J. Math. Soc. Japan
Vol. 47, No. 1, 1995

Braid relations, meta-abelianizations and the symbols $\{p, -1\}$ in $K_2(2, \boldsymbol{Z}[1/p])$

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(Received Oct. 3, 1991)
(Revised June 10, 1993)

1. Introduction.

In [5], a finite presentation of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ was given for a prime number p, where two braid relations appeared. The purpose of this paper is to study the group structures of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ and $K_2(2, \boldsymbol{Z}[1/p])$. To do this, first we will look at the braid groups (cf. [1]). Then we obtain the following.

$$(1) \qquad \mathrm{St}\left(2, \boldsymbol{Z}\left[\frac{1}{p}\right]\right)^{mab} \cong \begin{cases} Z_3 \ltimes (Z_2 \times Z_2) & \text{if } p=2; \\ Z_8 \ltimes Z_3 & \text{if } p=3; \\ Z_{p^2-1} \ltimes (Z \times Z) & \text{otherwise.} \end{cases}$$

(2) If $p \neq 2, 3, 5, 11$, then $\{p, -1\}^2 \neq 1$ in $K_2(2, \boldsymbol{Z}[1/p])$.

(3) If $p \neq 2, 3, 5, 11$, then $K_2(2, \boldsymbol{Z}[1/p]) \not\cong Z \times Z_{p-1}$.

It is already known that

$$(*) \qquad K_2(2, \boldsymbol{Z}_S) \cong Z \times \prod_{p \in S} Z_{p-1}$$

if S is one of {*the first n successive prime numbers*} with $n \geqq 1$, $\{3\}$, $\{2, 5\}$, $\{2, 3, 7\}$, $\{2, 3, 11\}$, $\{2, 3, 5, 11\}$, $\{2, 3, 13\}$, $\{2, 3, 7, 13\}$, $\{2, 3, 17\}$, and $\{2, 3, 5, 19\}$, where $\boldsymbol{Z}_S = \boldsymbol{Z}[1/p]_{p \in S}$ (cf. [4], [5]). One might expect that (*) holds for every set S of prime numbers. But the above (3) tells us that (*) is not true in general.

Here, we fix our notation as follows. Let $\boldsymbol{Z}$ be the ring of rational integers, $\boldsymbol{Q}$ the field of rational numbers, and $\boldsymbol{R}$ the field of real numbers. For elements x, y in a group, we set $x^y = yxy^{-1}$. For subgroups H_1, H_2 of a group G we denote by $[H_1, H_2]$ the subgroup of G generated by $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ for all $h_1 \in H_1$, $h_2 \in H_2$. Then, put $G' = [G, G]$, $G'' = [G', G']$, $G^{ab} = G/G'$, $G^{mab} = G/G''$ and $G'^{ab} = G'/G''$. We use Z_m for the cyclic group of order m, and Z for Z_∞. If a group H acts on another group K, the semi-direct product of H and K is denoted by $H \ltimes K$. And $\langle generators | relations \rangle$ means a group presentation as usual.

2. Braid relations and meta-abelianizations.

The presentation of St(2, $\boldsymbol{Z}[1/p]$), established in [**5**], tells us that braid relations are very important to study St(2, $\boldsymbol{Z}[1/p]$) and its meta-abelianization. Therefore, we want to see what happens in braid groups. The n-braid group B_n has the presentation defined by the generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and the defining relations ($n\geqq 2$):

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

for $1\leqq i\leqq n-2$, and

$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

for $1\leqq i, j\leqq n-1$ with $|i-j|>1$ (cf. [**1**]). Then we obtain the following.

THEOREM 1. (1) $B_n^{ab}\cong Z$.

(2) $B_n'=[B_n, B_n']$.

(3)

$$B_n'^{ab} \cong \begin{cases} Z\times Z & \textit{if } n=3, 4; \\ 1 & \textit{otherwise.} \end{cases}$$

(4)

$$\begin{aligned} B_n^{mab} &\cong B_n^{ab}\ltimes B_n'^{ab} \\ &\cong \begin{cases} Z\ltimes(Z\times Z) & \textit{if } n=3, 4; \\ Z & \textit{otherwise.} \end{cases} \end{aligned}$$

