J. Math. Soc. Japan
Vol. 47, No. 1, 1995

On the wellposed Cauchy problem for some dispersive equations

By Shigeo TARAMA

(Received April 15, 1992)
(Revised June 23, 1993)

1. Introduction.

The forward Cauchy problem for the operator with real coefficients $Hu(t, x) =\partial_t u(t, x)+a(x)\partial_x^2 u(t, x)+b(x)\partial_x u(t, x)+c(x)u(t, x)$ with the datum on a line $t=0$ is L^2 and H^∞-wellposed if and only if $a(x)\leqq 0$.

We consider the same problem for the operator with real coefficients

$$(1.1)\quad Au(t, x)=\partial_t u(t, x)+\partial_x^3 u(t, x)+a(x)\partial_x^2 u(t, x)+b(x)\partial_x u(t, x)+c(x)u(t, x).$$

which is obtained by adding the dispersive term $\partial_x^3 u(t, x)$ to $Hu(t, x)$. Our problem is under which conditions on the coefficient $a(x)$ the forward Cauchy problem for $Au(t, x)$ is L^2 or H^∞-wellposed.

Similar problems arise for the Schrödinger type operator

$$Su(t, x)=\partial_t u(t, x)+i\partial_x^2 u(t, x)+A(x)\partial_x u(t, x)+B(x)u(t, x).$$

In this case, the following condition on the imaginary part of $A(x)$: $\Im A(x)$ is necessary and sufficient for the L^2[resp. H^∞]-wellposedness;

There exists some constant C satisfying

$$\left|\int_x^y \Im A(x)dx\right| \leqq C \quad \left[\text{resp.} \left|\int_x^y \Im A(x)dx\right| \leqq C\log(|x-y|+2)\right]$$
$$\text{for any}\quad x, y\in \boldsymbol{R},$$

while for the operator $\partial_t u(t, x)+A(x)\partial_x u(t, x)+B(x)u(t, x)$ the necessary and sufficient condition is $\Im A(x)=0$ (see W. Ichinose [**1**] and [**2**], S. Mizohata [**4**] and J. Takeuchi [**6**]).

In the following, we consider only real-valued functions and operators with real coefficients with some obvious exceptions.

Now we formulate the forward Cauchy problem for the operator A defined by (1.1):

For the given datum $g(x)$ and right-hand side $f(t, x)$ of the equation, find a solution $u(t, x)$ satisfying

$$
\text{(C)} \qquad \begin{cases} Au(t,\ x) = f(t,\ x) & (t,\ x) \in [0,\ T] \times \boldsymbol{R} \\ u(0,\ x) = g(x) & x \in \boldsymbol{R} \end{cases}
$$

where T is some positive number and all the coefficients $a(x)$, $b(x)$, and $c(x)$ of A belong to B^∞: the space of bounded C^∞ functions defined on $\boldsymbol{R}=(-\infty,\ +\infty)$ with bounded derivatives of any order.

THEOREM 1. *If the Cauchy problem* (C) *is* L^2 [*resp.* H^∞]*-wellposed, then the following inequality holds with some positive constant* K

$$
\text{(N)} \qquad K \geqq \int_x^y a(s)ds
$$

$$
\left[resp. \right.
$$

$$
\text{(N}^\infty\text{)} \qquad \left. K \log(y-x+2) \geqq \int_x^y a(s)ds \right]
$$

for any $x,\ y \in \boldsymbol{R}$ *satisfying* $y \geqq x$.

Theorem 1 follows from the same arguments as that for the Schrödinger type operator (see S. Mizohata [**4**, Lecture VII] or J. Takeuchi [**7**]). Therefore we mention only the form of the asymptotic solution which we use in the proof of Theorem 1: that is

$$
u(t,\ x,\ \xi) = \exp\Big(\sqrt{-1}(\xi^3 t + \xi x) + \frac{1}{3}\int_x^{x+3\xi^2 t} a(s)ds\Big)
$$

$$
\times \Big(u_0(x+3\xi^2 t) + \sum_{n \geqq 1} u_n(t,\ x+3\xi^2 t,\ \xi)\Big)
$$

where $u_0(x)$ is a suitably chosen function and for $n \geqq 1$ $u_n(t,\ x,\ \xi)$ are defined successively by

$$
\partial_t u_n(t,\ x,\ \xi) = \xi P_1(x,\ x-3\xi^2 t,\ \partial_x) u_{n-1}(t,\ x,\ \xi) + P_0(x,\ x-3\xi^2 t,\ \partial_x) u_{n-1}(t,\ x)
$$

where $P_1(x,\ x-3\xi^2 t,\ \partial_x)$ and $P_0(x,\ x-3\xi^2 t,\ \partial_x)$ are the differential operators given by

$$
P_1(x,\ x-3\xi^2 t,\ \partial_x) = -\sqrt{-1} e^{-S}(3\partial_x^2 + 2a(x-3\xi^2 t)\partial_x + b(x-3\xi^2 t))e^S
$$

$$
P_0(x,\ x-3\xi^2 t,\ \partial_x) = -e^{-S}(\partial_x^3 + a(x-3\xi^2 t)\partial_x^2 + b(x-3\xi^2 t)\partial_x + c(x-3\xi^2 t))e^S
$$

$$
\text{with} \quad S = \frac{1}{3}\int_{x-3\xi^2 t}^x a(s)ds\,.
$$

On the other hand, concerning the sufficiency of the condition (N) or (N$^\infty$) we show two results.

