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Homological and dynamical study on certain groups of Lipschitz homeomorphisms of the circle

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Let $\mathcal{X}(\boldsymbol{R}/\boldsymbol{Z})$ be a real vector space of bounded integrable functions on the circle which is invariant under the composition of any Lipschitz homeomorphism of the circle. That is, for any $\varphi\in\mathcal{X}$ and any Lipschitz homeomorphism f of the circle, $\varphi\circ f\in\mathcal{X}$. Such a function space $\mathcal{X}$ gives rise to a group $G^{L,\mathcal{X}}(\boldsymbol{R}/\boldsymbol{Z})$ of Lipschitz homeomorphisms of the circle: an element of $G^{L,\mathcal{X}}(\boldsymbol{R}/\boldsymbol{Z})$ is a Lipschitz homeomorphism f of the circle such that $\log f'(x-0)$ belongs to $\mathcal{X}$. The verification of the fact that $G^{L,\mathcal{X}}(\boldsymbol{R}/\boldsymbol{Z})$ is a group is elementary ([**19**]).

The groups $G^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})=\mathrm{Diff}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ $(0<\alpha<1)$ are of course examples of such groups. This $G^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ is a subgroup of $G^{L,\mathcal{V}_{1/\alpha}}(\boldsymbol{R}/\boldsymbol{Z})$ defined in [**19**].

In order to define the group $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$, we need the notion of β-variation for a real number $\beta\geqq 1$. For a function φ on $\boldsymbol{R}/\boldsymbol{Z}$ and a finite subset $A=\{x_1,\cdots,x_k\}$ of $\boldsymbol{R}/\boldsymbol{Z}$, we put

$$v_\beta(\varphi, A)=\sup\sum_{j=1}^{k}|\varphi(x_j)-\varphi(x_{j-1})|^\beta,$$

where $x_1,\cdots,x_k=x_0$ is in the cyclic order. Then we put

$$V_\beta(\varphi)=\sup v_\beta(\varphi, A),$$

where the supremum is taken over all finite subsets A of $\boldsymbol{R}/\boldsymbol{Z}$. We call it the β-variation of φ. The functions on $\boldsymbol{R}/\boldsymbol{Z}$ whose β-variations are bounded form a linear space $\mathcal{V}_\beta(\boldsymbol{R}/\boldsymbol{Z})$ with the β-pseudonorm $|||\ |||_\beta$ defined by

$$|||\varphi|||_\beta=V_\beta(\varphi)^{1/\beta}.$$

We define $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$ to be the group of Lipschitz homeomorphisms f with compact support such that $\log f'(x-0)$ exist as elements of $\mathcal{V}_\beta(\boldsymbol{R}/\boldsymbol{Z})$.

Then it is easy to see that $G^{L,\mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$ contains both $G^{1+1/\beta}(\boldsymbol{R}/\boldsymbol{Z})$ and the group of class P ([8]) which is denoted by $G^{L,\mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})$ in this paper. Note that $G^{L,\mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})$ contains the group $PL(\boldsymbol{R}/\boldsymbol{Z})$ of piecewise linear homeomorphisms whose homological property is rather well known by the work of Greenberg ([**7**]).

In this paper, we study dynamical properties and homological properties of $\boldsymbol{G}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ and $\boldsymbol{G}^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})$.

As a dynamical property, it is interesting to know whether there exist the Denjoy-Pixton actions ([**1**], [**13**]). A Denjoy action is an action Φ of $\boldsymbol{Z}^k$ ($k\geqq 1$) on the circle without fixed points but with a wandering interval. A wandering interval is an interval I such that $\Phi(\lambda)(I)$ ($\lambda\in \boldsymbol{Z}^k$) is a disjoint collection of intervals. On the other hand, a Pixton action is an action Φ of $\boldsymbol{Z}^k$ ($k\geqq 2$) on the circle with fixed points and with a wandering interval.

Denjoy ([**1**]) proved that the Denjoy homeomorphism ($\boldsymbol{Z}$ action) does not exist in $\boldsymbol{G}^{L\cdot \mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})$. Herman showed that Denjoy homeomorphisms exist in $\boldsymbol{G}^{1+\alpha}$ ($\alpha<1$) ([**8**], the existence of Denjoy homeomorphisms for this case is attributed to Sergeraert).

It has been known to specialists that there are no Pixton actions in $\boldsymbol{G}_c^{L\cdot \mathcal{V}_1}(\boldsymbol{R})$ and that Kopell's lemma ([**10**]) holds. (I was taught this fact by Vlad Sergiescu.) In § 1, we first show that there is a $\boldsymbol{Z}^k$ Pixton action in $\boldsymbol{G}_c^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R})$ ($\beta>k-1$). As a simple smoothing of this action, we obtain a $\boldsymbol{Z}^k$ Pixton action in $\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ ($\alpha<1/k$). The verification is rather easy in this case and this is used in § 2. By a similar construction, we obtain $\boldsymbol{Z}^k$ Denjoy actions in $\boldsymbol{G}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ ($\alpha<1/k$).

By a little more careful construction, we obtain a $\boldsymbol{Z}^k$ Pixton action in $\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ ($\alpha<1/(k-1)$). In particular, we have a $\boldsymbol{Z}^2$ Pixton action in $\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ ($\alpha<1$) and we have a sharp estimate on the regularity. This gives examples of foliations of class $C_c^{1+\alpha}(\boldsymbol{R})$ ($1/2<\alpha<1$) with a topological behavior different from the foliations of class C^2. For $\boldsymbol{Z}^2$ Pixton actions in $\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ ($1/2<\alpha<1$), the Hurder-Katok-Godbillon-Vey class ([**9**]) is defined and it is very interesting to know whether this class vanishes.

In [**17**], we used the existence of C^1 Pixton actions of arbitrarily big rank to show the acyclicity of the group $\boldsymbol{G}_c^1(\boldsymbol{R})=\mathrm{Diff}_c^1(\boldsymbol{R})$ of C^1 diffeomorphisms of $\boldsymbol{R}$ with compact support. The same method of proof shows that the existence of well controlled Pixton actions implies the vanishing of the homology in small dimensions and we show in § 2 that as β tends to ∞, $B\boldsymbol{G}_c^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R})^\delta$ and $B\boldsymbol{G}_c^{1+1/\beta}(\boldsymbol{R})^\delta$ become acyclic in arbitrarily large degrees. Hence the classifying spaces for the codimension 1 foliations of corresponding transverse regularities become contractible in arbitrarily large degrees. Note that the Godbillon-Vey invariant ([**6, 9, 5, 3, 19**]) exists and nontrivial in $H^2(B\boldsymbol{G}_c^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R})^\delta;\boldsymbol{R})$ and $H^2(B\boldsymbol{G}_c^{1+1/\beta}(\boldsymbol{R})^\delta;\boldsymbol{R})$ ($\beta<2$). We do not know whether there are nontrivial cohomology classes for $B\boldsymbol{G}_c^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R})^\delta$ and $B\boldsymbol{G}_c^{1+1/\beta}(\boldsymbol{R})^\delta$ ($\beta\geqq 2$).

We could not calculate the homology of $B\boldsymbol{G}^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})^\delta$ ($\beta>1$) except for the 1 dimensional homology. We show in § 2, $H_1(B\boldsymbol{G}^{L\cdot \mathcal{V}_\beta}(\boldsymbol{R}/\boldsymbol{Z})^\delta;\boldsymbol{Z})=0$ ($\beta>1$). Note that $H_1(B\boldsymbol{G}^{L\cdot \mathcal{V}_1}(\boldsymbol{R}/\boldsymbol{Z})^\delta;\boldsymbol{Z})$ is nontrivial and very big, and $H_1(B\boldsymbol{G}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})^\delta;\boldsymbol{Z})=0$ ($\alpha<1$) by the results of [**12**].

In the 2 dimensional homology, $H_2(BG_c^{L,CV_1}(\boldsymbol{R})^\delta;\boldsymbol{Z})$ is very big because the first homology is highly nontrivial ([**12**]) and there are at least two nontrivial Godbillon-Vey invariants ([**19**]). However, the image of this group in $H_2(BG_c^{L,CV_\beta}(\boldsymbol{R})^\delta;\boldsymbol{Z})$ $(\beta>1)$ does not contain the effect of the 1 dimensional homology and hence $H_2(BG_c^{L,CV_\beta}(\boldsymbol{R})^\delta;\boldsymbol{Z})$ would be easier to understand. In this direction, we can show that the image of $H_2(BPL_c(\boldsymbol{R})^\delta;\boldsymbol{Z})$ in $H_2(BG_c^{L,CV_\beta}(\boldsymbol{R})^\delta;\boldsymbol{Z})$ $(1\leqq\beta<2)$ is isomorphic to $\boldsymbol{R}$ and this isomorphism is given by the discrete Godbillon-Vey invariant ([**5**]). We include the proof of this fact in Appendix. The proof is rather independent of the body of paper and we use some results shown in [**20**]. This fact is used in characterizing the Godbillon-Vey invariant in terms of foliated cobordisms and perturbation of foliations ([**21**]).

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§ 1. Centralizer of Lipschitz homeomorphisms.

The theorem of Denjoy in [**1**] describes the centralizer of a homeomorphism in $G^{L,CV_1}(\boldsymbol{R}/\boldsymbol{Z})$ which has an irrational rotation number. On the other hand, Kopell's Lemma in [**10**] describes the centralizer of a C^2 contraction on the line. Both of these results state that the centralizers are cyclically or linearly ordered. These are important results for the investigation of 1 dimensional dynamics.

These results fail if the homeomorphisms are less regular. Herman ([8]) gave $C^{1+\alpha}$ $(\alpha<1)$ examples of the Denjoy actions and Pixton ([**13**]) gave C^1 examples of non-linearly-ordered centralizers.

We first show that there are $\boldsymbol{Z}^k$ Pixton actions in $G_c^{L,CV_\beta}(\boldsymbol{R})$, where $\beta>k-1$.

