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Multiple stochastic integrals appearing in the stochastic Taylor expansions

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Introduction.

Let $V_0, V_1, \cdots, V_n$ be smooth vector fields on $\boldsymbol{R}^d$ (in general may be a smooth manifold) and we consider the stochastic differential equation (abbr. SDE) on $\boldsymbol{R}^d$:

$$\begin{aligned} dX_t &= \sum_{i=0}^{n} V_i(X_t) \circ dB_t^i \\ X_0 &= x \end{aligned} \tag{0.1}$$

where $(B_t^1, \cdots, B_t^n)$ is the n-dimensional Brownian motion starting from $0 \in \boldsymbol{R}^d$, $B_t^0 = t$ and the symbol $\circ$ denotes the Stratonovich integral. Let us denote by $X(t, x)$ the solution to this SDE. (Here let us suppose some appropriate condition under which the SDE (0.1) has a unique and global solution.) Then as $t \downarrow 0$, $X(t, x)$ is expanded as follows:

$$X(t, x) \sim x + \sum_{m=1}^{\infty} \sum_{i_1, \cdots, i_m = 0}^{n} B_t^{i_1 \cdots i_m} V_{i_1} \cdots V_{i_m}(x). \tag{0.2}$$

This is called the stochastic Taylor expansion and has a sense as an asymptotic expansion, and generally does not converge in probability for given $t > 0$. In the expansion, $B_t^{i_1 \cdots i_m}$ is a multiple stochastic integral for $B_t^{i_1}, \cdots, B_t^{i_m}$ defined by

$$B_t^{i_1 \cdots i_m} = \int_0^t \circ dB_{s_m}^{i_m} \int_0^{s_m} \circ dB_{s_{m-1}}^{i_{m-1}} \cdots \circ dB_{s_2}^{i_2} \int_0^{s_2} \circ dB_{s_1}^{i_1}. \tag{0.3}$$

When we study the asymptotic problem of quantity relative to $X(t, x)$ such as heat kernel, the expansion (0.2) is basic and there is a routine as follows: We decompose $X(t, x)$ as $X(t, x) = F(t, x) + R(t, x)$ such that $F(t, x)$ is a finite expansion in (0.2) cut in the m_0-th term (m_0 is chosen large enough in advance) and $R(t, x)$ is the remainder, and then show $R(t, x)$ to be actually negligible in an appropriate sense and hence reduce the problem to that for $F(t, x)$, i.e., a finite system $B_t^{i_1 \cdots i_m}$, $0 \leqq i_1, \cdots, i_m \leqq n$, $1 \leqq m \leqq m_0$.

In this paper, we are interested in an infinite system $B_t^{i_1 \cdots i_m}$, $0 \leqq i_1, \cdots, i_m \leqq$

n, $m \geqq 1$ and would like to know what it means. Since the expansion (0.2) is not convergent in a usual sense, we forget that V_i is a vector field on $\boldsymbol{R}^d$ and regard it as a variable (or an indeterminate). Then (0.2) is a formal power series in variables $V_0, V_1, \cdots, V_n$, i.e., in (0.2), $V_{i_1} \cdots V_{i_m}$ is a monomial of degree m and $\sum_{i_1, \cdots, i_m=0}^{n} B_t^{i_1 \cdots i_m} V_{i_1} \cdots V_{i_m}$ is a homogeneous polynomial of degree m and the expansion (0.2) is the sum of these homogeneous polynomials.

In general let $\mathcal{A}_m$ be the subspace of homogeneous polynomials of degree m. Then the algebra $\mathcal{A}$ of polynomials is the direct sum $\sum_{m=0}^{\infty} \oplus \mathcal{A}_m$ and the closure $\bar{\mathcal{A}}$ with respect to a pseudo norm (cf. Section 1.2) is the algebra of formal power series. Equipped with a bracket product (which is defined as usual), $\mathcal{A}$ is a Lie algebra and so is $\bar{\mathcal{A}}$ ($\mathcal{A}$ is a Lie subalgebra of $\bar{\mathcal{A}}$). Let $\mathcal{L}$ be a Lie subalgebra of $\mathcal{A}$ generated by $V_0, V_1, \cdots, V_n$ and $\bar{\mathcal{L}}$ be its closure in $\bar{\mathcal{A}}$. If $\mathcal{L}_m$ is the subspace spanned by brackets of order m of $V_0, V_1, \cdots, V_n$, then $\mathcal{L}$ is the direct sum $\sum_{m=1}^{\infty} \oplus \mathcal{L}_m$.

In Theorem 1 and 2 below it is stated that $1 + \sum_{m=1}^{\infty} \sum_{i_1, \cdots, i_m=0}^{n} B_t^{i_1 \cdots i_m} V_{i_1} \cdots V_{i_m} \in \bar{\mathcal{A}}$ is an exponential of an $\bar{\mathcal{L}}$-valued process $U_t = \sum_{m=1}^{\infty} U_{m,t}$ where $U_{m,t} \in \mathcal{L}_m$, i.e.,

$$1 + \sum_{m=1}^{\infty} \sum_{i_1, \cdots, i_m=0}^{n} B_t^{i_1 \cdots i_m} V_{i_1} \cdots V_{i_m} = \exp U_t \tag{0.4}$$

and that each $U_{m,t}$ is given explicitly by the formula in terms of $B_t^{j_1 \cdots j_m}$, $0 \leqq j_1, \cdots, j_m \leqq n$ and brackets of order m of $V_0, V_1, \cdots, V_n$. (Here exp is a mapping of the closure $\overline{\sum_{m=1}^{\infty} \oplus \mathcal{A}_m}$ into $\bar{\mathcal{A}}$ defined by (1.10).) This is shown algebraically by appealing to Friedrichs' theorem and Specht-Wever theorem, but the proof is not hard. We remark that the expression (0.4) already appears in [**F-CN**], though the explicit formula of $U_{m,t}$ is not obtained.

Next we give another description of U_t. By means of the Campbell-Hausdorff formula, a multiplication in $\bar{\mathcal{L}}$ is introduced (cf. (1.16)) and with this $\bar{\mathcal{L}}$ has a group structure. Although $\bar{\mathcal{L}}$ is of infinite dimension, it is regarded as a "Lie group" and its "invariant algebra" is itself. Corresponding to a variable V_i, an "invariant vector field" R_i on $\bar{\mathcal{L}}$ can be defined (cf. (1.20)). Similarly as (0.1), we consider the SDE on $\bar{\mathcal{L}}$:

$$\begin{aligned} dU_t &= \sum_{i=0}^{n} R_i(U_t) \circ dB_t^i \\ U_0 &= 0. \end{aligned} \tag{0.5}$$

Then, in Theorem 3 below, it is stated that the preceding $U_t = \sum_{m=1}^{\infty} U_{m,t}$ is the unique solution of the SDE (0.5). Since $\bar{\mathcal{L}}$ is in fact not a Lie group, the problem is modified into that of finite dimension and the proof is precisely done as follows: Let $\mathcal{L}[1, m] := \sum_{k=1}^{m} \oplus \mathcal{L}_k$. $\mathcal{L}[1, m]$ is clearly a finite dimensional subspace of $\bar{\mathcal{L}}$. Moreover a multiplication in $\bar{\mathcal{L}}$ makes it a usual Lie group

and the restriction $R_i^{\{m\}}$ of R_i to $\mathcal{L}[1, m]$ is an invariant vector field. Hence the SDE (0.5) (replacing R_i by $R_i^{\{m\}}$) is valid and it is shown that $\sum_{k=1}^{m} U_{k,t}$ is the unique solution of the modified SDE. We remark that in [**B**], Ben Arous introduces $R_i^{\{m\}}$ and considers this SDE.

Finally, as an application of those theorems, we present the representation formula (abbr. RF) of solutions to SDE's with nilpotent coefficients, i.e., of the solution $X(t, x)$ to the SDE (0.1) in which the vector fields $V_0, V_1, \cdots, V_n$ generate a nilpotent Lie subalgebra $\mathfrak{h}$. Following Yamato [**Y**], we consider the system of first order partial differential equations on $\boldsymbol{R}^d$:

$$\begin{aligned} R_i^{\{m\}} h(\cdot, x) &= V_i(h(\cdot, x)) \qquad i = 0, 1, \cdots, n \\ h(0, x) &= x \end{aligned} \tag{0.6}$$

where m is the order of nilpotency of $\mathfrak{h}$, i.e., a positive integer such that $\mathfrak{h}^{m+1}$ (=the bracket of order $m+1$ of $\mathfrak{h}$)$=\{0\}$. This solution is given explicitly by the formula (3.3). (In [**Y**], this is only implicitly determined.) In Theorem 4 below, the RF of $X(t, x)$ is obtained from the conclusion of Theorem 3 and the formula (3.3).

So far we have considered such as (0.1) the SDE's with respect to Brownian motions. But, B_t^i being replaced by a continuous semimartingale M_t^i with $M_0^i = 0$, the above results remain to hold. In particular M_t^i being deterministic, the RF mentioned above is just that of solutions to ordinary differential equations (abbr. ODE) which is stated in Strichartz [**S**] (cf. Theorem 5).

The problem finding the RF for SDE's has been already studied by Yamato [**Y**], Kunita [**K1**], [**K2**], Fliess and Normand-Cyrot [**F-NC**], Ben Arous [**B**], Hu [**H**] and Castell [**C**]. Among them, in [**K1**], [**F-NC**], [**H**] and [**C**], the RF for ODE's is based on first, and then from the transfer principle (due to Malliavin) or from the approximation theorem the stochastic version of the RF is obtained. In contrast to this we directly derive it without the transfer principle. Our approach seems to be sharp and simpler. We remark that in [**H**] and [**C**] the explicit formula of $U_{m,t}$ in (0.4) is obtained in finding the RF for SDE's

The organization of this paper is as follows: Section 1 is a section for some preliminaries. There, notion and notations necessary for stating and proving the above theorems (Theorem 1~5) are presented. Also some facts (Fact 1~5) which are like as lemmas for these theorems are presented. Although they can be seen in some standard text books, we give their proofs for the completeness. In Section 2, our main results (Theorem 1~3) are stated and proved. By virtue of much efforts in Section 1 the proofs are not hard. As an application of the results, in Section 3 the RF for SDE's and ODE's (Theorem 4 and 5) are obtained.

1. Some preliminaries.

What we will state in this section can be partly seen in other papers [**T1**] and [**T2**]. We here state them minutely and intelligibly.

In the following (except in the proof of Theorem 5 below) let us fix $n\in\boldsymbol{N}$.

Let $\boldsymbol{E}:=\{0, 1, \cdots, n\}$. Let $\mathcal{A}(\boldsymbol{E})$ and $\mathcal{L}(\boldsymbol{E})$ be the free algebra (over $\boldsymbol{R}$) and the free Lie algebra (over $\boldsymbol{R}$) generated by $\boldsymbol{E}$, respectively. They are here interpreted as follows: $\mathcal{A}(\boldsymbol{E})$ is an algebra of polynomials in variables (or indeterminates) 0, 1, $\cdots$, n over $\boldsymbol{R}$. Thus $\mathcal{A}(\boldsymbol{E})\supset\boldsymbol{R}$, $1(\in\boldsymbol{R})$ is a unit and in general $\mathcal{A}(\boldsymbol{E})$ is non-commutative. If, for $X, Y\in\mathcal{A}(\boldsymbol{E})$ set the bracket product $[X, Y]:=XY-YX$ as usual, then $\mathcal{A}(\boldsymbol{E})$ is a Lie algebra over $\boldsymbol{R}$ and $\mathcal{L}(\boldsymbol{E})$ is a Lie subalgebra of $\mathcal{A}(\boldsymbol{E})$ generated by $\boldsymbol{E}$. Moreover the freeness of $\mathcal{L}(\boldsymbol{E})$ and $\mathcal{A}(\boldsymbol{E})$ means that

(i) given a Lie algebra $\mathcal{L}$ over $\boldsymbol{R}$ and γ: $\boldsymbol{E}\to\mathcal{L}$, γ has a unique homomorphic extension of $\mathcal{L}(\boldsymbol{E})$ into $\mathcal{L}$,

(ii) given an algebra $\mathcal{A}$ over $\boldsymbol{R}$ and α: $\boldsymbol{E}\to\mathcal{A}$, α has a unique homomorphic extension of $\mathcal{A}(\boldsymbol{E})$ into $\mathcal{A}$ such that $\alpha([X, Y])=[\alpha(X), \alpha(Y)]$ for any X, $Y\in\mathcal{L}(\boldsymbol{E})$.

