J. Math. Soc. Japan
Vol. 47, No. 4, 1995

Hessian manifolds of constant Hessian sectional curvature

By Hirohiko SHIMA

(Received Jan. 14, 1994)

Introduction.

Let M be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M is said to be a *Hessian metric* if g is locally expressed by $g=D^2u$ where u is a local smooth function. We call such a pair (D, g) a *Hessian structure* on M and a triple (M, D, g) a *Hessian manifold* [**5**]-[**8**]. Geometry of Hessian manifolds is deeply related to Kählerian geometry and affine differential geometry. In [**1**] Amari showed that a manifold consisting of a smooth family of probability distributions admits dual affine connections and proposed geometry of statistical manifolds (i.e., manifolds with dual affine connections). The notion of a Hessian structure is the same that dual affine connections are flat. It is known that many important smooth families of probability distributions admit Hessian structures (dual flat affine connections).

In section 1 we define Hessian sectional curvatures (which correspond to holomorphic sectional curvatures for Kählerian manifolds) and study fundamental properties of spaces of constant Hessian sectional curvature. In section 2 we construct Hessian manifolds of constant Hessian sectional curvatures. We see in section 3 that certain smooth families of probability distributions are Hessian manifolds of constant Hessian sectional curvature. Chen and Ogiue [**4**] characterized Kählerian manifolds of constant holomorphic sectional curvature in terms of Chern classes. We give in section 4 a similar characterization of the spaces of constant Hessian sectional curvature by affine Chern classes. In the last section 5 we define the notion of affine Chern classes for flat affine manifolds, which correspond to Chern classes for complex manifolds.

1. Spaces of constant Hessian sectional curvature.

Let M be a Hessian manifold with Hessian structure (D, g). We express various geometric concepts for the Hessian structure (D, g) in terms of affine coordinate systems $\{x^1, \cdots, x^n\}$ with respect to D, i.e., $Ddx^i=0$.

(i) The Hessian metric;

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

(ii) Let γ be a tensor field of type (1, 2) defined by

$$\gamma(X, Y) = \nabla_X Y - D_X Y,$$

where ∇ is the Riemannian connection for g. Then we have

$$\gamma^i{}_{jk} = \Gamma^i{}_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k},$$

$$\gamma_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2} \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k},$$

$$\gamma_{ijk} = \gamma_{jik} = \gamma_{kji},$$

where $\Gamma^i{}_{jk}$ are the Christoffel's symbols of ∇.

(iii) Define a tensor field S of type (1, 3) by

$$S = D\gamma,$$

and call it the *Hessian curvature tensor* for (D, g). Then we have

$$S^i{}_{jkl} = \frac{\partial \gamma^i{}_{jl}}{\partial x^k},$$

$$S_{ijkl} = \frac{1}{2} \frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^r} \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s},$$

$$S_{ijkl} = S_{ilkj} = S_{kjil} = S_{jilk} = S_{klij}.$$

(iv) The Riemannian curvature tensor for ∇;

$$R^i{}_{jkl} = \gamma^i{}_{rk} \gamma^r{}_{jl} - \gamma^i{}_{rl} \gamma^r{}_{jk},$$

$$R_{ijkl} = \frac{1}{2}(S_{jikl} - S_{ijkl}).$$

(v) We denote by v the volume element determined by g. Define a closed 1-form α and a symmetric bilinear form β by

$$D_X v = \alpha(X) v, \qquad \beta = D\alpha.$$

Then

$$\alpha_i = \frac{1}{2} \frac{\partial \log \det [g_{kl}]}{\partial x^i} = \gamma^r{}_{ri},$$

$$\beta_{ij} = \frac{1}{2} \frac{\partial^2 \log \det [g_{kl}]}{\partial x^i \partial x^j} = S^r{}_{rij}.$$

We call α and β the *first Koszul form* and the *second Koszul form* for (D, g) respectively. The second Koszul form β fills the role of the Ricci tensor for a Kählerian manifold.

DEFINITION 1. Let $\mathcal{S}$ be an endomorphism of the space of contravariant symmetric tensor fields of degree 2 defined by

$$\mathcal{S}(\xi)^{ik} = S^i{}_j{}^k{}_l \xi^{jl} .$$

Then $\mathcal{S}$ is a symmetric operator. In fact we have

$$\begin{aligned}\langle \mathcal{S}(\xi), \eta \rangle &= S^i{}_j{}^k{}_l \xi^{jl} \eta_{ik} = S_{ijkl} \xi^{jl} \eta^{ik} \\ &= S_{jilk} \eta^{ik} \xi^{jl} = (S^j{}_i{}^l{}_k \eta^{ik}) \xi_{jl} = \langle \xi, \mathcal{S}(\eta) \rangle ,\end{aligned}$$

where $\langle , \rangle$ is the inner product by g.

DEFINITION 2. For a non-zero contravariant symmetric tensor ξ_x of degree 2 at x we set

$$h(\xi_x) = \frac{\langle \mathcal{S}(\xi_x), \xi_x \rangle}{\langle \xi_x, \xi_x \rangle} ,$$

and call it the *Hessian sectional curvature in the direction* ξ_x.

DEFINITION 3. A Hessian manifold (M, D, g) is said to be a *space of constant Hessian sectional curvature* c if $h(\xi_x)$ is a constant c for all contravariant symmetric tensor ξ_x at x and for all point $x \in M$.

THEOREM 1. *Let (M, D, g) be a Hessian manifold of dimension $\geqq 2$. If the Hessian sectional curvature $h(\xi_x)$ depends only on x, then (M, D, g) is of constant Hessian sectional curvature. (M, D, g) is of constant Hessian sectional curvature c if and only if*

$$S_{ijkl} = \frac{c}{2}(g_{ij} g_{kl} + g_{il} g_{kj}) .$$

PROOF. Put $h(x) = h(\xi_x)$. Since

$$\langle \mathcal{S}(\xi_x), \xi_x \rangle = h(x) \langle \xi_x, \xi_x \rangle ,$$

for all contravariant symmetric tensor ξ_x at x and since $\mathcal{S}$ is symmetric we have

$$\mathcal{S}(\xi_x) = h(x) \xi_x ,$$

for all ξ_x. Set

$$T^i{}_j{}^k{}_l = S^i{}_j{}^k{}_l - \frac{h(x)}{2}(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j) ,$$

where δ^i_j is Kronecker's delta. Then we see

$$T^i{}_j{}^k{}_l = T^k{}_j{}^i{}_l = T^i{}_l{}^k{}_j$$

$$T^i{}_j{}^k{}_l \xi_x{}^{jl} = 0\,,$$

for all ξ_x. Thus we have

$$0 = T_{ijkl}(a^j b^l + a^l b^j)(c^i d^k + c^k d^i) = 4T_{ijkl} a^j b^l c^i d^k\,,$$

for all tangent vectors a^i, b^i, c^i, d^i at x, and so $T^i{}_j{}^k{}_l = 0$. Hence we obtain

$$S^i{}_j{}^k{}_l = \frac{h(x)}{2}(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j)\,,$$

$$S^i{}_{jkl} = \frac{h(x)}{2}(\delta^i_j g_{kl} + \delta^i_l g_{jk})\,.$$

By differentiating $S^i{}_{jkl}$ we have

$$\frac{\partial S^i{}_{jkl}}{\partial x^r} = \frac{1}{2}\frac{\partial h}{\partial x^r}(\delta^i_j g_{kl} + \delta^i_l g_{jk}) + h(\delta^i_j \gamma_{klr} + \delta^i_l \gamma_{kjr})\,.$$