THEOREM 2. *If the coefficient* $a(x)$ *satisfies* (N), *then the Cauchy problem* (C) *is* L^2*-wellposed.*

And

THEOREM 3. *If the coefficient $a(x)$ satisfies* (N^{∞}), *then the Cauchy problem* (C) *is H^{∞}-wellposed.*

Here we say that the Cauchy problem (C) is X-wellposed if and only if for any datum $g(x)$ in X and any right-hand side $f(t, x)$ which is X-valued continuous function on $[0, T]$, that is to say, $f(t, x)\in C([0, T], X)$, there exists one and only one solution $u(t, x)\in C([0, T], X)$ of the problem (C) satisfying the following estimate: For any continuous semi-norm $\rho_1(\cdot)$ on X, there exists a continuous semi-norm $\rho_2(\cdot)$ on X satisfying:

$$\rho_1(u(t, \cdot)) \leqq \rho_2(g(\cdot))+\int_0^t \rho_2(f(s, \cdot))ds$$

for any $t\in[0, T]$ where the semi-norm $\rho_2(\cdot)$ is chosen independently of $g(x)$, $f(t, x)$ and $u(t, x)$.

We denote by L^2 the space of square integrable functions on $\boldsymbol{R}=(-\infty, +\infty)$ with the inner product $(v(x), w(x))=\int_{\boldsymbol{R}} v(x)w(x)dx$ and the norm $\|f(\cdot)\|=\{(f(x), f(x))\}^{1/2}$ and by H^l the space defined by $\{f\in L^2 ; \|f(\cdot)\|_l<+\infty\}$ with the norm $\|\cdot\|_l$ where $\|f(\cdot)\|_l$ is defined by $\left(\int_{\boldsymbol{R}} |\langle\xi\rangle^l \hat{f}(\xi)|^2 d\xi\right)^{1/2}$ with the Fourier transform $\hat{f}(\xi)$ of $f(x)$ and $\langle\xi\rangle=\sqrt{1+|\xi|^2}$. We put $H^{\infty}=\bigcap_{-\infty<l<\infty} H^l$.

The rest of this paper is devoted to the proofs of Theorem 2 and 3. The outline of the proofs is the following. We transform the given operator A to an operator with a non-positive coefficient $a(x)$ for which the Cauchy problem (C) is L^2, H^{∞} and Schwartz space S-wellposed, by the change of unknown function $u(t, x)=\exp(D(x))v(t, x)$ with some appropriate function $D(x)$ (see Sec. 2). For the proof of Theorem 3, we need another observation that, roughly speaking, the regularity of solution implies the decay of solution in the space variable (see Sec. 3).

In the following we denote by C or suffixed C_* an arbitrary constant which may be different according to the contexts and $\langle x\rangle$ denotes $\sqrt{1+|x|^2}$.

2. The proof of Theorem 2.

The proofs of Theorem 2 and 3 are based on the following proposition.

PROPOSITION 4. *If the coefficient $a(x)$ of the operator A defined by* (1.1) *is non-positive, then the forward Cauchy problem* (C) *is L^2, H^{∞} and S-wellposed.*

PROOF. Let $v(x)\in S$ or $v(x)\in H^{\infty}$. Noting $a(x)\partial_x^2 v(x)=\partial_x a(x)\partial_x v(x)-a'(x)\partial_x v(x)$, we have

$$(\partial_x^3 v(x)+a(x)\partial_x^2 v(x)+b(x)\partial_x v(x)+c(x)v(x),\ v(x))$$
$$= -(a(x)\partial_x v(x),\ \partial_x v(x))+\Big(\Big\{\frac{1}{2}a''(x)-\frac{1}{2}b'(x)+c(x)\Big\}v(x),\ v(x)\Big).$$

Hence under $a(x)\leqq 0$, for any real λ

$$(\lambda v(x)+Bv(x),\ v(x)) \geqq (\lambda-C)\|v(x)\|^2,$$

where we put $Bv(x)=\partial_x^3 v(x)+a(x)\partial_x^2 v(x)+b(x)\partial_x v(x)+c(x)v(x)$.

On the other hand, from the ellipticity of B, we see

$$\{v(x)\in L^2;\ Bv(x)\in L^2\} = H^3$$

The above facts and S. Mizohata [**5**, Ch. 6, Sec. 4] imply that $-B$ is a generator of C^0 semi-group on L^2. Then the problem (C) is L^2-wellposed. We see also the problem (C) is H^∞-wellposed, because $v(x)\in H^\infty$ is equivalent to "$B^l v(x)\in L^2$ for any $l\geqq 0$". The S-wellposedness follows almost directly from the results in Kato [**3**, Section 8 and Appendix A.3]. Indeed we have only to show that $-TBT^{-1}$ is a generator of C^0 semigroup on L^2 with its domain H^3, where $Tv(x)=((-\partial_x^2)^N+\langle x\rangle^N)v(x)$ with $N\geqq 1$. This claim follows from

$$TBT^{-1}\equiv B+2Na'(x)\partial_x(-\partial_x^2)^N T^{-1}$$
$$(2Na'(x)\partial_x(-\partial_x^2)^N T^{-1})^*+(2Na'(x)\partial_x(-\partial_x^2)^N T^{-1})\equiv 0$$

mod L^2 bounded operators. ■

Under the condition (N) we construct some function $d(x)\in B^\infty$ satisfying,

(2.1) $$d(x)\geqq a(x) \qquad x\in \boldsymbol{R}$$

(2.2) $$\int_0^x d(y)dy \text{ is bounded in } \boldsymbol{R}.$$

This property (2.2) means that the multiplication by $\exp\Big(\int_0^x d(y)dy/3\Big)$ is an automorphism of L^2 and of H^∞. Then the L^2 or H^∞-wellposedness of the Cauchy problem (C) for the operator $\tilde{A}$ defined by

$$\tilde{A}u(t,x)=\exp\Big(\frac{1}{3}\int_0^x d(y)dy\Big)A\exp\Big(-\frac{1}{3}\int_0^x d(y)dy\Big)u(t,x)$$

(2.3) $$=\partial_t u(t,x)+\partial_x^3 u(t,x)+(-d(x)+a(x))\partial_x^2 u(t,x)$$
$$+\tilde{b}(x)\partial_x u(t,x)+\tilde{c}(x)u(t,x)$$

with some $\tilde{b}(x)$ and $\tilde{c}(x)\in B^\infty$

is equivalent to that of the Cauchy problem for A. And according to Proposition 4, (2.1) and (2.3), the Cauchy problem (C) for $\tilde{A}$ is L^2-wellposed. Therefore the proof of Theorem 2 is completed if we can construct a function $d(x)$ verifying (2.1) and (2.2).

