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A generalization of H-surfaces and a certain duality

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§ 1. Introduction.

In 1970 Lawson [2] showed that for a simply connected Riemann surface M there exists a bijective correspondence between minimal immersions of M into S^3 and isometric immersions of M into $\boldsymbol{R}^3$ with constant mean curvature $(\neq 0)$.

As a generalization of surfaces of constant mean curvature $H\neq 0$ (in abbreviation H-surfaces), we can consider solutions of

$$\Delta f=2H\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y}, \tag{1.1}$$

where (x, y) is an isothermal coordinate system,

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

and

$$\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y}=\begin{pmatrix}\dfrac{\partial f^2}{\partial x}\dfrac{\partial f^3}{\partial y}-\dfrac{\partial f^2}{\partial y}\dfrac{\partial f^3}{\partial x}\\ \dfrac{\partial f^3}{\partial x}\dfrac{\partial f^1}{\partial y}-\dfrac{\partial f^3}{\partial y}\dfrac{\partial f^1}{\partial x}\\ \dfrac{\partial f^1}{\partial x}\dfrac{\partial f^2}{\partial y}-\dfrac{\partial f^1}{\partial y}\dfrac{\partial f^2}{\partial x}\end{pmatrix}$$

(see § **2**).

In this paper we shall show the following generalization of Lawson's result.

THEOREM. *Let M be a simply connected Riemann surface. Then there exists a bijective correspondence between*

$$\{\varphi : M\longrightarrow S^3 \mid \varphi \textit{ is a harmonic map}\}/SO(4)$$

and

$$\{f : M\longrightarrow \boldsymbol{R}^3 \mid f \textit{ satisfies } (1.1)\}/SO(3)\ltimes \boldsymbol{R}^3 .$$

If f is isometric so is φ and vice versa. This part of correspondence is exactly one cited above in [2] by Lawson. Therefore our proof gives an alternative proof of Lawson's result in [2].

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§ 2. Preliminaries.

Let (M, g), (N, h) be Riemannian manifolds. A map φ from M into N is called harmonic if it extremizes the energy functional

$$E(\varphi)=\frac{1}{2}\int_M |d\varphi|^2 dVol_M$$

on every compact subdomain of M. The Euler-Lagrange equation for E is

$$g^{jk}\left\{\frac{\partial^2\varphi^\alpha}{\partial x^j \partial x^k}-{}^M\Gamma^l_{jk}\frac{\partial\varphi^\alpha}{\partial x^l}+{}^N\Gamma^\alpha_{\beta\gamma}(\varphi)\frac{\partial\varphi^\beta}{\partial x^j}\frac{\partial\varphi^\gamma}{\partial x^k}\right\}=0$$

in local coordinates, where the Γ are the Christoffel symbol of M and N.

Now let M be a simply connected Riemann surface and N a compact Lie group G with its Lie algebra $\mathfrak{g}$.

Let ζ be the Maurer-Cartan form on G and α a $\mathfrak{g}$-valued 1-form on M.

Since M is simply connected, $\alpha=\varphi^*\zeta$ for some map $\varphi: M\to G$ if and only if α satisfies the Maurer-Cartan equation

$$d\alpha+\frac{1}{2}[\alpha\wedge\alpha]=0\,, \tag{2.1}$$

where $[\alpha\wedge\alpha](X, Y)=2[\alpha(X), \alpha(Y)]$ for $X, Y\in TM$. φ is unique up to left multiplication by an element of G.

And $\varphi: M\to G$ is harmonic if and only if $\alpha=\varphi^*\zeta$ is co-closed, i.e.,

$$d^*\alpha=0 \tag{2.2}$$

(see [3], [4]).

So if we consider harmonic maps from M into G, it suffices to consider $\mathfrak{g}$-valued 1-forms α on M satisfying (2.1) and (2.2).

For harmonic map theory we refer to [1] as a survey article.

Let M be a Riemann surface. For a 2-form θ on $\boldsymbol{R}^3$ such that $d\theta=2Hdx^1\wedge dx^2\wedge dx^3$ ($H\in\boldsymbol{R}\setminus\{0\}$), we call $f: M\to\boldsymbol{R}^3$ is an H-surface (a surface of constant mean curvature H) if it is conformal and extremizes the functional

$$\Phi_H(f)=E(f)+\int_M f^*\theta$$

on every compact subdomain of M.

In an isothermal coordinate system a simple computation shows that the Euler-Lagrange equation for Φ_H is (1.1). So we can consider solutions of (1.1) as a generalization of H-surfaces. For geometric meaning of Φ_H see [**5**, p. 180].