On the other hand we have

$$\frac{\partial S^i{}_{jkl}}{\partial x^r} = \frac{\partial^2 \gamma^i{}_{jl}}{\partial x^r \partial x^k} = \frac{\partial S^i{}_{jrl}}{\partial x^k} = \frac{1}{2}\frac{\partial h}{\partial x^k}(\delta^i_j g_{rl} + \delta^i_l g_{jr}) + h(\delta^i_j \gamma_{rlk} + \delta^i_l \gamma_{rjk})\,.$$

These imply

$$\frac{\partial h}{\partial x^r}(\delta^i_j g_{kl} + \delta^i_l g_{jk}) = \frac{\partial h}{\partial x^k}(\delta^i_j g_{rl} + \delta^i_l g_{jr})\,.$$

Multiplying g^{kl} and contracting k, l and i, j we obtain

$$n(n+1)\frac{\partial h}{\partial x^r} = (n+1)\frac{\partial h}{\partial x^r}\,.$$

Thus $h(x)$ is a constant c and

$$S_{ijkl} = \frac{c}{2}(g_{ij} g_{kl} + g_{il} g_{kj})\,.$$

It is easy to see that, if the above condition is satisfied, then (M, D, g) is of constant Hessian sectional curvature c. ■

COROLLARY 2. *If a Hessian manifold* (M, D, g) *is a space of constant Hessian sectional curvature* c, *then the Riemannian manifold* (M, g) *is a space of constant sectional curvature* $-c/4$.

The proof immediately follows from Theorem 1 and (iv).

COROLLARY 3. *Let* (M, D, g) *be a simply connected Hessian manifold of constant Hessian sectional curvature* c. *If the Riemannian metric* g *is complete, then* $c \geqq 0$.

PROOF. Suppose $c<0$. Then (M, g) is a simply connected complete Riemannian manifold of constant sectional curvature $-c/4$, so (M, g) is isometric to the sphere of radius $2/\sqrt{-c}$. On the other hand by a theorem of Yagi [8] [10], a simply connected Hessian manifold (M, D, g) with complete Riemannian metric g is isomorphic to a domain in $\boldsymbol{R}^n$. This is a contradiction. ∎

DEFINITION 4. A Hessian manifold (M, D, g) is said to be *Hessian-Einstein* if $\beta=\lambda g$ holds.

COROLLARY 4. *A Hessian manifold of constant Hessian sectional curvature c is Hessian-Einstein;* $\beta=\{(n+1)c/2\}g$.

We define a tensor field W by

$$W^i{}_{jkl}=S^i{}_{jkl}-\frac{1}{n+1}(\delta^i_j\beta_{kl}+\delta^i_l\beta_{kj}).$$

This tensor field W is similar to the projective curvature tensor for a Kählerian manifold.

THEOREM 5. *(M, D, g) is of constant Hessian sectional curvature if and only if $W=0$.*

PROOF. Suppose M is a space of constant Hessian sectional curvature c. It follows from Theorem 1

$$S^i{}_{jkl}=\frac{c}{2}(\delta^i_j g_{kl}+\delta^i_l g_{kj}).$$

Using the above formula and (v) we have

$$\beta_{kl}=S^i{}_{ikl}=\frac{c}{2}(n+1)g_{kl}.$$

Thus we have

$$\begin{aligned} W^i{}_{jkl} &= S^i{}_{jkl}-\frac{1}{n+1}(\delta^i_j\beta_{kl}+\delta^i_l\beta_{kj}) \\ &= S^i{}_{jkl}-\frac{c}{2}(\delta^i_j g_{kl}+\delta^i_l g_{kj}) \\ &= 0. \end{aligned}$$

Conversely suppose $W=0$. Then

$$S_{ijkl}=\frac{1}{n+1}(g_{ij}\beta_{kl}+g_{il}\beta_{kj}).$$

Since $S_{ijkl}=S_{klij}$ we get

$$g_{ij}\beta_{kl}+g_{il}\beta_{kj}=g_{kl}\beta_{ij}+g_{kj}\beta_{il}.$$

By multiplication g^{ij} and by contraction we obtain

$$\beta_{kl} = \frac{\beta^i{}_i}{n} g_{kl} .$$

Hence we have

$$S_{ijkl} = \frac{\beta^r{}_r}{n(n+1)}(g_{ij}g_{kl}+g_{il}g_{kj}) .$$

∎

THEOREM 6. *We have*

$$TrS^2 \geqq \frac{2}{n(n+1)}(Tr\beta)^2 ,$$

where $Tr\beta=\beta^i{}_i$. *The equality holds if and only if* (M, D, g) *is of constant Hessian sectional curvature.*

PROOF. As in the proof of Theorem 1 we set

$$T^i{}_j{}^k{}_l = S^i{}_j{}^k{}_l - \frac{Tr\beta}{n(n+1)}(\delta^i_j\delta^k_l+\delta^i_l\delta^k_j) .$$

Then we have

$$\begin{aligned} T^i{}_j{}^k{}_l T_i{}^j{}_k{}^l &= S^i{}_j{}^k{}_l S_i{}^j{}_k{}^l - \frac{2Tr\beta}{n(n+1)} S_i{}^j{}_k{}^l(\delta^i_j\delta^k_l+\delta^i_l\delta^k_j) \\ &\quad + \left\{\frac{Tr\beta}{n(n+1)}\right\}^2 (\delta^i_j\delta^k_l+\delta^i_l\delta^k_j)(\delta^j_i\delta^l_k+\delta^l_i\delta^j_k) \\ &= TrS^2 - \frac{2Tr\beta}{n(n+1)}(S_i{}^i{}_k{}^k+S_i{}^j{}_j{}^i)+\frac{2(Tr\beta)^2}{n(n+1)} \\ &= TrS^2 - \frac{2}{n(n+1)}(Tr\beta)^2 . \end{aligned}$$

Thus

$$TrS^2 \geqq \frac{2}{n(n+1)}(Tr\beta)^2 ,$$

and the equality holds if and only if

$$S^i{}_j{}^k{}_l = \frac{Tr\beta}{n(n+1)}(\delta^i_j\delta^k_l+\delta^i_l\delta^k_j) .$$

∎

2. Constructions of Hessian manifolds of constant Hessian sectional curvature.

In this section we shall construct, for each constant c, a Hessian manifold with constant Hessian sectional curvature c. We now recall the following result due to Yagi [8][10]. Let (M, D, g) be a simply connected Hessian manifold. If g is complete, then (M, D, g) is isomorphic to $(\Omega, \tilde{D}, \tilde{D}^2\varphi)$, where Ω is a convex domain in $\boldsymbol{R}^n$, $\tilde{D}$ is the canonical flat connection on $\boldsymbol{R}^n$ and φ is a smooth convex function on Ω.