1.1. For each $b\in\boldsymbol{Z}_+:=\{0, 1, 2, \cdots\}$, set

$$\boldsymbol{E}_b:=\begin{cases}\{1\}\ (\text{where } 1\in\boldsymbol{R}) & \text{if } b=0\\ \{i_1\cdots i_b;\ i_1, \cdots, i_b\in\boldsymbol{E}\} & \text{if } b\in\boldsymbol{N}\end{cases}$$

and for $a, b\in\boldsymbol{Z}_+$ such that $a\leqq b$

$$\boldsymbol{E}[a, b]:=\boldsymbol{E}_a\cup\cdots\cup\boldsymbol{E}_b$$
$$\boldsymbol{E}[a, \infty):=\boldsymbol{E}_a\cup\boldsymbol{E}_{a+1}\cup\cdots.$$

$\boldsymbol{E}[0, \infty)$ form a basis of $\mathcal{A}(\boldsymbol{E})$. Let $\mathcal{A}_b$, $\mathcal{A}[a, b]$ and $\mathcal{A}[a, \infty)$ be the subspaces spanned by $\boldsymbol{E}_b$, $\boldsymbol{E}[a, b]$ and $\boldsymbol{E}[a, \infty)$, respectively.

For $I\in\boldsymbol{E}[1, \infty)$, $|I|:=$ the degree of a monomial I, i.e., $|i_1\cdots i_b|:=b$, and $[I]\in\mathcal{L}(\boldsymbol{E})$ is defined as follows:

$$[i_1\cdots i_b]:=\begin{cases}i_1 & \text{if } b=1\\ [[\cdots[[i_1, i_2], i_3], \cdots, i_{b-1}], i_b] & \text{if } b\geqq 2.\end{cases}\tag{1.1}$$

$\{[I];\ I\in\boldsymbol{E}[1, \infty)\}$ belong to a $\boldsymbol{Z}$-module in $\mathcal{A}(\boldsymbol{E})$ generated by $\boldsymbol{E}[1, \infty)$. So $c_I^J\in\boldsymbol{Z}$, I, $J\in\boldsymbol{E}[1, \infty)$ are uniquely determined by $[I]=\sum_{J\in\boldsymbol{E}[1,\infty)}c_I^J J$ (c_I^J is a coefficient of J in expanding $[I]$, and hence $c_I^J=0$ if $|J|\neq|I|$). Then it is clear that

(i) $c_i^j=\delta_{ij}$ $\quad i, j\in\boldsymbol{E}_1=\boldsymbol{E}$

(ii) $\operatorname{rank}(c_I^J)_{I, J\in\boldsymbol{E}_b}=:r_b\geqq 1$ $\quad$ for each $b\in\boldsymbol{N}$.

By (ii) we can take a subset $\boldsymbol{G}_b \subset \boldsymbol{E}_b$ such that $\#\boldsymbol{G}_b = r_b$ and the row vectors $(c_I^J)_{J \in \boldsymbol{E}_b}$, $I \in \boldsymbol{G}_b$ are linearly independent.

For $a, b \in \boldsymbol{N}$ such that $a \leqq b$, let $\mathcal{L}_b$, $\mathcal{L}[a, b]$ and $\mathcal{L}[a, \infty)$ be the subspaces of $\mathcal{L}(\boldsymbol{E})$ spanned by $\{[I];\ I \in \boldsymbol{E}_b\}$, $\{[I];\ I \in \boldsymbol{E}[a, b]\}$ and $\{[I];\ I \in \boldsymbol{E}[a, \infty)\}$, respectively. Then, taken $\boldsymbol{G}_b$ above, $\{[I];\ I \in \boldsymbol{G}_b\}$ form a basis of $\mathcal{L}_b$. Conversely if $\boldsymbol{G}_b \subset \boldsymbol{E}_b$ has this property, then $\#\boldsymbol{G}_b = r_b$ and it has the preceding property. Although the choice of such a $\boldsymbol{G}_b$ is not unique, we choose one arbitrarily and fix it in the following.

EXAMPLE 1. For $b=1$, $\boldsymbol{G}_1 = \boldsymbol{E}_1 = \boldsymbol{E}$. For $b=2, 3$, we can take

$$\boldsymbol{G}_2 = \{ij;\ 0 \leqq i < j \leqq n\}$$

$$\boldsymbol{G}_3 = \{ijk, jki;\ 0 \leqq i < j < k \leqq n\} \cup \{jkj, jkk;\ 0 \leqq j < k \leqq n\}.$$

For $a, b \in \boldsymbol{N}$ such that $a \leqq b$, similarly as $\boldsymbol{E}[a, b]$ and $\boldsymbol{E}[a, \infty)$, set

$$\boldsymbol{G}[a, b] := \boldsymbol{G}_a \cup \cdots \cup \boldsymbol{G}_b$$

$$\boldsymbol{G}[a, \infty) := \boldsymbol{G}_a \cup \boldsymbol{G}_{a+1} \cup \cdots.$$

Since $\{[I];\ I \in \boldsymbol{G}[1, \infty)\}$ form a basis of $\mathcal{L}[1, \infty)$, there exist $(e_J^I)_{I \in \boldsymbol{G}[1,\infty),\, J \in \boldsymbol{E}[1,\infty)} \subset \boldsymbol{Z}$ such that

$$[J] = \sum_{I \in \boldsymbol{G}[1,\infty)} e_J^I [I] \qquad \text{for each} \quad J \in \boldsymbol{E}[1, \infty)$$

where $e_J^I = 0$ if $|I| \neq |J|$. If $\mathcal{A}$ is an algebra over $\boldsymbol{R}$ and $\alpha: \boldsymbol{E} \to \mathcal{A}$, then a homomorphic extension α has the same property as above, i.e.,

$$\alpha([J]) = \sum_{I \in \boldsymbol{G}[1,\infty)} e_J^I \alpha([I]) \qquad \text{for each} \quad J \in \boldsymbol{E}[1, \infty). \tag{1.2}$$

Similarly as $[I]$, define $\tau(I) \in \mathcal{L}(\boldsymbol{E})$ for $I \in \boldsymbol{E}[1, \infty)$ as follows (cf. (1.1)):

$$\tau(i_1 \cdots i_b) := \begin{cases} i_1 & \text{if} \quad b = 1 \\ [i_1, [i_2, \cdots, [i_{b-2}, [i_{b-1}, i_b]] \cdots]] & \text{if} \quad b \geqq 2. \end{cases} \tag{1.3}$$

Clearly $\tau(i_1 \cdots i_b) = (-1)^{b-1}[i_b \cdots i_1]$. Since $\mathcal{A}[1, \infty)$ is spanned by $\boldsymbol{E}[1, \infty)$, mappings of $\boldsymbol{E}[1, \infty)$ into $\mathcal{L}(\boldsymbol{E})$: $I \mapsto [I]$, $I \mapsto \tau(I)$ are uniquely extended to linear mappings of $\mathcal{A}[1, \infty)$ into $\mathcal{L}(\boldsymbol{E})$. We denote them by the same symbols $[\ ]$, τ. Then it is known as the Specht-Wever theorem (cf. [**J**]) that

$$X \in \mathcal{L}_b \underset{\text{iff}}{\Longleftrightarrow} [X] = bX \tag{1.4}$$

$$X \in \mathcal{L}_b \underset{\text{iff}}{\Longleftrightarrow} \tau(X) = bX. \tag{1.5}$$

1.2. The algebra $\mathcal{A}(\boldsymbol{E})$ is a graded algebra with $\mathcal{A}_b$ as the subspace of homogeneous elements of degree b:

$$\mathcal{A}(\boldsymbol{E}) = \sum_{b=0}^{\infty} \oplus \mathcal{A}_b$$

$$\mathcal{A}_b \mathcal{A}_c \subset \mathcal{A}_{b+c} \qquad b, c \in \boldsymbol{Z}_+ .$$

So it can be extended to the algebra $\overline{\mathcal{A}(\boldsymbol{E})}$ of formal power series in variables $0, 1, \cdots, n$ over $\boldsymbol{R}$. More specifically the elements of $\overline{\mathcal{A}(\boldsymbol{E})}$ are the expressions $\sum_{b=0}^{\infty} X_b = X_0 + X_1 + X_2 + \cdots$, $X_b \in \mathcal{A}_b$ such that $\sum_{b=0}^{\infty} X_b = \sum_{b=0}^{\infty} Y_b$ if and only if $X_b = Y_b$, $b \in \boldsymbol{Z}_+$. The algebraic structure (i.e., addition, scalar multiplication and multiplication) in $\overline{\mathcal{A}(\boldsymbol{E})}$ is defined as follows:

$$\sum_{b=0}^{\infty} X_b + \sum_{b=0}^{\infty} Y_b := \sum_{b=0}^{\infty} (X_b + Y_b)$$

$$\lambda \sum_{b=0}^{\infty} X_b := \sum_{b=0}^{\infty} \lambda X_b \qquad \lambda \in \boldsymbol{R}$$

$$\Big(\sum_{b=0}^{\infty} X_b\Big)\Big(\sum_{b=0}^{\infty} Y_b\Big) := \sum_{b=0}^{\infty} Z_b \quad \text{where} \quad Z_b := X_b Y_0 + \cdots + X_0 Y_b .$$

Let $\overline{\mathcal{A}[b, \infty)}$ be the subset of all elements of the form $X_b + X_{b+1} + \cdots$. It is a two-sided ideal in $\overline{\mathcal{A}(\boldsymbol{E})}$. We introduce a pseudo norm $|\cdot|$ in $\overline{\mathcal{A}(\boldsymbol{E})}$ by

$$|X| := 2^{-\max\{b; X \in \overline{\mathcal{A}[b, \infty)}\}} .$$

It is clear that

$$X \in \overline{\mathcal{A}[b, \infty)} \underset{\text{iff}}{\Longleftrightarrow} |X| \leqq 2^{-b} \tag{1.6}$$

$$|X+Y| \leqq |X| \vee |Y| \tag{1.7}$$

$$|\lambda X| \leqq |X| \qquad \lambda \in \boldsymbol{R} \tag{1.8}$$

$$|XY| \leqq |X||Y| . \tag{1.9}$$

Then $\overline{\mathcal{A}(\boldsymbol{E})}$ is complete as a linear space equipped with a pseudo norm $|\cdot|$ and the subalgebra $\mathcal{A}(\boldsymbol{E})$ is dense in it.