These facts are probably well-known to many people already, but there seems to be no good reference. Indeed, the above results for $n=2, 3$ are well-known or almost trivial, and those for $n\geqq 5$ follow from the fact that

$$[\sigma_{i-1}, \sigma_i][\sigma_i, \sigma_{i+1}] \equiv [\sigma_i, \sigma_{i+1}][\sigma_{i-1}, \sigma_i] \qquad \bmod B_n''$$

implies

$$[\sigma_{i-1}, \sigma_i] \equiv [\sigma_{i+1}, \sigma_i] \qquad \bmod B_n''$$

and

$$[\sigma_i, \sigma_{i+1}] \equiv [\sigma_1, \sigma_2]^{\pm 1} \qquad \bmod B_n''.$$

Using the natural (folding) homomorphism of B_4 onto B_3, one can easily establish the results for $n=4$. Here we pick up some essence which will be used later.

Let $B_3=\langle a, b\,|\,aba=bab\rangle$ be the 3-braid group. Then $a=b[a, b]$. Therefore,

$$\begin{aligned} b^3[b^{-2}ab^2, b][a, b] &= b[a, b]bb[a, b] \\ &= aba \\ &= bab \\ &= bb[a, b]b \\ &= b^3[b^{-1}ab, b] \end{aligned}$$

and hence $[b^{-1}ab, b]=[b^{-2}ab^2, b][a, b]$. As a fact, we know

$$B_3^{ab} = \langle b \bmod B_3' \rangle \cong Z\,,$$

$$B_3'^{ab} = \langle [a, b], [b^{-1}ab, b] \bmod B_3'' \rangle \cong Z \times Z\,,$$

and

$$\begin{aligned} B_3^{mab} &\cong B_3^{ab} \ltimes B_3'^{ab} \\ &\cong Z \ltimes (Z \times Z). \end{aligned}$$

The action of b on $Z \times Z$ $(=Z \oplus Z)$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

(with respect to a basis $\{u_1=[a, b]=\begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2=[b^{-1}ab, b]=\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$) and, then, $b^3ub^{-3}=u^{-1}$ for all $u \in Z \times Z$. In particular,

$$[a, b] \equiv [b^{-3}ab^3, b]^{-1} \bmod B_3''\,.$$

Note that

$$(aba)^2 = b^3[b^{-1}ab, b]\cdot b^3[b^{-1}ab, b] \equiv b^6 \bmod B_3''\,.$$

We will frequently use these relations in B_3 or in B_3^{mab} applying the canonical homomorphism θ_i of B_3 into $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ with $a \mapsto x_i$, $b \mapsto y_i$ for each $i=1, 2$ (for example, see Propositions 4, 5, 6).

3. The group St(2, $\boldsymbol{Z}[1/p]$).

Let G be the Steinberg group, usually denoted by $\mathrm{St}(2, \boldsymbol{Z}[1/p])$, of rank one over $\boldsymbol{Z}[1/p]$, where p is a prime number. Then G has the following finite presentation:

(generators)

$$x_1,\ x_2,\ y_1,\ y_2,$$

(defining relations)

$$x_iy_ix_i = y_ix_iy_i \quad (i=1, 2), \qquad x_1 = y_2^p\,, \qquad x_2 = y_1^p\,,$$

$$[(x_1y_1x_1)^2,\ y_2]=[(x_2y_2x_2)^2,\ y_1]=1$$

(cf. [**5**]). Taking $x_1=y_1=1$ and $x_2=y_2=p$, we obtain a homomorphism of G onto $Z_{p^2-1}\cong \boldsymbol{Z}/(p^2-1)\boldsymbol{Z}$, and we see that $G^{ab}\cong Z_{p^2-1}$. By the definition of the Steinberg symbol $\{p,\ -1\}$ (cf. [**2**], [**3**], [**5**]),

$$\begin{aligned}\{p,\ -1\} &= \{p,\ -1\}_\alpha \\ &= \left\{\frac{1}{p},\ -1\right\}_\alpha \\ &= h_\alpha\left(-\frac{1}{p}\right)h_\alpha\left(\frac{1}{p}\right)^{-1}h_\alpha(-1)^{-1} \\ &= w_\alpha\left(-\frac{1}{p}\right)w_\alpha\left(-\frac{1}{p}\right)w_\alpha(1)^2 \\ &= w_\alpha(1)^2w_{-\alpha}(p)^2 \\ &= (x_1y_1x_1)^2(x_2y_2x_2)^{-2}\,.\end{aligned}$$

The above isomorphism implies that $\{p,\ -1\}$ mod G' is corresponding to $6(1-p)$ in $Z_{p^2-1}\cong \boldsymbol{Z}/(p^2-1)\boldsymbol{Z}$. Hence, we obtain the following.