§ 3. Duality.

We consider $\boldsymbol{R}^3$ as a Lie group under addition. Then its Lie algebra is $\boldsymbol{R}^3$. The Maurer-Cartan form η on $\boldsymbol{R}^3$ is given by

$$\eta=\begin{pmatrix}1\\0\\0\end{pmatrix}dx^1+\begin{pmatrix}0\\1\\0\end{pmatrix}dx^2+\begin{pmatrix}0\\0\\1\end{pmatrix}dx^3 .$$

Let M be a simply connected Riemann surface and β an $\boldsymbol{R}^3$-valued 1-form on M. Since M is simply connected, $\beta=f^*\eta$ for some $f: M\to\boldsymbol{R}^3$ if and only if β satisfies

$$d\beta=0 . \tag{3.1}$$

f is unique up to addition by an element of $\boldsymbol{R}^3$.

LEMMA 1. *$f: M\to\boldsymbol{R}^3$ satisfies (1.1) if and only if $\beta=f^*\eta$ satisfies*

$$d^*\beta+H*(\beta\wedge\beta)=0 , \tag{3.2}$$

where $*$ is the Hodge star operator on M and $\beta\wedge\beta=2a\wedge b dx\wedge dy$ for $\beta=a dx+b dy$ $(a, b\in\boldsymbol{R}^3)$.

PROOF. We take an isothermal coordinate system on M.
Since

$$\begin{aligned}\beta&=f^*\eta\\&=(f^*\eta)\Big(\frac{\partial}{\partial x}\Big)dx+(f^*\eta)\Big(\frac{\partial}{\partial y}\Big)dy\\&=\eta\Big(\sum_{j=1}^{3}\frac{\partial f^j}{\partial x}\frac{\partial}{\partial x^j}\Big)dx+\eta\Big(\sum_{j=1}^{3}\frac{\partial f^j}{\partial y}\frac{\partial}{\partial x^j}\Big)dy\\&=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy ,\\ d^*\beta&=-*d*\beta\\&=-*d*\Big(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\Big)\\&=-*d\Big(\frac{\partial f}{\partial x}dy-\frac{\partial f}{\partial y}dx\Big)\end{aligned}$$

$$= -*\Big(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\Big)dx\wedge dy$$

$$= -\Big(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\Big)*(dx\wedge dy).$$

And

$$\beta\wedge\beta = \Big(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\Big)\wedge\Big(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\Big)$$

$$= \Big(\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y}\Big)dx\wedge dy+\Big(\frac{\partial f}{\partial y}\wedge\frac{\partial f}{\partial x}\Big)dy\wedge dx$$

$$= 2\Big(\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y}\Big)dx\wedge dy\,. \quad \blacksquare$$

Note that an element of $\boldsymbol{R}^3$ acts on solutions of (1.1) as a parallel displacement and an element of a Lie group G acts on harmonic maps into G. We are now in a position to prove the main lemma.

LEMMA 2. *Let M be a simply connected Riemann surface. Then there exists a bijective correspondence between*

$$\{\varphi: M \longrightarrow SU(2)\,|\,\varphi \text{ is a harmonic map}\}/SU(2)$$

and

$$\{f: M \longrightarrow \boldsymbol{R}^3\,|\,f \text{ satisfies } (1.1)\}/\boldsymbol{R}^3\,.$$

PROOF. It is easy to see that equations (3.1) and (3.2) are equivalent to

$$d^*(*\beta) = 0\,, \tag{3.3}$$

$$d(*\beta)-H(*\beta\wedge*\beta) = 0\,. \tag{3.4}$$

We define a bijective map

$$\iota: \boldsymbol{R}^3 \longrightarrow \mathfrak{su}(2)$$

by

$$\iota\left(\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}\right) = \begin{pmatrix} ix^1 & -x^2+ix^3 \\ x^2+ix^3 & -ix^1 \end{pmatrix}$$

for

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \boldsymbol{R}^3\,.$$

Then we have $\iota(a\wedge b)=-[\iota(a), \iota(b)]/2$ for $a, b\in\boldsymbol{R}^3$. ι is extended to a map from $\boldsymbol{R}^3$-valued 1-forms on M into $\mathfrak{su}(2)$-valued 1-forms on M. We also denote it by ι.

