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A class of Riemannian manifolds with integrable geodesic flows

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Introduction.

The purpose of the present paper is to introduce a notion of geodesic flows—*simple integrability*. In a word, a simply integrable geodesic flow is a geodesic flow which can be integrated by a single quadratic function. A remarkable property of simple integrability is the duality: To a Riemannian manifold with simply integrable geodesic flow, there corresponds, through certain conformal change of the metric, such another Riemannian manifold. To be more precise, let $(M, \boldsymbol{g})$ be an n-dimensional Riemannian manifold $(n\geqq 2)$. For a tensor field ι of type (1, 1) on M such that the determinant $\sigma_n(\iota)$ is positive on M, we introduce tensor fields ι_p of type (1, 1) as follows:

$$\iota_p=\sigma_n(\iota)^{-(p-1)/(n-1)}\sum_{s=0}^{p-1}(-1)^s\sigma_s(\iota)\iota^{p-s}, \qquad p=1, \cdots, n .$$

Here we view ι as endomorphisms of tangent spaces, ι^{p-s} are the compositions iterated $p-s$ times, $\sigma_0(\iota)=1$, and $\sigma_s(\iota)$ denote the elementary symmetric polynomials of the eigenvalues of ι, of degree s.

DEFINITION. We say that the geodesic flow of $(M, \boldsymbol{g})$ is *simply integrable* if there exists a symmetric tensor field ι of type (1, 1) with $\sigma_n(\iota)>0$ such that the n functions f_p on $T(M)$ defined by $f_p(X)=\boldsymbol{g}(\iota_p(X), X)$, $p=1, \cdots, n$, form a complete involutive set, i.e., are functionally independent and every Poisson bracket $\{f_p, f_q\}$ vanishes. We then call ι the *generating tensor field*. Note that simple integrability implies complete integrability in Liouville's sense, because $f_n(X)=(-1)^{n+1}\boldsymbol{g}(X, X)$.

The Riemannian manifolds with simply integrable geodesic flows have the following characteristic property.

MAIN THEOREM. *Suppose that the geodesic flow of* $(M, \boldsymbol{g})$ *is simply integrable with generating tensor field* ι. *Let* $\tilde{\boldsymbol{g}}=\sigma_n(\iota)^{-1/(n-1)}\boldsymbol{g}$ *be the conformal change of the metric. Then the geodesic flow of* $(M, \tilde{\boldsymbol{g}})$ *is simply integrable, and the generating tensor field is given by* ι^{-1}.

A typical example of Riemannian manifolds with simply integrable geodesic flows is the base space of the Riemannian submersion:

$$(SO(n+1), \boldsymbol{g}) \longrightarrow (SO(n)\backslash SO(n+1), \check{\boldsymbol{g}}),$$

where $\boldsymbol{g}$ is a Mishchenko-Dikii-Manakov-Fomenko metric (a left invariant metric on $SO(n+1)$ giving completely integrable geodesic flows, see § 4). Using the fact that the base space $(SO(n)\backslash SO(n+1), \check{\boldsymbol{g}})$ is conformally equivalent to an ellipsoid, by our main theorem we know

THEOREM 4.2. *The geodesic flow of an ellipsoid with distinct axes is simply integrable.*

Hence, we obtain a geometric proof of the classical

THEOREM (Jacobi). *The geodesic flow of an ellipsoid with distinct axes is completely integrable in the sense of Liouville.*

Indeed, the attempt to understand the relation between $(SO(n)\backslash SO(n+1), \check{\boldsymbol{g}})$ and the ellipsoid from the view point of complete integrability of geodesic flows was the motivation leading to Main Theorem. The proof of Main Theorem (in § 3) is straightforward, i.e., is done through the local expressions for $\{\ \}=0$ (§ 1) and ι_p (§ 2) with respect to suitable orthonormal vector fields. In those expressions, the key fact is that the n quadratic integrals f_p are fiberwise commutative, that is, can be written as the diagonal forms in the common orthonormal frame. As applications of Main Theorem, in § 4 we discuss some Riemannian metrics on S^n. In Appendix, for convenience, after recalling some notions of symplectic geometry, we give a Riemannian expression of Poisson bracket.

We should mention the recent work of Kiyohara [**K**]. He studies Riemannian manifolds whose geodesic flows have Poisson commutative, fiberwise commutative quadratic integrals, and gives the classification of such manifolds. Our Riemannian manifolds belong to Kiyohara's Liouville manifolds. In our class, the quadratic integrals are described explicitly by a single tensor field (generating tensor field).

Through this paper all manifolds and tensor fields are assumed to be of class C^∞ unless otherwise stated.

§ 1. Local expressions for Poisson commutativity.

Let μ, ν be symmetric tensor fields of type (1, 1) on M, and assume that μ, ν are commutative at every point as endomorphisms of tangent spaces. Define f_μ, f_ν: $T(M)\to\boldsymbol{R}$ by $f_\mu(X)=\boldsymbol{g}(\mu(X), X)$, $f_\nu(X)=\boldsymbol{g}(\nu(X), X)$. Now suppose that we can find orthonormal vector fields $X_1, X_2, \cdots, X_n$ defined on an open set U

of M such that each X_i is the common eigenvector of μ, ν belonging to the eigenvalues μ_i, ν_i. Put $d_{ij}^k = \boldsymbol{g}(\nabla_{X_i} X_j, X_k)$. The following lemma gives us the expression for the Poisson commutativity $\{f_\mu, f_\nu\}=0$ (with respect to the symplectic structure on $T(M)$ determined by $\boldsymbol{g}$).

LEMMA 1.1. *A necessary and sufficient condition for* $\{f_\mu, f_\nu\}=0$ *on* $T(U)$ *is that* μ_i, ν_j *and* d_{ij}^k *satisfy*

(C1) $$\mu_i X_i(\nu_i) - \nu_i X_i(\mu_i) = 0 \qquad \text{for } i=1, \cdots, n,$$

(C2) $$\mu_j X_j(\nu_i) - \nu_j X_j(\mu_i) + 2d_{ii}^j(\mu_j \nu_i - \mu_i \nu_j) = 0 \qquad \text{for any } i \neq j, \text{ and}$$

(C3) $$d_{ij}^k(\nu_i(\mu_k - \mu_j) - \mu_i(\nu_k - \nu_j)) + d_{ki}^j(\nu_k(\mu_j - \mu_i) - \mu_k(\nu_j - \nu_i)) + d_{jk}^i(\nu_j(\mu_i - \mu_k) - \mu_j(\nu_i - \nu_k)) = 0 \qquad \text{for any } i \neq j \neq k \neq i.$$

PROOF. Apply Lemma 1 in Appendix.

Since the Hamiltonian vector field sgrad $(1/2)\boldsymbol{g}(X, X)$ is the geodesic flow, taking $\nu=\mathrm{Id}$, we have immediately the following.

