, $y\in\bar{D}$ and $2|y-z|\leqq|x-z|<1$, then

$$\Theta(x, y)\approx\Theta(x, z)\approx\theta_z(|x-z|)$$

with constant of comparison depending only on D.

PROOF. Let $0<r<R<1$. The maximum principle yields

$$\sup_{D\cap\partial B(z,R)} K(\cdot, z)\leqq\sup_{D\cap\partial B(z,r)} K(\cdot, z).$$

By Lemma 7 we have

$$K(A_R(z), z) \leqq MK(A_r(z), z)$$

with M independent of z, r and R. This, together with Lemma 8, yields

$$\theta_z(R) = \frac{K(A_R(z), z)}{g(A_R(z))} \leqq M\frac{K(A_r(z), z)}{g(A_r(z))} = M\theta_z(r).$$

Suppose $x, y \in \bar{D}$ and $2|y-z| \leqq |x-z| < 1$. By the continuity we may assume that $x, y \in D$. Let $r = |x-z|$. Observe that $K(\cdot, y)$ and g are harmonic on $D \cap B(z, 2r) \setminus B(z, (1/2)r)$ and vanish on $\partial D \cap B(z, 2r) \setminus B(z, (1/2)r)$. By an elementary geometrical observation and the Lemma 6 we have

$$\frac{K(\cdot, y)}{K(A_r(z), y)} \approx \frac{g}{g(A_r(z))}$$

on $D \cap \partial B(z, r)$. Hence

$$\Theta(x, y) = \frac{K(x, y)}{g(x)} \approx \frac{K(A_r(z), z)}{g(A_r(z))} = \Theta(A_r(z), z) = \theta_z(r).$$

The above comparison holds particularly for $y=z$, whence the lemma follows.

LEMMA 10. *Let $z \in \partial D$, $x \in D$ and $y \in \bar{D}$.*

(i) *If $2|y-z| \leqq |x-z| < 1$, then $K(x, y) \approx K(x, z)$ with constant of comparison independent of z, x and y.*

(ii) *If $2|x-z| \leqq |y-z| < 1$, then $K(x, y) \leqq MK(x, z)$ with M independent of z, x and y.*

PROOF. The first assertion readily follows from Lemma 9 and the definitions of K and Θ. Suppose $2|x-z| \leqq |y-z| < 1$. Changing the roles of x and y, we obtain from Lemma 9 that

$$\Theta(x, y) \approx \Theta(y, z) \approx \theta_z(|y-z|) \leqq M\theta_z(|x-z|) \approx \Theta(x, z).$$

Hence $K(x, y) \leqq MK(x, z)$. The lemma follows.

PROOF OF LEMMA 2. Let $x, y \in D$ and $z \in \partial D$. Suppose

$$(6) \qquad \begin{aligned} |x-y| &\geqq \rho\delta(x), \\ |y-z| &\leqq \gamma\delta(y). \end{aligned}$$

Without loss of generality we may assume that $|x-z| < 1$ and $|y-z| < 1$. Suppose first $2|y-z| \leqq |x-z|$. Then Lemma 10 (i) yields $K(x, y) \approx K(x, z)$. In particular we have the required estimate. Suppose next $2|x-z| \leqq |y-z|$. Then Lemma 10 (ii) yields $K(x, y) \leqq MK(x, z)$. Thus we have again the required estimate. Finally suppose $(1/2)|y-z| \leqq |x-z| \leqq 2|y-z|$. In view of (6), we have

(7) $$|x-y| \geqq M|y-z|.$$

Let $r=4M|y-z|$ and $y_0=A_r(z)$. Then $4|y-z|=M^{-1}r\leqq|y_0-z|\leqq r=4M|y-z|$ by the Corkscrew condition. Hence (6) and (7) yield

(8) $$|y-y_0| \leqq (4M+1)|y-z| \leqq M' \min\{|x-y|, \delta(y)\}.$$

Observe that $G(x, \cdot)$ and g are positive and harmonic in $D'=D\setminus\{x, 0\}$. In view of (8) and the Harnack chain condition, we can find a Harnack chain from y to y_0 in D' with length independent of x, y and y_0. Hence it follows from the Harnack principle that $K(x, y)\approx K(x, y_0)$. Since $2|x-z|\leqq 4|y-z|\leqq|y_0-z|$, it follows from Lemma 10 (ii) that

$$K(x, y) \approx K(x, y_0) \leqq MK(x, z).$$

The lemma is proved.

6. Proof of Theorem 4.

First we note the following characterization of minimal thinness which readily follows from [5, Theorem XV.6, Theorem XV.8 and Theorem XV.9].