Now, for $X \in \overline{\mathcal{A}[1, \infty)}$ we define $\exp X$, $\log(1+X) \in \overline{\mathcal{A}(\boldsymbol{E})}$ by

$$\exp X := 1 + \sum_{m=1}^{\infty} \frac{X^m}{m!}, \qquad \log(1+X) := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} X^m . \tag{1.10}$$

The series converge with respect to $|\cdot|$. Clearly

$$\exp \log(1+X) = 1+X, \qquad \log \exp X = X \tag{1.11}$$

$$|\exp X - \exp Y|, \ |\log(1+X) - \log(1+Y)| \leqq |X-Y| . \tag{1.12}$$

Let $\overline{\mathcal{L}(\boldsymbol{E})}$ and $\overline{\mathcal{L}[a, \infty)}$ be the closures of $\mathcal{L}(\boldsymbol{E})$ and $\mathcal{L}[a, \infty)$ in $\overline{\mathcal{A}(\boldsymbol{E})}$, respectively. Then $\overline{\mathcal{L}(\boldsymbol{E})}$ is a Lie subalgebra of $\overline{\mathcal{A}(\boldsymbol{E})}$ and $\overline{\mathcal{L}[a, \infty)}$ is an ideal of $\overline{\mathcal{L}(\boldsymbol{E})}$. Also $\overline{\mathcal{L}[a, \infty)}$ is the subset of all elements of the form $X_a + X_{a+1} + \cdots$, $X_b \in \mathcal{L}_b$, and thus $\overline{\mathcal{L}[a, \infty)} \subset \overline{\mathcal{A}[a, \infty)}$. Moreover the following holds:

FACT 1. *For* $X, Y\in\overline{\mathcal{L}(\boldsymbol{E})}$, $\log(\exp X\exp Y)\in\overline{\mathcal{L}(\boldsymbol{E})}$ *and it is given by the Campbell-Hausdorff formula*:

$$\log(\exp X\exp Y)=\sum_{b=1}^{\infty}c_b(X,Y) \tag{1.13}$$

where $c_b(X,Y)\in\overline{\mathcal{L}[b,\infty)}$ *is defined by*

$$c_b(X,Y):=\frac{1}{b}\sum_{m=1}^{b}\sum_{\substack{p_1,q_1,\cdots,p_m,q_m\geq 0;\\ p_1+q_1,\cdots,p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=b}}\frac{(-1)^{m-1}}{m}\frac{(\mathrm{ad}X)^{p_1}(\mathrm{ad}Y)^{q_1}\cdots(\mathrm{ad}X)^{p_m}(\mathrm{ad}Y)^{q_m}}{p_1!\,q_1!\cdots p_m!\,q_m!}.$$

Here $(\mathrm{ad}X)Y:=[X,Y]$ *and* $\mathrm{ad}X:=X$.

PROOF. For details see [**J**]. Let δ be the diagonal mapping of $\mathcal{A}(\boldsymbol{E})$, i.e., the homomorphism of $\mathcal{A}(\boldsymbol{E})$ into $\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E})$ such that $\delta(i)=1\otimes i+i\otimes 1$. Then $X\in\mathcal{A}(\boldsymbol{E})$ belongs to $\mathcal{L}(\boldsymbol{E})$ if and only if $\delta(X)=1\otimes X+X\otimes 1$. This criterion is known as Friedrichs' theorem. Similarly as $\mathcal{A}(\boldsymbol{E})$, $\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E})$ is a graded algebra with $(\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E}))_b:=\mathcal{A}_b\otimes\mathcal{A}_0+\cdots+\mathcal{A}_0\otimes\mathcal{A}_b$ as the subspace of homogeneous elements of degree b. Thus we can construct the algebra $\overline{\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E})}$ as $\overline{\mathcal{A}(\boldsymbol{E})}$. Then the diagonal mapping δ has an extension to a homomorphism of $\overline{\mathcal{A}(\boldsymbol{E})}$ into $\overline{\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E})}$. Indeed, if, for $X=\sum_{b=0}^{\infty}X_b(X_b\in\mathcal{A}_b)$, we define

$$\delta(X):=\sum_{b=0}^{\infty}\delta(X_b)$$

then this δ is the desired one. Here note that $\delta(X)\in(\mathcal{A}(\boldsymbol{E})\otimes\mathcal{A}(\boldsymbol{E}))_b$ if $X\in\mathcal{A}_b$. By the definition of δ and Friedrichs' theorem it is clear that for $X\in\overline{\mathcal{A}(\boldsymbol{E})}$

$$X\in\overline{\mathcal{L}(\boldsymbol{E})}\underset{\text{iff}}{\Longleftrightarrow}\delta(X)=1\otimes X+X\otimes 1. \tag{1.14}$$

We proceed to the proof of Fact 1. First of all note that $[1\otimes X, Y\otimes 1]=0$ for $X, Y\in\overline{\mathcal{A}(\boldsymbol{E})}$ and

$$\exp(X\otimes 1)=(\exp X)\otimes 1,\qquad \exp(1\otimes X)=1\otimes(\exp X)$$
$$\log((1+X)\otimes 1)=\log(1+X)\otimes 1,\qquad \log(1\otimes(1+X))=1\otimes\log(1+X).$$

Let $X, Y\in\overline{\mathcal{L}(\boldsymbol{E})}$. Then by the above note and (1.14)

$$\begin{aligned}\delta(\exp X\exp Y)&=\delta(\exp X)\delta(\exp Y)\\&=\exp\delta(X)\exp\delta(Y)\\&=\exp(1\otimes X+X\otimes 1)\exp(1\otimes Y+Y\otimes 1)\\&=\exp(1\otimes X)\exp(X\otimes 1)\exp(1\otimes Y)\exp(Y\otimes 1)\\&=(1\otimes\exp X)(\exp X\otimes 1)(1\otimes\exp Y)(\exp Y\otimes 1)\end{aligned}$$

$$= (1\otimes \exp X)(1\otimes \exp Y)(\exp X\otimes 1)(\exp Y\otimes 1)$$
$$= (1\otimes \exp X \exp Y)(\exp X \exp Y\otimes 1).$$

Hence

$$\begin{aligned}\delta(\log \exp X \exp Y) &= \log\delta(\exp X \exp Y)\\ &= \log(1\otimes \exp X \exp Y)(\exp X \exp Y\otimes 1)\\ &= \log(1\otimes \exp X \exp Y)+\log(\exp X \exp Y\otimes 1)\\ &= 1\otimes \log \exp X \exp Y+\log \exp X \exp Y\otimes 1.\end{aligned}$$

By (1.14) this shows that $\log \exp X \exp Y\in\overline{\mathcal{L}(\boldsymbol{E})}$.

Next we show the formula (1.13). For $X, Y\in\overline{\mathcal{A}[1, \infty)}$, a direct calculation shows

$$\log \exp X \exp Y = \sum_{b=1}^{\infty} Z_b$$

where

$$Z_b := \sum_{m=1}^{b} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_1, q_1, \cdots, p_m, q_m\geq 0;\\ p_1+q_1, \cdots, p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=b}} \frac{X^{p_1}Y^{q_1}\cdots X^{p_m}Y^{q_m}}{p_1!\, q_1!\cdots p_m!\, q_m!} \in \overline{\mathcal{A}[b, \infty)}.$$

In particular, if $X, Y\in\boldsymbol{E}$, $Z_b\in\mathcal{A}_b (b\in\boldsymbol{N})$. In this case, since $\sum_{b=1}^{\infty} Z_b\in\overline{\mathcal{L}(\boldsymbol{E})}$, $Z_b\in\mathcal{L}_b$, and hence by (1.5)

$$Z_b = \frac{1}{b}\tau(Z_b).$$

This suggests that for two indeterminates X, Y

$$\sum_{m=1}^{b} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_1, q_1, \cdots, p_m, q_m\geq 0;\\ p_1+q_1, \cdots, p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=b}} \frac{X^{p_1}Y^{q_1}\cdots X^{p_m}Y^{q_m}}{p_1!\, q_1!\cdots p_m!\, q_m!}$$
$$= \frac{1}{b}\sum_{m=1}^{b} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_1, q_1, \cdots, p_m, q_m\geq 0;\\ p_1+q_1, \cdots, p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=b}} \frac{(\mathrm{ad}X)^{p_1}(\mathrm{ad}Y)^{q_1}\cdots(\mathrm{ad}X)^{p_m}(\mathrm{ad}Y)^{q_m}}{p_1!\, q_1!\cdots p_m!\, q_m!}.$$

Consequently, applying this formula for $X, Y\in\overline{\mathcal{L}(\boldsymbol{E})}$ and substituting it into the preceding expression, we obtain the formula (1.13). ■

REMARK 1. By the same reason as above, it can be verified that for $X, Y\in\overline{\mathcal{L}(\boldsymbol{E})}$

$$\sum_{\substack{p, q\geq 0;\\ p+q=b}} \frac{X^pY^q}{p!\, q!} = \sum_{m=1}^{b} \frac{1}{m!} \sum_{\substack{b_1, \cdots, b_m\geq 1;\\ b_1+\cdots+b_m=b}} c_{b_1}(X, Y)\cdots c_{b_m}(X, Y). \tag{1.15}$$

By virtue of Fact 1, we introduce a multiplication $\cdot$ in $\overline{\mathcal{L}(\boldsymbol{E})}$ by

$$X\cdot Y := \log(\exp X \exp Y) \qquad X, Y\in\overline{\mathcal{L}(\boldsymbol{E})}. \tag{1.16}$$

Then $\overline{\mathcal{L}(\boldsymbol{E})}$ has a group structure equipped with this multiplication, that is, it holds that

(i) $(X\cdot Y)\cdot Z=X\cdot(Y\cdot Z)$,

(ii) the unit element$=0$, i.e., $0\cdot X=X\cdot 0=X$,

(iii) the inverse element of $X=-X$, i.e., $(-X)\cdot X=X\cdot(-X)=0$.

For $X\in\overline{\mathcal{L}(\boldsymbol{E})}$ and $I\in\boldsymbol{E}[1,\infty)$, $X\cdot t[I]\in\overline{\mathcal{L}(\boldsymbol{E})}(t\in\boldsymbol{R})$. By differentiating it at 0 component wise, $(d/dt)X\cdot t[I]|_{t=0}\in\overline{\mathcal{L}(\boldsymbol{E})}$ and it is computed as follows:

FACT 2. *For each* $X\in\overline{\mathcal{L}(\boldsymbol{E})}$ *and* $I\in\boldsymbol{E}[1,\infty)$

$$\frac{d}{dt}X\cdot t[I]\Big|_{t=0}=[I]+\sum_{m=1}^{\infty}\frac{b_m}{m!}(\mathrm{ad}(-X))^m[I]. \tag{1.17}$$

Here $\{b_m\}_{m=1}^{\infty}$ *are the Bernoulli numbers, i.e., it is a sequence determined by the Taylor expansion*:

$$\frac{z}{e^z-1}=1+\sum_{m=1}^{\infty}\frac{b_m}{m!}z^m \qquad |z|<2\pi$$

and

$$(\mathrm{ad}X)^mY:=[\underbrace{X,[\cdots,[X}_{m},[X,Y]]\cdots]] \qquad X,Y\in\overline{\mathcal{L}(\boldsymbol{E})}.$$

PROOF. By the Campbell-Hausdorff formula (1.13)

$$X\cdot t[I]=\sum_{b=1}^{\infty}\frac{1}{b}\sum_{m=1}^{b}\frac{(-1)^{m-1}}{m}$$
$$\sum_{\substack{p_1,q_1,\cdots,p_m,q_m\geq 0;\\ p_1+q_1,\cdots,p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=b}}t^{q_1+\cdots+q_m}\frac{(\mathrm{ad}X)^{p_1}(\mathrm{ad}[I])^{q_1}\cdots(\mathrm{ad}X)^{p_m}(\mathrm{ad}[I])^{q_m}}{p_1!\,q_1!\cdots p_m!\,q_m!}$$

and so, by differentiating it at 0

$$\frac{d}{dt}X\cdot t[I]\Big|_{t=0}=[I]+\sum_{b=2}^{\infty}\frac{1}{b}\sum_{m=1}^{b}\sum_{k=1}^{m}$$
$$\sum_{\substack{p_1\geq 1,\cdots,p_{k-1}\geq 1,\,p_{k+1}\geq 1,\cdots,p_m\geq 1;\\ p_1+\cdots+p_m=b-1}}\frac{(-1)^{m-1}}{m}\frac{(\mathrm{ad}X)^{p_1+\cdots+p_k}(\mathrm{ad}[I])(\mathrm{ad}X)^{p_{k+1}+\cdots+p_m}}{p_1!\cdots p_m!}.$$