COROLLARY 1.2. *A necessary and sufficient condition that the function* f_μ *be the first integral for the geodesic flow of* $(U, \boldsymbol{g}|_U)$ *is that the eigenvalues* μ_i *satisfy*

(C1) $$X_i(\mu_i) = 0 \qquad \text{for } i=1, \cdots, n,$$

(C2) $$X_j(\mu_i) - 2d_{ii}^j(\mu_j - \mu_i) = 0 \qquad \text{for any } i \neq j, \text{ and}$$

(C3) $$d_{ij}^k(\mu_k - \mu_j) + d_{ki}^j(\mu_j - \mu_i) + d_{jk}^i(\mu_i - \mu_k) = 0 \qquad \text{for any } i \neq j \neq k \neq i.$$

PROPOSITION 1.3. *If* f_μ, f_ν *are the first integrals for the geodesic flow of* $(U, \boldsymbol{g}|_U)$ *then the conditions* (C1), (C2) *for* $\{f_\mu, f_\nu\}=0$ *in Lemma* 1.1 *are satisfied.*

PROOF. From (C1), (C2) in Corollary 1.2 we get (C1), (C2) in Lemma 1.1.

§ 2. Basic properties of the derived tensor fields ι_p.

Let ι be a symmetric tensor field of type (1, 1) on M. Let $\sigma_s(\iota)$ denote the elementary symmetric polynomials of the eigenvalues of ι, of degree s, and put $\sigma_0(\iota)=1$. Assume that the determinant $\sigma_n(\iota)>0$. Denote by ι^{-1} the tensor field defined by $(\iota^{-1})_x = (\iota_x)^{-1} : T_x(M) \to T_x(M)$. The *derived tensor fields* ι_p, $p=1, \cdots, n$, of ι are defined to be the symmetric tensor fields of type (1, 1)

$$\iota_p = \sigma_n(\iota)^{-(p-1)/(n-1)} \sum_{s=0}^{p-1} (-1)^s \sigma_s(\iota) \iota^{p-s}.$$

We define $\iota_0=0$ for convenience. Note that $\iota_1=\iota$, $\iota_n=(-1)^{n+1}$ Id, where Id denotes the identity tensor field.

LEMMA 2.1. *For each $p=1, \cdots, n$, the derived tensor field $(\iota^{-1})_p$ of ι^{-1} has the following two expressions*:

$$(\iota^{-1})_p = (-1)^{n+1}\iota_{n+1-p}\circ\iota^{-1},$$

$$(\iota^{-1})_p = (-1)^{n+1}(e^{2u}\iota_{n-p}+(-1)^{n-p}e^{2(n-p)u}\sigma_{n-p}(\iota)\,\mathrm{Id}),$$

where $e^{2u}=(\sigma_n(\iota))^{-1/(n-1)}$.

PROOF. By definition we have

$$(\iota^{-1})_p = (\sigma_n(\iota^{-1}))^{-(p-1)/(n-1)}\sum_{s=0}^{p-1}(-1)^s\sigma_s(\iota^{-1})\iota^{-(p-s)}.$$

Using the fact $\sigma_s(\iota^{-1})=\sigma_{n-s}(\iota)/\sigma_n(\iota)$, and replacing s by $t=n-s$ we have

$$(\iota^{-1})_p = (-1)^n\sigma_n(\iota)^{(p-n)/(n-1)}\sum_{t=n}^{n-p+1}(-1)^t\sigma_t(\iota)\iota^{n-t-p}.$$

Hence by the Cayley-Hamilton theorem we obtain

$$(\iota^{-1})_p = (-1)^{n+1}\sigma_n(\iota)^{(p-n)/(n-1)}\sum_{t=0}^{n-p}(-1)^t\sigma_t(\iota)\iota^{n-p-t}$$

$$= (-1)^{n+1}\sigma_n(\iota)^{(p-n)/(n-1)}\left(\sum_{t=0}^{n-p-1}(-1)^t\sigma_t(\iota)\iota^{n-p-t}+(-1)^{n-p}\sigma_{n-p}(\iota)\mathrm{Id}\right).$$

Thus the definitions of ι_{n-p}, ι_{n+1-p} yield our desired formulas.

PROPOSITION 2.2. *Let $X_1, \cdots, X_n$ be orthonormal vector fields on an open set U of M such that $\iota(X_i)=\varepsilon_i X_i$ for each $i=1, \cdots, n$, where $\varepsilon_i: U\to\boldsymbol{R}$. Then each X_i is simultaneously the eigenvector field for ι_p, $(\iota^{-1})_p$, $p=1, \cdots, n$, and if we denote by $\varepsilon_{p,i}$, $\tilde{\varepsilon}_{p,i}$ the eigenvalues of ι_p, $(\iota^{-1})_p$ to which X_i belongs, respectively, then we have two expressions for $\tilde{\varepsilon}_{p,i}$;*

(E1) $$\tilde{\varepsilon}_{p,i} = (-1)^{n+1}\varepsilon_{n+1-p,i}\varepsilon_i^{-1},$$

(E2) $$\tilde{\varepsilon}_{p,i} = (-1)^{n+1}(e^{2u}\varepsilon_{n-p,i}+(-1)^{n-p}e^{2(n-p)u}\sigma_{n-p}(\iota)),$$

and hence

(E3) $$\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,j} = (-1)^{n+1}e^{2u}(\varepsilon_{n-p,i}-\varepsilon_{n-p,j}),$$

where $e^{2u}=(\sigma_n(\iota))^{-1/(n-1)}$.

PROOF. Immediate from Lemma 2.1.

§ 3. Proof of Main Theorem.

We prepare two lemmas. The first allows us to apply assertions in § 1, and the second tells us how d_{ij}^k's are transformed by a conformal change of the metric.

LEMMA 3.1. *Let a symmetric tensor field μ of type* (1, 1) *be given on M. Then there exists an open dense subset M_0 of M such that any point of M_0 has a neighborhood on which μ is diagonalized.*

PROOF. Elementary.

LEMMA 3.2. *Let $X_1, X_2, \cdots, X_n$ be the orthonormal vector fields defined on an open set U in M. Let $\tilde{\boldsymbol{g}}=e^{2u}\boldsymbol{g}$ be a conformal change of $\boldsymbol{g}$, and put $\tilde{X}_i=e^{-u}X_i$, $i=1, \cdots, n$. Let $d_{ij}^k=\boldsymbol{g}(\nabla_{X_i}X_j, X_k)$, and $\tilde{d}_{ij}^k=\tilde{\boldsymbol{g}}(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j, \tilde{X}_k)$, where $\tilde{\nabla}$ denotes the covariant derivative with respect to $\tilde{\boldsymbol{g}}$. Then we have*

$$\tilde{d}_{ii}^j = e^{-u}(d_{ii}^j - X_j(u)) \qquad \textit{for any } i\neq j, \textit{ and}$$

$$\tilde{d}_{ij}^k = e^{-u}d_{ij}^k \qquad \textit{for any } i\neq j\neq k\neq i.$$

PROOF. This is a direct consequence of the formula

$$\boldsymbol{g}(\tilde{\nabla}_X Y, Z) = \boldsymbol{g}(\nabla_X Y, Z)+X(u)\boldsymbol{g}(Y, Z)+Y(u)\boldsymbol{g}(X, Z)-\boldsymbol{g}(X, Y)Z(u),$$

for $X, Y, Z\in T(M)$ (see [**Be**]).