For this purpose we prepare one lemma.

LEMMA 5. *Let $l(t)$ be a non-negative and non-decreasing function on $[1, +\infty)$. If two sequences of non-negative numbers $\{A_n^+\}_{n=1,2,\dots}$ and $\{A_n^-\}_{n=1,2,3,\dots}$ satisfy*

$$l(n-m+1) \geqq \sum_{k=m}^{n} (A_k^+ - A_k^-) \qquad \text{for} \quad n \geqq m \geqq 1,$$

then there exists a sequence $\{\theta_n\}_{n=1,2,\dots}$ in $[0, 1]$ such that for any n

$$l(n) \geqq \sum_{k=1}^{n} (A_k^+ - \theta_k A_k^-) \geqq 0. \tag{2.4}$$

PROOF. We put

$$\theta_1 = \begin{cases} \dfrac{A_1^+}{A_1^-} & \text{if} \quad A_1^+ - A_1^- < 0, \\ 1 & \text{if} \quad A_1^+ - A_1^- \geqq 0. \end{cases}$$

For $n \geqq 2$, we define θ_n inductively by

$$\theta_n = \frac{\sum_{k=1}^{n-1}(A_k^+ - \theta_k A_k^-) + A_n^+}{A_n^-}$$

$$\text{if} \quad \sum_{k=1}^{n-1} (A_k^+ - \theta_k A_k^-) + A_n^+ - A_n^- < 0$$

$$\theta_n = 1$$

$$\text{if} \quad \sum_{k=1}^{n-1} (A_k^+ - \theta_k A_k^-) + A_n^+ - A_n^- \geqq 0.$$

We see inductively that for any $n \geqq 1$

$$\sum_{k=1}^{n} (A_k^+ - \theta_k A_k^-) \geqq 0. \tag{2.5}$$

Thus $\sum_{k=1}^{n-1}(A_k^+ - \theta_k A_k^-) + A_n^+ - A_n^- < 0$ implies $A_n^- > 0$ and $0 \leqq \theta_n < 1$. Hence $\theta_n \in [0, 1]$ and

$$\sum_{k=1}^{n} (A_k^+ - \theta_k A_k^-) = 0 \qquad \text{if} \quad \theta_n < 1. \tag{2.6}$$

If $\theta_k = 1$ for $1 \leqq k \leqq n$, the left inequality of (2.4) is evident. Otherwise, let k_0 be the maximum of k satisfying $1 \leqq k \leqq n$ and $\theta_k < 1$, then by (2.6) and $\theta_k = 1$ for $k > k_0$ we see that

$$\sum_{k=1}^{n} (A_k^+ - \theta_k A_k^-) = \sum_{k=k_0+1}^{n} (A_k^+ - A_k^-)$$

$$\leqq l(n-k_0)$$

and, since $l(t)$ is non-decreasing,

$$\leqq l(n).$$

In the case $k_0 = n$, the same inequality follows from (2.6) and $l(n) \geqq 0$. ■

Now we assume the coefficient $a(x)$ satisfies (N). Put for $n=1, 2, \cdots$

$$A_n^+ = \int_{n-1}^{n} a^+(x)dx, \qquad A_n^- = \int_{n-1}^{n} a^-(x)dx \tag{2.7}$$

where $a^+(x)=\max\{a(x), 0\}$ and $a^-(x)=\max\{-a(x), 0\}$.

Then the condition (N) implies for $n>m$

$$\sum_{k=m}^{n}\left(A_k^+ - A_k^-\right) \leqq K.$$

Lemma 5 asserts that there exist a sequence $\{\theta_n\}_{n=1,2,\dots}$ in $[0, 1]$ such that for any $n \geqq 1$

$$0 \leqq \sum_{k=1}^{n}\left(A_k^+ - \theta_k A_k^-\right) \leqq K. \tag{2.8}$$

Similarly, by setting for $n=1, 2, \cdots$

$$B_n^+ = \int_{-n}^{-n+1} a^+(x)dx \qquad B_n^- = \int_{-n}^{-n+1} a^-(x)dx \tag{2.9}$$

we see that there exists a sequence $\{\delta_n\}_{n=1,2,\dots}$ in $[0, 1]$ such that for any $n \geqq 1$

$$0 \leqq \sum_{k=1}^{n}\left(B_k^+ - \delta_k B_k^-\right) \leqq K. \tag{2.10}$$

We define a function $h(x)$ on $\boldsymbol{R}$ by

$$h(x) = \begin{cases} \theta_n & x \in [n-1, n) \\ \delta_n & x \in [-n, -n+1) \end{cases}$$

for $n=1, 2, \cdots$. And we put $\tilde{a}(x)=a^+(x)-h(x)a^-(x)$. Then we have

$$a^+(x) \geqq \tilde{a}(x) \geqq a(x) \tag{2.11}$$

and

$$K+M \geqq \int_0^x \tilde{a}(y)dy \geqq -M \qquad x \geqq 0 \tag{2.12}$$

$$M \geqq \int_0^x \tilde{a}(y)dy \geqq -K-M \qquad x \leqq 0 \tag{2.13}$$

with $M=\sup_{x\in \boldsymbol{R}}|a(x)|$.

Indeed (2.11) follows from $0 \leqq h(x) \leqq 1$ and $a^-(x) \geqq 0$.

Since for any integer $n \geqq 1$

$$\int_0^n \tilde{a}(x)dx = \sum_{k=1}^{n}\left(A_k^+ - \theta_k A_k^-\right)$$

and for $t \geqq 0$

$$\left|\int_n^{n+t} \tilde{a}(x)dx\right| \leqq t \sup_{x\in \boldsymbol{R}}|a(x)|,$$

we see (2.12) from (2.8). Similarly (2.13) follows from (2.10).