In the right expression, note that if either $k\leqq m-2$, or $k=m-1$ and $p_m\geqq 2$, $(\mathrm{ad}X)^{p_{k+1}+\cdots+p_m}=0$. Hence this is further computed as follows:

$$\frac{d}{dt}X\cdot t[I]\Big|_{t=0}=[I]+\sum_{b=2}^{\infty}\frac{1}{b}\Bigg(\sum_{1\leq m\leq b}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_1\geq 1,\cdots,p_{m-1}\geq 1;\\ p_1+\cdots+p_m=b-1}}\frac{(\mathrm{ad}X)^{p_1+\cdots+p_m}[I]}{p_1!\cdots p_m!}$$
$$+\sum_{2\leq m\leq b}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_1\geq 1,\cdots,p_{m-2}\geq 1,\,p_m=1;\\ p_1+\cdots+p_m=b-1}}\frac{(\mathrm{ad}X)^{p_1+\cdots+p_{m-1}}(\mathrm{ad}[I])X}{p_1!\cdots p_m!}\Bigg)$$

$$= [I] + \sum_{b=2}^{\infty} \frac{1}{b} \Bigg(\sum_{1 \leq m \leq b} \sum_{\substack{p_1 \geq 1, \cdots, p_{m-1} \geq 1; \\ p_1 + \cdots + p_m = b-1}} \frac{(-1)^{m-1}}{m} \frac{1}{p_1! \cdots p_m!}$$

$$- \sum_{2 \leq m \leq b} \sum_{\substack{p_1 \geq 1, \cdots, p_{m-2} \geq 1, p_m = 1; \\ p_1 + \cdots + p_m = b-1}} \frac{(-1)^{m-1}}{m} \frac{1}{p_1! \cdots p_m!} \Bigg) (\mathrm{ad}X)^{b-1}[I].$$

Here the last equality is due to the fact: $(\mathrm{ad}[I])X = -(\mathrm{ad}X)[I]$. Thus, setting $\{c_m\}_{m=1}^{\infty}$ by

$$c_m := \frac{(-1)^m m!}{m+1} \Bigg(\sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{p_1 \geq 1, \cdots, p_{k-1} \geq 1; \\ p_1 + \cdots + p_k = m}} \frac{1}{p_1! \cdots p_k!} \tag{1.18}$$

$$- \sum_{k=2}^{m+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{p_1 \geq 1, \cdots, p_{k-2} \geq 1, p_k = 1; \\ p_1 + \cdots + p_k = m}} \frac{1}{p_1! \cdots p_k!} \Bigg),$$

we have

$$\frac{d}{dt} X \cdot t[I] \Big|_{t=0} = [I] + \sum_{m=1}^{\infty} \frac{c_m}{m!} (\mathrm{ad}(-X))^m [I].$$

To obtain (1.17), it remains to show that c_m is the Bernoulli number. But this is done in Lemma 1 below, so that the proof is complete. ■

LEMMA 1. *A sequence* $\{c_m\}_{m=1}^{\infty}$ *defined by* (1.18) *is the Bernoulli numbers.*

PROOF. Note the Taylor expansion of $(e^z-1)^{k-1}e^z$ $(k \in \boldsymbol{N})$:

$$(e^z-1)^{k-1}e^z = \sum_{m=0}^{\infty} \Bigg(\sum_{\substack{p_1 \geq 1, \cdots, p_{k-1} \geq 1; \\ p_1 + \cdots + p_k = m}} \frac{1}{p_1! \cdots p_k!} \Bigg) z^m \qquad z \in \boldsymbol{C}.$$

By this and Cauchy's integral formula, we observe

$$\sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{p_1 \geq 1, \cdots, p_{k-1} \geq 1; \\ p_1 + \cdots + p_k = m}} \frac{1}{p_1! \cdots p_k!}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=r} \sum_{k=1}^{m+1} \frac{(-1)^{k-1}}{k} (e^{\zeta}-1)^{k-1} e^{\zeta} \zeta^{-m-1} d\zeta$$

$$\sum_{k=2}^{m+1} \frac{(-1)^{k-1}}{k} \sum_{\substack{p_1 \geq 1, \cdots, p_{k-2} \geq 1, p_k = 1; \\ p_1 + \cdots + p_k = m}} \frac{1}{p_1! \cdots p_k!}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=r} \sum_{k=2}^{m+1} \frac{(-1)^{k-1}}{k} (e^{\zeta}-1)^{k-2} e^{\zeta} \zeta^{-m} d\zeta$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=r} \sum_{k=2}^{m+1} \frac{(-1)^{k-1}}{k} \Big(\frac{1}{k-1} (e^{\zeta}-1)^{k-1} \Big)' \zeta^{-m} d\zeta$$

$$= \frac{m}{2\pi\sqrt{-1}} \int_{|\zeta|=r} \sum_{k=2}^{m+1} \frac{(-1)^{k-1}}{k(k-1)} (e^{\zeta}-1)^{k-1} \zeta^{-m-1} d\zeta$$

$$= \frac{m}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\sum_{k=2}^{m+1}\left(\frac{(-1)^{k-1}}{k-1}+\frac{(-1)^k}{k}\right)(e^\zeta-1)^{k-1}\zeta^{-m-1}d\zeta$$

$$= \frac{m}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\left(1+\sum_{k=1}^{m+1}\frac{(-1)^k}{k}(e^\zeta-1)^{k-1}e^\zeta-\frac{(-1)^{m+1}}{m+1}(e^\zeta-1)^{m+1}\right)\zeta^{-m-1}d\zeta$$

$$= \frac{m}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\sum_{k=1}^{m+1}\frac{(-1)^k}{k}(e^\zeta-1)^{k-1}e^\zeta\zeta^{-m-1}d\zeta.$$

Here $r\in(0, \log 2)$. Hence (1.18) is simplified:

$$c_m = \frac{(-1)^m m!}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\sum_{k=1}^{m+1}\frac{(-1)^{k-1}}{k}(e^\zeta-1)^{k-1}e^\zeta\zeta^{-m-1}d\zeta. \tag{1.19}$$

But note that

$$\int_{|\zeta|=r}(e^\zeta-1)^{k-1}e^\zeta\zeta^{-m-1}d\zeta = 0 \quad \text{if} \quad k\geqq m+2,$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(e^\zeta-1)^k = \log e^\zeta = \zeta \quad \text{on} \quad \{|\zeta|<\log 2\}.$$

Therefore (1.19) is further simplified:

$$c_m = \frac{(-1)^m m!}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\zeta\frac{e^\zeta}{e^\zeta-1}\zeta^{-m-1}d\zeta$$

$$= \frac{(-1)^m m!}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\zeta\frac{1}{1-e^{-\zeta}}\zeta^{-m-1}d\zeta$$

$$= \frac{m!}{2\pi\sqrt{-1}}\int_{|\zeta|=r}\frac{\zeta}{e^\zeta-1}\zeta^{-m-1}d\zeta$$

and this shows that $c_m=b_m$, $m\in\boldsymbol{N}$. ■

Before concluding this subsection, we present the following definition:

DEFINITION 1. For each $i\in\boldsymbol{E}$, define a mapping R_i of $\overline{\mathcal{L}(\boldsymbol{E})}$ into itself by

$$R_i(X) := \frac{d}{dt}X\cdot ti\Big|_{t=0}. \tag{1.20}$$

By Fact 2

$$R_i(X) = i+\sum_{m=1}^{\infty}\frac{b_m}{m!}(\mathrm{ad}(-X))^m i \qquad \text{(cf. [B])}. \tag{1.21}$$

1.3. Throughout this subsection let $b\in\boldsymbol{N}$ be fixed. $\overline{\mathcal{A}[b+1, \infty)}$ was a two-sided ideal of $\overline{\mathcal{A}(\boldsymbol{E})}$ and $\overline{\mathcal{L}[b+1, \infty)}$ was an ideal of $\overline{\mathcal{L}(\boldsymbol{E})}$. Clearly

$$\begin{aligned}&\overline{\mathcal{A}[1, \infty)}/\overline{\mathcal{A}[b+1, \infty)} \cong \mathcal{A}[1, b],\\ &\overline{\mathcal{L}(\boldsymbol{E})}/\overline{\mathcal{L}[b+1, \infty)} \cong \mathcal{L}[1, b].\end{aligned} \tag{1.22}$$

$\boldsymbol{E}[1, b]$ and $\{[I]; I\in\boldsymbol{G}[1, b]\}$ are bases of $\mathcal{A}[1, b]$ and $\mathcal{L}[1, b]$, respectively. Thus, on $\mathcal{A}[1, b]$ and $\mathcal{L}[1, b]$ the global coordinates are naturally introduced, so that they become smooth manifolds. Let us here denote the coordinates by $(y^I)_{I\in\boldsymbol{E}[1,b]}$ on $\mathcal{A}[1, b]$ and $(u^I)_{I\in\boldsymbol{G}[1,b]}$ on $\mathcal{L}[1, b]$.

Let $X-X', Y-Y'\in\overline{\mathcal{A}[b+1, \infty)}$ (where $X, X', Y, Y'\in\overline{\mathcal{A}[1, \infty)}$). Then, by (1.6), (1.7), (1.9) and (1.12), so are $XY-X'Y'$, $\exp X-\exp X'$ and $\log(1+X)-\log(1+X')$, and hence $\log\exp X\exp Y-\log\exp X'\exp Y'\in\overline{\mathcal{A}[b+1, \infty)}$. In particular, if $X, X', Y, Y'\in\overline{\mathcal{L}(\boldsymbol{E})}$, $X\cdot Y-X'\cdot Y'\in\overline{\mathcal{L}[b+1, \infty)}$ and thus $R_i(X)-R_i(X')\in\overline{\mathcal{L}[b+1, \infty)}$ $(i\in\boldsymbol{E})$. By these observations, we define the following:

DEFINITION 2. (i) For $X, Y\in\mathcal{L}[1, b]$, a product Z of X and Y is a unique element of $\mathcal{L}[1, b]$ such that

$$X\cdot Y-Z\in\overline{\mathcal{L}[b+1, \infty)} \tag{1.23}$$

where $X\cdot Y$ is a product in $\overline{\mathcal{L}(\boldsymbol{E})}$ of X and Y. We denote Z by the same symbol $X\cdot Y$.

(ii) Smooth mappings $\Phi^{(b)}$ of $\mathcal{L}[1, b]\times\mathcal{A}[1, b]$ into $\mathcal{A}[1, b]$ and $\varphi^{(b)}$ of $\mathcal{L}[1, b]$ into $\mathcal{A}[1, b]$ are defined as follows: For $X\in\mathcal{L}[1, b]$ and $Y\in\mathcal{A}[1, b]$, $\Phi^{(b)}(X, Y)$ and $\varphi^{(b)}(X)$ are unique elements of $\mathcal{A}[1, b]$ such that

$$\begin{aligned}&1+\Phi^{(b)}(X, Y)-(\exp X)(1+Y)\in\overline{\mathcal{A}[b+1, \infty)}\\&1+\varphi^{(b)}(X)-\exp X\in\overline{\mathcal{A}[b+1, \infty)}.\end{aligned} \tag{1.24}$$

(iii) For each $i\in\boldsymbol{E}$, a smooth mapping $R_i^{(b)}$ of $\mathcal{L}[1, b]$ into itself is defined as follows: For $X\in\mathcal{L}[1, b]$, $R_i^{(b)}(X)$ is a unique element of $\mathcal{L}[1, b]$ such that

$$R_i(X)-R_i^{(b)}(X)\in\overline{\mathcal{L}[b+1, \infty)}. \tag{1.25}$$

By definition, it is clear that

$$\varphi^{(b)}(X)=\Phi^{(b)}(X, 0) \tag{1.26}$$

$$\begin{aligned}&\Phi^{(b)}(X, \Phi^{(b)}(Y, Z))=\Phi^{(b)}(X\cdot Y, Z)\\&\text{in particular}\quad \Phi^{(b)}(X, \varphi^{(b)}(Y))=\varphi^{(b)}(X\cdot Y)\end{aligned} \tag{1.27}$$

$$R_i^{(b)}(X)=\frac{d}{dt}X\cdot ti\Big|_{t=0}\qquad i\in\boldsymbol{E}. \tag{1.28}$$

Moreover the following holds:

FACT 3. *For each $I\in\boldsymbol{E}[1, b]$, $X\in\mathcal{L}[1, b]$ and $i\in\boldsymbol{E}$*

$$y^I\circ(1+\varphi^{(b)}(X))i=\sum_{J\in\boldsymbol{G}[1,b]}\frac{\partial}{\partial u^J}y^I\circ\varphi^{(b)}(X)u^J\circ R_i^{(b)}(X). \tag{1.29}$$

PROOF. For $X, Y\in\overline{\mathcal{A}(\boldsymbol{E})}$ let us write $X\sim Y$ if and only if $X-Y\in\overline{\mathcal{A}[b+1, \infty)}$.