The heart of the proof of Main theorem is the following proposition. To state this, let ι be a symmetric tensor field of type (1, 1) with $\sigma_n(\iota)>0$, and ι_p the derived tensor fields of ι. Define $f_p: T(M)\to\boldsymbol{R}$ by $f_p(X)=\boldsymbol{g}(\iota_p(X), X)$, $p=1, \cdots, n$. Put $\tilde{\boldsymbol{g}}=\sigma_n(\iota)^{-1/(n-1)}\boldsymbol{g}$. Denote by $\{\ \}^{\sim}$ the Poisson bracket with respect to the symplectic structure on $T(M)$ determined by $\tilde{\boldsymbol{g}}$. Define $\tilde{f}_p: T(M)\to\boldsymbol{R}$ by $\tilde{f}_p(X)=\tilde{\boldsymbol{g}}((\iota^{-1})_p(X), X)$, $p=1, \cdots, n$, where $(\iota^{-1})_p$ denote the derived tensor fields of ι^{-1}.

PROPOSITION 3.3. *Fix $p, q=1, \cdots, n$. If*

$$\{f_1, f_n\} = \{f_{n+1-p}, f_n\} = \{f_{n-p}, f_n\} = 0,$$

then we have

$$\{\tilde{f}_p, \tilde{f}_n\}^{\sim} = 0.$$

If in addition

$$\{f_{n+1-q}, f_n\} = \{f_{n-q}, f_n\} = \{f_{n-p}, f_{n-q}\} = 0,$$

then we have

$$\{\tilde{f}_p, \tilde{f}_q\}^{\sim} = 0.$$

PROOF. Let $X_1, \cdots, X_n$ be orthonormal vector fields defined on an open set

U such that $\iota(X_i)=\varepsilon_i X_i$ for each $i=1, \cdots, n$, where $\varepsilon_i: U\to \boldsymbol{R}$. By Lemma 3.1 it suffices to prove that our formulas hold on $T(U)$. For this purpose, we denote by $\varepsilon_{r,i}$, $\tilde{\varepsilon}_{r,i}$ the eigenvalues of ι_r, $(\iota^{-1})_r$ to which X_i belongs respectively, as in Proposition 2.2. Moreover, put $\tilde{X}_i=e^{-u}X_i$, $i=1, \cdots, n$, where $e^{2u}=(\sigma_n(\iota))^{-1/(n-1)}$, and let $\tilde{d}^k_{ij}=\tilde{\boldsymbol{g}}(\tilde{\nabla}_{\tilde{X}_i}\tilde{X}_j, \tilde{X}_k)$ as in Lemma 3.2. Now, by Corollary 1.2, in order to prove the former part of our proposition it suffices to verify

$$(\tilde{\mathrm{C}}1) \qquad \tilde{X}_i(\tilde{\varepsilon}_{p,i})=0 \quad \text{for } i=1, \cdots, n,$$

$$(\tilde{\mathrm{C}}2) \qquad \tilde{X}_j(\tilde{\varepsilon}_{p,i})-2\tilde{d}^j_{ii}(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})=0 \quad \text{for any } i\neq j, \text{ and}$$

$$(\tilde{\mathrm{C}}3) \qquad \tilde{d}^k_{ij}(\tilde{\varepsilon}_{p,k}-\tilde{\varepsilon}_{p,j})+\tilde{d}^j_{ki}(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})+\tilde{d}^i_{jk}(\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,k})=0 \quad \text{for any } i\neq j\neq k\neq i.$$

First we shall verify ($\tilde{\mathrm{C}}1$). From the assumption $\{f_1, f_n\}=\{f_{n+1-p}, f_n\}=0$ and (C1) in Corollary 1.2 it follows that $X_i(\varepsilon_i)=X_i(\varepsilon_{n-1-p,i})=0$. Hence using (E1) of Proposition 2.2 we get $\tilde{X}_i(\tilde{\varepsilon}_{p,i})=0$. Next, to prove ($\tilde{\mathrm{C}}2$) it suffices to prove $\tilde{X}_j\log(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})=-2\tilde{d}^j_{ii}$, which can be written as $\tilde{X}_j\log((-1)^{n+1}e^{2u}(\varepsilon_{n-p,j}-\varepsilon_{n-p,i}))=-2\tilde{d}^j_{ii}$ by (E3) of Proposition 2.2. On the other hand, from (C1), (C2) for $\{f_{n-p}, f_n\}=0$ in Corollary 1.2 we have $X_j\log(\varepsilon_{n-p,j}-\varepsilon_{n-p,i})=-2d^j_{ii}$. Consequently, the relation $\tilde{d}^j_{ii}=e^{-u}(d^j_{ii}-X_j(u))$ gives us ($\tilde{\mathrm{C}}2$). Similarly, using (E3) of Proposition 2.2, (C3) for $\{f_{n-p}, f_n\}=0$ in Corollary 1.2 and the relation $\tilde{d}^k_{ij}=e^{-u}d^k_{ij}$ in Lemma 3.2, we obtain ($\tilde{\mathrm{C}}3$). Thus the former part is proved.

We proceed to the latter part. We already know that $\{\tilde{f}_p, \tilde{f}_n\}^\sim=\{\tilde{f}_q, \tilde{f}_n\}^\sim=0$. Hence by Lemma 1.1 and Proposition 1.3 it suffices to prove:

$$(\tilde{\mathrm{C}}3) \qquad \tilde{d}^k_{ij}(\tilde{\varepsilon}_{q,i}(\tilde{\varepsilon}_{p,k}-\tilde{\varepsilon}_{p,j})-\tilde{\varepsilon}_{p,i}(\tilde{\varepsilon}_{q,k}-\tilde{\varepsilon}_{q,j}))+\tilde{d}^j_{ki}(\tilde{\varepsilon}_{q,k}(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})-\tilde{\varepsilon}_{p,k}(\tilde{\varepsilon}_{q,j}-\tilde{\varepsilon}_{q,i}))$$
$$+\tilde{d}^i_{jk}(\tilde{\varepsilon}_{q,j}(\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,k})-\tilde{\varepsilon}_{p,j}(\tilde{\varepsilon}_{q,i}-\tilde{\varepsilon}_{q,k}))=0 \quad \text{for any } i\neq j\neq k\neq i.$$

Denote by L the left hand side of ($\tilde{\mathrm{C}}3$), and put $\phi_p=(-1)^{n-p}e^{2(n-p)u}\sigma_{n-p}(\iota)$ for simplicity of notation. Then by (E2) in Proposition 2.2 we have $\tilde{\varepsilon}_{p,i}=(-1)^{n+1}\cdot(e^{2u}\varepsilon_{n-p,i}+\phi_p)$. Using the relations $\tilde{d}^k_{ij}=e^{-u}d^k_{ij}$, we see that

$$\begin{aligned}(-1)^{n+1}e^u L = {} & d^k_{ij}((e^{2u}\varepsilon_{n-q,i}+\phi_q)(\tilde{\varepsilon}_{p,k}-\tilde{\varepsilon}_{p,j})-(e^{2u}\varepsilon_{n-p,i}+\phi_p)(\tilde{\varepsilon}_{q,k}-\tilde{\varepsilon}_{q,j}))\\ & +d^j_{ki}((e^{2u}\varepsilon_{n-q,k}+\phi_q)(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})-(e^{2u}\varepsilon_{n-p,k}+\phi_p)(\tilde{\varepsilon}_{q,j}-\tilde{\varepsilon}_{q,i}))\\ & +d^i_{jk}((e^{2u}\varepsilon_{n-q,j}+\phi_q)(\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,k})-(e^{2u}\varepsilon_{n-p,j}+\phi_p)(\tilde{\varepsilon}_{q,i}-\tilde{\varepsilon}_{q,k})).\end{aligned}$$