PROPOSITION 2. (1) $\{p,\ -1\}\in G'$ *if and only if* $p=2,\ 5$.
(2) $\{p,\ -1\}^2\in G'$ *if and only if* $p=2,\ 3,\ 5,\ 11$.

Next we need compute several relations of commutators.

PROPOSITION 3.

$$[x_1,\ x_2]\equiv[y_2,\ y_1] \quad \mod G''\,.$$

PROOF. Since $y_1^{-1}x_1=[x_1,\ y_1]$ and $y_2^{-1}x_2=[x_2,\ y_2]$, we see

$$\begin{aligned}y_1^{-1}y_2^{-1}x_1x_2 &\equiv y_1^{-1}x_1y_2^{-1}x_2 \\ &\equiv [x_1,\ y_1][x_2,\ y_2] \\ &\equiv [x_2,\ y_2][x_1,\ y_1] \\ &\equiv y_2^{-1}x_2y_1^{-1}x_1 \\ &\equiv y_2^{-1}y_1^{-1}x_2x_1 \qquad \mod G'',\end{aligned}$$

and $[x_1,\ x_2]\equiv[y_2,\ y_1]$ mod G''. □

PROPOSITION 4. (1)

$$[x_1, x_2] \equiv \begin{cases} [x_1, y_1][y_1x_1y_1^{-1}, y_1] & \text{if } p=2; \\ [y_1x_1y_1^{-1}, y_1]^2 & \text{if } p=3; \\ [x_1, y_1] & \text{if } p\equiv 1 \bmod 6; \\ [y_1^2x_1y_1^{-2}, y_1] & \text{if } p\equiv 5 \bmod 6; \end{cases} \quad \bmod G''.$$

(2)

$$[x_2, x_1] \equiv \begin{cases} [x_2, y_2][y_2x_2y_2^{-1}, y_2] & \text{if } p=2; \\ [y_2x_2y_2^{-1}, y_2]^2 & \text{if } p=3; \\ [x_2, y_2] & \text{if } p\equiv 1 \bmod 6; \\ [y_2^2x_2y_2^{-2}, y_2] & \text{if } p\equiv 5 \bmod 6; \end{cases} \quad \bmod G''.$$

PROOF. (1)

$$\begin{aligned} [x_1, x_2] &= [x_1, y_1^p] \\ &= [x_1, y_1^{p-1}][y_1^{p-1}x_1y_1^{-(p-1)}, y_1] \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= [x_1, y_1][y_1x_1y_1^{-1}, y_1]\cdots[y_1^{p-2}x_1y_1^{-(p-2)}, y_1][y_1^{p-1}x_1y_1^{-(p-1)}, y_1]. \end{aligned}$$

Hence, we obtain (1). Similarly (2) can be shown. □

For $p=2, 3$, we get the following special result.

PROPOSITION 5.

(1) *If* $p=2$, *then* $[x_1, y_1]^2\equiv[x_2, y_2]^2\equiv 1 \bmod G''$.

(2) *If* $p=3$, *then* $[y_i^{-1}x_iy_i, y_i]\equiv[x_i, y_i]^2 \bmod G''$, *and* $[x_i, y_i]^3\equiv 1 \bmod G''$ *for* $i=1, 2$.

PROOF. Let $p=2$. Then,

$$\begin{aligned} [y_1^2x_1y_1^{-2}, y_1]^{-1}[y_1x_1y_1^{-1}, y_1] &\equiv [x_1, y_1] \\ &\equiv [y_2^2, y_1] \\ &\equiv [y_2, y_1]^{y_2}\cdot[y_2, y_1] \\ &\equiv [x_1, x_2]^{y_2}\cdot[x_1, x_2] \\ &\equiv [x_1, y_1]^{x_2}\cdot[y_1x_1y_1^{-1}, y_1]^{x_2}\cdot[x_1, y_1][y_1x_1y_1^{-1}, y_1] \\ &\equiv [y_1^2x_1y_1^{-2}, y_1][y_1^3x_1y_1^{-3}, y_1][x_1, y_1][y_1x_1y_1^{-1}, y_1] \\ &\equiv [y_1^2x_1y_1^{-2}, y_1][y_1x_1y_1^{-1}, y_1] \bmod G'' \end{aligned}$$

and $[y_1^2x_1y_1^{-2}, y_1]^2\equiv 1 \bmod G''$, hence, $[x_1, y_1]^2\equiv 1 \bmod G''$. Similarly, we obtain $[x_2, y_2]^2\equiv 1 \bmod G''$.