THEOREM E. *Let $y\in\partial D$ and $E\subset D$. Then the following are equivalent:*
(i) *E is minimally thin at y.*
(ii) *There is a nonnegative superharmonic function u on D such that*

$$\liminf_{\substack{x\to y\\ x\in D}} \frac{u(x)}{g(x)} < \liminf_{\substack{x\to y\\ x\in E}} \frac{u(x)}{g(x)}.$$

(iii) *There is a Green potential p on D such that*

$$\liminf_{\substack{x\to y\\ x\in D}} \frac{p(x)}{g(x)} < \liminf_{\substack{x\to y\\ x\in E}} \frac{p(x)}{g(x)}.$$

(iv) *There is a Green potential p on D such that*

$$\liminf_{\substack{x\to y\\ x\in D}} \frac{p(x)}{g(x)} < \liminf_{\substack{x\to y\\ x\in E}} \frac{p(x)}{g(x)} = \infty.$$

PROOF OF THEOREM 4 (SUFFICIENCY). Suppose there is a positive harmonic function H such that

$$\liminf_{\substack{x\to y\\ x\in D}} \frac{H(x)}{g(x)} < \liminf_{\substack{x\to y\\ x\in E}} \frac{H(x)}{g(x)}.$$

Then by Lemma 1 there is ρ, $0<\rho<1$, such that the above inequality with E_ρ replacing E holds. Hence, Theorem E, (ii)$\Rightarrow$(i), implies that E_ρ is minimally thin at y. By Theorem 1 E is minimally thin at y for harmonic functions.

For the necessity of Theorem 4 we need to consider a version of the above theorem in the context of harmonic functions. To this end we give a refinement of Lemma 3. We extend the notation $E_\rho=\bigcup_{x\in E}B(x, \rho\delta(x))$ for $\rho\geqq 1$.

LEMMA 11. *Let h, E, ρ, u, F and η be as in Lemma 3. Then the measure μ given in Lemma 3 satisfies $K\mu\leqq M_0u$ on $D\setminus E_{3\eta+2\rho}$.*

PROOF. Let $x\in D\setminus E_{3\eta+2\rho}$. Let x_j^i and z_j^i be as in the proof of Lemma 3. We observe that

$$|x-z_j^i| \geqq |x-x_j^i|-|x_j^i-z_j^i| \geqq (3\eta+2\rho-\eta)\delta(x_j^i) \geqq 2(\eta+\rho)\delta(x_j^i) > 2|y-z_j^i|.$$

Hence, Lemma 10 (i) with $z=z_j^i$ yields $K(x, y)\approx K(x, z_j^i)$. Let ν_i, μ_i and μ be the measures as in the proof of Lemma 3. The above inequality implies that $K\mu_i(x)\approx K\nu_i(x)$. Hence

$$K\mu(x) \leqq M\sum_{i=1}^{N} K\nu_i(x) \leqq M\sum_{i=1}^{N} \hat{R}_h^{E_\rho^i}(x) \leqq M_0u(x).$$

The lemma follows.

PROOF OF THEOREM 4 (NECESSITY). Suppose E is minimally thin at y for harmonic functions. Let $0<\rho<1<\eta$. We take $\alpha>6\eta+4\rho$ and let $\beta=(\alpha-6\eta-4\rho)/(1+3\eta+2\rho)$. By Lemma 5 we may assume that $E\subset D\setminus\Gamma_\alpha(y)$. We observe that

$$E_{3\eta+2\rho}\cap\Gamma_\beta(y)=\emptyset. \tag{9}$$

In fact, if $z\in E_{3\eta+2\rho}$, then by an elementary calculation $|z-y|\geqq(1+\alpha-3\eta-2\rho)\cdot(1+3\eta+2\rho)^{-1}\delta(z)=(1+\beta)\delta(z)$. This means that $z\notin\Gamma_\beta(y)$. Thus (9) follows. Since E_ρ is minimally thin at y by Theorem 1, it follows from Theorem E, (i)$\Rightarrow$(iv), that there is a Green potential p such that

$$\liminf_{\substack{x\to y\\ x\in D}}\frac{p(x)}{g(x)} < \liminf_{\substack{x\to y\\ x\in E_\rho}}\frac{p(x)}{g(x)} = \infty. \tag{10}$$

By Lemma 5 the nontangential region $\Gamma_\beta(y)$ is not minimally thin at y. Hence it follows from Theorem E, (iii)$\Rightarrow$(i), that

$$\liminf_{\substack{x\to y\\ x\in\Gamma_\beta(y)}}\frac{p(x)}{g(x)} = \liminf_{\substack{x\to y\\ x\in D}}\frac{p(x)}{g(x)} < \infty.$$

Let

$$c > M_0\liminf_{\substack{x\to y\\ x\in\Gamma_\beta(y)}}\frac{p(x)}{g(x)},$$

where M_0 is the constant in Lemma 11. By (10) we can choose $\delta>0$ such that $p\geqq cg$ on $E_\rho\cap B(y, \delta)$. By Lemmas 3 and 11 with $u=p$ and (9) we find a positive harmonic function H such that $H\geqq cg$ on $E_\rho\cap B(y, \delta)$ and $H\leqq M_0p$ on

$\Gamma_\beta(y)$. Hence

$$\liminf_{\substack{x\to y\\ x\in E}} \frac{H(x)}{g(x)} \leqq M_0 \liminf_{\substack{x\to y\\ x\in\Gamma_\beta(y)}} \frac{p(x)}{g(x)} < c \leqq \liminf_{\substack{x\to y\\ x\in E_\rho}} \frac{H(x)}{g(x)} \leqq \liminf_{\substack{x\to y\\ x\in E}} \frac{H(x)}{g(x)}.$$

Thus Theorem 4 is proved.

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