Now we regularize $\tilde{a}(x)$ by the convolution with a smooth non-negative function $\rho(x)$ supported in $[-1/2, 1/2]$ satisfying $\int_{\boldsymbol{R}}\rho(x)dx=1$. And we define the desired function $d(x)$ by

$$d(x)=\int_{\boldsymbol{R}}\tilde{a}(y)\rho(x-y)dy-\int_{\boldsymbol{R}}a(y)\rho(x-y)dy+a(x). \tag{2.14}$$

Then $d(x)$ satisfies the followings

$$d(x)\geqq a(x) \qquad x\in\boldsymbol{R} \tag{2.15}$$

and with the same constants in (2.12) and (2.13)

$$K+4M\geqq\int_0^x d(y)dy\geqq -4M \qquad \text{for } x\geqq 0, \tag{2.16}$$

$$4M\geqq\int_0^x d(y)dy\geqq -4M-K \qquad \text{for } x\leqq 0. \tag{2.17}$$

PROOFS OF (2.15), (2.16) AND (2.17). From (2.11) and $\rho(x)\geqq 0$ follows (2.15). We have also from (2.11)

$$|\tilde{a}|\leqq M. \tag{2.18}$$

Thus (2.16) is valid for $0\leqq x\leqq 1$. Taking into account the support of $\rho(x)$ and (2.12) and (2.18), we have for $1\leqq x$

$$K+M\geqq\int_{\boldsymbol{R}}\rho(z)\left(\int_0^{x-z}\tilde{a}(y)dy\right)dz\geqq -M$$

and

$$\left|\int_{\boldsymbol{R}}\rho(z)\left(\int_{-z}^0\tilde{a}(y)dy\right)dz\right|\leqq M.$$

Then from

$$\int_0^x\left(\int_{\boldsymbol{R}}\tilde{a}(z)\rho(y-z)dz\right)dy=\int_{\boldsymbol{R}}\rho(z)\left(\int_0^{x-z}\tilde{a}(y)dy+\int_{-z}^0\tilde{a}(y)dy\right)dz$$

we see that for $x\geqq 0$

$$K+2M\geqq\int_0^x\left(\int_{\boldsymbol{R}}\tilde{a}(z)\rho(y-z)dz\right)dy\geqq -2M. \tag{2.19}$$

Noting $\int_{\boldsymbol{R}}a(z)\rho(y-z)dz-a(y)=\int_{\boldsymbol{R}}\rho(z)(a(y-z)-a(y))dz$,

$$\begin{aligned}&\left|\int_0^x\left(\int_{\boldsymbol{R}}a(z)\rho(y-z)dz-a(y)\right)dy\right|\\&=\left|\int_{\boldsymbol{R}}\rho(z)\left(-\int_{x-z}^x a(y)dy+\int_{-z}^0 a(y)dy\right)dz\right|\\&\leqq 2M.\end{aligned}$$

From the above estimate and (2.19) we have (2.16). The estimate (2.17) follows similarly.

Now we have seen the existence of a function $d(x)$ verifying the properties (2.1) and (2.2), the proof of Theorem 2 is completed.

The above arguments show also the following.

PROPOSITION 6. *If the coefficient* $a(x)$ *of* A *satisfies the assumptions* (N^∞), *then there exists a function* $d(x)\in B^\infty$ *verifying*

$$d(x) \geqq a(x) \tag{2.20}$$

$$K\log(x+2)+4M \geqq \int_0^x d(y)dy \geqq -4M \qquad for \quad x \geqq 0 \tag{2.21}$$

and

$$4M \geqq \int_0^x d(y)dy \geqq -K\log(|x|+2)-4M \qquad for \quad x \leqq 0. \tag{2.22}$$

Indeed we apply Lemma 5 with $l(t)=K\log(t+2)$ to the sequences $\{A_n^\pm\}_{n=1,2,\dots}$ and $\{B_n^\pm\}_{n=1,2,\dots}$ defined by (2.7) and (2.9). Then the arguments drawing (2.15), (2.16) and (2.17) show that $d(x)$ defined by (2.14) satisfies (2.20), (2.21) and (2.22).

3. The proof of Theorem 3.

First of all, we show the uniqueness of solutions under the assumptions of Theorem 3. From Proposition 6 follows the existence of a function $d(x)$ verifying (2.20), (2.21) and (2.22). From (2.21), (2.22) and $d(x)\in B^\infty$ we see that the multiplication by $\exp\left(\int_0^x d(y)dy/3\right)$ is an automorphism of S. On the other hand by (2.20) and Proposition 4 we see the S-well posedness of the Cauchy problem (C) for the operator $\tilde{A}$ defined by (2.3). Hence the Cauchy problem for the operator A itself is S-wellposed. This claim is also valid for the backward Cauchy problem for the formal adjoint A^* of A;

$$\begin{aligned} A^*v(t, x) &= f(t, x) \qquad (t, x) \in [0, T]\times \boldsymbol{R} \\ v(T, x) &= g(x) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} A^*v(t, x) = &-\partial_t v(t, x)-\partial_x^3 v(t, x)+\partial_x^2(a(x)v(t, x)) \\ &-\partial_x(b(x)v(t, x))+c(x)v(t, x), \end{aligned}$$

because by the change of the variable $t=T-s$ we have the forward Cauchy problem for the operator which satisfies the assumptions of Theorem 3.

Hence by the duality method we see the uniqueness of solutions in $C([0, T], S')$ for the forward Cauchy problem for A (see S. Mizohata [**5**, Proof of Theorem 4.2]).

In the following we show the existence and estimate of solutions.

PROPOSITION 7. *Let $e(x)$ be a smooth function such that*

$$\frac{d^l}{dx^l}e(x)\ (l=1,2,\cdots)\quad \text{and}\quad \exp(e(x))\frac{d}{dx}e(x)$$

are bounded on $\boldsymbol{R}$.