THEOREM (1.1). *Let k be a positive integer and β a real number greater than $k-1$. There is a homomorphism $\Phi:\boldsymbol{Z}^k\to G_c^{L,CV_\beta}(\boldsymbol{R})$ with an open interval U such that $\Phi(\lambda)(U)$ $(\lambda\in\boldsymbol{Z}^k)$ is a disjoint collection of intervals.*

PROOF. Let ℓ be a positive integer. Let $u_{k,\ell}:\boldsymbol{Z}\to\boldsymbol{R}$ be the function defined by

$$u_{k,\ell}(n)=(\ell+|n|)^{-k-\varepsilon}.$$

This $u_{k,\ell}$ is a strictly decreasing function on $|n|$. For

$$\lambda=\sum_{i=1}^{k}\lambda_i e_i=(\lambda_1,\cdots,\lambda_k)\in\boldsymbol{Z}^k,$$

put

$$a_\lambda = u_{k,\ell}(\max_i\{|\lambda_i|\}).$$

We consider the intervals of length a_λ. For $m \leqq k$ and $(\lambda_1, \cdots, \lambda_m) \in \boldsymbol{Z}^m$, put

$$L_{(\lambda_1, \cdots, \lambda_m)} = \sum_{\lambda \in \{(\lambda_1, \cdots, \lambda_m)\} \times \boldsymbol{Z}^{k-m}} a_\lambda .$$

Then the total length $L_{(\ )}$ is bounded. In fact,

$$\begin{aligned} L_{(\ )} &= \sum_{\lambda \in \boldsymbol{Z}^k} a_\lambda \\ &\leqq \ell^{-k-\varepsilon} + \sum_{n=1}^{\infty} \{(2n+1)^k - (2n-1)^k\}(\ell+n)^{-k-\varepsilon} \\ &\leqq \ell^{-k-\varepsilon} + 2^k k \varepsilon^{-1} \ell^{-\varepsilon} < \infty . \end{aligned}$$

We divide the interval $[0, L_{(\ )}]$ into the intervals

$$I_{(\lambda_1)} = [\sum_{\lambda_1' \leqq \lambda_1 - 1} L_{(\lambda_1')}, \sum_{\lambda_1' \leqq \lambda_1} L_{(\lambda_1')}] \quad (\lambda_1 \in \boldsymbol{Z})$$

of length $L_{(\lambda_1)}$. Then we divide this interval $I_{(\lambda_1)}$ into the intervals $I_{(\lambda_1, \lambda_2)}$ of length $L_{(\lambda_1, \lambda_2)}$. We continue this process and obtain the partition of $[0, L_{(\ )}]$ into the intervals $I_{(\lambda_1, \cdots, \lambda_k)}$ of length $L_{(\lambda_1, \cdots, \lambda_k)} = a_{(\lambda_1, \cdots, \lambda_k)}$.

We define the homeomorphism $\Phi(e_i)$ of $\boldsymbol{R}$ as follows. $\Phi(e_i)$ is the identity on $\boldsymbol{R} - [0, L_{(\ )}]$ and is the similarity transformation which sends the interval $I_{(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_k)}$ onto the interval $I_{(\lambda_1, \cdots, \lambda_i+1, \cdots, \lambda_k)}$ on $I_{(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_k)}$. Then the homeomorphism $\Phi(e_i)$ commutes with the homeomorphism $\Phi(e_j)$, and we obtain a homeomorphism $\Phi : \boldsymbol{Z}^k \to \mathrm{Homeo}_c(\boldsymbol{R})$. We put $U = I_{(0, \cdots, 0)}$, then $\Phi(\lambda)(U)$ $(\lambda \in \boldsymbol{Z}^k)$ is a disjoint collection of intervals.

Now we show that $\Phi(e_i)$ $(i = 1, \cdots, k)$ is an element of $\boldsymbol{G}^{L, \mathcal{V}_\beta}$.

The derivative $(\Phi(e_i))'$ on the interval $I_{(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_k)}$ is equal to

$$\frac{a_{(\lambda_1, \cdots, \lambda_i+1, \cdots, \lambda_k)}}{a_{(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_k)}} = \begin{cases} 1 & \text{if } \left|\lambda_i + \dfrac{1}{2}\right| < \max\limits_{j \neq i}\{|\lambda_j|\}, \\ \dfrac{u_{k,\ell}(|\lambda_i + 1|)}{u_{k,\ell}(|\lambda_i|)} & \text{otherwise.} \end{cases}$$

Note that the intervals $I_{(\lambda_1, \cdots, \lambda_{i-1})}$ are invariant under $\Phi(e_i)$ and the their endpoints are the fixed points of $\Phi(e_i)$. Moreover at the endpoints of the interval $I_{(\lambda_1, \cdots, \lambda_{k-1})}$, the derivative $(\Phi(e_i))'$ exists and equal to 1.

We need to estimate $V_\beta(\log(\Phi(e_i))')$. Let A be a finite subset of $\boldsymbol{R}$. By adding at most 2 points between the subsequent points of A, we obtain A' such that $A' \cap I_{(\lambda_1, \cdots, \lambda_{k-1})} \neq \emptyset$ implies $\partial I_{(\lambda_1, \cdots, \lambda_{k-1})} \subset A'$. By the argument of Proposition (2.12) of [**19**],

$$v_\beta(\log(\Phi(e_i))', A) \leqq 3^{\beta-1} v_\beta(\log(\Phi(e_i))', A').$$

On the interval $I_{(\lambda_1,\dots,\lambda_{k-1})}$, $\log(\Phi(e_i))'$ is described as follows. First we look at $\log(\Phi(e_i))'$ for $i<k$. If $|\lambda_i+1/2|<\max_{j\neq i, j<k}\{|\lambda_j|\}$, then

$$\log(\Phi(e_i))'=0 \qquad \text{on } I_{(\lambda_1,\dots,\lambda_{k-1})}.$$

If $|\lambda_i+1/2|>\max_{j\neq i, j<k}\{|\lambda_j|\}$, then

$$(\Phi(e_i))'=\begin{cases} 1 & \text{on } I_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k)} \text{ for } |\lambda_k|>\left|\lambda_i+\frac{1}{2}\right| \text{ and} \\ \dfrac{u_{k,\ell}(|\lambda_i+1|)}{u_{k,\ell}(|\lambda_i|)} & \text{on } I_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k)} \text{ for } |\lambda_k|<\left|\lambda_i+\frac{1}{2}\right|. \end{cases}$$

Hence $v_\beta(\log(\Phi(e_i))', A')$ is smaller than the sum of

$$2|\log(u_{k,\ell}(|\lambda_i+1|))-\log(u_{k,\ell}(|\lambda_i|))|^\beta$$

over all $(\lambda_1, \cdots, \lambda_{k-1})$ such that $|\lambda_i+1/2|>\max_{j\neq i, j<k}\{|\lambda_j|\}$. This sum is equal to

$$\begin{aligned} &2\sum_{n=0}^{\infty}(n+1)^{k-2}2|\log(u_{k,\ell}(n+1))-\log(u_{k,\ell}(n))|^\beta \\ &=4\sum_{n=0}^{\infty}(n+1)^{k-2}\left|(-k-\varepsilon)\log\left(1+\frac{1}{\ell+n}\right)\right|^\beta \\ &\leqq 4(k+\varepsilon)^\beta\sum_{n=0}^{\infty}(n+1)^{k-2}\left(\frac{1}{\ell+n}\right)^\beta \\ &\leqq 4(k+\varepsilon)^\beta(\beta-k+1)^{-1}(\ell-1)^{k-1-\beta}<\infty, \end{aligned}$$

where we used $\beta>k-1$. Thus $\Phi(e_i)$ $(i<k)$ is an element of $G^{L,\mathcal{V}_\beta}$.

Now for $i=k$, we have

$$(\Phi(e_k))'=\begin{cases} \dfrac{u_{k,\ell}(|\lambda_k+1|)}{u_{k,\ell}(|\lambda_k|)} & \text{on } I_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k)} \text{ for } \left|\lambda_k+\frac{1}{2}\right|>\max_{j<k}\{|\lambda_j|\} \text{ and} \\ 1 & \text{on } I_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k)} \text{ for } \left|\lambda_k+\frac{1}{2}\right|<\max_{j<k}\{|\lambda_j|\}. \end{cases}$$

Note that $\log(\Phi(e_i))'$ is 0 at the endpoints of $I_{(\lambda_1,\dots,\lambda_{k-1})}$ and does not increase outside of $I_{(\lambda_1,\dots,\lambda_{k-1},\lambda_k)}$ for $|\lambda_k+1/2|<\max_{j<k}\{|\lambda_j|\}$ where the value is 0. Since the maximal value and the minimal value on $I_{(\lambda_1,\dots,\lambda_{k-1})}$ are

$$\pm\{\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}))-\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}+1))\},$$

$v_\beta(\log(\Phi(e_i))', A')$ is smaller than the sum of

$$(2^\beta+2)|\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}))-\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}+1))|^\beta$$

over all $(\lambda_1, \cdots, \lambda_{k-1})$. Now we have

$$(2^{\beta}+2)\sum_{(\lambda_1,\cdots,\lambda_{k-1})\in Z^{k-1}}|\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}))-\log(u_{k,\ell}(\max_{j<k}\{|\lambda_j|\}+1))|^{\beta}$$

$$=(2^{\beta}+2)\sum_{(\lambda_1,\cdots,\lambda_{k-1})\in Z^{k-1}}\left|(-k-\varepsilon)\log\left(1+\frac{1}{\ell+\max_{j<k}\{|\lambda_j|\}}\right)\right|^{\beta}$$

$$\leqq(2^{\beta}+2)\sum_{(\lambda_1,\cdots,\lambda_{k-1})\in Z^{k-1}}(k+\varepsilon)^{\beta}\left|\frac{1}{\ell+\max_{j<k}\{|\lambda_j|\}}\right|^{\beta}$$

$$\leqq(2^{\beta}+2)(k+\varepsilon)^{\beta}\left\{\ell^{-\beta}+\sum_{n=1}^{\infty}\{(2n+1)^{k-1}-(2n-1)^{k-1}\}(\ell+n)^{-\beta}\right\}$$

$$\leqq(2^{\beta}+2)(k+\varepsilon)^{\beta}\{\ell^{-\beta}+2^{k-1}(k-1)(\beta-k+1)^{-1}\ell^{k-1-\beta}\}.$$

Since $\beta>k-1$, this sum is bounded. Thus $\Phi(e_k)$ is an element of $G^{L,\mathcal{V}_\beta}$.

REMARK. The estimates above shows that the above action would not be in $G^{L,\mathcal{V}_{k-1}}$ and its smoothing defined later would not be in $G^{1+1/(k-1)}$. For $k=2$, it is known that there are no such homomorphisms $Z^2\to G_c^{L,1}(R)$. For the group of C^2 diffeomorphisms, this fact has been known as Kopell's Lemma ([**10**]). I was told by Vlad Sergiescu that this lemma holds for $G^{L,\mathcal{V}_1}$. Here is a sketch of the proof of that fact.

PROPOSITION (1.2). *If there exists a homomorphism* $\Phi: Z^2\to G_c^{L,\mathcal{V}_\beta}(R)$ *with an open interval* U *such that* $\Phi(\lambda)(U)$ $(\lambda\in Z^2)$ *are disjoint, then* $\beta>1$.