Next, suppose $p=3$. Then,

$$\begin{aligned}[x_1, y_1] &\equiv [y_2^3, y_1]\\ &\equiv [y_2, y_1]^{y_2^2}\cdot[y_2^2, y_1]\\ &\equiv [y_2, y_1]^{y_2^2}\cdot[y_2, y_1]^{y_2}\cdot[y_2, y_1]\\ &\equiv [x_1, x_2]^{y_2^2}\cdot[x_1, x_2]^{y_2}\cdot[x_1, x_2]\\ &\equiv ([y_1x_1y_1^{-1}, y_1]^{x_2^2}\cdot[y_1x_1y_1^{-1}, y_1]^{x_2}\cdot[y_1x_1y_1^{-1}, y_1])^2\\ &\equiv ([y_1x_1y_1^{-1}, y_1]^{y_1^6}\cdot[y_1x_1y_1^{-1}, y_1]^{y_1^3}\cdot[y_1x_1y_1^{-1}, y_1])^2\\ &\equiv [y_1x_1y_1^{-1}, y_1]^2 \bmod G''.\end{aligned}$$

Therefore, $[y_1^{-1}x_1y_1, y_1]\equiv[x_1, y_1]^2 \bmod G''$, and

$$\begin{aligned}[x_1, y_1] &\equiv [y_1^{-1}x_1y_1, y_1][x_1, y_1]^{-1}\\ &\equiv [y_1^{-2}x_1y_1^2, y_1]\\ &\equiv [y_1^{-1}x_1y_1, y_1]^2\\ &\equiv [x_1, y_1]^4 \bmod G''.\end{aligned}$$

Hence, $[x_1, y_1]^3\equiv 1 \bmod G''$. Similarly we obtain $[y_2^{-1}x_2y_2, y_2]\equiv[x_2, y_2]^2 \bmod G''$, and $[x_2, y_2]^3\equiv 1 \bmod G''$. □

On the other hand,

$$\begin{aligned}y_1^{p^2-1} &\equiv y_1^{p^2}y_1^{-1}\\ &\equiv x_2^py_1^{-1}\\ &\equiv (y_2[x_2, y_2])^py_1^{-1}\\ &\equiv y_2^p[y_2^{-(p-1)}x_2y_2^{p-1}, y_2][y_2^{-(p-2)}x_2y_2^{p-2}, y_2]\cdots[y_2^{-1}x_2y_2, y_2][x_2, y_2]y_1^{-1}\\ &\equiv x_1[y_2^{-(p-1)}x_2y_2^{p-1}, y_2][y_2^{-(p-2)}x_2y_2^{p-2}, y_2]\cdots[y_2^{-1}x_2y_2, y_2][x_2, y_2]y_1^{-1}\\ &\equiv \left\{\begin{array}{ll} x_1[y_2^{-1}x_2y_2, y_2][x_2, y_2]y_1^{-1} & \text{if } p=2;\\ x_1[y_2^{-1}x_2y_2, y_2]^2y_1^{-1} & \text{if } p=3;\\ x_1[x_2, y_2]y_1^{-1} & \text{if } p\equiv 1 \bmod 6;\\ x_1[y_2^{-2}x_2y_2^2, y_2]y_1^{-1} & \text{if } p\equiv 5 \bmod 6;\end{array}\right. \bmod G''.\end{aligned}$$

If $p=2$, then

$$\begin{aligned} y_1^3 &\equiv x_1[y_2^{-2}x_2y_2^2,\ y_2]y_1^{-1} \\ &\equiv [x_2,\ y_2][y_1x_1y_1^{-1},\ y_1] \\ &\equiv x_1^{-1}[y_2^2x_2y_2^{-2},\ y_2][y_1^2x_1y_1^{-2},\ y_1]x_1 \\ &\equiv x_1^{-1}[x_2,\ x_1][x_1,\ x_2]x_1 \\ &\equiv 1 \mod G''. \end{aligned}$$