This can be written as

$$(-1)^{n+1}e^u L = e^{2u}S_1+\phi_q S_2-\phi_p S_3,$$

where

$$S_1 = d^k_{ij}(\varepsilon_{n-q,i}(\tilde{\varepsilon}_{p,k}-\tilde{\varepsilon}_{p,j})-\varepsilon_{n-p,i}(\tilde{\varepsilon}_{q,k}-\tilde{\varepsilon}_{q,j}))$$
$$+d^j_{ki}(\varepsilon_{n-q,k}(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})-\varepsilon_{n-p,k}(\tilde{\varepsilon}_{q,j}-\tilde{\varepsilon}_{q,i}))$$
$$+d^i_{jk}(\varepsilon_{n-q,j}(\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,k})-\varepsilon_{n-p,j}(\tilde{\varepsilon}_{q,i}-\tilde{\varepsilon}_{q,k})),$$
$$S_2 = d^k_{ij}(\tilde{\varepsilon}_{p,k}-\tilde{\varepsilon}_{p,j})+d^j_{ki}(\tilde{\varepsilon}_{p,j}-\tilde{\varepsilon}_{p,i})+d^i_{jk}(\tilde{\varepsilon}_{p,i}-\tilde{\varepsilon}_{p,k}),$$
$$S_3 = d^k_{ij}(\tilde{\varepsilon}_{q,k}-\tilde{\varepsilon}_{q,j})+d^j_{ki}(\tilde{\varepsilon}_{q,j}-\tilde{\varepsilon}_{q,i})+d^i_{jk}(\tilde{\varepsilon}_{q,i}-\tilde{\varepsilon}_{q,k}).$$

First we contend that $S_1=0$. In fact, by (E3) of Proposition 2.2 we have

$$(-1)^{n+1}e^{-2u}S_1 = d^k_{ij}(\varepsilon_{n-q,i}(\varepsilon_{n-p,k}-\varepsilon_{n-p,j})-\varepsilon_{n-p,i}(\varepsilon_{n-q,k}-\varepsilon_{n-q,j}))$$
$$+d^j_{ki}(\varepsilon_{n-q,k}(\varepsilon_{n-p,j}-\varepsilon_{n-p,i})-\varepsilon_{n-p,k}(\varepsilon_{n-q,j}-\varepsilon_{n-q,i}))$$
$$+d^i_{jk}(\varepsilon_{n-q,j}(\varepsilon_{n-p,i}-\varepsilon_{n-p,k})-\varepsilon_{n-p,j}(\varepsilon_{n-q,i}-\varepsilon_{n-q,k})).$$

Then (C3) in Lemma 1.1 for the assumption $\{f_{n-p}, f_{n-q}\}=0$ shows that each term of the last formula vanishes. Hence $S_1=0$. Next we contend $S_2=0$. Indeed, the relations $\tilde{d}^k_{ij}=e^{-u}d^k_{ij}$ and the condition (C3) for $\{\tilde{f}_p, \tilde{f}_n\}^{\sim}=0$ in Corollary 1.2 give $S_2=0$. Similarly $S_3=0$. This completes the proof of $L=0$. Proposition 3.3 is proved.

PROOF OF MAIN THEOREM. Suppose that the geodesic flow of $(M, \boldsymbol{g})$ is simply integrable with generating tensor field ι. We shall prove that the geodesic flow of $(M, \tilde{\boldsymbol{g}})$, $\tilde{\boldsymbol{g}}=\sigma_n(\iota)^{-1/(n-1)}\boldsymbol{g}$, is simply integrable with the generating tensor field ι^{-1}. From Proposition 3.3 it follows that the n functions $\tilde{f}_p: T(M)\to\boldsymbol{R}$, $\tilde{f}_p(X)=\tilde{\boldsymbol{g}}((\iota^{-1})_p(X), X)$, are Poisson commutative (with respect to the symplectic structure determined by $\tilde{\boldsymbol{g}}$). It remains to prove the functional independence of $\tilde{f}_p$, $p=1, \cdots, n$. In other words, we have to prove that the set consisting of $X\in T(M)$ such that the rank of

$$(\tilde{F}_*)_X : T_X(T(M)) \longrightarrow T_{\tilde{F}(X)}(\boldsymbol{R}^n)$$

is less than n has no interior point, where $\tilde{F}_*$ denotes the induced mapping of

$$\tilde{F}=(\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_n): T(M) \longrightarrow \boldsymbol{R}^n.$$

Owing to Lemma 3.1, it suffices to prove that any open set U where ι is diagonalized, the set of $X\in T(U)$ such that the rank of

$$(\tilde{F}_*)_X : T_X(T(U)) \longrightarrow T_{\tilde{F}(X)}(\boldsymbol{R}^n)$$

is less than n has no interior point. As before, let $X_1, X_2, \cdots, X_n$ be the orthonormal vector fields (with respect to $\boldsymbol{g}$) on U such that $\iota(X_i)=\varepsilon_i X_i$ with $\varepsilon_i: U\to\boldsymbol{R}$, $i=1, \cdots, n$. We may assume that $\varepsilon_1, \cdots, \varepsilon_m>0$, $\varepsilon_{m+1}, \cdots, \varepsilon_n<0$ for some m. Let $\varepsilon_{p,i}$ denote the eigenvalues of ι_p. Then by (E1) of Proposition

2.2, we have

$$\tilde{f}_p(X)=(-1)^{n+1}\sigma_n(\iota)^{-1/(n-1)}\sum_{i=1}^{n}\varepsilon_{n+1-p,i}\varepsilon_i^{-1}z_i^2 \qquad \text{for } X=\sum_{i=1}^{n} z_iX_i\in T(U).$$

Define $g_p: U\times \boldsymbol{R}^n\to\boldsymbol{R}$ by

$$g_p(x, z_1, \cdots, z_n)=\sum_{i=1}^{n}\varepsilon_{p,i}(x)\frac{|\varepsilon_i(x)|}{\varepsilon_i(x)}z_i{}^2.$$

We contend that $g_1, g_2, \cdots, g_n$ are functionally independent. This will prove the functional independence of $\tilde{f}_1, \cdots, \tilde{f}_n: T(U)\to\boldsymbol{R}$, because we have the relation

$$\rho\circ\tilde{F}\circ\phi(X)=(-1)^{n+1}(g_1(x, z_1, \cdots, z_n), g_2(x, z_1, \cdots, z_n), \cdots, g_n(x, z_1, \cdots, z_n))$$

for $X=\sum_{i=1}^{n} z_iX_i\in T_x(U)$, where $\phi: T(U)\to T(U)$, $\rho: \boldsymbol{R}^n\to\boldsymbol{R}^n$ are diffeomorphisms defined by

$$\phi\Big(\sum_{i=1}^{n} z_iX_i\Big)=\sigma_n(\iota)^{1/2(n-1)}\sum_{i=1}^{n}\sqrt{|\varepsilon_i|}z_iX_i, \quad \rho(w_1, w_2, \cdots, w_n)=(w_n, w_{n-1}, \cdots, w_1).$$