SKETCH OF THE PROOF. First we choose a basis for Z^2. Let $e_2\in Z^2$ be the element such that $\Phi(e_2)(U)$ is the nearest interval to U among those intervals $\Phi(\lambda)(U)$ $(\lambda\in Z^2)$ which are contained in (b,∞), where $U=(a,b)$. It is clear that e_2 is primitive and one can choose e_2 as an element of a basis. Put

$$a_1=\inf\bigcup_{m\in Z}\Phi(me_2)(U)\quad\text{and}\quad b_1=\sup\bigcup_{m\in Z}\Phi(me_2)(U).$$

Then a_1 and b_1 are fixed points of $\Phi(e_2)$. The other element e_1 of the basis is taken so that $\Phi(e_1)(a_1,b_1)$ is the nearest interval to (a_1,b_1) among those intervals $\Phi(\lambda)((a_1,b_1))$ $(\lambda\in Z^2)$ which are contained in (b_1,∞). Then we see that the action of Z^2 looks similar to the example given in Theorem (1.1).

Since $\Phi(\lambda_1e_1)(a_1,b_1)$ $(\lambda_1\in Z)$ is a disjoint collection of intervals, by the proof of Denjoy ([**1**]), $V_1(\log(\Phi(\lambda_1e_1))'|(a_1,b_1))$ $(\lambda_1\in Z)$ is bounded by $V_1(\log(\Phi(e_1))')$. If we take λ_2 large enough, the variation $V_1(\log(\Phi(\lambda_2e_2))'|(a_1,b_1))$ is bigger than $3V_1(\log(\Phi(e_1))')$. Hence the variation $V_1(\log(\Phi(\lambda_2e_2))'|\Phi(\lambda_1e_1)(a_1,b_1))$ is bigger than $V_1(\log(\Phi(e_1))')$. Hence the variation $V_1(\log(\Phi(\lambda_2e_2))')$ is not bounded.

We are going to construct a Z^k Pixton action in $G_c^{1+\alpha}(R)$ by smoothing the above Z^k Pixton action in Theorem (1.1). First we introduce a family of diffeomorphisms which we use to perform the smoothing, and establish some useful estimates.

Let $\xi(x)(\partial/\partial x)$ be a C^∞ vector field on $[0, 1]$ such that

$$\xi(x) = x \qquad \text{near } 0$$

$$\xi(x) = 0 \qquad \text{on } \left[\frac{1}{2}, 1\right] \text{ and}$$

$$\left|\frac{\partial \xi}{\partial x}\right| \leqq 1 .$$

Let $\varphi_t(x)$ be the solution of the differential equation

$$\begin{cases} \dfrac{d\varphi_t}{dt}(x) = \xi(\varphi_t(x)) \\ \varphi_0(x) = x . \end{cases}$$

Consider the diffeomorphism $x \mapsto b\cdot\varphi_t(x/a)$ which sends the interval $[0, a]$ onto the interval $[0, b]$. The derivative of this diffeomorphism is equal to b/a on $[a/2, a]$ and is equal to $(b/a)e^t$ at 0. For real numbers a', a, b' and b such that $a'<0<a$ and $b'<0<b$, let $\phi_{b',b}^{a',a}: [0, a] \to [0, b]$ be the diffeomorphism defined by

$$\phi_{b',b}^{a',a}(x) = b\cdot\varphi_{\log(b'a/a'b)}\left(\frac{x}{a}\right).$$

Then it is easy to check that for real numbers a', a, b', b, c and c' such that $a'<0<a$, $b'<0<b$ and $c'<0<c$,

$$\phi_{c',c}^{b',b}\phi_{b',b}^{a',a} = \phi_{c',c}^{a',a} .$$

To show the Hölder continuity, we use the following estimates. First note that

$$\left(\frac{d}{dt}\frac{\partial\varphi_t}{\partial x}\right)(x) = \frac{\partial\xi}{\partial x}(\varphi_t(x))\frac{\partial\varphi_t}{\partial x}(x).$$

Hence

$$\log\left(\frac{\partial\varphi_t}{\partial x}\right)(x) = \int_0^t \frac{d}{ds}\log\left(\frac{\partial\varphi_s}{\partial x}\right)(x)ds = \int_0^t \frac{\partial\xi}{\partial x}(\varphi_s(x))ds .$$

Since $|(\partial\xi/\partial x)(x)| \leqq 1$, we see that

$$\frac{\partial\varphi_t}{\partial x}(x) \leqq e^t \quad \text{and} \quad |\varphi_t(x_2)-\varphi_t(x_1)| \leqq e^t|x_2-x_1| .$$

Now

$$\log\frac{\partial\phi_{b',b}^{a',a}}{\partial x}(x) = \log\frac{b}{a} + \log\left(\left(\frac{\partial}{\partial x}\right)\varphi_{\log(b'a/a'b)}\right)\left(\frac{x}{a}\right)$$

and

$$\left|\log\left(\left(\frac{\partial}{\partial x}\right)\varphi_{\log(b'a/a'b)}\right)\right| \leqq \left|\log\frac{b'a}{a'b}\right| = \left|\log\frac{b'}{a'} - \log\frac{b}{a}\right|.$$

Suppose that $|\partial^2\xi/\partial x^2| \leqq C$ for a positive real number C. Then we have

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial x}\right)\log\frac{\partial\psi_{b',b}^{a',a}}{\partial x}(x)\right| \\
&\quad = \left|\left(\frac{\partial}{\partial x}\right)\left(\log\left(\left(\frac{\partial}{\partial x}\right)\varphi_{\log(b'a/a'b)}\right)\left(\frac{x}{a}\right)\right)\right| \\
&\quad = \frac{1}{a}\left|\int_0^{\log(b'a/a'b)}\frac{\partial^2\xi}{\partial x^2}\left(\varphi_s\left(\frac{x}{a}\right)\right)\frac{\partial\varphi_s}{\partial x}\left(\frac{x}{a}\right)ds\right| \\
&\quad \leqq \frac{C}{a}\left|\int_0^{\log(b'a/a'b)} e^s ds\right| \\
&\quad = \frac{C}{a}\left|\frac{b'a}{a'b}-1\right|.
\end{aligned}$$

Let $\chi_b^a : [0, a]\to[0, b]$ be the diffeomorphism defined by

$$\chi_b^a(x) = \begin{cases} \psi_{-1,\,b/2}^{-1,\,a/2}(x) & \text{for } x\in[0, a/2], \\ b-\psi_{-1,\,b/2}^{-1,\,a/2}(a-x) & \text{for } x\in[a/2, a]. \end{cases}$$

Then for positive real numbers a, b and c,

$$\chi_c^b\chi_b^a = \chi_c^a$$

and

$$\left|\left(\frac{\partial}{\partial x}\right)\log\frac{\partial\chi_b^a}{\partial x}(x)\right| \leqq \frac{2C}{a}\left|\frac{a}{b}-1\right|.$$

The definition of the smoothing $\Psi : \boldsymbol{Z}^k\to G_c^{1+\alpha}(\boldsymbol{R})$ of $\Phi : \boldsymbol{Z}^k\to G_c^{L\cdot\mathcal{V}_\beta}(\boldsymbol{R})$ is as follows. This smoothing is similar to that in [**13**].

To define $\Psi(e_i)$ $(1\leqq i\leqq k)$, we replace the similarity transformation sending the interval $I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)}$ onto the interval $I_{(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k)}$ in the definition of $\Phi(e_i)$, by the diffeomorphism $\chi_b^a(x-w)+w'$ if

$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)} = [w, w+a] \quad\text{and}\quad I_{(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k)} = [w', w'+b].$$

Then the above equality $\chi_c^b\chi_b^a=\chi_c^a$ implies that the homeomorphism $\Psi(e_i)$ commutes with the homeomorphism $\Psi(e_j)$ $(i\neq j)$, and we obtain a homomorphism $\Psi : \boldsymbol{Z}^k\to \mathrm{Homeo}_c(\boldsymbol{R})$. We put $U=I_{(0,\dots,0)}$, then $\Psi(\lambda)(U)$ $(\lambda\in\boldsymbol{Z}^k)$ are disjoint.

Now we examine the regularity of this action and we obtain the following proposition.

PROPOSITION (1.3). *The above smoothing Ψ is a $\boldsymbol{Z}^k$ Pixton action of class $1+\alpha$ for any $\alpha<1/k$.*

PROOF. We look at $\log(\Psi(e_i))'$. By definition, $\log(\Psi(e_i))'$ has the same value as that of $\log(\Phi(e_i))'$ on the middle two quarters of the interval $I_{(\lambda_1,\dots,\lambda_k)}$, and it is zero at the endpoints of $I_{(\lambda_1,\dots,\lambda_k)}$.

Hence for x_1 and x_2 on $I_{(\lambda_1,\dots,\lambda_k)}$, we have

$$\left|\frac{\log(\Psi(e_i))'(x_2)-\log(\Psi(e_i))'(x_1)}{x_2-x_1}\right|\leqq\frac{2C}{a}\left|\frac{a}{b}-1\right|.$$

with a, b being $u_{k,\ell}(n)$ or $u_{k,\ell}(n+1)$. Hence this is smaller than

$$\frac{2C}{u_{k,\ell}(n+1)}\left|\frac{u_{k,\ell}(n)}{u_{k,\ell}(n+1)}-1\right|.$$

Thus

$$\frac{|\log(\Psi(e_i))'(x_2)-\log(\Psi(e_i))'(x_1)|}{|x_2-x_1|^{\alpha}}$$

$$\leqq\frac{2C}{u_{k,\ell}(n+1)}\left|\frac{u_{k,\ell}(n)}{u_{k,\ell}(n+1)}-1\right|u_{k,\ell}(n)^{1-\alpha}$$

$$\leqq 2C(n+1+\ell)^{k+\varepsilon}\left|\left(1+\frac{1}{n+\ell}\right)^{k+\varepsilon}-1\right|(n+1+\ell)^{-(k+\varepsilon)(1-\alpha)}.$$

The last term is bounded if $\alpha<1/(k+\varepsilon)$. Since $\log(\Psi(e_i))'=0$ at the endpoints of intervals, for x_1 and x_2 $(x_1<x_2)$ which are not on the same $I_{(\lambda_1,\dots,\lambda_{k-1})}$, we can find x_1' and x_2' such that $x_1\leqq x_1'\leqq x_2'\leqq x_2$, x_1 and x_1' are on an interval $I_{(\lambda_1,\dots,\lambda_{k-1})}$, x_2 and x_2' are on another interval $I_{(\lambda_1,\dots,\lambda_{k-1}')}$, and $\log(\Psi(e_i))'(x_1')=\log(\Psi(e_i))'(x_2')=0$. Hence it is easy to see that $\log(\Psi(e_i))'$ is α-Hölder.