A. The case $c=0$.

It is obvious that the Euclidean space $(\boldsymbol{R}^n, \tilde{D}, g=(1/2)\tilde{D}^2\{\sum(x^i)^2\})$ is a simply connected Hessian manifold of constant Hessian sectional curvature 0.

B. The case $c>0$.

THEOREM 7. *Let Ω be a domain in $\boldsymbol{R}^n$ given by*

$$x^n > \frac{c}{2}\sum_{i=1}^{n-1}(x^i)^2,$$

where c is a positive constant, and let φ be a smooth function on Ω defined by

$$\varphi = -\frac{1}{c}\log\left\{x^n - \frac{c}{2}\sum_{i=1}^{n-1}(x^i)^2\right\}.$$

Then $(\Omega, \tilde{D}, g=\tilde{D}^2\varphi)$ is a simply connected Hessian manifold of positive constant Hessian sectional curvature c. As Riemannian manifold (Ω, g) is isometric to the hyperbolic space $(\boldsymbol{H}(-c/4), g)$ of constant sectional curvature $-c/4$;

$$\boldsymbol{H} = \{(\xi^1, \cdots, \xi^{n-1}, \xi^n) \in \boldsymbol{R}^n \mid \xi^n > 0\},$$

$$g = \frac{1}{(\xi^n)^2}\left\{\sum_{i=1}^{n-1}(d\xi^i)^2 + \frac{4}{c}(d\xi^n)^2\right\}.$$

PROOF. It is easy to see that Ω is a convex domain in $\boldsymbol{R}^n$. Set $\eta_i=\partial\varphi/\partial x^i$. Then

$$\eta_i = \begin{cases} x^i\left\{x^n - \frac{c}{2}\sum_{r=1}^{n-1}(x^r)^2\right\}^{-1} & 1\leqq i\leqq n-1 \\ -c^{-1}\left\{x^n - \frac{c}{2}\sum_{r=1}^{n-1}(x^r)^2\right\}^{-1} & i=n, \end{cases}$$

$$[g_{ij}] = c\begin{bmatrix} \eta_i\eta_j - \delta_{ij}\eta_n & \eta_i\eta_n \\ \eta_n\eta_j & \eta_n\eta_n \end{bmatrix}.$$

Put

$$\xi^i = \begin{cases} x^i & 1 \leqq i \leqq n-1, \\ \left\{x^n - \frac{c}{2}\sum_{r=1}^{n-1}(x^r)^2\right\}^{1/2} & i=n. \end{cases}$$

Then we have

$$-\infty < \xi^i < \infty \quad (1 \leqq i \leqq n-1), \quad \xi^n > 0,$$

$$g = \frac{1}{(\xi^n)^2}\left\{\sum_{i=1}^{n-1}(d\xi^i)^2 + \frac{4}{c}(d\xi^n)^2\right\}.$$

Thus (M, g) is isometric to the hyperbolic space $(\boldsymbol{H}(-c/4), g)$. Consider matrices $\gamma_i = [\gamma_{ijl}]$ and $\gamma^i = [\gamma^i_{jl}]$. By (ii) we obtain

$$\gamma_i = \frac{1}{2}\frac{\partial}{\partial x^i}[g_{jl}] = \frac{c}{2}\begin{bmatrix} g_{ij}\eta_l + g_{il}\eta_j - \delta_{jl}g_{in} & g_{ij}\eta_n + g_{in}\eta_j \\ g_{in}\eta_l + g_{il}\eta_n & 2g_{in}\eta_n \end{bmatrix},$$

and so

$$\gamma^i = \frac{c}{2}\begin{bmatrix} \delta^i_j\eta_l + \delta^i_l\eta_j - \delta_{jl}\delta^i_n & \delta^i_j\eta_n + \delta^i_n\eta_j \\ \delta^i_n\eta_l + \delta^i_l\eta_n & 2\delta^i_n\eta_n \end{bmatrix}.$$

Setting $S^i{}_k = \partial\gamma^i/\partial x^k$ we obtain

$$[S^i{}_{jkl}] = S^i{}_k = \frac{\partial \gamma^i}{\partial x^k} = \frac{c}{2}\begin{bmatrix} \delta^i_j g_{kl} + \delta^i_l g_{kj} & \delta^i_j g_{kn} + \delta^i_n g_{kj} \\ \delta^i_n g_{kl} + \delta^i_l g_{kn} & 2\delta^i_n g_{kn} \end{bmatrix}.$$

Hence

$$S^i{}_{jkl} = \frac{c}{2}(\delta^i_j g_{kl} + \delta^i_l g_{kj}),$$

and $(\Omega, \tilde{D}, g = \tilde{D}^2\varphi)$ is of positive constant Hessian sectional curvature c. ■

C. The case $c<0$.

THEOREM 8. *Let φ be a smooth function on $\boldsymbol{R}^n$ defined by*

$$\varphi = -\frac{1}{c}\log\left(\sum_{i=1}^{n} e^{-cx^i} + 1\right),$$

where c is a negative constant. Then $(\boldsymbol{R}^n, \tilde{D}, g = \tilde{D}^2\varphi)$ is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. The Riemannian manifold $(\boldsymbol{R}^n, g)$ is isometric to a domain of the sphere $\sum_{i=1}^{n+1}\xi_i^2 = -4/c$ defined by $\xi_i > 0$ for all i.