By (1.27) and (1.24)

$$1+\varphi^{(b)}(X\cdot ti)\sim \exp X\exp ti\,.$$

Note that $(d/dt)\exp X\exp ti\,|_{t=0}=(\exp X)i\sim(1+\varphi^{(b)}(X))i$ and that by (1.28)

$$\text{RHS of (1.29)}=\frac{d}{dt}\,y^I\circ\varphi^{(b)}(X\cdot ti)\Big|_{t=0}.$$

Hence combined these we obtain (1.29). ■

We may think of $R_i^{(b)}\in C^\infty(\mathcal{L}[1,\,b]\to\mathcal{L}[1,\,b])$ as a vector field on $\mathcal{L}[1,\,b]$ by setting

$$R_i^{(b)}=\sum_{J\in\boldsymbol{G}[1,\,b]}u^J\circ R_i^{(b)}\frac{\partial}{\partial u^J}. \tag{1.30}$$

Here we use the same symbol $R_i^{(b)}$ to denote the vector field. Equipped with a multiplication $\cdot$ introduced in Definition 2, (i), $\mathcal{L}[1,\,b]$ has a group structure. By (1.13) it is easy to check that a mapping $(X,\,Y)\mapsto X\cdot Y^{-1}=X\cdot(-Y)$ is smooth. Hence $(\mathcal{L}[1,\,b],\,\cdot)$ becomes a Lie group. Let L_X be the left translation by $X(\in\mathcal{L}[1,\,b])$ and $\mathfrak{h}_b$ be the left invariant Lie algebra of $\mathcal{L}[1,\,b]$. Then

$$(R_i^{(b)})_X=(L_X)_*\Big(\frac{\partial}{\partial u^i}\Big)_0\qquad X\in\mathcal{L}[1,\,b] \tag{1.31}$$

and hence $R_i^{(b)}\in\mathfrak{h}_b$. Because by (1.28) and (1.30)

$$\begin{aligned}(L_X)_*\Big(\frac{\partial}{\partial u^i}\Big)_0 f&=\Big(\frac{\partial}{\partial u^i}\Big)_0 f\circ L_X\\&=\frac{d}{dt}f(X\cdot ti)\Big|_{t=0}\\&=\sum_{J\in\boldsymbol{G}[1,\,b]}u^J\circ R_i^{(b)}(X)\frac{\partial}{\partial u^J}f(X)\\&=(R_i^{(b)})_X f\qquad f\in C^\infty(\mathcal{L}[1,\,b]).\end{aligned}$$

By the freeness of $\mathcal{L}(\boldsymbol{E})$ the mapping $\boldsymbol{E}\ni i\mapsto R_i^{(b)}\in\mathfrak{h}_b$ is extended to a homomorphism γ of $\mathcal{L}(\boldsymbol{E})$ into $\mathfrak{h}_b$. It is clear that

$$\gamma([i_1\cdots i_m])=[[\cdots[[R_{i_1}^{(b)},\,R_{i_2}^{(b)}],\,R_{i_3}^{(b)}],\cdots,\,R_{i_{m-1}}^{(b)}],\,R_{i_m}^{(b)}].$$

Let us denote it by $R_{[i_1\cdots i_m]}^{(b)}$. Then the following fact holds:

FACT 4. *For* $I\in\boldsymbol{G}[1,\,b]$

$$(R_{[I]}^{(b)})_X=(L_X)_*\Big(\frac{\partial}{\partial u^I}\Big)_0\qquad X\in\mathcal{L}[1,\,b] \tag{1.32}$$

and for $I\in\boldsymbol{E}[b+1,\,\infty)$

$$R_{[I]}^{(b)}=0. \tag{1.33}$$

PROOF. First of all note that (1.32) is equivalent to (1.32)′ or (1.32)″:

$$R^{(b)}_{[I]} = \sum_{J\in \boldsymbol{G}[1,b]} \frac{d}{dt} u^J \circ (X\cdot t[I])\Big|_{t=0} \frac{\partial}{\partial u^J} \tag{1.32)′}$$

$$(R^{(b)}_{[I]})_0 = \Big(\frac{\partial}{\partial u^I}\Big)_0. \tag{1.32)″}$$

For convenience set a smooth function $R^{(b),J}_{[I]}$ on $\mathcal{L}[1, b]$ ($J\in \boldsymbol{G}[1, b]$) by

$$R^{(b),J}_{[I]}(X) := \frac{d}{dt} u^J \circ (X\cdot t[I])\Big|_{t=0}. \tag{1.34}$$

We show (1.32) by induction on the degree of I. When $I\in \boldsymbol{G}_1$, (1.32) is obvious because it is just (1.31). Next let $1\leqq c\leqq b-1$ and suppose that (1.32) (so (1.32)′ or (1.32)″) is true for any $I\in \boldsymbol{G}[1, c]$. Let $J\in \boldsymbol{E}_c$ and $j\in \boldsymbol{E}$. By (1.2)

$$R^{(b)}_{[J]} = \sum_{I\in \boldsymbol{G}_c} e^I_J R^{(b)}_{[I]}.$$

By this and the assumption of induction, we observe

$$\begin{aligned} R^{(b)}_{[J_j]} &= [R^{(b)}_{[J]}, R^{(b)}_j] \\ &= \sum_{I\in \boldsymbol{G}_c} e^I_J [R^{(b)}_{[I]}, R^{(b)}_j] \\ &= \sum_{I\in \boldsymbol{G}_c} e^I_J \Big[\sum_{K\in \boldsymbol{G}[1,b]} R^{(b),K}_{[I]} \frac{\partial}{\partial u^K}, \sum_{L\in \boldsymbol{G}[1,b]} R^{(b),L}_j \frac{\partial}{\partial u^L}\Big] \qquad (1.35) \\ &= \sum_{I\in \boldsymbol{G}_c} e^I_J \sum_{K,L\in \boldsymbol{G}[1,b]} \Big\{R^{(b),K}_{[I]} \Big(\frac{\partial}{\partial u^K} R^{(b),L}_j\Big) \frac{\partial}{\partial u^L} - R^{(b),L}_j \Big(\frac{\partial}{\partial u^L} R^{(b),K}_{[I]}\Big) \frac{\partial}{\partial u^K}\Big\}. \end{aligned}$$

By (1.34) and (1.17)

$$\sum_{K\in \boldsymbol{G}[1,b]} R^{(b),K}_{[I]}(X)[K] \sim [I] + \sum_{m=1}^{\infty} \frac{b_m}{m!} (\mathrm{ad}(-X))^m [I] \tag{1.36}$$

and this implies that

$$\begin{aligned} &R^{(b),K}_{[I]}(0) = \delta^K_I, \qquad R^{(b),L}_j(0) = \delta^L_j \\ &\frac{\partial}{\partial u^L} R^{(b),K}_{[I]}(0) = u^K \circ \frac{1}{2}[[L], [I]] \\ &\frac{\partial}{\partial u^K} R^{(b),L}_j(0) = u^L \circ \frac{1}{2}[[K], j]. \end{aligned}$$

Hence a tangent vector $(R^{(b)}_{[J_j]})_0$ at 0 is computed:

$$\begin{aligned} (R^{(b)}_{[J_j]})_0 &= \sum_{I\in \boldsymbol{G}_c} e^I_J \sum_{K,L\in \boldsymbol{G}[1,b]} \Big\{\delta^K_I u^L \circ \frac{1}{2}[[K], j] \Big(\frac{\partial}{\partial u^L}\Big)_0 \\ &\qquad - \delta^L_j u^K \circ \frac{1}{2}[[L], [I]] \Big(\frac{\partial}{\partial u^K}\Big)_0\Big\} \\ &= \sum_{I\in \boldsymbol{G}_c} e^I_J \sum_{K\in \boldsymbol{G}[1,b]} u^K \circ [Ij] \Big(\frac{\partial}{\partial u^K}\Big)_0 \end{aligned}$$

$$= \sum_{K\in G[1,b]} \Big(\sum_{I\in G_c} e_J^I u^K \circ [Ij] \Big) \Big(\frac{\partial}{\partial u^K} \Big)_0$$

$$= \sum_{K\in G[1,b]} e_{Jj}^K \Big(\frac{\partial}{\partial u^K} \Big)_0 .$$

Here the last equality is seen from the expression:

$$\begin{aligned} \sum_{K\in G[1,b]} \Big(\sum_{I\in G_c} e_J^I u^K \circ [Ij] \Big)[K] &= \sum_{I\in G_c} e_J^I \sum_{K\in G[1,b]} (u^K \circ [Ij])[K] \\ &= \sum_{I\in G_c} e_J^I [Ij] \\ &= [\sum_{I\in G_c} e_J^I [I], j] \\ &= [Jj] \\ &= \sum_{K\in G[1,b]} e_{Jj}^K [K] . \end{aligned}$$

Thus we obtain that for $J\in \boldsymbol{E}_{c+1}$

$$(R_{[J]}^{(b)})_0 = \sum_{K\in G[1,b]} e_J^K \Big(\frac{\partial}{\partial u^K} \Big)_0$$

and, in particular, for $J\in \boldsymbol{G}_{c+1}$

$$(R_{[J]}^{(b)})_0 = \Big(\frac{\partial}{\partial u^J} \Big)_0$$

since $e_J^K = \delta_J^K$. This is just (1.32)″ and so we see that (1.32) holds for any $I \in \boldsymbol{G}_{c+1}$. Consequently we have (1.32).

Next we show (1.33). By (1.36), it is easy to see that for $I\in \boldsymbol{G}_b$ and $i\in \boldsymbol{E}$

$$R_{[I]}^{(b),K}(X) = \delta_I^K, \qquad \frac{\partial}{\partial u^I} R_i^{(b),K}(X) = 0 \qquad K \in \boldsymbol{G}[1, b].$$

By this, (1.35) implies that $R_{[Jj]}^{(b)} = 0$ for $J\in \boldsymbol{E}_b$ and $j\in \boldsymbol{E}$, and hence (1.33) follows. ■

1.4. We start this subsection with the following definition:

DEFINITION 3. Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{R}$ and $\mathfrak{g}^m$, $m\in \boldsymbol{N}$ be a decreasing sequence of ideals in $\mathfrak{g}$ defined by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^m := [\mathfrak{g}, \mathfrak{g}^{m-1}]$ $(m \geqq 2)$. Then we say that $\mathfrak{g}$ is b-nilpotent if and only if $\mathfrak{g}^{b+1} = \{0\}$.

Fact 4 tells us that the left invariant Lie algebra $\mathfrak{h}_b$ of $\mathcal{L}[1, b]$ is b-nilpotent.