As for the $\boldsymbol{Z}^k$ Denjoy actions on the circle, the usual construction ([8], [**13**]) is as follows. Take an injective homomorphism $\rho:\boldsymbol{Z}^k\to\boldsymbol{R}/\boldsymbol{Z}$ and a family of intervals I_λ $(\lambda\in\boldsymbol{Z}^k)$ with the sum of lengths being bounded. Then insert the interval I_λ at $\rho(\lambda)(0)$ and extend the homomorphisms to the inserted intervals. If we use the above intervals used in Theorem (1.1), and affine extensions on the intervals, this Denjoy action is of class $C^{L,\mathcal{V}_\beta}$ for $\beta>k$. Its smoothing in the way of Proposition (1.3) is of class $C^{1+\alpha}$ for $\alpha<1/k$. Thus we obtain the following proposition.

PROPOSITION (1.4). *Let k be a positive integer and α a positive real number smaller than $1/k$. There is a homomorphism $\Psi:\boldsymbol{Z}^k\to\boldsymbol{G}^{1+\alpha}(\boldsymbol{R}/\boldsymbol{Z})$ without fixed points with an open interval U such that $\Psi(\lambda)(U)$ $(\lambda\in\boldsymbol{Z}^k)$ is a disjoint collection of intervals.*

The Hölder regularity for the $\boldsymbol{Z}^k$ Pixton action obtained in Proposition (1.3) is not the optimal one. In fact we show in Theorem (1.5) below that there is a $\boldsymbol{Z}^k$ Pixton action in $\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ $(\alpha<1/(k-1))$. Thus for $k=2$, we have a sharp estimate for the existence of $\boldsymbol{Z}^2$ Pixton actions.

On the other hand, the Hölder regularity for the $\boldsymbol{Z}^k$ Denjoy action obtained in Proposition (1.4) seems optimal. Note that in the smoothing of Denjoy action, the derivative at the end-points of the interval I_λ should be 1 and this makes the difference between the constructions of the Pixton actions and the Denjoy actions.

In $C^{1+\alpha}$ actions $(1/2<\alpha\leqq 1)$, there seems no Denjoy actions but there exist $\boldsymbol{Z}^2$ Pixton actions. For $\boldsymbol{Z}^2$ Pixton actions, the Hurder-Katok-Godbillon-Vey class ([**9**]) is defined and it is very interesting to know whether this class vanishes.

It is also interesting that there are $\boldsymbol{Z}^3$ Pixton actions for $\alpha<1/2$. For, we need to use $\boldsymbol{Z}^3$ Pixton actions in the next section to show the 2 acyclicity of $G_c^{1+\alpha}(\boldsymbol{R})$ $(\alpha<1/3)$, and this estimate $\alpha<1/3$ seems to be replaced by $\alpha<1/2$ by a development of technical constructions.

THEOREM (1.5). *Let k be a positive integer and α a positive real number smaller than $1/(k-1)$. There is a homomorphism $\Psi: \boldsymbol{Z}^k \to G_c^{1+\alpha}(\boldsymbol{R})$ with an open interval U such that $\Psi(\lambda)(U)$ $(\lambda\in\boldsymbol{Z}^k)$ is a disjoint collection of intervals.*

PROOF. This action is obtained by smoothing another action.

Let ℓ be an integer greater than k and let ε be a positive real number. For

$$\lambda=\sum_{i=1}^{k}\lambda_i e_i=(\lambda_1, \cdots, \lambda_k)\in\boldsymbol{Z}^k ,$$

put

$$\delta_\lambda=\left(\ell^2+\sum_{i=1}^{k}\lambda_i^2\right)^{1/2} \quad\text{and}\quad b_\lambda=(\delta_\lambda)^{-k-\varepsilon} .$$

We consider the intervals I_λ of length b_λ and define $L_{(\lambda_1,\dots,\lambda_m)}$ as in the proof of Theorem (1.1). We see that the total length $L_{(\ )}$ is bounded. For,

$$(\delta_{\lambda_1,\dots,\lambda_k})^2\geqq(1/2)(\delta_{|\lambda_1|+1,\dots,|\lambda_k|+1})^2$$

and $L_{(\ )}$ is smaller than the integral

$$2^{-(k+\varepsilon)/2}\int_{\boldsymbol{R}^k}(\ell^2+\sum x_i^2)^{-(k+\varepsilon)/2}dx_1\cdots dx_k$$

$$=2^{-(k+\varepsilon)/2}\int_{S^{k-1}}d\,\mathrm{vol}_{S^{k-1}}\int_0^\infty r^{k-1}(\ell+r^2)^{-(k+\varepsilon)/2}<\infty .$$

Then we define the homeomorphisms $\Phi(e_i)$ as in the proof of Theorem (1.1), and Φ is again a homomorphism to $G_c^{L,\mathcal{V}\beta}(\boldsymbol{R})$ $(\beta>k-1)$. Now by smoothing this Φ in the following way, we obtain a homomorphism to $G_c^{1+\alpha}(\boldsymbol{R})$ $(\alpha<1/(k-1))$.

To define $\Psi(e_i)$ $(i<k)$, we replace the similarity transformation sending the interval $I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)}$ onto the interval $I_{(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k)}$ in the definition of $\Phi(e_i)$, by the diffeomorphism $\phi_{b',b}^{a',a}(x-w)+w'$ if

$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1)}=[w+a',\ w],$$
$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)}=[w,\ w+a],$$
$$I_{(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k-1)}=[w'+b',\ w'],\quad\text{and}$$
$$I_{(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k)}=[w',\ w'+b].$$

To define $\Psi(e_k)$, we replace the similarity transformation sending the interval $I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)}$ onto the interval $I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k+1)}$ in the definition of $\Phi(e_i)$, by the diffeomorphism $\phi^{a',a}_{-a',b}(x-w)+w+a$ if

$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1)}=[w+a',\ w],$$
$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k)}=[w,\ w+a],\quad\text{and}$$
$$I_{(\lambda_1,\dots,\lambda_i,\dots,\lambda_k+1)}=[w+a,\ w+a+b].$$

Then the above equality $\phi^{b',b}_{c',c}\phi^{a',a}_{b',b}=\phi^{a',a}_{c',c}$ implies that the homomorphism $\Psi(e_i)$ commutes with the homeomorphism $\Psi(e_j)$ $(i\neq j)$, and we obtain a homeomorphism $\Psi: \boldsymbol{Z}^k\to\mathrm{Homeo}_c(\boldsymbol{R})$. We put $U=I_{(0,\dots,0)}$, then $\Psi(\lambda)(U)$ $(\lambda\in\boldsymbol{Z}^k)$ are disjoint.

Note that the smoothing of Φ in Theorem (1.1) by this way is not as regular as the present one. The reason is that the values of the derivatives of the previous homeomorphisms $\Phi(e_i)$ change on special endpoints of intervals $I_{(\lambda_1,\dots,\lambda_m)}$, while those of the present ones change on all the intervals. This gives the effect of raising the regularity.

We show that $\Psi(e_i)$ $(i=1,\cdots,k)$ is $C^{1+\alpha}$.

We look at $\log(\Psi(e_i))'$. By definition, $\log(\Psi(e_i))'$ has the same value as that of $\log(\Phi(e_i))'$ on the bigger half of the interval $I_{(\lambda_1,\dots,\lambda_k)}$, and on the smaller half of $I_{(\lambda_1,\dots,\lambda_k)}$, the value depends on the values of $\log(\Phi(e_i))'$ on $I_{(\lambda_1,\dots,\lambda_k)}$ and on $I_{(\lambda_1,\dots,\lambda_k-1)}$. If they are the same, $\log(\Psi(e_i))'$ is constant on the interval $I_{(\lambda_1,\dots,\lambda_k)}$.

On the interval $I_{(\lambda_1,\dots,\lambda_{k-1})}$, $\log(\Psi(e_i))'$ for $i<k$ is described as follows.

First we look at $\log(\Psi(e_i))'$ for $i<k$. Since

$$(\Psi(e_i))'=\frac{\partial}{\partial x}\phi^{-b\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1,\ b\lambda_1,\dots,\lambda_i,\dots,\lambda_k}_{-b\lambda_1,\dots,\lambda_i+1,,\dots,\lambda_k-1,b\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k}\qquad\text{on } I_{(\lambda_1,\dots,\lambda_k)},$$

for x_1 and x_2 on $I_{(\lambda_1,\dots,\lambda_k)}$, we have

$$\left|\frac{\log(\Psi(e_i))'(x_2)-\log(\Psi(e_i))'(x_1)}{x_2-x_1}\right|$$
$$\leqq\frac{C}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}\left|\frac{b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k-1}b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1}b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k}}-1\right|$$
$$=\frac{C}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}\left|\left(\frac{(\delta_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})^2}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k})^2}\right)^{-(k+\varepsilon)/2}-1\right|$$
$$=\frac{C}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}\left|\left(\frac{(1+2\lambda_i)(1-2\lambda_k)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k})^2}+1\right)^{-(k+\varepsilon)/2}-1\right|.$$

Here

$$(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})^2\left|\left(\frac{(1+2\lambda_i)(1-2\lambda_k)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k})^2}+1\right)^{-(k+\varepsilon)/2}-1\right|$$

is bounded, say by a positive real number B. Hence we have

$$\begin{aligned}&\frac{|\log(\Psi(e_i))'(x_2)-\log(\Psi(e_i))'(x_1)|}{|x_2-x_1|^\alpha}\\ &\leqq \frac{C}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}} b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}{}^{1-\alpha}(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})^{-2}B\\ &= BC(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})^{-\alpha(-k-\varepsilon)-2}\,.\end{aligned}$$

The last term is bounded if $\alpha<1/(k-1)\leqq 2/k$. One might expect a bigger regularity because of this calculation. But it cannot have a bigger regularity than stated in our theorem because the β-norm of $\Psi(e_i)$ is not bounded for $\beta\leqq 1/(k-1)$ by the proof of Theorem (1.1). The estimate becomes a little complicated when x_1 and x_2 belong to $I_{(\lambda_1,\dots,\lambda_{k-1})}$.

We use frequently the following lemma.

LEMMA (1.6). *Let f be a function on* $\boldsymbol{R}$. *For $x_1<x_2<x_3$, if*

$$\frac{|f(x_2)-f(x_1)|}{|x_2-x_1|^\alpha}\leqq K \quad \textit{and} \quad \frac{|f(x_3)-f(x_2)|}{|x_3-x_2|^\alpha}\leqq K\,,$$

then

$$\frac{|f(x_3)-f(x_1)|}{|x_3-x_1|^\alpha}\leqq 2^{1-\alpha}K\,.$$

If x_1 belongs to $I_{(\lambda_1,\dots,\lambda_k)}$ and x_2 belongs to $I_{(\lambda_1,\dots,\lambda'_k)}$, then

$$(\Psi(e_i))'(x_1)=\frac{\partial}{\partial x}\phi^{-b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1},\ b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}_{-b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k-1},\ b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k}} \quad \text{and}$$

$$(\Psi(e_i))'(x_2)=\frac{\partial}{\partial x}\phi^{-b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'-1},\ b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'}}_{-b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k'-1},\ b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k'}}\,.$$

If $\lambda_k'=\lambda_k\pm 1$, then taking x_3 to be the common endpoint of the two intervals, we use Lemma (1.6) to get the desired estimate. If $|\lambda_k'-\lambda_k|\leqq 2$, we can use Lemma (1.6) again to get the desired estimate.