PROOF. Put

$$\eta_i = \begin{cases} \partial\varphi/\partial x^i = e^{-cx^i}(\sum_{j=1}^n e^{-cx^j} + 1)^{-1} & 1 \leqq i \leqq n \\ (\sum_{j=1}^n e^{-cx^j} + 1)^{-1} & i = n+1, \end{cases}$$

$$\xi_i = 2\left(-\frac{\eta_i}{c}\right)^{1/2}.$$

Then

$$\xi_1^2+\cdots+\xi_n^2+\xi_{n+1}^2=-\frac{4}{c}, \qquad \xi_i>0,$$

$$d\xi_{n+1}=-\sum_{i=1}^{n}\frac{\xi_i}{\xi_{n+1}}d\xi_i,$$

$$x^i=-\frac{2}{c}\log\frac{\xi_i}{\xi_{n+1}}.$$

Thus we have

$$\begin{aligned} g &= \sum_{i,j=1}^{n}\frac{\partial\eta_i}{\partial x^j}dx^i dx^j=-\frac{c}{4}\sum_{i,j=1}^{n}\frac{\partial\xi_i^2}{\partial x^j}dx^i dx^j \\ &= -\frac{c}{2}\sum_{i=1}^{n}\xi_i\Big(\sum_{j=1}^{n}\frac{\partial\xi_i}{\partial x^j}dx^j\Big)dx^i=-\frac{c}{2}\sum_{i=1}^{n}(\xi_i d\xi_i)dx^i \\ &= \sum_{i=1}^{n}(\xi_i d\xi_i)d\log\frac{\xi_i}{\xi_{n+1}}=\sum_{i=1}^{n}(\xi_i d\xi_i)\frac{\xi_{n+1}}{\xi_i}\frac{\xi_{n+1}d\xi_i-\xi_i d\xi_{n+1}}{\xi_{n+1}^2} \\ &= \sum_{i=1}^{n}(d\xi_i)^2-\Big(\sum_{i=1}^{n}\frac{\xi_i}{\xi_{n+1}}d\xi_i\Big)d\xi_{n+1}=\sum_{i=1}^{n}(d\xi_i)^2+(d\xi_{n+1})^2. \end{aligned}$$

Hence $(\boldsymbol{R}^n, g)$ is isometric to a domain of the sphere $\sum_{i=1}^{n+1}\xi_i^2=-4/c$ defined by $\xi_i>0$. On the other hand we obtain

$$g_{ij}=c(-\delta_{ij}\eta_i+\eta_i\eta_j).$$

Using the same notation as in the proof of Theorem 7, we get

$$\gamma_i=\frac{1}{2}\frac{\partial}{\partial x^i}[g_{jl}]=\frac{c}{2}[-\delta_{jl}g_{ij}+g_{ij}\eta_l+g_{il}\eta_j],$$

$$\gamma^i=\frac{c}{2}[-\delta_{jl}\delta_j^i+\delta_j^i\eta_l+\delta_l^i\eta_j].$$

Thus

$$[S^i{}_{jkl}]=S^i{}_k=\frac{\partial\gamma^i}{\partial x^k}=\frac{c}{2}[\delta_j^i g_{kl}+\delta_l^i g_{kj}].$$

Hence $(\boldsymbol{R}^n, \check{D}, g=\tilde{D}^2\varphi)$ is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. ■

3. Families of the multivariate normal distributions and the multinomial distributions.

In this section we see that Hessian manifolds of constant Hessian sectional curvature are realized as smooth families of probability distributions.

The family of normal distributions of dimension n. Let $\mathcal{N}_n$ be the family of n-variate normal distributions having the density functions

$$p(x, \mu, \Sigma) = \{(\det \Sigma)(2\pi)^n\}^{-1/2} \exp\left\{-\frac{1}{2}{}^t(x-\mu)\Sigma^{-1}(x-\mu)\right\}$$

with mean vectors $\mu=[\mu_i]$ and positive definite variance-covariance matrices $\Sigma=[\sigma_{ij}]$. Then $\mathcal{N}_n$ can be identified with

$$\{(\mu, \Sigma) \mid \mu \in \boldsymbol{R}^n, \Sigma \in \mathcal{P}^n\},$$

where $\boldsymbol{R}^n$ and $\mathcal{P}^n$ denote the vector space of all n-dimensional column vectors and the set of all positive definite symmetric (n, n) matrices respectively. Then the Fisher information metric $\tilde{g}$ on $\mathcal{N}_n$ is expressed by (cf. [**9**])

$$\tilde{g}\left(\frac{\partial}{\partial\mu_i}, \frac{\partial}{\partial\mu_j}\right) = \sigma^{ij}, \qquad \tilde{g}\left(\frac{\partial}{\partial\mu_i}, \frac{\partial}{\partial\sigma_{jk}}\right) = 0,$$

$$\tilde{g}\left(\frac{\partial}{\partial\sigma_{ij}}, \frac{\partial}{\partial\sigma_{kl}}\right) = \sigma^{il}\sigma^{jk} + \sigma^{ik}\sigma^{jl},$$

where σ^{ij} is the (i, j)-component of Σ^{-1}. Let $\tilde{D}$ be the flat affine connection on $\mathcal{P}^n$ such that $\tilde{D}d\sigma_{ij}=0$ and let $\tilde{\varphi}$ be a smooth function on $\mathcal{P}^n$ defined by

$$\tilde{\varphi}(\Sigma) = -\frac{1}{2}\log\det\Sigma.$$

Then $(\tilde{D}, \tilde{D}^2\varphi)$ is a Hessian structure on $\mathcal{P}^n$ and

$$\tilde{D}^2\tilde{\varphi}\left(\frac{\partial}{\partial\sigma_{ij}}, \frac{\partial}{\partial\sigma_{kl}}\right) = \sigma^{il}\sigma^{jk} + \sigma^{ik}\sigma^{jl}.$$

Thus we have:

A. *For each fixed* $\Sigma_0 \in \mathcal{P}^n$, *the subfamily of* $\mathcal{N}_n$ *given by*

$$\{(\mu, \Sigma_0) \mid \mu \in \boldsymbol{R}^n\}$$

is a simply connected Hessian manifold of constant Hessian sectional curvature 0.

B. For each fixed $c>0$ and $\mu_0 \in \boldsymbol{R}^n$, consider a subfamily Ω of $\mathcal{N}_n$ given by

$$\left\{(\mu_0, \Sigma) \mid \Sigma = \begin{bmatrix} 1 & 0 \cdots 0 & \sigma^1 \\ 0 & 1 \cdots 0 & \sigma^2 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 1 & \sigma^{n-1} \\ \sigma^1 & \sigma^2 \cdots \sigma^{n-1} & \frac{2}{c}\sigma^n \end{bmatrix} \in \mathcal{P}^n\right\}.$$

Identifying (μ_0, Σ) and Σ we set

$$\varphi(\Sigma)=-\frac{2}{c}\tilde{\varphi}(\Sigma)$$
$$=-\frac{1}{c}\log\left\{\frac{2}{c}\sigma^n-\sum_{i=1}^{n-1}(\sigma^i)^2\right\}.$$

Denoting by D the restriction of $\tilde{D}$ to Ω, $(\Omega, D, g=D^2\varphi)$ *is a simply connected Hessian manifold of constant Hessian sectional curvature* c (cf. Theorem 7).