If the coefficient $a(x)$ of A is non-positive and the datum and the right hand side of the problem (C) *satisfy: for some integer $N\geqq 0$*

$$\exp(ke(x))g(x)\in H^\infty$$

$$\exp(ke(x))f(t,x)\in C([0,T],H^\infty)\qquad \text{for any}\quad k=0,1,2,\cdots,N$$

then the solution $u(t,x)$ of the Cauchy problem (C) *satisfies*

$$\exp(ke(x))u(t,x)\in C([0,T],H^\infty)\qquad k=0,1,2,\cdots,N$$

and for any integer $l\geqq 0$ and any $t\in[0,T]$

$$\text{(3.2)}\qquad \|\exp(Ne(x))u(t,x)\|_l$$

$$\leqq C_l\sum_{0\leq j\leq N,\,2j+k\leq 2N+l}\left(\|\exp(je(x))g(x)\|_k+\int_0^t\|\exp(je(x))f(s,x)\|_k ds\right)$$

PROOF. Proposition 4 asserts L^2 and H^∞-wellposedness of the Cauchy problem (C). Thus $u(t,x)\in C([0,T],H^\infty)$.

Set $u_{1,0}(t,x)=\exp(e(x))u(t,x)$. Then $u_{1,0}(t,x)$ satisfies the equation;

$$\text{(3.3)}\qquad \partial_t u_{1,0}(t,x)+\partial_x^3 u_{1,0}(t,x)+a(x)\partial_x^2 u_{1,0}(t,x)+\tilde{b}(x)\partial_x u_{1,0}(t,x)$$

$$+\tilde{c}(x)u_{1,0}(t,x)=3\exp(e(x))e'(x)\partial_x^2 u(t,x)+\exp(e(x))f(t,x)$$

$$\text{(3.4)}\qquad u_{1,0}(0,x)=\exp(e(x))g(x)$$

with some $\tilde{b}(x)$ and $\tilde{c}(x)\in B^\infty$, where we used

$$\partial_x^3 u(t,x)+3e'(x)\partial_x^2 u(t,x)$$

$$=\exp(-e(x))\{\partial_x^3 u_{1,0}(t,x)+k_1(x)\partial_x u_{1,0}(t,x)+k_0(x)u_{1,0}(t,x)\}$$

$$\text{with}\quad k_1(x)=-3\{e'(x)\}^2-3e''(x)$$

$$k_0(x)=2\{e'(x)\}^3-e'''(x)$$

and the assumption of $e(x)$.

From the assumptions on $f(t,x)$, $g(x)$ and $e(x)$ and $u(t,x)\in C([0,T],H^\infty)$, the right hand side of (3.3) belongs to $C([0,T],L^2)$ and $u_{1,0}(0,x)\in H^\infty$.

Since $a(x)\leqq 0$, the Cauchy problem (3.3) and (3.4) is L^2-wellposed. The above mentioned uniqueness in $C([0,T],S')$ implies

$$u_{1,0}(t,x)\in C([0,T],L^2)$$

and

$$\|u_{1,0}(t,\ x)\| \leqq C_{1,0}\left\{\|\exp(e(x))g(x)\|+\int_0^t(\|\partial_x^2 u(s,\ x)\|+\|\exp(e(x))f(s,\ x)\|)ds\right\}.$$

Since the original Cauchy problem for $u(t, x)$ is also L^2-wellposed and $u(t, x)$ and $f(t, x)\in C([0, T], H^\infty)$ and $g(x)\in H^\infty$, we have

$$\|\partial_x^2 u(t,\ x)\| \leqq C_{0,2}\sum_{k=0}^{2}\left(\|\partial_x^k g(x)\|+\int_0^t\|\partial_x^k f(s,\ x)\|ds\right).$$

Thus

$$\|u_{1,0}(t,\ x)\| \leqq C_{1,0}\sum_{2j+k\leqq 2}\left\{\|\exp(je(x))\partial_x^k g(x)\|+\int_0^t\|\exp(je(x))\partial_x^k f(s,\ x)\|ds\right\}.$$

Inductively we can show similar claim for $u_{1,m}(t, x)=\exp(e(x))\partial_x^m u(t, x)$ with $m\geqq 1$. Generally we have, by the induction, for $l=0, 1, \cdots, N$ and $m\geqq 0$,

$$u_{l,m}(t,\ x)=\exp(le(x))\partial_x^m u(t,\ x)\in C([0,\ T],\ L^2),$$

and

$$\|u_{l,m}(t,\ x)\|^2 \leqq C_{l,m}\sum_{\substack{2j+k\leqq 2l+m\\ j\leqq l}}\left\{\|\exp(je(x))\partial_x^k g(x)\|^2+\int_0^t\|\exp(je(x))\partial_x^k f(s,\ x)\|^2ds\right\}. \quad \blacksquare$$

In the following we use Proposition 7 in the case $e(x)=\log\langle x-p\rangle$ with an integer p. That is to say;

COROLLARY 8. *If the supports of $g(x)\in H^\infty$ and $f(t, x)\in C([0, T], H^\infty)$ are compact, the solution $u(t, x)$ of the problem* (C) *for the operator A with a non-positive coefficient $a(x)$ satisfies: for any integers $N\geqq 0$, $l\geqq 0$ and p and any $t\in [0, T]$*

$$\|\langle x-p\rangle^N u(t,\ x)\|_l \leqq C_{l,N}\sum_{0\leqq j\leqq N,\ 2j+k\leqq 2N+l}\left(\|\langle x-p\rangle^j g(x)\|_k+\int_0^t\|\langle x-p\rangle^j f(s,\ x)\|_k ds\right) \tag{3.5}$$

where the constant $C_{l,N}$ is independent of p.

We remark that Corollary 8 is already shown in T. Kato [**3**, Section 8].
We use also the following lemma owing to T. Kato [**3**, Section 10].

LEMMA 9. *Under the assumptions of Corollary 8 on $g(x)$, $f(t, x)$ and the coefficient $a(x)$ of A, the solution $u(t, x)$ of the problem* (C) *satisfies: for any $b\geqq 0$*

$$e^{bx}u(t,\ x)\in C([0,\ T],\ H^\infty)$$

$$\|e^{bx}u(t,\ x)\|_l \leqq C\left(\|e^{bx}g(x)\|_l+\int_0^t\|e^{bx}f(s,\ x)\|_l ds\right) \tag{3.6}$$

for any integer $l \geqq 0$ *and any* $t \in [0, T]$.