Let $b\in \boldsymbol{N}$ be fixed as the preceding subsection. Let G be a connected Lie group and $\mathcal{R}$ be the right invariant Lie algebra of G. Suppose that we are given $A_i \in \mathcal{R}$, $i\in \boldsymbol{E}$ and that a Lie subalgebra $\mathfrak{g}$ generated by them is b-nilpotent. Let γ: $\mathcal{L}(\boldsymbol{E}) \to \mathfrak{g}$ be a homomorphic extension of the mapping $\boldsymbol{E} \ni i \mapsto A_i \in \mathfrak{g}$. Then the following fact holds:

FACT 5. *For* $X, Y\in\mathcal{L}[1, b]$

$$\exp\gamma(X\cdot Y)=(\exp\gamma(Y))(\exp\gamma(X)). \tag{1.37}$$

Here exp *is the exponential mapping of* $\mathcal{R}$ *into* G.

PROOF. Fix $X, Y\in\mathcal{L}[1, b]$. Note that $\exp\gamma(tX\cdot tY)$ and $\exp\gamma(tY)\exp\gamma(tX)$ are analytic functions in t. Thus it suffices to show that for any $f\in C^\infty(G)$ and $m\in\boldsymbol{N}$

$$\frac{d^m}{dt^m}f(\exp\gamma(tX\cdot tY))\Big|_{t=0}=\frac{d^m}{dt^m}f(\exp\gamma(tY)\exp\gamma(tX))\Big|_{t=0}. \tag{1.38}$$

By (1.13)

$$tX\cdot tY\sim\sum_{\mu=1}^{\infty}t^\mu c_\mu(X, Y).$$

By the b-nilpotency of $\mathfrak{g}$, this implies that

$$\gamma(tX\cdot tY)=\sum_{\mu=1}^{\infty}t^\mu c_\mu(\gamma(X), \gamma(Y)). \tag{1.39}$$

Here, for $A, B\in\mathcal{R}$, $c_\mu(A, B)\in\mathcal{R}$ is defined by

$$c_\mu(A, B):=\frac{1}{\mu}\sum_{m=1}^{\mu}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_1, q_1, \cdots, p_m, q_m\geq 0;\\ p_1+q_1, \cdots, p_m+q_m\geq 1\\ p_1+q_1+\cdots+p_m+q_m=\mu}}\frac{(\mathrm{ad}\,A)^{p_1}(\mathrm{ad}\,B)^{q_1}\cdots(\mathrm{ad}\,A)^{p_m}(\mathrm{ad}\,B)^{q_m}}{p_1!\,q_1!\cdots p_m!\,q_m!}.$$

Since $c_\mu(\gamma(X), \gamma(Y))=0$ for $\mu\geqq b+1$ (by the b-nilpotency of $\mathfrak{g}$), RHS of (1.39) is a finite sum.

Let $f\in C^\infty(G)$. In general we know that for $A\in\mathcal{R}$, $g\in G$, $t\in\boldsymbol{R}$ and $l\in\boldsymbol{Z}_+$

$$\begin{aligned}f((\exp tA)g)&=\sum_{p=0}^{l}\frac{t^p}{p!}(A^pf)(g)\\&\quad+\int_0^t dt_1\int_0^{t_1}dt_2\cdots dt_l\int_0^{t_l}(A^{l+1}f)((\exp t_{l+1}A)g)dt_{l+1}.\end{aligned} \tag{1.40}$$

Let $A=\gamma(tX\cdot tY)$, $g=e$ and $t=1$ in (1.40). Then, by (1.39), we have that for each $l\in\boldsymbol{Z}_+$

$$\begin{aligned}&f(\exp\gamma(tX\cdot tY))\\&=\sum_{p=0}^{l}\frac{1}{p!}\sum_{\mu_1, \cdots, \mu_p\geq 1}t^{\mu_1+\cdots+\mu_p}(c_{\mu_1}(\gamma(X), \gamma(Y))\cdots c_{\mu_p}(\gamma(X), \gamma(Y))f)(e)+O(t^{l+1})\end{aligned}$$

and hence, for $m\in\boldsymbol{N}$

$$\begin{aligned}&\frac{1}{m!}\frac{d^m}{dt^m}f(\exp\gamma(tX\cdot tY))\Big|_{t=0}\\&\qquad=\sum_{k=1}^{m}\frac{1}{k!}\sum_{\substack{m_1, \cdots, m_k\geq 1;\\ m_1+\cdots+m_k=m}}(c_{m_1}(\gamma(X), \gamma(Y))\cdots c_{m_k}(\gamma(X), \gamma(Y))f)(e).\end{aligned}$$

By the same way as above (this time the formula (1.40) is used twice), we have that for each $l\in\boldsymbol{Z}^+$

$$f(\exp\gamma(tY)\exp\gamma(tX))=\sum_{p,q=0}^{l}\frac{t^{p+q}}{p!q!}(\gamma(X)^p\gamma(Y)^qf)(e)+O(t^{l+1})$$

and hence, for $m\in\boldsymbol{N}$

$$\frac{1}{m!}\frac{d^m}{dt^m}f(\exp\gamma(tY)\exp\gamma(tX))\Big|_{t=0}=\sum_{\substack{p,q\geqq 0;\\ p+q=m}}\frac{1}{p!q!}(\gamma(X)^p\gamma(Y)^qf)(e).$$

Here recall (1.15). This suggests that as a differential operator on G

$$\sum_{\substack{p,q\geqq 0;\\ p+q=m}}\frac{\gamma(X)^p\gamma(Y)^q}{p!q!}=\sum_{k=1}^{m}\frac{1}{k!}\sum_{\substack{m_1,\cdots,m_k\geqq 1;\\ m_1+\cdots+m_k=m}}c_{m_1}(\gamma(X),\gamma(Y))\cdots c_{m_k}(\gamma(X),\gamma(Y)).$$

Consequently combining all the above, we obtain (1.38) immediately. ∎

2. Main results.

Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space. For each $i\in\boldsymbol{E}$ let $(M_t^i)_{t\geqq 0}$ be a continuous $(\mathcal{F}_t)$-local semimartingale such that $M_0^i=0$.

DEFINITION 4. We define a system $(M_t^I)_{t\geqq 0}$, $I\in\boldsymbol{E}[1,\infty)$ of continuous $(\mathcal{F}_t)$-local semimartingales successively: For $I=i_1\cdots i_m\in\boldsymbol{E}[1,\infty)$

$$M_t^I:=\begin{cases}M_t^{i_1} & \text{if } m=1\\ \int_0^t M_s^{i_1\cdots i_{m-1}}\circ dM_s^{i_m} & \text{if } m\geqq 2.\end{cases}$$

Here the symbol $\circ$ denotes the Stratonovich stochastic integral.

First of all we note the following formula. This plays important roles in the following theorems. The proof is easily done by using the chain rule (Itô's formula). (Here we remark that this formula appears in appendix of [***B***].)

LEMMA 2. *Let $m, m_1, \cdots, m_b\in\boldsymbol{N}$ be such that $m_1+\cdots+m_b=m$. Then for each $i_1\cdots i_m\in\boldsymbol{E}[1,\infty)$*

$$M_t^{i_1\cdots i_{m_1}}M_t^{i_{m_1+1}\cdots i_{m_1+m_2}}\cdots M_t^{i_{m_1+\cdots+m_{b-1}+1}\cdots i_{m_1+\cdots+m_b}}=\sum_{\substack{\sigma\in\mathfrak{S}_m;\sigma(1)<\cdots<\sigma(m_1)\\ \sigma(m_1+1)<\cdots<\sigma(m_1+m_2)\\ \cdots\cdots\cdots\cdots\\ \sigma(m_1+\cdots+m_{b-1}+1)<\cdots<\sigma(m_1+\cdots+m_b)}}M_t^{i_{\sigma^{-1}(1)}\cdots i_{\sigma^{-1}(m)}}.$$

Here $\mathfrak{S}_m$ is the symmetric group of degree m.

THEOREM 1. $\log(1+\sum_{I\in\boldsymbol{E}[1,\infty)}M_t^I I)\in\overline{\mathcal{L}(\boldsymbol{E})}$ *for each* $t\geqq 0$.

PROOF. For simplicity set

$$Y_t:=1+\sum_{I\in\boldsymbol{E}[1,\infty)}M_t^I I\in\overline{\mathcal{A}(\boldsymbol{E})},\qquad X_t=\log Y_t\in\overline{\mathcal{A}[1,\infty)}.$$

For the proof, it suffices to show that

$$\delta(Y_t) = Y_t \otimes Y_t \tag{2.1}$$

where δ is the diagonal mapping of $\overline{\mathcal{A}(\boldsymbol{E})}$. Indeed, since $Y_t = \exp X_t$

$$\begin{aligned}
\delta(Y_t) &= \exp \delta(X_t) \\
Y_t \otimes Y_t &= \exp X_t \otimes \exp X_t \\
&= (1 \otimes \exp X_t)(\exp X_t \otimes 1) \\
&= \exp(1 \otimes X_t) \exp(X_t \otimes 1) \\
&= \exp(1 \otimes X_t + X_t \otimes 1) \ .
\end{aligned}$$

Hence by (1.14), (2.1) is equivalent to that $X_t \in \overline{\mathcal{L}(\boldsymbol{E})}$.

Now $\delta(Y_t)$ is computed as follows:

$$\begin{aligned}
\delta(Y_t) &= \delta\Big(1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I I\Big) \\
&= 1 \otimes 1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I \delta(I) \\
&= 1 \otimes 1 + \sum_{m=1}^{\infty} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} \delta(i_1) \cdots \delta(i_m) \\
&= 1 \otimes 1 + \sum_{m=1}^{\infty} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} (i_1 \otimes 1 + 1 \otimes i_1) \cdots (i_m \otimes 1 + 1 \otimes i_m) \\
&= 1 \otimes 1 + \sum_{m=1}^{\infty} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} ((i_1 \cdots i_m) \otimes 1 + 1 \otimes (i_1 \cdots i_m) \\
&\quad + \sum_{p=1}^{m-1} \sum_{\substack{\sigma \in \mathfrak{S}_m; \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(m)}} i_{\sigma(1)} \cdots i_{\sigma(p)} \otimes i_{\sigma(p+1)} \cdots i_{\sigma(m)}) \\
&= 1 \otimes 1 + \sum_{m=1}^{\infty} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} ((i_1 \cdots i_m) \otimes 1 + 1 \otimes (i_1 \cdots i_m)) \\
&\quad + \sum_{m=2}^{\infty} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} \sum_{p=1}^{m-1} \sum_{\substack{\sigma \in \mathfrak{S}_m; \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(m)}} i_{\sigma(1)} \cdots i_{\sigma(p)} \otimes i_{\sigma(p+1)} \cdots i_{\sigma(m)}.
\end{aligned}$$