Note that $f_p: T(M)\to\boldsymbol{R}$, $f_p(X)=\boldsymbol{g}(\iota_p(X), X)$, can be written on $U\times\boldsymbol{R}^n\cong T(U)$ as

$$f_p(x, z_1, \cdots, z_n)=\sum_{i=1}^{n}\varepsilon_{p,i}(x)z_i^2.$$

Hence

$$g_p(x, z)=f_p(x, z^*), \qquad \text{where } z=(z_1, \cdots, z_n),\ z^*=(z_1, \cdots, z_m, iz_{m+1}, \cdots, iz_n).$$

Then we see that any minor of the Jacobian matrix $\partial(g_1, \cdots, g_n)/\partial(x_1, \cdots, x_n, z_1, \cdots, z_n)$ is equal to $(i)^r$ times the corresponding minor of $\partial(f_1, \cdots, f_n)/\partial(x_1, \cdots, x_n, z_1, \cdots, z_n)$ at (x, z^*) for some r. Therefore the functional independence of $f_1, \cdots, f_n$ implies the functional independence of $g_1, \cdots, g_n$. This completes the proof of Main Theorem.

§4. Examples.

We shall give a family of Riemannian metrics on the n-sphere S^n whose geodesic flows are simply integrable. We need to prepare some notations. Let $SO(n+1)$ be the special orthogonal group of degree $n+1$, and for $1\leqq i\neq j\leqq n+1$, let e_{ij} denote the $(n+1)\times(n+1)$ matrix whose (i, j) component is 1, (j, i) component is -1, and others are 0. Let e_{ii} denote the zero matrix, for convenience. As usual, we regard e_{ij} as the tangent vectors of $SO(n+1)$ at the unit element. Let $\boldsymbol{g}_0$ be the bi-invariant metric on $SO(n+1)$ defined so that e_{ij}, $1\leqq i<j\leqq n+1$, are orthonormal. In order to define a left invariant metric on $SO(n+1)$, let A be an $(n+1)\times(n+1)$ diagonal matrix with positive and distinct diagonal elements $a_1, a_2, \cdots, a_{n+1}$, and put $E_{ij}=(1/\sqrt{a_ia_j})e_{ij}$. The Mishchenko-Dikii-Manakov-Fomenko metric (MDMF metric) on $SO(n+1)$ is defined to be the left

invariant metric $\boldsymbol{g}$ such that the left translations of E_{ij}, $1 \leqq i < j \leqq n+1$, are orthonormal vector fields on $SO(n+1)$. It is useful to introduce a symmetric linear mapping $\Lambda: so(n+1) \to so(n+1)$, $\Lambda(e_{ij}) = a_i a_j e_{ij}$, of the Lie algebra of $SO(n+1)$. We regard Λ as a left invariant symmetric tensor field of type (1, 1) on $SO(n+1)$. Clearly, $\boldsymbol{g}(*, *) = \boldsymbol{g}_0(\Lambda(*), *)$. Now, let M^n denote the space $SO(n) \backslash SO(n+1)$ of right cosets. Let $\check{g}$ be the Riemannian metric on M^n defined by the requirement that the natural mapping $\pi: SO(n+1) \to M^n$ is a Riemannian submersion of $(SO(n+1), \boldsymbol{g})$ (see [**O**]). Let $\check{\Lambda}$ be the tensor field of type (1, 1) on M defined by

$$\check{\Lambda}(X) = \mathrm{proj}_{\mathscr{H}}(\Lambda(\bar{X})) \qquad \text{for } X \in T(M),$$

where $\mathrm{proj}_{\mathscr{H}}$ denotes the projection to the horizontal subspaces, which are identified with the tangent spaces of M, and $\bar{X}$ denotes any horizontal lift of X. Obviously, the tensor field $\check{\Lambda}$ is well defined and symmetric. We are ready to state our theorem.

THEOREM 4.1. *The geodesic flow of* $(M^n, \check{\boldsymbol{g}})$ *is simply integrable with the generating tensor field* $\check{\Lambda}$.

Using the fact that $(M^n, \sigma_n(\check{\Lambda})^{-1/(n-1)}\check{\boldsymbol{g}})$ is isometric to the ellipsoid E^n (Proposition 4.7) and applying our main theorem, we get at once the following.

THEOREM 4.2. *The geodesic flow of the ellipsoid* E^n *is simply integrable, and hence in particular completely integrable in Liouville's sense.*

The complete integrability of the geodesic flow of E^n is classical since Jacobi, and another proof is known by [**Mo**].

In order to prove Theorem 4.1, we first recall Mishchenko-Dikii-Manakov-Fomenko's theorem. For $p=1, \cdots, n$, let $\Lambda_{(p)}$ be the left invariant symmetric tensor field of type (1, 1) on $SO(n+1)$ defined by

$$\Lambda_{(p)}(e_{ij}) = \frac{a_i a_j (a_i{}^p - a_j{}^p)}{a_i - a_j} e_{ij}, \qquad 1 \leqq i < j \leqq n+1.$$

Clearly, $\Lambda_{(1)} = \Lambda$. Define $f_{(p)}: T(SO(n+1)) \to \boldsymbol{R}$ by

$$f_{(p)}(Z) = \boldsymbol{g}(\Lambda_{(p)}(Z), Z), \qquad Z \in T(SO(n+1)).$$

THEOREM (Mishchenko-Dikii-Manakov-Fomenko). *The* n *functions* $f_{(p)}$ *on the tangent bundle* $T(SO(n+1))$ *of the Riemannian manifold* $(SO(n+1), \boldsymbol{g})$, $p = 1, \cdots, n$, *are Poisson commutative.*

REMARK. Among $\Lambda_{(p)}$'s and the identity tensor field Id there is the following relation:

$$(-1)^{n+1}\sigma_{n+1}(A)\,\mathrm{Id}$$
$$= \Lambda_{(n)}-\sigma_1(A)\Lambda_{(n-1)}+\cdots+(-1)^s\sigma_s(A)\Lambda_{(n-s)}+\cdots+(-1)^{n-1}\sigma_{n-1}(A)\Lambda_{(1)},$$

where $\sigma_s(A)$ denote the elementary symmetric polynomials of $a_1, a_2, \cdots, a_{n+1}$. Hence, we have

$$(-1)^{n+1}\sigma_{n+1}(A)\|\ \|^2$$
$$= f_{(n)}-\sigma_1(A)f_{(n-1)}+\cdots+(-1)^s\sigma_s(A)f_{(n-s)}+\cdots+(-1)^{n-1}\sigma_{n-1}(A)f_{(1)},$$

where $\|\ \|^2$ denotes the function on $T(SO(n+1))$ defined by $\|Z\|^2=\boldsymbol{g}(Z, Z)$

The derived tensor fields $(\check{\Lambda})_p$ of $\check{\Lambda}$ are related to $\Lambda_{(p)}$ as follows.

PROPOSITION 4.3. *We have*

$$(\check{\Lambda})_p(X)=\sigma_{n+1}(A)^{-(p-1)/(n-1)}\sum_{s=1}^{p}(-1)^{p-s}\sigma_{p-s}(A)\,\mathrm{proj}_{\mathscr{H}}(\Lambda_{(s)}(\bar{X})) \quad \textit{for } X\in T(M),$$

where $\mathrm{proj}_{\mathscr{H}}$, $\bar{X}$ *are as in the definition of* $\check{\Lambda}$.