With a smooth function $\chi(x)$ satisfying $1 \geqq \chi(x) \geqq 0$, $\chi(x)=1$ for $x \geqq 1$, and $\chi(x)=0$ for $x \leqq 0$, we define a partition of unity $\{\phi_n(x)\}_{n=0,\pm 1,\pm 2,\dots}$ defined by

$$\phi_n(x) = \chi(x-n+1) - \chi(x-n).$$

We decompose the datum $g(x) \in H^\infty$ and the right-hand side $f(t, x) \in C([0, T], H^\infty)$ of the problem (C) in the following way. Let for $n=0, \pm 1, \pm 2, \cdots$

$$\begin{aligned} g_n(x) &= \phi_n(x) g(x) \\ f_n(t, x) &= \phi_n(x) f(t, x). \end{aligned} \tag{3.7}$$

As we remarked at the beginning of this section, under (N^∞) the problem (C) is S-wellposed. There exists the solution $u_n(t, x) \in C([0, T], S)$ of the problem

$$(C_n) \qquad \begin{cases} Au_n(t, x) = f_n(t, x) & (t, x) \in [0, T] \times \boldsymbol{R} \\ u_n(0, x) = g_n(x) & x \in \boldsymbol{R}. \end{cases}$$

PROPOSITION 10. *There exists an integer* $N \geqq 0$ *such that for any integers* $l \geqq 0$, m *and* n *and any* $t \in [0, T]$

$$\|\phi_m(x) u_n(t, x)\|_l \leqq C_l \langle m-n \rangle^{-2} \left(\|g_n(x)\|_{l+N} + \int_0^t \|f_n(s, x)\|_{l+N} ds \right) \tag{3.8}$$

where the constant C_l *is independent of* m *and* n.

Using Proposition 10, we show that the sum $\sum_{n=-\infty}^{+\infty} u_n(t, x)$ is a solution of the problem (C) belonging to $C([0, T], H^\infty)$.

First we remark that for any $h(x) \in H^\infty$ and any integers $l \geqq 0$ and s and t satisfying $s \leqq t$,

$$\left\| \sum_{n=s}^{t} \phi_n(x) h(x) \right\|_l \leqq C_1 \left(\sum_{n=s}^{t} \|\phi_n(x) h(x)\|_l^2 \right)^{1/2} \leqq C_2 \left\| \sum_{n=s-1}^{t+1} \phi_n(x) h(x) \right\|_l \tag{3.9}$$

with positive constants C_1 and C_2 which are independent of s and t, where we used the following properties of $\phi_n(x)$:

$$\begin{aligned} \frac{d^p}{dx^p}(\phi_m(x)h(x)) \frac{d^p}{dx^p}(\phi_n(x)h(x)) &= 0 \qquad \text{for} \quad |n-m| \geqq 2, \\ \phi_n(x) &= 0 \qquad \text{for} \quad |x-n| \geqq 1 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \sum_{n=s-1}^{t+1} \phi_n(x) &= \chi(x-s+2) - \chi(x-t-1) \\ &= 1 \qquad \text{on} \quad [s-1, t+1]. \end{aligned} \tag{3.11}$$

From (3.9) and (3.11) we see that

$$\left(\sum_{n=-\infty}^{+\infty}\|g_n(x)\|_{l+N}^2\right)^{1/2} \leqq C\|g(x)\|_{l+N}, \tag{3.12}$$

$$\left(\sum_{n=-\infty}^{+\infty}\|f_n(t, x)\|_{l+N}^2\right)^{1/2} \leqq C\|f(t, x)\|_{l+N}. \tag{3.13}$$

From (3.13) we see that the l^2-valued function on $[0, T]$

$$t \longrightarrow \{\|f_n(t, x)\|_{l+N}\}_{n\in Z}$$

is continuous, where l^2 is a space of all square summable sequences $\{a_n\}_{n\in Z}$. Thus we have from (3.13)

$$\left\{\sum_{n=-\infty}^{+\infty}\left(\int_0^t\|f_n(s, x)\|_{l+N}ds\right)^2\right\}^{1/2} \leqq C\int_0^t\|f(t, x)\|_{l+N}ds. \tag{3.14}$$

From (3.12) and (3.14) we have

$$\left\{\|g_n(x)\|_{l+N}+\int_0^T\|f_n(s, x)\|_{l+N}ds\right\}_{n\in Z} \in l^2.$$

On the other hand for any fixed m

$$\{\langle m-n\rangle^{-2}\}_{n\in Z} \in l^2.$$

Hence we see from (3.8) that, for any m, $\sum_{n=-\infty}^{+\infty}\phi_m(x)u_n(t, x)$ is an absolutely convergent series in $C([0, T], H^l)$. Taking (3.11) into account, we see that for any $k\geqq 0$

$$\sum_{n=-\infty}^{+\infty}u_n(t, x) \in C([0, T], H^l((-k, k))).$$

Let $u(t, x)=\sum_{n=-\infty}^{+\infty}u_n(t, x)$. Noting that $\sum_{n=-\infty}^{+\infty}g_n(x)=g(x)$ and $\sum_{n=-\infty}^{+\infty}f_n(t, x)=f(t, x)$, we see from (C_n) that $u(t, x)$ is a solution of the problem (C).