By Lemma 2, the last expression is further computed:

$$\begin{aligned}
&\sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} \sum_{p=1}^{m-1} \sum_{\substack{\sigma \in \mathfrak{S}_m; \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(m)}} i_{\sigma(1)} \cdots i_{\sigma(p)} \otimes i_{\sigma(p+1)} \cdots i_{\sigma(m)} \\
&= \sum_{p=1}^{m-1} \sum_{\substack{\sigma \in \mathfrak{S}_m; \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(m)}} \sum_{i_1 \cdots i_m} M_t^{i_{\sigma(\sigma^{-1}(1))} \cdots i_{\sigma(\sigma^{-1}(m))}} i_{\sigma(1)} \cdots i_{\sigma(p)} \otimes i_{\sigma(p+1)} \cdots i_{\sigma(m)} \\
&= \sum_{p=1}^{m-1} \sum_{i_1 \cdots i_m} \Big(\sum_{\substack{\sigma \in \mathfrak{S}_m; \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(m)}} M_t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(m)}} \Big) i_1 \cdots i_p \otimes i_{p+1} \cdots i_m
\end{aligned}$$

$$= \sum_{p=1}^{m-1} \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_p} M_t^{i_{p+1} \cdots i_m} i_1 \cdots i_p \otimes i_{p+1} \cdots i_m$$

$$= \sum_{|I|+|J|=m} M_t^I M_t^J I \otimes J .$$

Hence substituting it into the expression of $\delta(Y_t)$, we see

$$\begin{aligned}\delta(Y_t) &= 1 \otimes 1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I (I \otimes 1 + 1 \otimes I) + \sum_{m \geq 2} \sum_{|I|+|J|=m} M_t^I M_t^J I \otimes J \\ &= 1 \otimes 1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I (I \otimes 1 + 1 \otimes I) + \sum_{I, J \in \boldsymbol{E}[1,\infty)} M_t^I M_t^J I \otimes J \\ &= Y_t \otimes Y_t\end{aligned}$$

and (2.1) is obtained. ■

By Theorem 1, we can take $X_m(t) \in \mathcal{L}_m$, $m \in \boldsymbol{N}$ such that

$$\log\Big(1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I I\Big) = \sum_{m=1}^{\infty} X_m(t). \tag{2.2}$$

This LHS can be written down as follows:

$$\log\Big(1 + \sum_{I \in \boldsymbol{E}[1,\infty)} M_t^I I\Big) = \sum_{m=1}^{\infty} \sum_{I \in \boldsymbol{E}_m} \Big(\sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} \sum_{\substack{I_1, \cdots, I_b \in \boldsymbol{E}[1,\infty); \\ I = I_1 \cdots I_b}} M_t^{I_1} \cdots M_t^{I_b} \Big) I.$$

Hence combined these it is obtained that

$$X_m(t) = \sum_{I \in \boldsymbol{E}_m} \Big(\sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} \sum_{\substack{I_1, \cdots, I_b \in \boldsymbol{E}[1,\infty); \\ I = I_1 \cdots I_b}} M_t^{I_1} \cdots M_t^{I_b} \Big) I,$$

and thus by (1.4) we have

$$X_m(t) = \frac{1}{m} \sum_{I \in \boldsymbol{E}_m} \Big(\sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} \sum_{\substack{I_1, \cdots, I_b \in \boldsymbol{E}[1,\infty); \\ I = I_1 \cdots I_b}} M_t^{I_1} \cdots M_t^{I_b} \Big) [I].$$

Now let us apply Lemma 2 to this RHS. Then

$$\begin{aligned}X_m(t) &= \frac{1}{m} \sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} \sum_{i_1 \cdots i_m} \\ &\quad \Big(\sum_{\substack{m_1, \cdots, m_b \geq 1; \\ m_1 + \cdots + m_b = m}} M_t^{i_1 \cdots i_{m_1}} M_t^{i_{m_1+1} \cdots i_{m_1+m_2}} \cdots M_t^{i_{m_1+\cdots+m_{b-1}+1} \cdots i_{m_1+\cdots+m_b}} \Big) [i_1 \cdots i_m] \\ &= \frac{1}{m} \sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} \sum_{i_1 \cdots i_m} \\ &\quad \Big(\sum_{\substack{m_1, \cdots, m_b \geq 1; \\ m_1 + \cdots + m_b = m}} \sum_{\substack{\sigma \in \mathfrak{S}_m ; \sigma(1) < \cdots < \sigma(m_1) \\ \sigma(m_1+1) < \cdots < \sigma(m_1+m_2) \\ \cdots\cdots\cdots\cdots \\ \sigma(m_1+\cdots+m_{b-1}+1) < \cdots < \sigma(m_1+\cdots+m_b)}} M_t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(m)}} \Big) [i_1 \cdots i_m]\end{aligned}$$

$$= \frac{1}{m} \sum_{i_1 \cdots i_m} \sum_{\sigma \in \mathfrak{S}_m} \left(\sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} d(m, b, \sigma) \right) M_t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(m)}} [i_1 \cdots i_m]$$

where $d(m, b, \sigma)$ is the number of $(m_1, \cdots, m_b) \in \boldsymbol{N}^b$ such that $m_1 + \cdots + m_b = m$, $\sigma(1) < \cdots < \sigma(m_1)$, $\sigma(m_1+1) < \cdots < \sigma(m_1+m_2)$, $\cdots$, $\sigma(m_1 + \cdots + m_{b-1}+1) < \cdots < \sigma(m_1 + \cdots + m_b)$. This value is computed by Strichartz [**S**]:

$$d(m, b, \sigma) = 1_{b \geq e(\sigma)+1} \binom{m-e(\sigma)-1}{b-e(\sigma)-1}. \tag{2.3}$$

Here $\sigma(b)$ is introduced in Definition 5 below. From (2.3), it is easy to see

$$\sum_{b=1}^{m} \frac{(-1)^{b-1}}{b} d(m, b, \sigma) = \frac{(-1)^{e(\sigma)}}{m \binom{m-1}{e(\sigma)}} \quad \text{(cf. [S])}.$$

Thus, substituting it into the preceding expression, we conclude that

$$X_m(t) = \sum_{i_1 \cdots i_m} \left(\sum_{\sigma \in \mathfrak{S}_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} M_t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(m)}} \right) [i_1 \cdots i_m] \tag{2.4}$$

$$= \sum_{i_1 \cdots i_m} M_t^{i_1 \cdots i_m} \sum_{\sigma \in \mathfrak{S}_m} \frac{(-1)^{e(\sigma)}}{m^2 \binom{m-1}{e(\sigma)}} [i_{\sigma(1)} \cdots i_{\sigma(m)}].$$

We return to (2.2). By this

$$1 + \sum_{m=1}^{\infty} \sum_{I \in \boldsymbol{E}_m} M_t^I I = \exp\left\{ \sum_{m=1}^{\infty} X_m(t) \right\}$$

$$= 1 + \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{m=1}^{\infty} X_m(t) \right)^p$$

$$= 1 + \sum_{m=1}^{\infty} \sum_{p=1}^{m} \frac{1}{p!} \sum_{\substack{m_1, \cdots, m_p \geq 1; \\ m_1 + \cdots + m_p = m}} X_{m_1}(t) \cdots X_{m_p}(t).$$

and hence

$$\sum_{I \in \boldsymbol{E}_m} M_t^I I = \sum_{p=1}^{m} \frac{1}{p!} \sum_{\substack{m_1, \cdots, m_p \geq 1; \\ m_1 + \cdots + m_p = m}} X_{m_1}(t) \cdots X_{m_p}(t).$$

Summarizing all the above, we obtain the following formulas as a by-product of Theorem 1. Before stating them, we introduce the following notations for simplicity:

DEFINITION 5. Let $m \in \boldsymbol{N}$. For $\sigma \in \mathfrak{S}_m$ and $I = i_1 \cdots i_m \in \boldsymbol{E}_m$, define

$$I \circ \sigma := i_{\sigma(1)} \cdots i_{\sigma(m)} \in \boldsymbol{E}_m$$

$$e(\sigma) := \#\{1 \leq j \leq m-1 \,;\, \sigma(j) > \sigma(j+1)\}.$$

THEOREM 2. *Define* $X_m(t)\in\mathcal{L}_m$, $m\in\boldsymbol{N}$ *as follows*:

$$X_m(t):=\sum_{I\in\boldsymbol{E}_m}\Big(\sum_{\sigma\in\mathfrak{S}_m}\frac{(-1)^{e(\sigma)}}{m^2\binom{m-1}{e(\sigma)}}M_t^{I\circ\sigma^{-1}}\Big)[I] \tag{2.5}$$

$$=\sum_{I\in\boldsymbol{E}_m}M_t^I\Big(\sum_{\sigma\in\mathfrak{S}_m}\frac{(-1)^{e(\sigma)}}{m^2\binom{m-1}{e(\sigma)}}[I\circ\sigma]\Big). \tag{2.6}$$

Then

$$\log\Big(1+\sum_{I\in\boldsymbol{E}[1,\infty)}M_t^I I\Big)=\sum_{m=1}^{\infty}X_m(t) \tag{2.7}$$

$$1+\sum_{I\in\boldsymbol{E}[1,\infty)}M_t^I I=\exp\Big\{\sum_{m=1}^{\infty}X_m(t)\Big\} \tag{2.8}$$

$$\sum_{I\in\boldsymbol{E}_m}M_t^I I=\sum_{p=1}^{m}\frac{1}{p!}\sum_{\substack{m_1,\cdots,m_p\geq 1;\\ m_1+\cdots+m_p=m}}X_{m_1}(t)\cdots X_{m_p}(t)\qquad m\in\boldsymbol{N}. \tag{2.9}$$

Next we give another characterization of this $X(t)=\sum_{m=1}^{\infty}X_m(t)$.

For this recall a mapping R_i of $\overline{\mathcal{L}(\boldsymbol{E})}$ into itself defined by (1.20) ($i\in\boldsymbol{E}$). We consider the following SDE on $\overline{\mathcal{L}(\boldsymbol{E})}$:

$$\begin{aligned}dX_t&=\sum_{i\in\boldsymbol{E}}R_i(X_t)\circ dM_t^i\\ X_0&=0.\end{aligned} \tag{2.10}$$

This SDE can be solved as follows: Written $X\in\overline{\mathcal{L}(\boldsymbol{E})}$ as $X=\sum_{m=1}^{\infty}X_m(X_m\in\mathcal{L}_m)$, (1.21) tells us that

$$R_i(X)=i+\sum_{m=1}^{\infty}\sum_{p=1}^{m}\frac{b_p}{p!}\sum_{\substack{m_1,\cdots,m_p\geq 1;\\ m_1+\cdots+m_p=m}}\mathrm{ad}(-X_{m_1})\cdots\mathrm{ad}(-X_{m_p})i.$$

Hence we see that the solution of (2.10) $=\sum_{m=1}^{\infty}X_m(t)$ where $X_m(t)\in\mathcal{L}_m$, $m\in\boldsymbol{N}$ are given by the recursion formula:

$$\begin{aligned}X_1(t)&=\sum_{i\in\boldsymbol{E}}M_t^i i\\ X_m(t)&=\sum_{i\in\boldsymbol{E}}\sum_{p=1}^{m-1}\frac{b_p}{p!}\\ &\qquad\sum_{\substack{m_1,\cdots,m_p\geq 1;\\ m_1+\cdots+m_p=m-1}}\int_0^t\mathrm{ad}(-X_{m_1}(s))\cdots\mathrm{ad}(-X_{m_p}(s))i\circ dM_s^i\qquad m\geqq 2.\end{aligned} \tag{2.11}$$

THEOREM 3. *The solution* (2.10) *coincides with* $\sum_{m=1}^{\infty}X_m(t)$ *where* $X_m(t)\in\mathcal{L}_m$, $m\in\boldsymbol{N}$ *are in Theorem* 2. *In other words, two definitions* (2.5) (*or* (2.6)) *and* (2.11) *of* $X_m(t)$, $m\in\boldsymbol{N}$ *agree with the other.*

We distinguish two definitions: Let $X^1_m(t)$ and $X^2_m(t)$ be defined by (2.5) (or (2.6)) and (2.11) respectively. Clearly by (2.8)

$$\exp\left\{\sum_{m=1}^{\infty} X^1_m(t)\right\} = 1+\sum_{I\in \boldsymbol{E}[1,\infty)} M^I_t I\,.$$

It is easy to check that the right expression is characterized by the SDE on $\overline{\mathcal{A}(\boldsymbol{E})}$:

$$\begin{aligned} dY_t &= \sum_{i\in \boldsymbol{E}} Y_t i\circ dM^i_t \\ Y_0 &= 1\,. \end{aligned} \tag{2.12}$$

Thus in order to prove Theorem 3, it suffices to show that $\exp\{\sum_{m=1}^{\infty} X^2_m(t)\}$ satisfies the SDE (2.12).