The proof will be given later in this section.

PROPOSITION 4.4. *Let W be a Riemannian manifold, and suppose that a compact Lie group G acts isometrically on W from the left. Suppose that for each point x of W, the mapping $G\ni g\mapsto gx\in W$ is an imbedding. Let $\pi: W\to G\backslash W$ be the Riemannian submersion to the quotient space. Let f, g be functions on $T(W)$ which are invariant by the induced action of G on $T(W)$. Denote by $\check{f}$, $\check{g}$ the functions on $T(G\backslash W)$ defined naturally by f, g, respectively. Then the flow generated by the Hamiltonian vector field* sgrad f *keeps the horizontal subspaces of $T(W)$ invariant. Moreover,* sgrad f *is invariant under the induced action of G on $T(W)$, and hence gives a vector field on $T(M)$, which coincides with* sgrad $\check{f}$. *Hence, if $\{f, g\}=0$, then $\{\check{f}, \check{g}\}=0$.*

PROOF. We shall prove only the invariance of the horizontal subspaces and the coincidence sgrad $f|$horizontal subspaces$=$sgrad $\check{f}$, since the others are obvious. Take orthonormal vector fields $X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+n}$ defined on an open subset U of W so that $X_1, \cdots, X_p$ are vertical, and $X_1, \cdots, X_{p+n}$ are invariant under the action of G, where $p=\dim G$, $p+n=\dim W$. Then for any integral curve $c: (-\varepsilon, \varepsilon)\to T(U)$ of sgrad f we have

$$\frac{d}{dt}\boldsymbol{g}(c(t), X_j)=\sum_{k,l=1}^{p+n}\frac{\partial f}{\partial p_k}c_{kj}^l\boldsymbol{g}(c(t), X_l)-\bar{X}_j(f), \qquad j=1, \cdots, p+n,$$

with the notation in the proof of Lemma 1 in Appendix. Since $X_1, \cdots, X_{p+n}$ are chosen G-invariant, we see that $c_{kj}^l=0$ if $p+1\leqq l\leqq p+n$, $1\leqq j\leqq p$. Moreover $\bar{X}_j(f)=0$, $j=1, \cdots, p$. Then by the uniqueness theorem of differential equations we conclude that if $c(0)$ is horizontal, then $c(t)$ is horizontal for any t. Thus the invariance of the horizontal subspaces under the flow is proved. The later part is now obvious, because if $c(t)$ is horizontal, then the differential equation for $c(t)$ is

$$\frac{d}{dt}\boldsymbol{g}(c(t), X_j) = \sum_{k,l=p+1}^{p+n} \frac{\partial f}{\partial p_k} c_{kj}^l \boldsymbol{g}(c(t), X_l) - \bar{X}_j(f), \qquad j=p+1, \cdots, p+n,$$

which shows that $c(t)$ satisfies the differential equation of sgrad $\check{f}$. Proposition 4.4 is proved.

PROOF OF THEOREM 4.1. We have to prove that the n functions f_p on $T(M)$, $f_p(X)=\check{\boldsymbol{g}}((\check{\Lambda})_p(X), X)$, $p=1, \cdots, n$, are Poisson commutative. From Proposition 4.3 we see that each f_p is expressed as a linear combination of $f_{(1)}$, $f_{(2)}, \cdots, f_{(n)}$, viewed as functions on $T(M)$, with constant coefficients. Then by MDMF's theorem and Proposition 4.4 we conclude that f_p, $p=1, \cdots, n$, are Poisson commutative.

The functional independence of f_p, $p=1, \cdots, n$, is verified as follows. Let $o=\pi(I)\in M$ be the image of the unit element I of $SO(n+1)$. We contend that the restrictions $f_p|=f_p|T_o(M)$ are functionally independent. This will prove the functional independence of f_p, because f_p are analytic functions on $T(M)$. Note that $\pi_{*I}(E_{i\,n+1})$, $1\leqq i\leqq n$, constitute the orthonormal basis $T_o(M)$, because $E_{i\,n+1}$, $1\leqq i\leqq n$ are the orthonormal basis of the horizontal subspace of $T_I(SO(n+1))$. Moreover $\pi_{*I}(E_{i\,n+1})$ are the eigenvectors of $\check{\Lambda}_o\colon T_o(M)\to T_o(M)$ with eigenvalues $a_i a_{n+1}$, respectively. Put $\alpha_i=a_i a_{n+1}$. Then, identifying $T_o(M)$ with $\boldsymbol{R}^n$ by means of the basis $\pi_{*I}(E_{i\,n+1})$, we have $f_p|(z_1, \cdots, z_n)=\sum_{i=1}^n \alpha_{p,i} z_i^2$, where $\alpha_{p,i}$ denote the eigenvalues of $(\check{\Lambda})_{p\,o}\colon T_o(M)\to T_o(M)$. Hence, the determinant of the Jacobian matrix $\partial(f_1|, \cdots, f_n|)/\partial(z_1, \cdots, z_n)$ is equal to $2^n z_1 z_2 \cdots z_n$ times the determinant of the matrix $(\alpha_{p,i}\,;\, 1\leqq p,\, i\leqq n)$. Using the fact that $\alpha_{p,i}$ are given by

$$\alpha_{p,i} = c_n^{-(p-1)/(n-1)} \sum_{s=0}^{p-1} (-1)^s c_s \alpha_i^{p-s}$$

with positive constants $c_1, \cdots, c_n$, we observe that $\det(\alpha_{p,i})$ is equal to nonzero constant times the Vandermonde determinant $\det(\alpha_i^p)$, which is nonzero because of the assumption $a_i\neq a_j$ $(i\neq j)$. Thus the functional independence of $f_1|, \cdots, f_n|$ is proved.

It remains to prove Proposition 4.3. We prepare two lemmas. First, to express the elementary symmetric polynomials $\sigma_s(\check{\Lambda})$ of eigenvalues of $\check{\Lambda}$, we introduce functions χ_p on $SO(n+1)$ defined by

$$\chi_p(x) = \sum_{i=1}^{n+1} a_i{}^p x_{n+1\,i}{}^2, \qquad x=(x_{ij})\in SO(n+1).$$

Clearly, $\chi_0=1$. Moreover, the functions χ_p may be considered as functions on

$M^n=SO(n)\backslash SO(n+1)$, and satisfy the relations

$$\chi_{n+1}-\sigma_1(A)\chi_n+\cdots+(-1)^i\sigma_i(A)\chi_{n+1-i}+\cdots+(-1)^{n+1}\sigma_{n+1}(A)=0,$$
$$\chi_n-\sigma_1(A)\chi_{n-1}+\cdots+(-1)^n\sigma_n(A)+(-1)^{n+1}\sigma_{n+1}(A)\chi_{-1}=0$$

with the elementary symmetric polynomials $\sigma_i(A)$ of $a_1, a_2, \cdots, a_{n+1}$.