Next we show that $u(t, x) \in C([0, T], H^l)$. Since $\sum_{m=-\infty}^{+\infty}\langle m\rangle^{-2}<+\infty$, Hausdorff-Young inequality and (3.8) imply that

$$\left\{\sum_{m=-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty}\|\phi_m(x)u_n(t, x)\|_l\right)^2\right\}^{1/2} \tag{3.15}$$

$$\leqq C\left\{\sum_{n=-\infty}^{+\infty}\left(\|g_n(x)\|_{l+N}+\int_0^t\|f_n(s, x)\|_{l+N}ds\right)^2\right\}^{1/2},$$

from (3.12) and (3.14)

$$\leqq C\left(\|g(x)\|_{l+N}+\int_0^t\|f(s, x)\|_{l+N}ds\right).$$

Since $\|\phi_m(x)u(t, x)\|_l\leqq\sum_{n=-\infty}^{+\infty}\|\phi_m(x)u_n(t, x)\|_l$, we have from (3.15)

$$\left(\sum_{m=-\infty}^{+\infty}\|\phi_m(x)u(t, x)\|_l^2\right)^{1/2} \leqq C\left(\|g(x)\|_{l+N}+\int_0^t\|f(s, x)\|_{l+N}ds\right). \tag{3.16}$$

For any $t\in[0, T]$, from (3.9), (3.11) and (3.16) we see that

(3.17) $$u(t, x) = \lim_{k\to+\infty} \sum_{m=-k}^{k} \phi_m(x)u(t, x) \in H^l$$

and

$$\|u(t, x)\|_l \leqq C\left(\|g(x)\|_{l+N} + \int_0^t \|f(s, x)\|_{l+N} ds\right).$$

The convergence of (3.17) is uniform on $[0, T]$, because, putting

$$A_m = C_l\left\{\sum_{n=-\infty}^{+\infty} \langle m-n\rangle^{-2}\left(\|g_n(x)\|_{l+N} + \int_0^T \|f_n(s, x)\|_{l+N} ds\right)\right\},$$

we see that

$$\|\phi_m(x)u(t, x)\|_l \leqq A_m \quad \text{on} \quad [0, T]$$

and

$$\sum_{m=-\infty}^{+\infty} A_m^2 < +\infty .$$

Hence $u(t, x) \in C([0, T], H^l)$.

Since the above argument is valid for any integer $l \geqq 0$, we see that $u(t, x) \in C([0, T], H^\infty)$ and that $u(t, x)$ is a solution of the problem (C) satisfying the desired estimates.

Therefore we have only to prove Proposition 10 in order to complete the proof of Theorem 3.

PROOF OF PROPOSITION 10. In the following we denote by C or a suffixed C_* a constant which is independent of m and n. We consider another problem for the operator $\tilde{A}$ defined by (2.3) with $d(x) = a(x)$, that is to say,

$$\begin{aligned}\tilde{A}u(t, x) &= \exp\left(\frac{1}{3}\int_0^x a(y)dy\right) A \exp\left(-\frac{1}{3}\int_0^x a(y)dy\right) u(t, x) \\ &= \partial_t u(t, x) + \partial_x^3 u(t, x) + \tilde{b}(x)\partial_x u(t, x) + \tilde{c}(x)u(t, x)\end{aligned}$$

with some $\tilde{b}(x)$ and $\tilde{c}(x) \in B^\infty$.

As $\exp\left(\int_0^x a(y)dy/3\right) g_n(x) \in S$ and $\exp\left(\int_0^x a(y)dy/3\right) f_n(t, x) \in C([0, T], S)$, there exists a solution $v_n(t, x) \in C([0, T], S)$ of the problem

$$(\tilde{\mathrm{C}}_n) \quad \begin{cases} \tilde{A}v_n(t, x) = \exp\left(\dfrac{1}{3}\displaystyle\int_0^x a(y)dy\right) f_n(t, x) & (t, x) \in [0, T] \times \boldsymbol{R} \\ v_n(0, x) = \exp\left(\dfrac{1}{3}\displaystyle\int_0^x a(y)dy\right) g_n(x) & x \in \boldsymbol{R}. \end{cases}$$

From (N^∞), we have

$$K \log(x+2) \geqq \int_0^x a(y)dy \geqq -Mx \quad \text{for} \quad x \geqq$$

and

$$M|x| \geqq \int_0^x a(y)dy \geqq -K \log(|x|+2) \qquad \text{for} \quad x \leqq 0$$

with the constant K appearing in (N^∞) and $M=\sup_{x\in R}|a(x)|$.

Then

$$\text{(3.18)} \quad \begin{aligned} C\langle x\rangle^{K/3} &\geqq \exp\Big(-\frac{1}{3}\int_0^x a(y)dy\Big) \geqq e^{-M|x|/3} \qquad \text{for} \quad x \leqq 0\,, \\ e^{Mx/3} &\geqq \exp\Big(-\frac{1}{3}\int_0^x a(y)dy\Big) \geqq C\langle x\rangle^{-K/3} \qquad \text{for} \quad x \geqq 0\,. \end{aligned}$$

From Lemma 9 and the above estimates, we see that $\exp\Big(-\int_0^x a(y)dy/3\Big) \times v_n(t, x) \in C([0, T], S)$ and this is a solution of (C_n). Then the uniqueness of the solution implies that $u_n(t, x)=\exp\Big(-\int_0^x a(y)dy/3\Big)v_n(t, x)$.

As $g_n(x)$ and $f_n(t, x)$ vanish for $|x-n|\geqq 1$ and $a(x)\in B^\infty$, we see that for any integers N and $l\geqq 0$,

$$\Big\|\langle x-n\rangle^N \exp\Big(\frac{1}{3}\int_0^x a(y)dy\Big)g_n(x)\Big\|_l \leqq C \exp\Big(\frac{1}{3}\int_0^n a(y)dy\Big)\|g_n(x)\|_l\,,$$

$$\Big\|\langle x-n\rangle^N \exp\Big(\frac{1}{3}\int_0^x a(y)dy\Big)f_n(t, x)\Big\|_l \leqq C \exp\Big(\frac{1}{3}\int_0^n a(y)dy\Big)\|f_n(t, x)\|_l\,.$$

Thus it follows from Corollary 8 that

$$\text{(3.19)} \quad \begin{aligned} &\|\langle x-n\rangle^N v_n(t, x)\|_l \\ &\leqq C_{l,N} \exp\Big(\frac{1}{3}\int_0^n a(y)dy\Big)\Big(\|g_n(x)\|_{2N+l}+\int_0^t \|f_n(s, x)\|_{2N+l}ds\Big). \end{aligned}$$