If $p=3$, then

$$\begin{aligned} y_1^8 &\equiv x_1[x_2,\ y_2]y_1^{-1} \\ &\equiv x_1[x_1,\ y_1]^2y_1^{-1} \\ &\equiv [y_1x_1y_1^{-1},\ y_1]^2\cdot x_1y_1^{-1} \\ &\equiv [y_1x_1y_1^{-1},\ y_1]^2\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv 1 \mod G''. \end{aligned}$$

If $p\equiv 1 \mod 6$, then

$$\begin{aligned} y_1^{p^2-1} &\equiv x_1[y_1,\ x_1]y_1^{-1} \\ &\equiv x_1y_1x_1y_1^{-1}x_1^{-1}y_1^{-1} \\ &\equiv y_1x_1y_1y_1^{-1}x_1^{-1}y_1^{-1} \\ &\equiv 1 \mod G''. \end{aligned}$$

If $p\equiv 5 \mod 6$, then

$$\begin{aligned} y_1^{p^2-1} &\equiv y_2^{(p-2)}[x_2,\ y_2]y_2^{-(p-2)}\cdot x_1y_1^{-1} \\ &\equiv [x_2,\ y_2]^{-1}[y_1x_1y_1^{-1},\ y_1] \\ &\equiv y_2^{-2}[y_2^2x_2y_2^{-2},\ y_2]^{-1}y_2^2\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv x_2^{-2}[y_1^2x_1y_1^{-2},\ y_1]x_2^2\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv y_1^{-2p}[y_1^2x_1y_1^{-2},\ y_1]y_1^{2p}\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv y_1^{-2(p-1)}[x_1,\ y_1]y_1^{2(p-1)}\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv y_1^{-2}[x_1,\ y_1]y_1^2\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv y_1[x_1,\ y_1]^{-1}y_1^{-1}\cdot[y_1x_1y_1^{-1},\ y_1] \\ &\equiv 1 \mod G''. \end{aligned}$$

Hence, we obtain the following.

PROPOSITION 6. $y_1^{p^2-1}\in G''$.

Summarizing the above results, we obtain the following two theorems.

THEOREM 7. (1)

$$\mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)^{ab} \cong Z_{p^2-1}.$$

(2)

$$\mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)' = \Big[\mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big),\ \mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)'\Big].$$

(3)

$$\mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)'^{ab} \cong \begin{cases} Z_2\times Z_2 & \text{if } p=2; \\ Z_3 & \text{if } p=3; \\ Z\times Z & \text{otherwise.} \end{cases}$$

(4)

$$\mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)^{mab} \cong \mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)^{ab} \ltimes \mathrm{St}\Big(2,\ \boldsymbol{Z}\Big[\frac{1}{p}\Big]\Big)'^{ab}$$

$$\cong \begin{cases} Z_3\ltimes (Z_2\times Z_2) & \text{if } p=2; \\ Z_8\ltimes Z_3 & \text{if } p=3; \\ Z_{p^2-1}\ltimes (Z\times Z) & \text{otherwise.} \end{cases}$$

PROOF. (1) is already discussed (see the part before Proposition 2). To obtain (2), it is enough to show that $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ is abelian modulo $[\mathrm{St}(2, \boldsymbol{Z}[1/p]), \mathrm{St}(2, \boldsymbol{Z}[1/p])']$, which is easy. We will establish (3), (4) at the same time. By Proposition 6, our meta-abelianization splits, that is, $\mathrm{St}(2, \boldsymbol{Z}[1/p])^{mab} \cong \mathrm{St}(2, \boldsymbol{Z}[1/p])^{ab} \ltimes \mathrm{St}(2, \boldsymbol{Z}[1/p])'^{ab}$. We define the groups M_p and the homomorphisms α_p of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ onto M_p by

$$M_2 = \langle \sigma,\ \tau_1,\ \tau_2 \mid \sigma^3=\tau_1^2=\tau_2^2=[\tau_1,\ \tau_2]=1,\ \sigma\tau_1\sigma^{-1}=\tau_1\tau_2,\ \sigma\tau_2\sigma^{-1}=\tau_1\rangle,$$

$$\alpha_2(x_1)=\sigma\tau_1,\quad \alpha_2(y_1)=\sigma,\quad \alpha_2(x_2)=\sigma^2,\quad \alpha_2(y_2)=\sigma^2\tau_1\tau_2,$$