We assume that $1<\lambda_k<\lambda_k'$ and show that we have a desired estimate. Then this implies a similar estimate for $-1>\lambda_k>\lambda_k'$ and two such estimates imply the general case by Lemma (1.6).

Note that, for $\lambda_k\geqq 0$, $\log(b_{\lambda_1,\dots,\lambda_i+1,\dots,\lambda_k})/(b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})$ is monotone in λ_k. Hence by the estimate on $\log(\partial/\partial x)\phi^{a',a}_{b',b}(x)$, we see that $|\log(\Psi(e_i))'(x_2)-\log(\Psi(e_i))'(x_1)|$ is smaller than

$$2\left|\log\frac{b_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k-1}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}}-\log\frac{b_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}\right|$$

$$=2\left|\log\frac{b_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k-1}b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}b_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'}}\right|$$

$$=2\left|\frac{k+\varepsilon}{2}\log\frac{(\delta_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})^2}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'})^2}\right|$$

$$=(k+\varepsilon)\left|\log\left(1+\frac{(2\lambda_i+1)(\lambda_k'^2-(\lambda_k-1)^2)}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'})^2}\right)\right|.$$

On the other hand, $x_2-x_1\geqq\sum_{\lambda_k=\lambda_k+1}^{\lambda_k'-1}b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}$.

First we estimate x_2-x_1 as follows.

$$|x_2-x_1|\leqq(\lambda_k'-\lambda_k-1)b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}\,.$$

We look at the following equality.

$$\left|\frac{(2\lambda_i+1)(\lambda_k'^2-(\lambda_k-1)^2)}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'})^2}\right||(\lambda_k'-\lambda_k-1)b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}|^{-\alpha}$$

$$=\left|\frac{2\lambda_i+1}{\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}}\frac{\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}{\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}}\frac{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})^2}{(\delta_{\lambda_1,\cdots,\lambda_i+1,\cdots,\lambda_k'})^2}\right|$$

$$\times\left|\frac{(\lambda_k'-\lambda_k+1)^\alpha}{(\lambda_k'-\lambda_k-1)^\alpha}\frac{(\lambda_k'-\lambda_k+1)^{1-\alpha}(\lambda_k'+\lambda_k-1)}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})^{1-\alpha}\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}\right|$$

$$\times|\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}|^{-\alpha-1+(k+\varepsilon)\alpha}\,.$$

This is bounded if $(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})/(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})$ is bounded and $(k-1+\varepsilon)\alpha<1$. Hence if $\lambda_k'\leqq 2\lambda_k$ or $(\lambda_k')^2\leqq 4(\delta_\lambda^2-\lambda_k^2)$, then we obtain the desired estimates.

Secondly, if $\lambda_k'\geqq 2\lambda_k$ and $(\lambda_k)^2\geqq\delta_\lambda^2-\lambda_k^2$, we estimate $|x_2-x_1|$ as follows. Note that $2^{1/2}n\geqq\delta_{\lambda_1,\cdots,\lambda_{k-1},n}$ for $\lambda_k\leqq n\leqq\lambda_k'$. Then we have

$$|x_2-x_1|\geqq\sum_{n=\lambda_k+1}^{\lambda_k'-1}2^{-(k+\varepsilon)/2}|n|^{-(k+\varepsilon)}$$

$$\geqq(k-1+\varepsilon)^{-1}2^{-(k+\varepsilon)/2}(|\lambda_k+1|^{-k+1-\varepsilon}-|\lambda_k'|^{-k+1-\varepsilon})$$

$$=(k-1+\varepsilon)^{-1}2^{-(k+\varepsilon)/2}\left(1-\left(\frac{\lambda_k+1}{\lambda_k'}\right)^{k-1+\varepsilon}\right)|\lambda_k+1|^{-k+1-\varepsilon}$$

$$\geqq(k-1+\varepsilon)^{-1}2^{-(k+\varepsilon)/2}\left(1-\left(\frac{1}{2}\right)^{k-1+\varepsilon}\right)|\lambda_k+1|^{-k+1-\varepsilon}\,.$$

Hence we obtain

$$\left|\frac{(2\lambda_i+1)(\lambda_k'^2-(\lambda_k-1)^2)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}})^2(\delta_{\lambda_1,\dots,\lambda_{i+1},\dots,\lambda_k'})^2}\right||\lambda_k+1|^{(-k+1-\varepsilon)(-\alpha)}$$

$$=\left|\frac{(2\lambda_i+1)(\lambda_k-1)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}})^2}\frac{\lambda_k+1}{\lambda_k-1}\frac{\lambda_k'^2-(\lambda_k-1)^2}{(\delta_{\lambda_1,\dots,\lambda_{i+1},\dots,\lambda_k'})^2}\right||\lambda_k+1|^{(-k+1-\varepsilon)(-\alpha)-1}.$$

This is bounded if $(k-1+\varepsilon)\alpha<1$.

Finally, if $\lambda_k'\geqq 2\lambda_k$, $(\lambda_k')^2\geqq 4(\delta_\lambda^2-\lambda_k^2)$ and $(\lambda_k)^2\leqq\delta_\lambda^2-\lambda_k^2$, we estimate $|x_2-x_1|$ as follows. Let λ_k'' be the smallest integer not less than $(\delta_\lambda^2-\lambda_k^2)^{1/2}$. Then $\lambda_k''<\lambda_k'/2+1$. Now we have

$$\begin{aligned}|x_2-x_1| &\geqq \sum_{n=\lambda_k''+1}^{\lambda_k'-1} 2^{-(k+\varepsilon)/2}|n|^{-(k+\varepsilon)}\\ &\geqq (k-1+\varepsilon)^{-1}2^{-(k+\varepsilon)/2}(|\lambda_k''+1|^{-k+1-\varepsilon}-|\lambda_k'|^{-k+1-\varepsilon})\\ &\geqq (k-1+\varepsilon)^{-1}2^{-(k+\varepsilon)/2}\left(1-\left(\frac{\lambda_k''+1}{\lambda_k'}\right)^{k-1+\varepsilon}\right)|\lambda_k''+1|^{-k+1-\varepsilon}.\end{aligned}$$

Here $(\lambda_k''+1)/\lambda_k'\leqq(\lambda_k'/2+2)/\lambda_k'\leqq 1/2+1/\ell<1$ if $\ell\geqq 3$. Hence we obtain

$$\left|\frac{(2\lambda_i+1)(\lambda_k'^2-(\lambda_k-1)^2)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}})^2(\delta_{\lambda_1,\dots,\lambda_{i+1},\dots,\lambda_k'})^2}\right||\lambda_k''+1|^{(-k+1-\varepsilon)(-\alpha)}$$

$$=\left|\frac{(2\lambda_i+1)(\lambda_k''+1)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}})^2}\frac{\lambda_k'^2-(\lambda_k-1)^2}{(\delta_{\lambda_1,\dots,\lambda_{i+1},\dots,\lambda_k'})^2}\right||\lambda_k''+1|^{(-k+1-\varepsilon)(-\alpha)-1}.$$

This is bounded if $(k-1+\varepsilon)\alpha<1$.

Now for x_1 and x_2 $(x_1<x_2)$ which are on distinct intervals $I_{(\lambda_1,\dots,\lambda_{k-1})}$ and $I_{(\lambda_1',\dots,\lambda_{k-1}')}$, we put x_1' and x_2' to be the upper endpoint of $I_{(\lambda_1,\dots,\lambda_{k-1})}$ and the lower endpoint of $I_{(\lambda_1',\dots,\lambda_{k-1}')}$, respectively. Since $\log(\Psi(e_i))'(x_1')=\log(\Psi(e_i))'(x_2')=0$, we obtain the desired estimate by Lemma (1.6).

If x_1 or x_2 does not belong to an interval $I_{(\lambda_1,\dots,\lambda_{k-1})}$, then $\log(\Psi(e_i))'$ vanishes there and we obtain the desired estimate.

Thus $\log(\Psi(e_i))'$ $(i<k)$ is α-Hölder.

Now for $i=k$, we have

$$(\Psi(e_k))'=\frac{\partial}{\partial x}\phi^{-b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}},\,b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}_{b-\lambda_1,\dots,\lambda_i,\dots,\lambda_k,\;\;b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k+1}}}\qquad\text{on } I_{(\lambda_1,\dots,\lambda_k)}.$$

For x_1 and x_2 on $I_{(\lambda_1,\dots,\lambda_k)}$, we have

$$\left|\frac{\log(\Psi(e_k))'(x_2)-\log(\Psi(e_k))'(x_1)}{x_2-x_1}\right|$$

$$\leqq\frac{C}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k}}\left|\frac{(b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k})^2}{b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k-1}}b_{\lambda_1,\dots,\lambda_i,\dots,\lambda_{k+1}}}-1\right|$$

$$=\frac{C}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}}\left|\left(\frac{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})^4}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k+1})^2}\right)^{-(k+\varepsilon)/2}-1\right|$$

$$=\frac{C}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}}\left|\left(\frac{4\lambda_k^2-2(\delta_\lambda)^2-1}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k+1})^2}+1\right)^{-(k+\varepsilon)/2}-1\right|.$$

Here

$$(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})^2\left|\left(\frac{4\lambda_k^2-2(\delta_\lambda)^2-1}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_{i+1},\cdots,\lambda_k})^2}+1\right)^{-(k+\varepsilon)/2}-1\right|$$

is bounded, say by a positive real number B'. Hence we have

$$\frac{|\log(\Psi(e_k))'(x_2)-\log(\Psi(e_k))'(x_1)|}{|x_2-x_1|^\alpha}$$

$$\leqq\frac{C}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}}b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}{}^{1-\alpha}(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})^{-2}B'$$

$$=B'C(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})^{-\alpha(-k-\varepsilon)-2}.$$

The last term is bounded if $\alpha<1/(k-1)\leqq 2/k$.