The family of multinomial distributions. Let $\mathcal{M}$ be the family of multinomial distributions having the density functions

$$p(x, \theta_1, \cdots, \theta_{n+1})=\frac{l\,!}{x_1!x_2!\cdots x_{n+1}!}\theta_1^{x_1}\theta_2^{x_2}\cdots\theta_{n+1}^{x_{n+1}},$$

where $\sum_{j=1}^{n+1}x_j=l$, $x_j=0, 1, \cdots, l$, and $\sum_{j=1}^{n+1}\theta_j=1$, $0<\theta_j<1$ and n, l are fixed positive integers. Setting

$$\psi(\theta_1, \cdots, \theta_n)=l\sum_{s=1}^{n+1}\theta_s\log\theta_s,$$

the Fisher information metric g is expressed by (cf. [**1**])

$$g\left(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right)=l\left(\delta_{ij}\frac{1}{\theta_i}+\frac{1}{\theta_{n+1}}\right)=\frac{\partial^2\psi}{\partial\theta_i\partial\theta_j},$$

where $i, j=1, \cdots, n$. Put

$$\xi^i=\frac{\partial\psi}{\partial\theta_i}=l\log\frac{\theta_i}{\theta_{n+1}}\qquad(i=1, \cdots, n).$$

Then

$$-\infty<\xi^i<+\infty,$$
$$\theta_i=\frac{e^{\xi^i/l}}{\sum_{r=1}^n e^{\xi^r/l}+1}\qquad(i=1, \cdots, n),$$
$$\theta_{n+1}=\frac{1}{\sum_{r=1}^n e^{\xi^r/l}+1}.$$

Setting

$$\varphi(\xi^1, \cdots, \xi^n)=l\log\left(\sum_{r=1}^n e^{\xi^r/l}+1\right),$$

we have

$$g\left(\frac{\partial}{\partial\xi^i}, \frac{\partial}{\partial\xi^j}\right)=\frac{\partial^2\varphi}{\partial\xi^i\partial\xi^j}.$$

Denote by D a flat affine connection on $\mathcal{M}$ such that $Dd\xi^i=0$. Then, by Theorem 8 we know:

C. *($\mathcal{M}$, D, g) is a simply connected Hessian manifold of constant Hessian sectional curvature* $-1/l$.

4. Characterizations of Hessian manifolds of constant Hessian sectional curvature by affine Chern classes.

In [4] B. Y. Chen and K. Ogiue characterized Kählerian manifolds of constant holomorphic sectional curvature in terms of Chern classes. In this section we give similar characterizations of Hessian manifolds of constant Hessian sectional curvature by affine Chern classes. For affine Chern classes, see the next section 5.

DEFINITION 5. We denote by c_k the k-th affine Chern form for the tangent bundle T over M with fiber metric g (see section 5 Definition 8) and call it the *k-th affine Chern form for* (M, D, g).

Since

$$A_g = 2(\gamma^i{}_{jl}dx^l)\otimes\Big(\frac{\partial}{\partial x^i}\otimes dx^j\Big),$$

$$\begin{aligned} B_g &= 2\Big(\frac{\partial \gamma^i{}_{jl}}{\partial x^k}dx^l\otimes dx^{\bar{k}}\Big)\otimes\Big(\frac{\partial}{\partial x^i}\otimes dx^j\Big) \\ &= 2(S^i{}_{j\bar{k}l}dx^l\otimes dx^{\bar{k}})\otimes\Big(\frac{\partial}{\partial x^i}\otimes dx^j\Big) \\ &= 2S\,, \end{aligned}$$

it follows

$$c_k = f_k(2S) = \frac{(-2)^k}{k!}\delta^{j_1\cdots j_k}_{i_1\cdots i_k}S^{i_1}_{j_1}\wedge\cdots\wedge S^{i_k}_{j_k}\,,$$

where $S^i_j = S^i{}_{j\bar{k}l}dx^l\otimes dx^{\bar{k}}$. Thus we have

$$c_1 = -2S^i{}_{i\bar{k}l}dx^l\otimes dx^{\bar{k}} = -2\beta\,,$$

$$c_2 = 2(S^i_i\wedge S^j_j - S^i_j\wedge S^j_i) = 2(\beta\wedge\beta - S^i_j\wedge S^j_i)\,.$$

We shall now prove the following.

PROPOSITION 9. *We have*

(1) $c_1^2\wedge g^{n-2} = 4(n-2)!\{(Tr\beta)^2 - Tr\beta^2\}v\otimes v$,

(2) $c_2\wedge g^{n-2} = 2(n-2)!\{(Tr\beta)^2 - 2Tr\beta^2 + TrS^2\}v\otimes v$.

PROOF. Let $\theta^1, \cdots, \theta^n$ be local orthonormal frame fields for the cotangent bundle T^* over M. We write

$$g=\sum_i \theta^i\otimes\theta^i ,$$

$$\beta=\sum_{i,j}\beta_{ij}\theta^i\otimes\theta^j .$$

Then we have

$$\begin{aligned}\beta^2\wedge g^{n-2}&=\sum\beta_{i_1j_1}\beta_{i_2j_2}(\theta^{i_1}\wedge\cdots\wedge\theta^{i_n})\otimes(\theta^{j_1}\wedge\cdots\wedge\theta^{j_n})\\&=(\sum_{\sigma\in S_n}\beta_{\sigma(1)\sigma(1)}\beta_{\sigma(2)\sigma(2)})v\otimes v-(\sum_{\sigma\in S_n}\beta_{\sigma(1)\sigma(2)}\beta_{\sigma(2)\sigma(1)})v\otimes v\\&=(n-2)!(\sum_{i\neq j}\beta_{ii}\beta_{jj})v\otimes v-(n-2)!(\sum_{i\neq j}\beta_{ij}^2)v\otimes v\\&=(n-2)!\{(\sum_i\beta_{ii})(\sum_j\beta_{jj})-\sum_{i,j}\beta_{ij}^2\}v\otimes v\\&=(n-2)!\{(Tr\beta)^2-Tr\beta^2\}v\otimes v ,\end{aligned}$$

where v is the volume element for g and S_n is the permutation group of $\{1, 2, \cdots, n\}$. This implies (1). Writing $S^i_j=S^i{}_{jpq}\theta^q\otimes\theta^p$ we have