As $\phi_m(x)$ vanishes for $|x-m|\geqq 1$, we have

$$\text{(3.20)} \quad \begin{aligned} &\Big\|\phi_m(x)\exp\Big(-\frac{1}{3}\int_0^x a(y)dy\Big)v_n(t, x)\Big\|_l \\ &\leqq C \exp\Big(-\frac{1}{3}\int_0^m a(y)dy\Big)\langle m-n\rangle^{-N}\|\phi_m(x)\langle x-n\rangle^N v_n(t, x)\|_l \\ &\leqq C \exp\Big(-\frac{1}{3}\int_0^m a(y)dy\Big)\langle m-n\rangle^{-N}\|\langle x-n\rangle^N v_n(t, x)\|_l\,. \end{aligned}$$

It follows from (3.19) and (3.20) that

$$\text{(3.21)} \quad \begin{aligned} &\Big\|\phi_m(x)\exp\Big(-\frac{1}{3}\int_0^x a(y)dy\Big)v_n(t, x)\Big\|_l \\ &\leqq C \exp\Big(\frac{1}{3}\int_m^n a(y)dy\Big)\langle m-n\rangle^{-N}\Big(\|g_n(x)\|_{2N+l}+\int_0^t \|f_n(s, x)\|_{2N+l}ds\Big). \end{aligned}$$

When $n\geqq m$, (N^∞) implies that

$$\exp\Big(\frac{1}{3}\int_m^n a(y)dy\Big)\langle m-n\rangle^{-N} \leqq C\langle m-n\rangle^{K/3-N}\,.$$

Thus, by taking $N-K/3 \geqq 2$, we see from (3.21) that for $n \geqq m$

$$\left\| \phi_m(x) \exp\left(-\frac{1}{3}\int_0^x a(y)dy\right) v_n(t, x) \right\|_l \leqq C\langle m-n\rangle^{-2}\left(\|g_n(x)\|_{2N+l} + \int_0^t \|f_n(s, x)\|_{2N+l} ds\right). \tag{3.22}$$

On the other hand, in the case where $m>n$, put

$$b = \frac{1}{3}\sup_{x\in\boldsymbol{R}} |a(x)| + 1. \tag{3.23}$$

From (3.7) and (3.10) it follows that

$$\left\| e^{bx} \exp\left(\frac{1}{3}\int_0^x a(y)dy\right) g_n(x) \right\|_l + \int_0^t \left\| e^{bx} \exp\left(\frac{1}{3}\int_0^x a(y)dy\right) f_n(s, x) \right\|_l ds$$

$$\leqq C \exp\left(\frac{1}{3}\int_0^n a(y)dy\right) e^{bn} \left(\|g_n(x)\|_l + \int_0^t \|f_n(s, x)\|_l ds\right).$$

Thus from the above estimate and Lemma 9, we see that

$$\|e^{bx} v_n(t, x)\|_l \leqq C \exp\left(\frac{1}{3}\int_0^n a(y)dy\right) e^{bn} \left(\|g_n(x)\|_l + \int_0^t \|f_n(s, x)\|_l ds\right). \tag{3.24}$$

From (3.10) we have

$$\left\| \phi_m(x) \exp\left(-\frac{1}{3}\int_0^x a(y)dy\right) v_n(t, x) \right\|_l$$

$$\leqq C \exp\left(-\frac{1}{3}\int_0^m a(y)dy\right) e^{-bm} \|\phi_m(x) e^{bx} v_n(t, x)\|_l$$

$$\leqq C \exp\left(-\frac{1}{3}\int_0^m a(y)dy\right) e^{-bm} \|e^{bx} v_n(t, x)\|_l .$$

The above estimate and (3.24) imply that

$$\left\| \phi_m(x) \exp\left(-\frac{1}{3}\int_0^x a(y)dy\right) v_n(t, x) \right\|_l \leqq C \exp\left(-\frac{1}{3}\int_n^m a(y)dy\right) e^{-b(m-n)} \left(\|g_n(x)\|_l + \int_0^t \|f_n(s, x)\|_l ds\right). \tag{3.25}$$

Since $m>n$, we see from (3.23) that

$$\exp\left(-\frac{1}{3}\int_n^m a(y)dy\right) e^{-b(m-n)} \leqq e^{-(m-n)} \leqq 2\langle n-m\rangle^{-2}.$$

Hence we see from (3.25) that (3.22) is also valid for $m>n$.

Since $u_n(t, x) = \exp\left(-\int_0^x a(y)dy/3\right) v_n(t, x)$, we obtain (3.8). ■

The author wishes to thank the referee for the invaluable remarks which helped to improve the results of the earlier version of the text.

References

[1] W. Ichinose, Some remarks on the Cauchy problem for Schrödinger type equations, Osaka J. Math., **21** (1984), 565-581.

[2] W. Ichinose, Sufficient condition on H_∞ well-posedness for Schrödinger type equations, Comm. Partial Differential Equations, **9-1** (1984), 33-48.

[3] T. Kato, On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation, Studies in Appl. Math. Ad. in Math. Suppl. Stud., **8** (1983), 93-128.

[4] S. Mizohata, On the Cauchy problem, Notes and Reports in Math., **3**, Academic Press, 1985.

[5] S. Mizohata, The theory of partial differential equations, Cambridge University Press, 1973.

[6] J. Takeuchi, A necessary condition for the well-posedness of the Cauchy problem for certain class of evolution equations, Proc. Japan Acad., **50** (1974), 133-137.

[7] J. Takeuchi, A necessary condition for H^∞-wellposedness of the Cauchy problem for linear partial differential operators of Schrödinger type, J. Math. Kyoto Univ., **25** (1985), 459-472.

Shigeo TARAMA
Department of Applied
Mathematics and Physics
Faculty of Engineering
Kyoto University
606-01 Kyoto
Japan