Then using the Hodge star operator $*$ we can define a "dual" map $\iota\circ *$. An easy computation shows that equations (3.3) and (3.4) are equivalent to

$$d^*(\iota(*\beta)) = 0\,, \tag{3.5}$$

$$d(\iota(*\beta)) + \frac{H}{2}[\iota(*\beta)\wedge\iota(*\beta)] = 0\,. \tag{3.6}$$

Since we may assume $H=1$ by rescaling, equation (3.6) is the Maurer-Cartan equation for $\mathfrak{su}(2)$-valued 1-forms on M and (3.5) is harmonic map equation (see (2.1) and (2.2)). So we can obtain a harmonic map from M into $SU(2)$. ∎

We identify $SU(2)$ with the standard unit 3-sphere S^3 by

$$\begin{pmatrix} y^0+iy^1 & -y^2+iy^3 \\ y^2+iy^3 & y^0-iy^1 \end{pmatrix} \in SU(2) \longleftrightarrow (y^0,\ y^1,\ y^2,\ y^3) \in S^3$$

and take a coordinate system $\{u^1,\ u^2,\ u^3\}$ with

$$u^j = \frac{y^j}{y^0} \qquad (j = 1,\ 2,\ 3)$$

in $\{(y^0,\ y^1,\ y^2,\ y^3)\,|\, y^0\neq 0\}\subset S^3$.

LEMMA 3. *The Maurer-Cartan form ζ on $SU(2)$ is given by*

$$\zeta = \begin{pmatrix} i\omega_1 & -\omega_2+i\omega_3 \\ \omega_2+i\omega_3 & -i\omega_1 \end{pmatrix},$$

where

$$\omega_1 = \frac{1}{1+\sum_{j=1}^3 (u^j)^2}(du^1 - u^3du^2 + u^2du^3)\,,$$

$$\omega_2 = \frac{1}{1+\sum_{j=1}^3 (u^j)^2}(du^2 - u^1du^3 + u^3du^1)\,,$$

$$\omega_3 = \frac{1}{1+\sum_{j=1}^3 (u^j)^2}(du^3 - u^2du^1 + u^1du^2)\,.$$

PROOF. It is an easy exercise. So we ommit the proof. ∎

If we consider $SO(3)\ltimes \boldsymbol{R}^3$ action on solutions of (1.1) and $SO(4)$ action on harmonic maps into $SU(2)\cong S^3$, we obtain the following main theorem.

THEOREM 4. *Let M be a simply connected Riemann surface. Then there exists a bijective correspondence between*

$$\{\varphi : M \longrightarrow S^3\,|\,\varphi \text{ is a harmonic map}\}/SO(4)$$

and

$$\{f: M \longrightarrow \boldsymbol{R}^3 \mid f \text{ satisfies } (1.1)\}/SO(3) \ltimes \boldsymbol{R}^3 .$$

PROOF. We may assume $H=1$. Let f be a solution of (1.1) and φ a corresponding harmonic map in Lemma 2. Then

$$-\frac{\partial f^j}{\partial y} = \varphi^* \omega_j\Big(\frac{\partial}{\partial x}\Big) \qquad (j=1, 2, 3) \tag{3.7}$$

$$\frac{\partial f^j}{\partial x} = \varphi^* \omega_j\Big(\frac{\partial}{\partial y}\Big) \qquad (j=1, 2, 3). \tag{3.8}$$

Put $u^j \circ \varphi = \varphi^j$ $(j=1, 2, 3)$. Then by Lemma 3,

$$-\frac{\partial f^1}{\partial y} = \Big(\frac{\partial \varphi^1}{\partial x} - \varphi^3 \frac{\partial \varphi^2}{\partial x} + \varphi^2 \frac{\partial \varphi^3}{\partial x}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}, \tag{3.9}$$

$$-\frac{\partial f^2}{\partial y} = \Big(\frac{\partial \varphi^2}{\partial x} - \varphi^1 \frac{\partial \varphi^3}{\partial x} + \varphi^3 \frac{\partial \varphi^1}{\partial x}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}, \tag{3.10}$$

$$-\frac{\partial f^3}{\partial y} = \Big(\frac{\partial \varphi^3}{\partial x} - \varphi^2 \frac{\partial \varphi^1}{\partial x} + \varphi^1 \frac{\partial \varphi^2}{\partial x}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}, \tag{3.11}$$

$$\frac{\partial f^1}{\partial x} = \Big(\frac{\partial \varphi^1}{\partial y} - \varphi^3 \frac{\partial \varphi^2}{\partial y} + \varphi^2 \frac{\partial \varphi^3}{\partial y}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}, \tag{3.12}$$

$$\frac{\partial f^2}{\partial x} = \Big(\frac{\partial \varphi^2}{\partial y} - \varphi^1 \frac{\partial \varphi^3}{\partial y} + \varphi^3 \frac{\partial \varphi^1}{\partial y}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}, \tag{3.13}$$

$$\frac{\partial f^3}{\partial x} = \Big(\frac{\partial \varphi^3}{\partial y} - \varphi^2 \frac{\partial \varphi^1}{\partial y} + \varphi^1 \frac{\partial \varphi^2}{\partial y}\Big) \frac{1}{1+\sum_{j=1}^3 (\varphi^j)^2}. \tag{3.14}$$