$$\begin{aligned}&S^i_j\wedge S^j_i\wedge g^{n-2}\\&=S^i{}_{jp_1q_1}S^j{}_{ip_2q_2}(\theta^{q_1}\wedge\theta^{q_2}\wedge\theta^{q_3}\wedge\cdots\wedge\theta^{q_n})\otimes(\theta^{p_1}\wedge\theta^{p_2}\wedge\theta^{p_3}\wedge\cdots\wedge\theta^{p_n})\\&=\sum_{\sigma\in S_n}\{S^i{}_{j\sigma(1)\sigma(1)}S^j{}_{i\sigma(2)\sigma(2)}-S^i{}_{j\sigma(1)\sigma(2)}S^j{}_{i\sigma(2)\sigma(1)}\}v\otimes v\\&=(n-2)!\sum_{p\neq q}\{S^i{}_{jpp}S^j{}_{iqq}-S^i{}_{jpq}S^j{}_{iqp}\}v\otimes v\\&=(n-2)!\{(\sum_p S^i{}_{jpp})(\sum_q S^j{}_{iqq})-\sum_{p,q}S^i{}_{jpq}S^j{}_{iqp}\}v\otimes v\\&=(n-2)!\{\beta^i{}_j\beta^j{}_i-S^i{}_j{}^k{}_lS^j{}_i{}^l{}_k\}v\otimes v\\&=(n-2)!\{Tr\beta^2-TrS^2\}v\otimes v .\end{aligned}$$

Hence we obtain

$$\begin{aligned}c_2\wedge g^{n-2}&=2(\beta^2-S^i_j\wedge S^j_i)\wedge g^{n-2}\\&=2(n-2)!\{(Tr\beta)^2-2Tr\beta^2+TrS^2\}v\otimes v .\end{aligned}$$

■

DEFINITION 6. Let $\omega=fv\otimes v\in\Omega^{n,n}$. According as f is everywhere positive, zero, negative, $\cdots$, ω is said to be positive, zero, negative, $\cdots$, and denoted by $\omega>0$, $\omega=0$, $\omega<0$, $\cdots$.

Suppose that (M, D, g) is an Einstein-Hessian manifold of dimension n. Then we know

$$\beta^i{}_j = \frac{Tr\beta}{n}\delta^i_j ,$$

and so

$$Tr\beta^2 = \beta^i{}_j\beta^j{}_i = \frac{(Tr\beta)^2}{n} .$$

Hence we have

$$c_1^2 \wedge g^{n-2} = 4(n-2)!\left\{\frac{n-1}{n}(Tr\beta)^2\right\}v\otimes v ,$$

$$c_2 \wedge g^{n-2} = 2(n-2)!\left\{\frac{n-2}{n}(Tr\beta)^2 + Tr\mathcal{S}^2\right\}v\otimes v .$$

Using these we obtain

$$\{-nc_1^2+2(n+1)c_2\}\wedge g^{n-2}$$
$$= 4(n+1)(n-2)!\left\{Tr\mathcal{S}^2 - \frac{2}{n(n+1)}(Tr\beta)^2\right\}v\otimes v .$$

By Theorem 5 we have:

THEOREM 10. *Let (M, D, g) be a Hessian-Einstein manifold of dimension n. Then we have*

$$\{-nc_1^2+2(n+1)c_2\}\wedge g^{n-2} \geqq 0 .$$

The equality holds if and only if (M, D, g) is of constant Hessian sectional curvature.

Similarly we know

$$\{-(n-2)c_1^2+2(n-1)c_2\}\wedge g^{n-2} = 4(n-1)!\,Tr\mathcal{S}^2 v\otimes v .$$

Thus we have:

THEOREM 11. *Let (M, D, g) be a Hessian-Einstein manifold of dimension n. Then we have*

$$\{-(n-2)c_1^2+2(n-1)c_2\}\wedge g^{n-2} \geqq 0 .$$

The equality holds if and only if (M, D, g) is of constant Hessian sectional curvature 0.

5. Affine Chern classes for flat vector bundles over flat affine manifolds.

Let F be a flat vector bundle of rank m over a flat affine manifold (M, D). We denote by $\Omega^{p,q}$ (resp. $\Omega^{p,q}(F\otimes F^*)$) the space of all smooth sections of $(\wedge^p T^*)\otimes(\wedge^q T^*)$ (resp. $(\wedge^p T^*)\otimes(\wedge^q T^*)\otimes F\otimes F^*$), where T^* is the cotangent bundle over M and F^* is the dual bundle of F. An element $\varphi\in\Omega^{p,q}(F\otimes F^*)$ is identified with (m, m) matrix (φ_j^i) where $\varphi_j^i \in \Omega^{p,q}$. For $\varphi=(\varphi_j^i)\in\Omega^{p,q}(F\otimes F^*)$, $\psi\in\Omega^{r,s}(F\otimes F^*)$ we define $[\varphi, \psi]\in\Omega^{p+r,q+s}(F\otimes F^*)$ by

$$[\varphi, \psi] = \varphi\wedge\psi-(-1)^{pr+qs}\psi\wedge\varphi\,,$$

that is

$$[\varphi, \psi]_j^i = \sum_k \varphi_k^i\wedge\psi_j^k-(-1)^{pr+qs}\sum_k \psi_k^i\wedge\varphi_j^k\,.$$

LEMMA 1. *Let $f(X_1, \cdots, X_k)$ be a $GL(m, \boldsymbol{R})$-invariant symmetric multilinear form on $\mathfrak{gl}(m, \boldsymbol{R})$. Then we have*

$$\sum_i(-1)^{(p_1+\cdots+p_i)p+(q_1+\cdots+q_i)q}f(\varphi_1, \cdots, [\varphi_i, \psi], \cdots, \varphi_k) = 0\,,$$

for $\varphi_i\in\Omega^{p_i,q_i}(F\otimes F^*)$, $\psi\in\Omega^{p,q}(F\otimes F^*)$.