As before we look at the case where x_1 belongs to $I_{(\lambda_1,\cdots,\lambda_k)}$ and x_2 belongs to $I_{(\lambda_1,\cdots,\lambda_k')}$. Then

$$(\Psi(e_k))'(x_1)=\frac{\partial}{\partial x}\phi^{-b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1},\,b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}}_{-b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k},\,b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k+1}}\quad\text{and}$$

$$(\Psi(e_k))'(x_2)=\frac{\partial}{\partial x}\phi^{-b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'-1},\,b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}_{-b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'},\,b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'+1}}.$$

It is sufficient to show that we have the desired estimate when $1<\lambda_k<\lambda_k'$. Since, for $\lambda_k\geqq 0$, $\log(b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k+1})/(b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})$ is monotone in λ_k, we see that $|\log(\Psi(e_k))'(x_2)-\log(\Psi(e_k))'(x_1)|$ is smaller than

$$2\left|\log\frac{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}}-\log\frac{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'+1}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}\right|$$

$$=2\left|\log\frac{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k}b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'}}{b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1}b_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'+1}}\right|$$

$$=2\left|\frac{k+\varepsilon}{2}\log\frac{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})^2}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'+1})^2}\right|$$

$$=(k+\varepsilon)\left|\log\left(1+\frac{(\lambda_k'-\lambda_k+1)(-2(\delta_\lambda)^2+2\lambda_k^2+2\lambda_k^2+2\lambda_k\lambda_k'+\lambda_k-\lambda_k'-1)}{(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})^2(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'+1})^2}\right)\right|.$$

Here, this is bounded if $(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k'})/(\delta_{\lambda_1,\cdots,\lambda_i,\cdots,\lambda_k-1})$ is bounded and $(k-1+\varepsilon)\alpha<1$. Hence if $\lambda_k'\leqq 2\lambda_k$ or $(\lambda_k')^2\leqq 4(\delta_\lambda^2-\lambda_k^2)$, then we obtain the desired estimates.

Secondly, if $\lambda_k'\geqq 2\lambda_k$ and $(\lambda_k)^2\geqq\delta_\lambda^2-\lambda_k^2$, we estimate $|x_2-x_1|$ as before and we have

$$\left|\frac{(\lambda_k'-\lambda_k+1)(-2(\delta_\lambda)^2+2\lambda_k^2+2\lambda_k\lambda_k'+\lambda_k-\lambda_k'-1)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'+1})^2}\right| |\lambda_k+1|^{(-k+1-\varepsilon)(-\alpha)}$$

$$=\left|\frac{\lambda_k'-\lambda_k+1}{\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k+1}}\frac{-2(\delta_\lambda)^2+2\lambda_k^2+2\lambda_k\lambda_k'+\lambda_k-\lambda_k'-1}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'+1})}\frac{\lambda_k+1}{\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1}}\right|$$

$$\times|\lambda_k+1|^{(-k+1-\varepsilon)(-\alpha)-1}.$$

This is bounded if $(k-1+\varepsilon)\alpha<1$.

Finally, if $\lambda_k'\geqq 2\lambda_k$, $(\lambda_k')^2\geqq 4(\delta_\lambda^2-\lambda_k^2)$ and $(\lambda_k)^2\leqq\delta_\lambda^2-\lambda_k^2$, we estimate $|x_2-x_1|$ as before using λ_k'', the smallest integer not less than $(\delta_\lambda^2-\lambda_k^2)^{1/2}$. Now we have

$$\left|\frac{(\lambda_k'-\lambda_k+1)(-2(\delta_\lambda)^2+2\lambda_k^2+2\lambda_k\lambda_k'+\lambda_k-\lambda_k'-1)}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})^2(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'+1})^2}\right| |\lambda_k''|^{(-k+1-\varepsilon)(-\alpha)}$$

$$=\left|\frac{\lambda_k'-\lambda_k+1}{\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k+1}}\frac{-2(\delta_\lambda)^2+2\lambda_k^2+2\lambda_k\lambda_k'+\lambda_k-\lambda_k'-1}{(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1})(\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k'+1})}\frac{\lambda_k''+1}{\delta_{\lambda_1,\dots,\lambda_i,\dots,\lambda_k-1}}\right|$$

$$\times|\lambda_k''+1|^{(-k+1-\varepsilon)(-\alpha)-1}.$$

This is bounded if $(k-1+\varepsilon)\alpha<1$.

Now for x_1 and x_2 $(x_1<x_2)$ which are on distinct intervals $I_{(\lambda_1,\dots,\lambda_{k-1})}$ and $I_{(\lambda_1',\dots,\lambda_{k-1}')}$, we obtain the desired estimate by Lemma (1.6) as before. If x_1 or x_2 does not belong to an interval $I_{(\lambda_1,\dots,\lambda_{k-1})}$, then the $\log(\Psi(e_k))'$ vanishes there and we obtain the desired estimate.

Thus $\log(\Psi(e_k))'$ is α-Hölder.

§2. Homology of groups Lipschitz homeomorphisms.

The following theorem was shown by Mather in [**12**]. This can also be proved by the method of [**15**].

Theorem (2.1). *If* $0<\alpha<1$, $G_c^{1+\alpha}(\boldsymbol{R})$ *is a perfect group.*

In this section we show a similar result for $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$.

Theorem (2.2). *If* $\beta>1$, $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ *is a perfect group.*

The strategy for showing these theorems is as follows. Let h_0 and h_1 be the similarity transformations $x\mapsto(x\pm1)/(2+\varepsilon)\mp1$ and put $U=(-\varepsilon/(2+\varepsilon),\varepsilon/(2+\varepsilon))=(a,b)$, where ε is a small positive real number which we choose later. Note that any element is conjugate to an element g with support in the U. For g, we construct G_0 and G_1 with support in $[-1,a]$ and $[b,1]$, respectively, such that $gG_0G_1=h_0^{-1}G_0h_0h_1^{-1}G_1h_1$. G_0G_1 is in fact a product of infinitely many conjugates of g divided into finitely many pieces and this is the reason for the above equality. Then g is written as a product of 2 commutators. What we really treat is the isotopy σ to the identity for the element g and the isotopy to the identity for gG_0G_1 is obtained as $I\sigma$ (the infinite iteration of σ) in [**15**].

For $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$, we need to show the existence of a nice isotopy to the identity for any element f of it. Then we show that the construction of an infinite iteration in [**15**] for such a nice isotopy $\sigma:[0,1]\to G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ belongs to $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ again. These can be carried out as follows.

For any element f of $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$, we can consider the following linear isotopy to the identity.

$$\varphi_t(x)=f(x)+t\cdot(x-f(x)).$$

In general, for elements $f_1,\cdots,f_m$ of $G_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ and $1\geqq t_1\geqq\cdots\geqq t_m\geqq 0$, put

$$\varphi_{(t_1,\cdots,t_m)}(x)=f_1\cdots f_m(x)+t_1\cdot(f_2\cdots f_m(x)-f_1\cdots f_m(x))+\cdots+t_m\cdot(x-f_m(x)).$$

We have the following lemma.

LEMMA (2.3). *There is a positive real number C depending on $f_1,\cdots,f_m$ such that*

$$\left\|\log\frac{\partial}{\partial x}(\varphi_{(t_1,\cdots,t_m)}\varphi_{(t'_1,\cdots,t'_m)}^{-1})\right\|_\beta\leqq C\sum_{k=1}^{m}|t_k-t'_k|.$$

PROOF. We would like to know the regularity of $(\partial/\partial x)(\varphi_{(t_1,\cdots,t_m)}\varphi_{(t'_1,\cdots,t'_m)}^{-1})$. Since

$$\begin{aligned}&\varphi_{(t_1,\cdots,t_m)}\varphi_{(t'_1,\cdots,t'_m)}^{-1}\\&=(\varphi_{(t_1,\cdots,t_m)}\varphi_{(t'_1,t_2,\cdots,t_m)}^{-1})(\varphi_{(t'_1,t_2,\cdots,t_m)}\varphi_{(t'_1,t'_2,t_3,\cdots,t_m)}^{-1})\cdots(\varphi_{(t'_1,\cdots,t'_{m-1},t_m)}\varphi_{(t'_1,\cdots,t'_m)}^{-1}),\end{aligned}$$

it is enough to look at the regularity of $(\partial/\partial x)(\varphi_{(t'_1,\cdots,t'_{k-1},t_k,t_{k+1},\cdots,t_m)}\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}^{-1})$. Put

$$\begin{aligned}\xi_k(t_1,\cdots,t_m,x)&=\Big(\frac{\partial}{\partial t_k}\varphi_{(t_1,\cdots,t_m)}\Big)(\varphi_{(t_1,\cdots,t_m)}^{-1}(x))\\&=(f_{k+1}\cdots f_m)(\varphi_{(t_1,\cdots,t_m)}^{-1}(x))-(f_k\cdots f_m)(\varphi_{(t_1,\cdots,t_m)}^{-1}(x)).\end{aligned}$$

Then

$$\begin{aligned}&\frac{\partial}{\partial x}\xi_k(t_1,\cdots,t_m,x)\\&=((f_{k+1}\cdots f_m)'-(f_k\cdots f_m)')(\varphi_{(t_1,\cdots,t_m)}^{-1}(x))\cdot\frac{\partial}{\partial x}\varphi_{(t_1,\cdots,t_m)}^{-1}(x)\\&=\frac{((f_{k+1}\cdots f_m)'-(f_k\cdots f_m)')(\varphi_{(t_1,\cdots,t_m)}^{-1}(x))}{((f_1\cdots f_m)'+t_1\cdot((f_2\cdots f_m)'-(f_1\cdots f_m)')+\cdots+t_m\cdot(1-f'_m))(\varphi_{(t_1,\cdots,t_m)}^{-1}(x))}.\end{aligned}$$

For $1\geqq t_1\geqq \cdots \geqq t_m\geqq 0$, we look at the family of mappings $(0, \infty)^m\to\boldsymbol{R}^m$ given by

$$\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{X_2-X_1}{X_1+t_1\cdot(X_2-X_1)+\cdots+t_m\cdot(1-X_m)} \\ \dfrac{X_3-X_2}{X_1+t_1\cdot(X_2-X_1)+\cdots+t_m\cdot(1-X_m)} \\ \vdots \\ \dfrac{1-X_m}{X_1+t_1\cdot(X_2-X_1)+\cdots+t_m\cdot(1-X_m)} \end{pmatrix}.$$

Then

$$0<\min X_i\leqq X_1+t_1\cdot(X_2-X_1)+\cdots+t_m\cdot(1-X_m)\leqq \max X_i$$

and in fact this is a family of diffeomorphisms onto

$$\{(Y_1, \cdots, Y_m)\in\boldsymbol{R}^m\,;\,1+(1-t_1)\cdot Y_1+\cdots+(1-t_m)\cdot Y_m>0\}.$$