$$M_3 = \langle \sigma,\ \tau \mid \sigma^8=\tau^3=1,\ \sigma\tau\sigma^{-1}=\tau^2\rangle,$$

$$\alpha_3(x_1)=\sigma\tau,\quad \alpha_3(y_1)=\sigma,\quad \alpha_3(x_2)=\sigma^3,\quad \alpha_3(y_2)=\sigma^3\tau,$$

$$M_p = \langle \sigma,\ \tau_1,\ \tau_2 \mid \sigma^{p^2-1}=[\tau_1,\ \tau_2]=1,\ \sigma\tau_1\sigma^{-1}=\tau_1\tau_2^{-1},\ \sigma\tau_2\sigma^{-1}=\tau_1\rangle,$$

$$\begin{cases} \alpha_p(x_1)=\sigma\tau_1,\ \alpha_p(y_1)=\sigma,\ \alpha_p(x_2)=\sigma^p,\ \alpha_p(y_2)=\sigma^p\tau_1 & \text{if } p\equiv 1 \bmod 6; \\ \alpha_p(x_1)=\sigma\tau_1,\ \alpha_p(y_1)=\sigma,\ \alpha_p(x_2)=\sigma^p,\ \alpha_p(y_2)=\sigma^p\tau_1^{-1}\tau_2 & \text{if } p\equiv 5 \bmod 6. \end{cases}$$

The homomorphism α_p induces a homomorphism $\tilde{\alpha}_p$ of $\mathrm{St}(2, \boldsymbol{Z}[1/p])^{mab}$ onto M_p. On the other hand, we find the homomorphism $\tilde{\beta}_p$ of M_p onto $\mathrm{St}(2, \boldsymbol{Z}[1/p])^{mab}$ defined by

$$\tilde{\beta}_p(\sigma) = y_1 \mod \mathrm{St}\left(2, \boldsymbol{Z}\left[\frac{1}{p}\right]\right)'',$$

$$\tilde{\beta}_p(\tau_1) = \tilde{\beta}_p(\tau) = [x_1, y_1] \mod \mathrm{St}\left(2, \boldsymbol{Z}\left[\frac{1}{p}\right]\right)''$$

and

$$\tilde{\beta}_p(\tau_2) = [y_1^{-1}x_1y_1, y_1] \mod \mathrm{St}\left(2, \boldsymbol{Z}\left[\frac{1}{p}\right]\right)''.$$

Then, we can see easily that $\tilde{\alpha}_p\tilde{\beta}_p = id.$ and $\tilde{\beta}_p\tilde{\alpha}_p = id.$. Now the fact

$$M_p \cong \begin{cases} Z_3 \ltimes (Z_2 \times Z_2) & \text{if } p=2; \\ Z_8 \ltimes Z_3 & \text{if } p=3; \\ Z_{p^2-1} \ltimes (Z \times Z) & \text{otherwise} \end{cases}$$

implies the theorem. □

THEOREM 8. (1) $\{2, -1\} = 1$ *in* $\mathrm{St}(2, \boldsymbol{Z}[1/2])$.

(2) $\{3, -1\}^2 = 1$ *in* $\mathrm{St}(2, \boldsymbol{Z}[1/3])$.

(3) $\{5, -1\} \in \mathrm{St}(2, \boldsymbol{Z}[1/5])''$.

(4) $\{11, -1\}^2 \in \mathrm{St}(2, \boldsymbol{Z}[1/11])''$.

(5) *If* $p \neq 2, 3, 5, 11$, *then* $\{p, -1\}^2 \notin \mathrm{St}(2, \boldsymbol{Z}[1/p])'$, *hence, in particular,* $\{p, -1\}^2 \neq 1$ *in* $K_2(2, \boldsymbol{Z}[1/p])$.

PROOF. (1) is well-known. (2) is given in [**5**]. (3), (4) and (5) are directly obtained from Proposition 2. □

The author is very grateful to Prof. A. Munemasa, who obtained the structure of $\mathrm{St}(2, \boldsymbol{Z}[1/p])'^{ab}$ in case of $p=2, 3, 5, 11$ using a computer (workstation) with the "Cayley" system. The machine quickly gave the answer for $p=2, 3, 5$, but it took a while for $p=11$. This output was very helpful.