PROOF. It is enough to prove the formula for $\varphi_i=\omega_iX_i$, $\psi=\omega Y$, where $\omega_i\in\Omega^{p_i,q_i}$, $\omega\in\Omega^{p,q}$, $X_i, Y\in\mathfrak{gl}(m, \boldsymbol{R})$. Since $[\omega_iX_i, \omega Y] = (\omega_i\wedge\omega)X_iY - (-1)^{p_ip+q_iq}\omega\wedge\omega_iYX_i$ we have

$$\begin{aligned}
&\sum_i(-1)^{(p_1+\cdots+p_i)p+(q_1+\cdots+q_i)q}f(\omega_1X_1, \cdots, [\omega_iX_i, \omega Y], \cdots, \omega_kX_k)\\
&=\sum_i(-1)^{(p_1+\cdots+p_i)p+(q_1+\cdots+q_i)q}\omega_1\wedge\cdots\wedge(\omega_i\wedge\omega)\wedge\cdots\wedge\omega_k f(X_1, \cdots, X_iY, \cdots, X_k)\\
&\quad-\sum_i(-1)^{(p_1+\cdots+p_{i-1})p+(q_1+\cdots+q_{i-1})q}\omega_1\wedge\cdots\wedge(\omega\wedge\omega_i)\wedge\cdots\wedge\omega_k f(X_1, \cdots, YX_i, \cdots, X_k)\\
&= (\omega\wedge\omega_1\wedge\cdots\wedge\omega_k)\sum_i\{f(X_1, \cdots, X_iY, \cdots, X_k)-f(X_1, \cdots, YX_i, \cdots, X_k)\}\\
&= (\omega\wedge\omega_1\wedge\cdots\wedge\omega_k)\sum_i f(X_1, \cdots, [X_i, Y], \cdots, X_k)\\
&=0\,.
\end{aligned}$$

■

For $\omega=\sum\omega_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(dx^{i_1}\wedge\cdots\wedge dx^{i_p})\otimes(dx^{\bar{j}_1}\wedge\cdots\wedge dx^{\bar{j}_q})\in\Omega^{p,q}$, we define $\partial\omega\in\Omega^{p+1,q}$, $\bar{\partial}\omega\in\Omega^{p,q+1}$ by

$$\partial\omega = \sum_i \frac{\partial\omega_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}}{\partial x^i}(dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p})\otimes(dx^{\bar{j}_1}\wedge\cdots\wedge dx^{\bar{j}_q}),$$

$$\bar{\partial}\omega = \sum_{\bar{j}} \frac{\partial\omega_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}}{\partial x^{\bar{j}}}(dx^{i_1}\wedge\cdots\wedge dx^{i_p})\otimes(dx^{\bar{j}}\wedge dx^{\bar{j}_1}\wedge\cdots\wedge dx^{\bar{j}_q}),$$

where $\{x^1, \cdots, x^n\}$ is an affine coordinate system for D. Then we have

$$\partial^2 = 0\,,\qquad \bar{\partial}^2 = 0\,.$$

Let h be a fiber metric on F. Using local frame fields such that the transition functions are all constant matrices in $GL(m, \boldsymbol{R})$, we define $A_h\in\Omega^{1,0}(F\otimes F^*)$ by

$$A_h = H(s)^{-1}\partial H(s),$$

where $s=\{s_1, \cdots, s_n\}$ is a local frame of F and $H(s)=[h(s_i, s_j)]$. This definition is independent of the choice of s. We define $B_h \in \Omega^{1,1}(F\otimes F^*)$by

$$B_h = \bar{\partial} A_h .$$

LEMMA 2. *We have*

(1) $\partial A_h = -A_h \wedge A_h$,

(2) $\partial B_h = -[B_h, A_h]$.

PROOF. Since $\partial H(s)^{-1} = -H(s)^{-1}(\partial H(s))H(s)^{-1}$, we have $\partial A_h = \partial H(s)^{-1} \wedge \partial H(s) = -H(s)^{-1}(\partial H(s))H(s)^{-1} \wedge \partial H(s) = -H(s)^{-1}\partial H(s) \wedge H(s)^{-1}\partial H(s) = -A_h \wedge A_h$. Using (1) we obtain $\partial B_h = \partial\bar{\partial} A_h = \bar{\partial}\partial A_h = \bar{\partial}(-A_h \wedge A_h) = -(\bar{\partial} A_h \wedge A_h + A_h \wedge \bar{\partial} A_h) = -(B_h \wedge A_h + A_h \wedge B_h) = -[B_h, A_h]$. ■

REMARK. The identity $\partial B_h = -[B_h, A_h]$ corresponds to the Bianchi identity.

Let $f(X_1, \cdots, X_k)$ be a $GL(m, \boldsymbol{R})$-invariant symmetric multilinear form on $\mathfrak{gl}(m, \boldsymbol{R})$. We set

$$f(B_h) = f(B_h, \cdots, B_h).$$

Then $f(B_h) \in \Omega^{k,k}$ and

LEMMA 3. *We have*

$$\partial f(B_h) = 0, \qquad \bar{\partial} f(B_h) = 0.$$

PROOF. Denoting by $x^i_j(X)$ the (i, j)-component of $X \in \mathfrak{gl}(m, \boldsymbol{R})$ we write

$$f(X_1, \cdots, X_k) = \sum c^{j_1 \cdots j_k}_{i_1 \cdots i_k} x^{i_1}_{j_1}(X_1) \cdots x^{i_k}_{j_k}(X_k).$$

Then $f(B_h) = \sum c^{j_1 \cdots j_k}_{i_1 \cdots i_k} B_{h\,j_1}^{\,i_1} \wedge \cdots \wedge B_{h\,j_k}^{\,i_k}$. Since $\bar{\partial} B_h = \bar{\partial}(\bar{\partial} A_h) = 0$, we have $\bar{\partial} f(B_h) = \sum c^{j_1 \cdots j_k}_{i_1 \cdots i_k} \sum_r (-1)^r B_{h\,j_1}^{\,i_1} \wedge \cdots \wedge \bar{\partial} B_{h\,j_r}^{\,i_r} \wedge \cdots \wedge B_{h\,j_k}^{\,i_k} = 0$. Applying Lemma 1, 2 we have

$$\begin{aligned}
\partial f(B_k) &= \sum c^{j_1 \cdots j_k}_{i_1 \cdots i_k} \sum_r (-1)^r B_{h\,j_1}^{\,i_1} \wedge \cdots \wedge \partial B_{h\,j_r}^{\,i_r} \wedge \cdots \wedge B_{h\,j_k}^{\,i_k} \\
&= \sum_r (-1)^r \sum c^{j_1 \cdots j_k}_{i_1 \cdots i_k} B_{h\,j_1}^{\,i_1} \wedge \cdots \wedge (-[B_h, A_h]^{i_r}_{j_r}) \wedge \cdots \wedge B_{h\,j_k}^{\,i_k} \\
&= -\sum_r (-1)^r f(B_h, \cdots, \overset{r}{[B_h, A_h]}, \cdots, B_h) \\
&= 0.
\end{aligned}$$

■

Let $\{h_t\}$ be a 1-parameter family of fiber metrics on F. We set

$$A_t = A_{h_t}, \qquad B_t = B_{h_t}.$$

Define $L_t \in F\otimes F^*$ by

$$L_t = H_t(s)^{-1}\frac{d}{dt}H_t(s),$$

where $H_t(s)=[h_t(s_i, s_j)]$. We put

$$f^*(B_t\,;\,L_t) = \sum_i f(B_t, \cdots, \overset{i}{L_t}, \cdots, B_t)\in\Omega^{k-1,k-1}.$$