The inverse mappings are

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{1-t_1\cdot Y_1-\cdots-t_m\cdot Y_m}{1+(1-t_1)\cdot Y_1+\cdots+(1-t_m)\cdot Y_m} \\ \dfrac{1+(1-t_1)\cdot Y_1-t_2\cdot Y_2-\cdots-t_m\cdot Y_m}{1+(1-t_1)\cdot Y_1+\cdots+(1-t_m)\cdot Y_m} \\ \vdots \\ \dfrac{1+(1-t_1)\cdot Y_1+\cdots+(1-t_{m-1})\cdot Y_{m-1}-t_m\cdot Y_m}{1+(1-t_1)\cdot Y_1+\cdots+(1-t_m)\cdot Y_m} \end{pmatrix}.$$

Since the set of parameters $\{(t_1, \cdots, t_m)\,;\,1\geqq t_1\geqq\cdots\geqq t_m\geqq 0\}$ is compact, this family of diffeomorphisms is uniformly Lipschitz on a bounded set. By composing with the exponential map $\log X_i\mapsto X_i$, and by taking a component, the mapping

$$(\log(f_1\cdots f_m)', \cdots, \log f'_m)\longmapsto \frac{((f_{k+1}\cdots f_m)'-(f_k\cdots f_m)')(\varphi_{(t_1,\cdots,t_m)}{}^{-1}(x))}{((f_1\cdots f_m)'+t_1\cdot((f_2\cdots f_m)'-(f_1\cdots f_m)')+\cdots+t_m\cdot(1-f'_m))(\varphi_{(t_1,\cdots,t_m)}{}^{-1}(x))}$$

is uniformly Lipschitz. Hence

$$\Big\|\frac{\partial}{\partial x}\xi_k(t'_1, \cdots, t'_{k-1}, s_k, t_{k+1}, \cdots, t_m, \varphi_{(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m)} \times\varphi_{(t'_1,\cdots,t'_{k-1},t_k',t_{k+1},\cdots,t_m)}{}^{-1}(x))\Big\|_\beta$$

is uniformly bounded.

On the other hand,

$$\log\frac{\partial}{\partial x}(\varphi_{(t'_1,\cdots,t'_{k-1},t_k,t_{k+1},\cdots,t_m)}\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}{}^{-1})(x)$$

$$=\log\Big(\frac{\partial}{\partial x}\varphi_{(t'_1,\cdots,t'_{k-1},t_k,t_{k+1},\cdots,t_m)}\Big)(\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}{}^{-1}(x))$$

$$-\log\Big(\frac{\partial}{\partial x}\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}\Big)(\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}{}^{-1}(x))$$

$$=\int_{t'_k}^{t_k}\frac{\partial}{\partial s_k}\log\Big(\frac{\partial}{\partial x}\varphi_{(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m)}\Big)(\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}{}^{-1}(x))ds_k$$

$$=\int_{t'_k}^{t_k}\frac{(f_{k+1}\cdots f_m)'-(f_k\cdots f_m)'}{\partial\varphi_{(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m)}/\partial x}(\varphi_{(t'_1,\cdots,t'_{k-1},t'_k,t_{k+1},\cdots,t_m)}{}^{-1}(x))ds_k$$

$$=\int_{t'_k}^{t_k}\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,t'_{k-1},s_k,t_{k+1},\cdots)}\varphi_{(\cdots,t'_{k-1},t'_k,t_{k+1},\cdots)}{}^{-1}(x))ds_k.$$

Since $\beta>1$, by the Hölder inequality,

$$\Big|\log\frac{\partial}{\partial x}(\varphi_{(\cdots,t_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1})(x_2)-\log\frac{\partial}{\partial x}(\varphi_{(\cdots,t_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1})(x_1)\Big|$$

$$\leqq\Big|\int_{t'_k}^{t_k}\Big|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_2))$$
$$-\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_1))\Big|ds_k\Big|$$

$$\leqq\Big|\int_{t'_k}^{t_k}\Big|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_2))$$
$$-\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_1))\Big|^\beta ds_k\Big|^{1/\beta}$$
$$\times\Big|\int_{t'_k}^{t_k}1ds_k\Big|^{1-1/\beta}.$$

For a finite set $\{x_1,\cdots,x_k\}$ of $\boldsymbol{R}$,

$$\sum_{j=1}^{k-1}\Big|\log\frac{\partial}{\partial x}(\varphi_{(\cdots,t_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1})(x_{j+1})-\log\frac{\partial}{\partial x}(\varphi_{(\cdots,t_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1})(x_j)\Big|^\beta$$

$$\leqq\sum_{j=1}^{k-1}\Big|\int_{t'_k}^{t_k}\Big|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_2))$$
$$-\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_1))\Big|ds_k\Big|^\beta$$

$$\leqq \sum_{j=1}^{k-1}\left|\left|\int_{t'_k}^{t_k}\left|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_2))\right.\right.\right.$$

$$\left.-\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_1))\right|^{\beta}ds_k\Big|^{1/\beta}$$

$$\times\left|\int_{t'_k}^{t_k}1ds_k\right|^{1-1/\beta}\Big|^{\beta}$$

$$\leqq \left|\int_{t'_k}^{t_k}\sum_{j=1}^{k-1}\left|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_2))\right.\right.$$

$$\left.\left.-\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x_1))\right|^{\beta}ds_k\right|$$

$$\times|t_k-t'_k|^{\beta-1}$$

$$\leqq \left|\int_{t'_k}^{t_k}\sup_{s_k}\left(\left\|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x))\right\|_{\beta}\right)^{\beta}ds_k\right|$$

$$\times|t_k-t'_k|^{\beta-1}$$

$$\leqq \sup_{s_k}\left(\left\|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x))\right\|_{\beta}\right)^{\beta}|t_k-t'_k|^{\beta}.$$

Hence

$$\left\|\log\frac{\partial}{\partial x}(\varphi_{(\cdots,t_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1})\right\|_{\beta}$$

$$\leqq \sup_{s_k}\left\|\frac{\partial}{\partial x}\xi_k(t'_1,\cdots,t'_{k-1},s_k,t_{k+1},\cdots,t_m,\varphi_{(\cdots,s_k,\cdots)}\varphi_{(\cdots,t'_k,\cdots)}{}^{-1}(x))\right\|_{\beta}|t_k-t'_k|.$$

PROOF OF THEOREM (2.2). We proceed using the construction given in [**15**]. We use the two similarity transformations $x\mapsto(x\pm1)/(2+\varepsilon)\mp1$ and put $U=(-\varepsilon/(2+\varepsilon),\ \varepsilon/(2+\varepsilon))$, where ε is a small positive real number which we choose later. Let f be an element of $G_c^{L,CV_\beta}(\boldsymbol{R})$ with support in U. Let $\sigma:[0,1]\to G_c^{L,CV_\beta}(\boldsymbol{R})$ be the path defined by $\sigma(t)(x)=f(x)+t\cdot(x-f(x))$. We are going to show that $\boldsymbol{I}\sigma$ defined in [**15**] is a path to the group of Lipschitz homeomorphisms. For the path σ, by Lemma (2.3), we have

$$\left\|\log\frac{\partial}{\partial x}(\sigma(t)\sigma(t')^{-1})\right\|_{\beta}\leqq C|t-t'|$$

for some positive real number C. We show that

$$\left\|\log\frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1})\right\|_{\beta}<\infty.$$

Let A be a finite subset of $[-1,1]$, then by adding at most 2 points between the subsequent points of A, we obtain A' such that for $\lambda\in \boldsymbol{Z}_+ * \boldsymbol{Z}_+$, $\bar{\Phi}(\lambda)(U)\cap A \neq \emptyset$ implies $\partial\bar{\Phi}(\lambda)(U)\subset A'$. Then we see that mesh $(A')\leqq$mesh (A) and

$$v_\beta\Big(\log \frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}),\ A\Big) \leqq 3^{\beta-1} v_\beta\Big(\log \frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}),\ A'\Big)$$

as in the proof of Proposition (2.9) in [**19**]. Then $A_\lambda = \bar{\Phi}(\lambda)^{-1}(\bar{\Phi}(\lambda)(\bar{U})\cap A')$ is a subset of $\bar{U}$ and

$$\begin{aligned} &v_\beta\Big(\log \frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}) \,|\, \bar{\Phi}(\lambda)(\bar{U}),\ \bar{\Phi}(\lambda)(\bar{U})\cap A'\Big) \\ &= v_\beta\Big(\log \frac{\partial}{\partial x}(\bar{\Phi}(\lambda)^{-1}((\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}) \,|\, \bar{\Phi}(\lambda)(\bar{U}))\bar{\Phi}(\lambda)),\ A_\lambda\Big) \\ &\leqq C^\beta 2^{-\beta l(\lambda)} . \end{aligned}$$

Hence

$$\begin{aligned} v_\beta\Big(\log \frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}),\ A\Big) &\leqq 3^{\beta-1} v_\beta\Big(\log \frac{\partial}{\partial x}(\boldsymbol{I}\sigma(0)(\boldsymbol{I}\sigma(1))^{-1}),\ A'\Big) \\ &\leqq 3^{\beta-1} \sum_{k=0}^{\infty} \sum_{l(\lambda)=k} C^\beta 2^{-\beta\cdot k} \\ &\leqq 3^{\beta-1} \sum_{k=0}^{\infty} C^\beta 2^{(1-\beta)k} \\ &< \infty . \end{aligned}$$

Thus by an argument as in [**15**], we see that f is a product of two commutators.

As we saw in §1, if β is big, there are Pixton actions of big rank in $\boldsymbol{G}_c^{L,\mathcal{CV}_\beta}(\boldsymbol{R})$. We prove the following theorems by an argument similar to that in [**17**].

THEOREM (2.4). *For any positive integer m, there is a positive real number β_m such that $B\boldsymbol{G}_c^{L,\mathcal{CV}_\beta}(\boldsymbol{R})$ is m-acyclic for $\beta>\beta_m$.*

THEOREM (2.5). *For any positive integer m, there is a real number $0<\alpha_m\leqq 1$ such that $B\boldsymbol{G}_c^{1+\alpha}(\boldsymbol{R})$ is m-acyclic for $0<\alpha<\alpha_m$.*

Here are estimates for the values of α_m and β_m which we obtained by our proof.

$$\alpha_1 = 1 , \qquad \beta_1 = 1 ,$$

$$\alpha_2 \geqq 1/3 , \qquad \beta_2 \leqq 3 ,$$

$$\alpha_m \geqq 1/(5\cdot 3^{m-2}), \qquad \beta_m \leqq 5\cdot 3^{m-2}-1 \quad (m\geqq 3).$$

PROOF OF THEOREM (2.4). We use the $\boldsymbol{Z}^N$ action constructed in the proof of Theorem (1.1). It is only necessary to check whether the chains constructed in [17] are chains of $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$. For a cube of $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$, we consider the sum of the simplices obtained by cutting the cube along the generalized diagonal and we consider the simplices of $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ obtained by the linear isotopy in Lemma (2.3). We perform the constructions in [17] for this sum Q.