LEMMA 4.5. *Fix* $x=(x_{ij})\in SO(n+1)$. *The characteristic polynomial* $P_{\check{\Lambda}}(\lambda)$ *of* $\check{\Lambda}$, *as an endomorphism of the tangent space* $T_{\pi(x)}(M)$, *is*

$$\begin{aligned}P_{\check{\Lambda}}(\lambda)=\lambda^n&+(\chi_{-1})^{-1}(\chi_1-\sigma_1(A))\lambda^{n-1}+(\chi_{-1})^{-2}(\chi_2-\sigma_1(A)\chi_1+\sigma_2(A))\lambda^{n-2}+\cdots\\&+(\chi_{-1})^{-s}(\chi_s-\sigma_1(A)\chi_{s-1}+\sigma_2(A)\chi_{s-2}-\sigma_3(A)\chi_{s-3}+\cdots+(-1)^s\sigma_s(A))\lambda^{n-s}\\&+\cdots+(-1)^n(\chi_{-1})^{-(n-1)}\sigma_{n+1}(A),\end{aligned}$$

where $\chi_i=\chi_i(x)$.

PROOF. As a basis (not necessarily orthonormal) of the horizontal subspace $\mathcal{H}_x$ of $T_x(SO(n+1))$, take $\Lambda^{-1}(\mathrm{ad}_{x^{-1}}e_{i\,n+1})$, $i=1,\cdots,n$. Here $\mathrm{ad}_{x^{-1}}e_{i\,n+1}=x^{-1}e_{i\,n+1}x$ are viewed as the tangent vectors $\in T_x(SO(n+1))$ by left translation. Then, for each $p=1,\cdots,n$, the linear mapping $\mathrm{proj}_{\mathcal{H}}\circ\Lambda_{(p)}|_{\mathcal{H}}\colon \mathcal{H}_x\to\mathcal{H}_x$ has the following matrix expression:

$$\Big(\frac{1}{\chi_{-1}}\sum_{s=0}^{p-1}(\chi_s(xA^{p-s}x^{-1})_{ij}-(xA^sx^{-1})_{n+1\,i}(xA^{p-s}x^{-1})_{n+1\,j})\,;\,1\leqq i,\ j\leqq n\Big).$$

(In order to get this matrix, use the facts that

$$g_0(\mathrm{ad}_{x^{-1}}e_{ij},\,e_{kl})=x_{(i,j)(k,l)},\qquad g(\mathrm{ad}_{x^{-1}}e_{ij},\,\mathrm{ad}_{x^{-1}}e_{kl})=(xAx^{-1})_{(i,j)(k,l)}$$

for $1\leqq i<j\leqq n+1$, $1\leqq k<l\leqq n+1$, $x\in SO(n+1)$, where $Q_{(i,j)(k,l)}$ denote the minors of degree 2 of matrix Q, and

$$\mathrm{proj}_{\mathcal{H}}(\mathrm{ad}_{x^{-1}}e_{i\,n+1})=\frac{1}{\chi_{-1}}\sum_{r=1}^{n+1}(xA^{-1}x^{-1})_{n+1\,r}\,\mathrm{ad}_{x^{-1}}e_{ir},\ i=1,\cdots,n.)$$

When $p=1$, in particular, the matrix expression of $\check{\Lambda}$ at $\pi(x)$ is

$$\Big(\frac{1}{\chi_{-1}}(xAx^{-1})_{ij}\,;\,1\leqq i,\ j\leqq n\Big).$$

From this matrix for $\check{\Lambda}$, using the fact that the elementary symmetric polynomials of $a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_{n+1}$ are given by

$$\sigma_s(a_1,\cdots,\check{a}_i,\cdots,a_{n+1})=\sum_{r=0}^{s}(-1)^r\sigma_{s-r}(A)a_i^r$$

we can get $P_{\check{\Lambda}}(\lambda)$.

Next, we express the tensor field $\mathrm{proj}_{\mathcal{H}}\circ\Lambda_{(p)}$ restricted to the horizontal subspaces in terms of $\check{\Lambda}$.

LEMMA 4.6. *For each* $s=1, \cdots, n$ *we have*

$$\mathrm{proj}_{\mathcal{H}}(\Lambda_{(s)}(\bar{X}))=\sum_{t=1}^{s}\chi_{-1}^{t-1}\chi_{s-t}\check{\Lambda}^{t}(X)$$

for any $X\in T(M)$ *and any horizontal lift* $\bar{X}$.

PROOF. Using the matrix expressions for $\mathrm{proj}_{\mathcal{H}}\circ\Lambda_{(p)}|_{\mathcal{H}}$ in the proof of Lemma 4.5, we obtain the relation

$$\mathrm{proj}_{\mathcal{H}}\circ\Lambda_{(p+1)}|_{\mathcal{H}}=\chi_{-1}\,\mathrm{proj}_{\mathcal{H}}\circ\Lambda_{(p)}\circ\mathrm{proj}_{\mathcal{H}}\circ\Lambda|_{\mathcal{H}}+\chi_{p}\,\mathrm{proj}_{\mathcal{H}}\circ\Lambda|_{\mathcal{H}}\,.$$

By induction we get the desired formula.

PROOF OF PROPOSITION 4.3. By Lemma 4.5 we have

$$(-1)^{s}\sigma_{s}(\check{\Lambda})=(\chi_{-1})^{-s}\sum_{t=0}^{s}(-1)^{t}\sigma_{t}(\Lambda)\chi_{s-t}\,.$$

In particular, $\sigma_{n}(\check{\Lambda})=(\chi_{-1})^{-(n-1)}\sigma_{n+1}(\Lambda)$. Hence, recalling the definition of $(\check{\Lambda})_{p}$ we get

$$(\check{\Lambda})_{p}(X)=\sigma_{n+1}(\Lambda)^{-(p-1)/(n-1)}(\chi_{-1})^{p-1}\sum_{\substack{0\leqq s\leqq p-1\\ 0\leqq t\leqq s}}(-1)^{t}\sigma_{t}(\Lambda)(\chi_{-1})^{-s}\chi_{s-t}\check{\Lambda}^{p-s}(X)\,.$$

On the other hand, by Lemma 4.6 and by introducing the indices $u=p-t$, $v=p-s$, we see that the right-hand side of the desired formula can be written as

$$\sigma_{n+1}(\Lambda)^{-(p-1)/(n-1)}\sum_{\substack{1\leqq s\leqq p\\ 1\leqq t\leqq s}}(-1)^{p-s}\sigma_{p-s}(\Lambda)\chi_{-1}^{t-1}\chi_{s-t}\check{\Lambda}^{t}(X)\,.$$

$$=\sigma_{n+1}(\Lambda)^{-(p-1)/(n-1)}\sum_{\substack{v\leqq u\leqq p-1\\ 0\leqq v\leqq p-1}}(-1)^{v}\sigma_{v}(\Lambda)\chi_{-1}^{p-u-1}\chi_{u-v}\check{\Lambda}^{p-u}(X)\,.$$

The last formula is equal to $(\check{\Lambda})_{p}(X)$ because of the fact $\{(s,t)\,|\,0\leqq s\leqq p-1,\ 0\leqq t\leqq s\}=\{(u,v)\,|\,v\leqq u\leqq p-1,\ 0\leqq v\leqq p-1\}$. Proposition 4.3 is proved.