Applying Lemma 1 and $\partial B_t=-[B_t, A_t]$ we have

$$\begin{aligned}\partial f^*(B_t\,;\,L_t) &= \sum_i\sum_{j<i}(-1)^{j-1}f(\cdots, \overset{j}{\partial B_t}, \cdots \overset{i}{L_t}, \cdots)\\ &\quad+\sum_i(-1)^{i-1}f(\cdots, \overset{i}{\partial L_t}, \cdots)+\sum_i\sum_{j>i}(-1)^{j-2}f(\cdots, \overset{i}{L_t}, \cdots\overset{j}{\partial B_t}, \cdots)\\ &= \sum_i\{\sum_{j<i}(-1)^{j}f(\cdots, \overset{j}{[B_t, A_t]}, \cdots, \overset{i}{L_t}, \cdots)\\ &\quad+(-1)^{i-1}f(\cdots, \overset{i}{[L_t, A_t]}, \cdots)+\sum_{j>i}(-1)^{j-1}f(\cdots, \overset{i}{L_t}, \cdots, \overset{j}{[B_t, A_t]}, \cdots)\}\\ &\quad-\sum_i(-1)^{i-1}f(\cdots, \overset{i}{[L_t, A_t]}, \cdots)+\sum_i(-1)^{i-1}f(\cdots, \overset{i}{\partial L_t}, \cdots)\\ &= \sum_i(-1)^{i-1}f(B_t, \cdots, \overset{i}{\partial L_t-[L_t, A_t]}, \cdots, B_t).\end{aligned}$$

We know

$$\partial L_t-[L_t, A_t] = \frac{d}{dt}A_t\,.$$

In fact we have

$$\begin{aligned}\partial L_t &= \partial\Big(H_t(s)^{-1}\frac{d}{dt}H_t(s)\Big)\\ &= -H_t(s)^{-1}(\partial H_t(s))H_t(s)^{-1}\frac{d}{dt}H_t(s)+H_t(s)^{-1}\partial\frac{d}{dt}H_t(s)\\ &= -A_tL_t+\frac{d}{dt}(H_t(s)^{-1}\partial H_t(s))+H_t(s)^{-1}\Big(\frac{d}{dt}H_t(s)\Big)H_t(s)^{-1}\partial H_t(s)\\ &= -A_tL_t+\frac{d}{dt}A_t+L_tA_t\\ &= [L_t, A_t]+\frac{d}{dt}A_t\,.\end{aligned}$$

Thus

$$\partial f^*(B_t\,;\,L_t) = \sum_i(-1)^{i-1}f\Big(B_t, \cdots, \overset{i}{\frac{d}{dt}A_t}, \cdots, B_t\Big).$$

We show

$$\frac{d}{dt}f(B_t)=\bar{\partial}\partial f^*(B_t\,;\,L_t).$$

Indeed, since $\bar{\partial}B_t=0$ we have

$$\begin{aligned}&\bar{\partial}\partial f^*(B_t\,;\,L_t)\\ &=\sum_i(-1)^{i-1}\sum_{j<i}(-1)^{j-1}f\Big(\cdots,\overset{j}{\bar{\partial}B_t},\cdots,\overset{i}{\frac{d}{dt}A_t},\cdots\Big)+\sum_i f\Big(\cdots,\overset{i}{\bar{\partial}\frac{d}{dt}A_t},\cdots\Big)\\ &\quad+\sum_i(-1)^{i-1}\sum_{j>i}(-1)^{j-2}f\Big(\cdots,\overset{i}{\frac{d}{dt}A_t},\cdots,\overset{j}{\bar{\partial}B_t},\cdots\Big)\\ &=\sum_i f\Big(B_t,\cdots,\overset{i}{\frac{d}{dt}B_t},\cdots,B_t\Big)\\ &=\frac{d}{dt}f(B_t,\cdots,B_t).\end{aligned}$$

Therefore we obtain

LEMMA 4. *Under the same notation as above we have*

$$f(B_1)-f(B_0)=\frac{1}{2}\bar{\partial}\partial\int_0^1 f^*(B_t\,;\,L_t)dt\,.$$

DEFINITION 7. We define a cohomology group $\hat{H}^k(M)$ by

$$\hat{H}^k(M)=\{\varphi\in\Omega^{k,k}\,|\,\partial\varphi=0,\ \bar{\partial}\varphi=0\}/\partial\bar{\partial}\Omega^{k-1,k-1}\,.$$

Let $f_k(X)$ be a $GL(m,\boldsymbol{R})$-invariant homogeneous polynomial of degree k defined by

$$\det(I-tX)=\sum_{k=0}^m t^k f_k(X),$$

where I is the unit matrix.

By Lemma 3 we may give

DEFINITION 8. We set $c_k(F,h)=f_k(B_h)$ and call it the *k-th affine Chern form* for a flat vector bundle (F,h) over M.

By Lemma 4 we have

THEOREM 12. *The class in $\hat{H}^k(M)$ represented by $c_k(F,h)$ is independent of the choice of h.*

DEFINITION 9. We denote by $\hat{c}_k(F)$ the class represented by $c_k(F,h)$ and call it the *k-th affine Chern class* for F.

References

[1] S. Amari, Differential-Geometric Methods in Statistics, Lecture Notes in Statist., **28**, Springer, 1985.

[2] C. Atkinson and A. F. Mitchell, Rao's distance measure, Sankhyā Ser. A, **43** (1981), 345-365.

[3] R. Bott and S. S. Chern, Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections, Acta Math., **114** (1965), 71-112.

[4] B. Y. Chen and K. Ogiue, Some characterizations of complex space forms in terms of Chern classes, Quart. J. Math. Oxford, **26** (1975), 459-464.

[5] H. Shima, Homogeneous Hessian manifolds, Ann. Inst. Fourier (Grenoble), **30-3** (1980), 91-128.

[6] H. Shima, Vanishing theorems for compact Hessian manifolds, Ann. Inst. Fourier (Grenoble), **36-3** (1986), 183-205.

[7] H. Shima, Hessian manifolds, Séminaire Gaston Darboux de géométrie et topologie différentielle, Université Montpellier, (1988-1989), pp. 1-48.

[8] H. Shima and K. Yagi, Geometry of compact Hessian manifolds, preprint.

[9] L. T. Skovgaard, A Riemannian geometry of the multivariate normal model, Scand. J. Statist., **11** (1984), 211-223.

[10] K. Yagi, Convexity of a Hessian manifold, Studia Humana et Natumalia, Faculty of Liberal Studies, Kyoto Prefectural University of Medicine, (1985), 1-2.

Hirohiko SHIMA
Department of Mathematics
Faculty of Science
Yamaguchi University
Yoshida, Yamaguchi 753
Japan