PROOF OF THEOREM 3. Write $X_m(t)=X^2_m(t)$ and let $X(t):=\sum_{m=1}^{\infty} X_m(t)$. $X(t)$ is the solution of (2.10). Fix $b\in \boldsymbol{N}$ arbitrarily. Let $X^{(b)}(t):=\sum_{m=1}^{b} X_m(t)\in \mathcal{L}[1, b]$. Recall $\varphi^{(b)}\in C^\infty(\mathcal{L}[1, b]\to \mathcal{A}[1, b])$ and $R^{(b)}_i\in C^\infty(\mathcal{L}[1, b]\to \mathcal{L}[1, b])$ (or $\in \mathcal{X}(\mathcal{L}[1, b])$). Since $X(t)\sim X^{(b)}(t)$ (this notation is in the proof of Fact 3),

$$1+\varphi^{(b)}(X^{(b)}(t)) \sim \exp X(t) \tag{2.13}$$

$$R^{(b)}_i(X^{(b)}(t)) \sim R_i(X(t))\,. \tag{2.14}$$

By (2.10) and (2.14), $X^{(b)}(t)$ is a solution of the SDE on $\mathcal{L}[1, b]$:

$$\begin{aligned} dX^{(b)}(t) &= \sum_{i\in \boldsymbol{E}} R^{(b)}_i(X^{(b)}(t))\circ dM^i_t \\ X^{(b)}(0) &= 0\,. \end{aligned} \tag{2.15}$$

Hence from this and (1.29) it follows that for each $I\in \boldsymbol{E}[1, b]$

$$\begin{aligned} dy^I\circ\varphi^{(b)}(X^{(b)}(t)) &= \sum_{i\in \boldsymbol{E}} R^{(b)}_i(y^I\circ\varphi^{(b)})(X^{(b)}(t))\circ dM^i_t \\ &= \sum_{i\in \boldsymbol{E}} y^I\circ(1+\varphi^{(b)}(X^{(b)}(t)))i\circ dM^i_t\,. \end{aligned}$$

This together with (2.13) implies that $\exp X(t)$ satisfies the SDE (2.12) up to the first b-th component. Finally letting $b\uparrow\infty$ we obtain the conclusion. ■

3. Applications to the SDE's with nilpotent coefficients.

In this section, as a consequence of our main results we present the representation formulas of solutions to SDE's and ODE's with nilpotent coefficients.

For this we first state the following fact: Let M be a smooth manifold, V_i, $i\in \boldsymbol{E}$ be smooth vector fields on M and $\mathfrak{h}$ be a Lie subalgebra of $\mathcal{X}(M)$ generated by them. Suppose that

(3.1) V_i is a complete vector field for each $i\in \boldsymbol{E}$

(3.2) $\mathfrak{h}$ is b-nilpotent.

Let $\beta: \mathcal{L}(\boldsymbol{E})\to\mathfrak{h}$ be the homomorphic extension of the mapping $i\mapsto V_i$. By (3.2) $\mathfrak{h}$ is of finite dimension and hence, by Palais [***P***], the statement of (3.1) is strengthened as follows: $\beta(X)\in\mathfrak{h}$ is complete for any $X\in\mathcal{L}(\boldsymbol{E})$. By virtue of this, we can define a smooth mapping h of $\mathcal{L}[1, b]\times M$ into M by

$$h(X, x) := \mathrm{Exp}(\beta(X))(x). \tag{3.3}$$

Then the following fact holds:

FACT 6. *For each $x\in M$, $h(\cdot, x)$ satisfies the following system of first order partial differential equations on M:*

$$R_i^{(b)}h(\cdot, x) = V_i(h(\cdot, x)) \qquad i\in\boldsymbol{E}, \tag{3.4}$$

or more precisely for $f\in C^\infty(M)$

$$R_i^{(b)}f(h(\cdot, x)) = (V_if)(h(\cdot, x)) \qquad i\in\boldsymbol{E}. \tag{3.5}$$

PROOF. By Palais [***P***], we can take a connected Lie group G and a smooth mapping φ of $G\times M$ into M such that

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x) \qquad g, h\in G, x\in M \tag{3.6}$$

$$\varphi(e, x) = x \tag{3.7}$$

$$\varphi^+ \text{ is an isomorphism of } \mathcal{R} \text{ onto } \mathfrak{h}. \tag{3.8}$$

Here $\mathcal{R}$ is the right invariant Lie algebra of G and φ^+ is defined by the following way: For $L\in\mathcal{R}$ and $x\in M$

$$\varphi^+(L)_x := \frac{d}{dt}\varphi(\exp tL, x)|_{t=0}.$$

For simplicity write $gx := \varphi(g, x)$ for $g\in G$, $x\in M$. By definition

$$\mathrm{Exp}(\varphi^+(L))(x) = (\exp L)x \qquad L\in\mathcal{R}, x\in M. \tag{3.9}$$

By (3.8) $\mathcal{R}$ is b-nilpotent. Moreover there exists a homomorphism γ of $\mathcal{L}(\boldsymbol{E})$ into $\mathcal{R}$ such that $\varphi^+(\gamma(X))=\beta(X)$ for any $X\in\mathcal{L}(\boldsymbol{E})$. Hence by this and (3.9)

$$h(X, x) = (\exp\gamma(X))x \qquad X\in\mathcal{L}[1, b], x\in M. \tag{3.10}$$

Here we recall Fact 5. From this and (3.10), it follows that for $X, Y\in\mathcal{L}[1, b]$ and $x\in M$

$$h(X\cdot Y, x) = (\exp\gamma(Y))h(X, x).$$

In particular, letting $Y=ti$ in the above we have

$$h(X\cdot ti, x) = \mathrm{Exp}(tV_i)(h(X, x)).$$

Consequently differentiating it at $t=0$, we obtain (3.4) at once. ■

As in the preceding section let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space and $(M_t^i)_{t\geq 0}$, $i\in \boldsymbol{E}$ be a system of continuous $(\mathcal{F}_t)$-local semimartingales with $M_0^i=0$.

Let $V_i\in\mathcal{X}(M)$, $i\in\boldsymbol{E}$ be as above and we consider the following SDE on M:

$$\begin{aligned} dX_t &= \sum_{i\in\boldsymbol{E}} V_i(X_t)\circ dM_t^i \\ X_0 &= x\,. \end{aligned} \tag{3.11}$$

As for the solution to the SDE (3.11), we have the following:

THEOREM 4. *The SDE* (3.11) *has a unique solution* $X(t, x)$ *and it is given explicitly by the formula*:

$$\begin{aligned} X(t, x) &= \exp\Big\{\sum_{I\in\boldsymbol{E}[1,b]}\sum_{\sigma\in\mathfrak{S}_{|I|}} \frac{(-1)^{e(\sigma)}}{|I|^2\binom{|I|-1}{e(\sigma)}} M_t^{I\circ\sigma^{-1}} V_{[I]}\Big\}(x) \\ &= \exp\Big\{\sum_{I\in\boldsymbol{E}[1,b]} M_t^I \sum_{\sigma\in\mathfrak{S}_{|I|}} \frac{(-1)^{e(\sigma)}}{|I|^2\binom{|I|-1}{e(\sigma)}} V_{[I\circ\sigma]}\Big\}(x)\,. \end{aligned}$$

Here, as $R_{[I]}^{(b)}$, *we set* $V_{[I]}:=\beta([I])$ *for* $I\in\boldsymbol{E}[1, \infty)$.

PROOF. Let $X^{(b)}(t)$ be the solution of the SDE (2.15) on $\mathcal{L}[1, b]$. Then from (3.4) it is easy to see that $h(X^{(b)}(t), x)$ satisfies the SDE (3.11). Also, as we saw in the proof of Theorem 3, $X^{(b)}(t)=\sum_{m=1}^b X_m(t)$ where $X_m(t)$ is given by (2.5) or (2.6). Hence by (3.3) $h(X^{(b)}(t), x)$ equals the above formulas. This shows the existence of solutions. By the general theory of SDE's the uniqueness is obvious. Consequently we complete the proof. ■

As a corollary to Theorem 4, we can state the following: Let $\mathfrak{g}$ be a Lie subalgebra of $\mathcal{X}(M)$ such that

(3.12) $\mathfrak{g}$ is of finite dimension

(3.13) every $V\in\mathfrak{g}$ is complete

(3.14) $\mathfrak{g}$ is b-nilpotent.

Suppose we are given an L_1^{loc}-function A of $[0, \infty)$ into $\mathfrak{g}$. This means that every component of A with respect to some (and so every) basis of $\mathfrak{g}$ is in L_1^{loc} as a real function on $[0, \infty)$. We now consider the following ODE on M:

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t, x(t)) \\ x(0) &= x\,. \end{aligned} \tag{3.15}$$

Here $A(t, x)$ denotes a tangent vector of $A(t)\in\mathfrak{X}(M)$ at $x\in M$. Then we have the following:

THEOREM 5. *The ODE* (3.15) *has a unique solution* $x(t, x)$ *and it is given explicitly by the formula*:

$$x(t, x) = \mathrm{Exp}(z(t))(x)$$

where

$$z(t) := \sum_{m=1}^{b} \sum_{\sigma\in\mathfrak{S}_m} \frac{(-1)^{e(\sigma)}}{m^2\binom{m-1}{e(\sigma)}}$$

$$\times \int_{0<s_1<\cdots<s_m<t} \cdots \int [[\cdots[A(s_{\sigma(1)}), A(s_{\sigma(2)})], \cdots, A(s_{\sigma(m-1)})], A(s_{\sigma(m)})]ds_1\cdots ds_m .$$

PROOF. Let $n+1 := \dim\mathfrak{g}$ where $n\geqq 0$. When $n=0$, i.e., $\dim\mathfrak{g}=1$, it is easily seen that

$$x(t, x) = \mathrm{Exp}\left(\int_0^t A(s)ds\right)(x)$$

satisfies the ODE (3.15). Hence we may suppose that $n\geqq 1$.

Let $V_0, V_1, \cdots, V_n$ be a basis of $\mathfrak{g}$. Then we can find an $a\in L_1^{\mathrm{loc}}([0, \infty)\to \boldsymbol{R}^{n+1})$ such that

$$A(t) = \sum_{i=0}^{n} a^i(t)V_i . \tag{3.16}$$

We set $M_t^i := \int_0^t a^i(s)ds$. This $(M_t^i)_{t\geq 0}$ is clearly a continuous semimartingale on some $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ (in fact it is deterministic and absolutely continuous in t), and (3.15) is rewritten as

$$dx(t) = \sum_{i=0}^{n} V_i(x(t))\circ dM_t^i$$

$$x(0) = x .$$

Therefore applying Theorem 4 to this SDE, we have

$$x(t, x) = \mathrm{Exp}\left\{\sum_{I\in\mathcal{E}[1,b]} M_t^I \sum_{\sigma\in\mathfrak{S}_{|I|}} \frac{(-1)^{e(\sigma)}}{|I|^2\binom{|I|-1}{e(\sigma)}} V_{[I_\sigma]}\right\}(x).$$

Here we note the following expression:

$$M_t^{i_1\cdots i_m}V_{[i_1\cdots i_m\circ\sigma]} = \int_{0<s_1<\cdots<s_m<t} \cdots \int [[\cdots[a^{i_{\sigma(1)}}(s_{\sigma(1)})V_{i_{\sigma(1)}}, a^{i_{\sigma(2)}}(s_{\sigma(2)})V_{i_{\sigma(2)}}],$$
$$\cdots], a^{i_{\sigma(m)}}(s_{\sigma(m)})V_{i_{\sigma(m)}}]ds_1\cdots ds_m .$$

Consequently, by combining this with (3.16), the preceding expression equals the desired one in Theorem 5, and the proof is complete. ∎

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