PROPOSITION 4.7 (cf. [**Br**]). *The Riemannian manifold* $(M^{n}, \sigma_{n}(\check{\Lambda})^{-1/(n-1)}\check{g})$ *is isometric to the ellipsoid*

$$E^{n}=\left\{(x_{1},\cdots,x_{n+1})\in\boldsymbol{R}^{n+1}\,\middle|\,\sum_{i=1}^{n+1}\frac{x_{i}^{2}}{a_{i}}=\left(\frac{1}{a_{1}\cdots a_{n+1}}\right)^{1/(n-1)}\right\}.$$

PROOF. Define $h_{i}: SO(n+1)\to\boldsymbol{R}$ by

$$h_{i}(x)=\frac{\sqrt{a_{i}}\,x_{n+1\,i}}{(a_{1}\cdots a_{n+1})^{1/2(n-1)}}\,.$$

Then $h=(h_{1},\cdots,h_{n+1}): SO(n+1)\to\boldsymbol{R}^{n+1}$ gives a mapping $\check{h}: M\to\boldsymbol{R}^{n+1}$. Using the fact

$$g(\mathrm{proj}_{\mathcal{V}}(E_{ij}),\ \mathrm{proj}_{\mathcal{V}}(E_{kl}))=C_{(i,j)(k,l)}$$

where $\mathrm{proj}_{\mathcal{V}}$ denotes the projection to the vertical subspace of $T_x SO(n+1)$, and $C_{(i,j)(k,l)}$ denotes the minor of degree 2 of the $n+1 \times n+1$ matrix

$$C = I_{n+1} - \frac{1}{\chi_{-1}}\left(\frac{x_{n+1\,i}x_{n+1\,j}}{\sqrt{a_i a_j}};\ 1 \leq i,\ j \leq n+1\right),$$

we see that $\check{h}$ gives the desired isometry.

Appendix A. Riemannian Expression of Poisson bracket.

We begin by recalling some notions of symplectic geometry (cf. [**F**]). The symplectic structure ω on $T(M)$ is defined to be the 2-form $\omega = 2d\theta$ (the multiplier 2 is put, since we view $d\theta$ as the bilinear mappings on the tangent spaces according to the definition of [**KN**]). Here θ is the canonical 1-form on $T(M)$, i.e., θ is defined by $\theta(\boldsymbol{X}) = g(\pi_*(\boldsymbol{X}), X)$, $\boldsymbol{X} \in T_X(T(M))$ with the induced mapping $\pi_* : T_X(T(M)) \to T_{\pi(X)}(M)$ of the projection $\pi : T(M) \to M$. For a function f on $T(M)$, the Hamiltonian vector field $\mathrm{sgrad}\, f$ on $T(M)$ is defined by the formula $\omega(\mathrm{sgrad}\, f, \boldsymbol{X}) = -df(\boldsymbol{X})$, $\boldsymbol{X} \in T(T(M))$. The Poisson bracket of two functions $f, g : T(M) \to \boldsymbol{R}$ is the function $\{f, g\} = \omega(\mathrm{sgrad}\, f, \mathrm{sgrad}\, g)$. The following lemma gives us the expression of the Poisson bracket of two quadratic functions on $T(M)$ in terms of covariant derivative ∇ of the Riemannian manifold M.

LEMMA 1. *Let μ, ν be symmetric tensor fields of type* (1, 1) *on M. Let $f, g : T(M) \to \boldsymbol{R}$ be the functions defined by $f(X) = \boldsymbol{g}(\mu(X), X)$, $g(X) = \boldsymbol{g}(\nu(X), X)$. Then we have*

$$\{f, g\}(X) = 2\{\boldsymbol{g}((\nabla\nu)(X; \mu(X)), X) - \boldsymbol{g}((\nabla\mu)(X; \nu(X)), X)\}, \quad X \in T(M).$$

PROOF. It suffices to prove our formula on each sufficiently small open subset of $T(M)$. Take n orthonormal vector fields $X_1, X_2, \cdots, X_n$ defined on an open set U of M. Put $c_{ij}^k = \boldsymbol{g}([X_i, X_j], X_k)$. Then by means of the isomorphism $\phi : T(U) \cong U \times \boldsymbol{R}^n$, $\phi(X) = (\pi(X), \boldsymbol{g}(X, X_1), \cdots, \boldsymbol{g}(X, X_n))$, we have $2n$ linearly independent vector fields $\bar{X}_1, \cdots, \bar{X}_n, \partial/\partial p_1, \cdots, \partial/\partial p_n$ on $T(U)$ such that

$$\bar{X}_i(\boldsymbol{g}(*, X_j)) = 0, \quad \partial/\partial p_i(\boldsymbol{g}(*, X_j)) = \delta_{ij}, \quad \pi_*(\bar{X}_i) = X_i,$$

$$\pi_*(\partial/\partial p_i) = 0, \qquad i, j = 1, \cdots, n.$$

Then $[\bar{X}_i, \bar{X}_j] = \sum_k c_{ij}^k \bar{X}_k$, $[\bar{X}_i, \partial/\partial p_j] = 0$, $[\partial/\partial p_i, \partial/\partial p_j] = 0$. With respect to this frame field the symplectic structure ω has the following expression: $\omega(\bar{X}_i, \partial/\partial p_j) = -\delta_{ij}$, $\omega(\partial/\partial p_i, \partial/\partial p_j) = 0$, $\omega(\bar{X}_i, \bar{X}_j) = -\sum_k c_{ij}^k p_k$ at $p_1 X_1 + \cdots + p_n X_n$. Using these formulas, for a function h on $T(M)$ we can express $\mathrm{sgrad}\, h$ as follows:

$$\operatorname{sgrad} h = \sum_i \frac{\partial h}{\partial p_i} \bar{X}_i - \sum_j \Big(\bar{X}_j(h) - \sum_{k,l} \frac{\partial h}{\partial p_k} c^l_{kj} p_l \Big) \frac{\partial}{\partial p_j} \quad \text{at } p_1 X_1 + \cdots + p_n X_n \in T(U).$$

Now we can prove our lemma on $T(U)$. For simplicity of notation, put $\mu_{ij} = \boldsymbol{g}(\mu(X_i), X_j)$, $\nu_{ij} = \boldsymbol{g}(\nu(X_i), X_j)$. Then $\{f, g\}$ is expressed as

$$\{f, g\} = 2 \sum_{r,s,t} \Big(\sum_i (\mu_{ir} X_i(\nu_{st}) - \nu_{ir} X_i(\mu_{st})) - 2 \sum_{i,j} c^t_{ij} \mu_{jr} \nu_{is} \Big) p_r p_s p_t$$

at $X = p_1 X_1 + \cdots + p_n X_n \in T(U)$. On the other hand, the usual tensor calculus yields

$$\boldsymbol{g}((\nabla \nu)(X ; \mu(X)), X) = \sum_{r,s,t} \Big(\sum_i \mu_{si} X_i(\nu_{rt}) - 2 \sum_{i,j} \mu_{si} d^j_{ir} \nu_{jt} \Big) p_r p_s p_t$$

for any tangent vector $X = p_1 X_1 + \cdots + p_n X_n$, where $d^k_{ij} = \boldsymbol{g}(\nabla_{X_i} X_j, X_k)$. Thus using the facts $c^k_{ij} = d^k_{ij} - d^k_{ji}$, $d^k_{ij} = -d^j_{ik}$, we obtain the desired formula at $X = p_1 X_1 + \cdots + p_n X_n$.

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