Let $A \in SO(3)$. Put $\begin{pmatrix} \tilde{f}^1 \\ \tilde{f}^2 \\ \tilde{f}^3 \end{pmatrix} = Af$ and $\begin{pmatrix} \tilde{\varphi}^1 \\ \tilde{\varphi}^2 \\ \tilde{\varphi}^3 \end{pmatrix} = A \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \end{pmatrix}$. Then a direct computation shows that in (3.9)-(3.14) we can replace f^i $(i=1, 2, 3)$ and φ^i $(i=1, 2, 3)$ with $\tilde{f}^i$ $(i=1, 2, 3)$ and $\tilde{\varphi}^i$ $(i=1, 2, 3)$. Note that $SU(2)$ is inclused in $SO(4)$ by

$$\begin{pmatrix} y^0+iy^1 & -y^2+iy^3 \\ y^2+iy^3 & y^0-iy^1 \end{pmatrix} \longmapsto \begin{pmatrix} y^0 & -y^2 & -y^1 & -y^3 \\ y^2 & y^0 & -y^3 & y^1 \\ y^1 & y^3 & y^0 & -y^2 \\ y^3 & -y^1 & y^2 & y^0 \end{pmatrix}.$$

Since for any element $A' \in SO(4)$ there exists a decomposition

$$A' = A'' \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

such that $A'' \in SU(2)$ and $A \in SO(3)$, the theorem is proved. ■

PROPOSITION 5. *In theorem* 4 *conformal* (*resp. isometric*) *harmonic maps correspond to conformal* (*resp. isometric*) *solutions of* (1.1).

PROOF. For $(y^0, y^1, y^2, y^3)\in S^3$ we use a coordinate system $\{v^1, v^2, v^3\}$ with

$$y^0=\frac{\sum_{j=1}^3(v^j)^2-1}{1+\sum_{j=1}^3(v^j)^2},$$

$$y^1=\frac{2v^1}{1+\sum_{j=1}^3(v^j)^2},$$

$$y^2=\frac{2v^2}{1+\sum_{j=1}^3(v^j)^2},$$

$$y^3=\frac{2v^3}{1+\sum_{j=1}^3(v^j)^2},$$

(this is a stereographic projection) in $\{(y^0, y^1, y^2, y^3)\in S^3 \mid y^0\neq 1\}$.

Then

$$\omega_j=\frac{\Omega_j}{(\sum_{k=1}^3(v^k)^2+1)^2} \qquad (j=1, 2, 3), \tag{3.15}$$

where

$$\Omega_1=2(-(v^1)^2+(v^2)^2+(v^3)^2-1)dv^1-4(v^3+v^1v^2)dv^2+4(v^2-v^3v^1)dv^3,$$

$$\Omega_2=2(-(v^2)^2+(v^3)^2+(v^1)^2-1)dv^2-4(v^1+v^2v^3)dv^3+4(v^3-v^1v^2)dv^1,$$

$$\Omega_3=2(-(v^3)^2+(v^1)^2+(v^2)^2-1)dv^3-4(v^2+v^3v^1)dv^1+4(v^1-v^2v^3)dv^2.$$

Let f be a solution of (1.1) for $H=1$ and φ a corresponding harmonic map. Put $v^j\circ\varphi=\varphi^j$ $(j=1, 2, 3)$. Then by (3.7), (3.8) and (3.15) a direct computation shows that

$$\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}=-\frac{4}{(\sum_{j=1}^3(\varphi^j)^2+1)^2}\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\varphi}{\partial y},$$

$$\left(\frac{\partial f}{\partial y}\right)^2=\frac{4}{(\sum_{j=1}^3(\varphi^j)^2+1)^2}\left(\frac{\partial\varphi}{\partial x}\right)^2,$$

$$\left(\frac{\partial f}{\partial x}\right)^2=\frac{4}{(\sum_{j=1}^3(\varphi^j)^2+1)^2}\left(\frac{\partial\varphi}{\partial y}\right)^2.$$

Since the metric of S^3 is given by

$$\frac{4}{(\sum_{j=1}^3(v^j)^2+1)^2}\sum_{j=1}^3 dv^j\otimes dv^j,$$

the proposition is proved. ■

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