4. The group $K_2(2, \boldsymbol{Z}[1/p])$.

For every $n>1$ and every commutative ring A with 1, we can define $K_2(n, A)$ in general. It is an important problem to determine the group structure of $K_2(n, A)$, which is much interesting when A is a Dedekind domain. As an example, we take a finite set, called S, of prime numbers, and define $\boldsymbol{Z}_S = \boldsymbol{Z}[1/p]_{p \in S}$. Then, it is well-known that

$$K_2(n, \boldsymbol{Z}_S) \cong Z_2 \times \prod_{p \in S} Z_{p-1}$$

for every $n>2$ and every S (cf. [**2**], [**3**], [**7**]). In case of $n=2$, not so many results are known. A set of generators of $K_2(2, \boldsymbol{Z}[1/p])$ for $p \leqq 29$ has been studied in [2].

For several S, we have confirmed

$$(*) \qquad K_2(2, \boldsymbol{Z}_S) \cong Z \times \prod_{p \in S} Z_{p-1}$$

(cf. [4], [5]). Therefore, one might imagine that (*) holds for every S. However, we present a counter result below. Here, we will study $K_2(2, \boldsymbol{Z}[1/p])$ using the same method as in Section 3.

The group $K_2(2, \boldsymbol{Z}[1/p])$ is the kernel of the homomorphism ϕ of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ onto $\mathrm{SL}(2, \boldsymbol{Z}[1/p])$ with

$$\phi(x_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \phi(y_1) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and

$$\phi(x_2) = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}, \qquad \phi(y_2) = \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}.$$

Then, the symbol $\{p, -1\}$ belongs to $K_2(2, \boldsymbol{Z}[1/p])$ and is corresponding to $6(1-p)$ in $Z_{p^2-1} \cong \boldsymbol{Z}/(p^2-1)\boldsymbol{Z}$. Now suppose $K_2(2, \boldsymbol{Z}[1/p]) \cong Z \times Z_{p-1}$. Then we choose generators $\sigma, \tau \in K_2(2, \boldsymbol{Z}[1/p])$, which generate Z and Z_{p-1} respectively. We write $\{-1, -1\} = \sigma^i \tau^j$. Let $\bar{\sigma}$ be the image of σ in the quotient group $K_2(2, \boldsymbol{Q})/\coprod_p Z_{p-1}$, whose generator is also denoted by $\{-1, -1\}$ (cf. [4]). Put $\bar{\sigma} = \{-1, -1\}^k$ for some k. Then $\bar{\sigma}^{p-1} = \{-1, -1\}^{k(p-1)} = \bar{\sigma}^{ik(p-1)}$, and hence $i = k = \pm 1$. Therefore we can choose $\{-1, -1\}$ instead of the above σ. Now we have $\{p, -1\}^{p-1} = \{-1, -1\}^m$ for some m. Taking the images of both sides under the canonical homomorphism of $\mathrm{St}(2, \boldsymbol{Z}[1/p])$ into $\mathrm{St}(2, \boldsymbol{R})$, we get $1 = \{-1, -1\}^m$ in $\mathrm{St}(2, \boldsymbol{R})$ and $m = 0$, hence $\{p, -1\}^{p-1} = 1$ in $\mathrm{St}(2, \boldsymbol{Z}[1/p])$. In particular, $\{p, -1\}^{p-1} \equiv 1 \bmod \mathrm{St}(2, \boldsymbol{Z}[1/p])'$. Therefore

$$6(1-p)(p-1) \equiv 0 \mod p^2 - 1,$$

and there is a positive integer k such that $6(p-1) = k(p+1)$. Then, this implies $(k, p) = (2, 2), (3, 3), (4, 5), (5, 11)$. Hence, we obtain the following.

THEOREM 9. (1) $K_2(2, \boldsymbol{Z}) \cong Z$.

(2) $K_2(2, \boldsymbol{Z}[1/2]) \cong Z$.

(3) $K_2(2, \boldsymbol{Z}[1/3]) \cong Z \times Z_2$.

(4) *If* $p \neq 2, 3, 5, 11$, *then* $K_2(2, \boldsymbol{Z}[1/p]) \not\cong Z \times Z_{p-1}$.

COROLLARY 10. *If* $p \neq 2, 3, 5, 11$, *then* $\{p, -1\}^{p-1} \neq 1$ *in* $K_2(2, \boldsymbol{Z}[1/p])$.

REMARK. Recently in [6] we established that $K_2(2, \boldsymbol{Z}[1/p]) \not\cong Z \times Z_{p-1}$ even if $p = 5, 11$.

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