If we show that $\boldsymbol{I}(\Lambda, \boldsymbol{m})Q$ is also a chain in $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$, then this shows certain acyclicity of $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$.

Let Q be a cube of dimension m. Then $\boldsymbol{I}(\Lambda, \boldsymbol{m})Q$ is constructed by using the action of the semigroup Λ of $\boldsymbol{Z}^N$, where $N=5\cdot 3^{m-2}$. In Λ there are $(k+1)^m$ elements of length k. By Lemma (2.3), the β norm of a piece corresponding to an element of length k of Λ is estimated by $mC/(k+1)$. Here the β norm means the maximum of the β norms of the logarithms of the first derivatives of the holonomies. Hence the β variation of the logarithms of the first derivatives of the holonomies of $\boldsymbol{I}(\Lambda, \boldsymbol{m})Q$ is estimated by $\sum_k (k+1)^m \cdot \{mC/(k+1)\}^\beta$. This converges if $\beta > m+1$. (Note that α is estimated by the dimension m here.)

When we treat locally degenerate chains, we need to repeat the above construction locally (see Theorem (5.1) of [17]). In that case, using the terminology of [17], if Q is supported in the family $\{Cl(\Phi_M(\lambda)(U_M))\,;\,\lambda\in\Theta\}$, the sum of β variations of $Q|\Phi_M(\lambda)(U_M)$ is estimated by the β variation of Q. Since the β variation of the construction for each $Q|\Phi_M(\lambda)(U_M)$ is estimated by the β variation of $Q|\Phi_M(\lambda)(U_M)$, the β variation of the construction for Q is bounded.

Thus the proof in [17] goes on and we see that if $\beta > 5\cdot 3^{m-2}-1$, then there exists a $\boldsymbol{Z}^{5\cdot 3^{m-2}}$ action given in Theorem (1.1) and we see that $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ is m acyclic.

In the case where, $m=2$ we can do better. If $\beta>2$, then there is a $\boldsymbol{Z}^3$ action given in Theorem (1.1). By the above argument, any 2-cycle is homologous in $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ to a locally degenerate cycle which is a sum of locally degenerate tori if $\beta>2+1=3$. Then the proof of Theorem (8.2) of [15] shows that these tori are homologous to zero. In fact, $I_{12}Q$ in [15] can be constructed by using the action of the semigroup $(\boldsymbol{Z}_+)^{*4}\subset(\boldsymbol{Z}_+)^{*2}$. In the semigroup $\boldsymbol{Z}_+^{*4}$, there are 2^{2k} elements of length k. By Lemma (2.3), the β norm of a piece corresponding to an element of length k of Λ is estimated by $2C/2^k$. Hence the β variation of $I_{12}Q$ is estimated by

$$\sum_k 2^{2k}\{2C/2^k\}^\beta = \sum_k 2^\beta C^\beta 2^{(2-\beta)k}\,.$$

This converges if $\beta>2$. Thus if $\beta>3$, $B\bar{G}_c^{L,\mathcal{V}_\beta}(\boldsymbol{R})$ is 2 acyclic.

PROOF OF THEOREM (2.5). We use the $\boldsymbol{Z}^N$ action given in Proposition (1.3).

(Even if we use Theorem (1.5) instead of Proposition (1.3), we could not get sharper the result until the present.) As in the proof of Theorem (2.4), it is only necessary to check whether the chains constructed in [**17**] are chains of $B\bar{\boldsymbol{G}}_c^{1+\alpha}(\boldsymbol{R})$. Note that there is an open interval U where the smoothed action $\Psi(\lambda)$ is affine and

$$(\partial/\partial x)\Psi(\lambda)|U=\ell^{N+\varepsilon}/(\ell+\max\{|\lambda_i|\})^{N+\varepsilon}\,.$$

Let Q be a cube of dimension m of $B\bar{\boldsymbol{G}}_c^{1+\alpha}(\boldsymbol{R})$. As before, $\boldsymbol{I}(\Lambda, \boldsymbol{m})Q$ is constructed by using the action of the semigroup Λ of $\boldsymbol{Z}^N$, where $N=5\cdot 3^{m-2}$. Then for $\lambda\in\Lambda$ of length k, the $C^{1+\alpha}$ norm of $\Phi(\lambda)Q_\lambda$ in [**17**] is estimated by

$$\frac{1}{k}\Big(\frac{\ell^{N+\varepsilon}}{(\ell+k)^{N+\varepsilon}}\Big)^{-\alpha}\|Q\|_{C^{1+\alpha}}\,.$$

Hence if $-1+\alpha(N+\varepsilon)<0$, then $\boldsymbol{I}(\Lambda, \boldsymbol{m})Q$ is of class $C^{1+\alpha}$. (Note that α is estimated by the rank N here.)

When we treat locally degenerate chains, we need to repeat the above construction locally (see Theorem (5.1) of [**17**]). However the estimate is similar.

Thus the proof in [**17**] goes on and we see that if $\alpha<1/(5\cdot 3^{m-2})$, then there exists a $\boldsymbol{Z}^{5\cdot 3^{m-2}}$ action given in Proposition (1.3) and we see that $B\bar{\boldsymbol{G}}_c^{1+\alpha}(\boldsymbol{R})$ is m acyclic.

In the case where, $m=2$ we can do better. If $\alpha<1/3$, then there is a $\boldsymbol{Z}^3$ action given in Proposition (1.3), and using this we see that any 2-cycle is homologous in $B\bar{\boldsymbol{G}}_c^{1+\alpha}(\boldsymbol{R})$ for $\alpha<1/3$ to a locally degenerate cycle which is a sum of locally degenerate tori. Then the proof of Theorem (8.2) of [**15**] shows that these tori are homologous to zero. In fact, as in the proof of Theorem (2.4), $I_{12}Q$ in [**15**] can be constructed by using the action of the semigroup $(\boldsymbol{Z}_+)^{*4}\subset(\boldsymbol{Z}_+)^{*2}$ as before. In the semigroup $(\boldsymbol{Z}_+)^{*4}$ there are 2^{2k} elements of length k and the derivative on U of these elements is $(2+\varepsilon)^{-2k}$. Hence the $C^{1+\alpha}$ norm of the part of length k of $I_{12}Q$ is estimated by

$$(2+\varepsilon)^{2k\alpha}2^{-k}\|Q\|_{C^{1+\alpha}}\,.$$

This tends to zero if $2\alpha-1<0$. Thus if $\alpha<1/3$, $B\bar{\boldsymbol{G}}_c^{1+\alpha}(\boldsymbol{R})$ is 2 acyclic.

Appendix. Piecewise linear Reeb foliations.

Let $PL_c(\boldsymbol{R})$ be the group of piecewise linear homeomorphisms of $\boldsymbol{R}$ with compact support. $PL_c(\boldsymbol{R})$ is a subgroup of $\boldsymbol{G}_c^{L\cdot\mathcal{V}_1}(\boldsymbol{R})$.

In this section, we show that the image of $H_2(BPL_c(\boldsymbol{R})^\delta;\boldsymbol{Z})$ in $H_2(B\boldsymbol{G}_c^{L\cdot\mathcal{V}_1}(\boldsymbol{R})^\delta;\boldsymbol{Z})$ is isomorphic to $\boldsymbol{R}$ and this isomorphism is given by the discrete Godbillon-Vey invariant ([**5**]).

Now the classifying space $B\bar{\Gamma}_1^{PL}$ for the piecewise linear foliations is described by Greenberg ([7]). The foliated cobordism group $\mathcal{F}\Omega_{3,1}^{PL}$ of transversely piecewise linear foliations, $\pi_3(B\bar{\Gamma}_1^{PL})$, $H_3(B\bar{\Gamma}_1^{PL}; \boldsymbol{Z})$ and $H_2(BPL_c(\boldsymbol{R})^\delta; \boldsymbol{Z})$ are isomorphic to each other and they are isomorphic to $\boldsymbol{R}\otimes_{\boldsymbol{Q}}\boldsymbol{R}$. Greenberg ([7]) showed that the generator $a\otimes b$ of $\mathcal{F}\Omega_{3,1}^{PL}$ is represented by the piecewise linear Reeb foliations of S^3 which is defined as follows. Consider the foliation of $\boldsymbol{R}^2\times[0, \infty)$ by planes $\boldsymbol{R}^2\times\{*\}$. This foliation is invariant under the similarity transformation with center (0, 0, 0) and with ratio e^a and the foliation induces a foliation of the solid torus

$$(\boldsymbol{R}^2\times[0, \infty)-(0, 0, 0))/(x, y, z)\sim e^a(x, y, z).$$

By attaching two such foliated solid tori, we obtain a piecewise linear Reeb foliations of S^3.

Let f_a be a piecewise linear homeomorphism of $\boldsymbol{R}$ with support in $[-1, 0]$ such that $\log f_a'(-0)=a$ and let g_b be a piecewise linear homeomorphisms of $\boldsymbol{R}$ with support in $[0, 1]$ such that $\log g_b'(+0)=b$. Then the piecewise linear Reeb foliation of S^3 whose compact toral leaf has the germs at 0 of f_a and g_b above as holonomies is mapped to the class of $(f_a, g_b)-(g_b, f_a)$ in $H_2(BPL_c(\boldsymbol{R})^\delta; \boldsymbol{Z})$ by the isomorphism. (In particular, the homology class of this 2-cycle does not depend on the choice of f_a and g_b, which is also a consequence of the fact that $PL_c(\boldsymbol{R})$ is a perfect group $(H_1(BPL_c(\boldsymbol{R})^\delta; \boldsymbol{Z})=0)$. [2])

It is easy to check that the value of the (discrete) Godbillon-Vey invariant on $(f_a, g_b)-(g_b, f_a)$ is equal to ab ([5]).

Now we show the following theorem.

THEOREM (A.1). *Let a, b, a', b' be real numbers such that $ab=a'b'$. Let f_a and $f_{a'}$ be piecewise linear homeomorphisms of $\boldsymbol{R}$ with support in $[-1, 0]$ such that $\log f_a'(-0)=a$ and $\log f_{a'}'(-0)=a'$, respectively, and let g_b and $g_{b'}$ be piecewise linear homeomorphisms of $\boldsymbol{R}$ with support in $[0, 1]$ such that $\log g_b'(+0)=b$ and $\log g_{b'}'(+0)=b'$. Then the 2-cycles $(f_a, g_b)-(g_b, f_a)$ and $(f_{a'}, g_{b'})-(g_{b'}, f_{a'})$ are homologous in $BG^{L.\mathcal{V}_1}$.*

REMARK. $GV((f_a, g_b)-(g_b, f_a))=ab=a'b'=GV((f_{a'}, g_{b'})-(g_{b'}, f_{a'}))$ but they are not homologous in $BPL_c(\boldsymbol{R})$ if $a\otimes_{\boldsymbol{Z}} b\neq a'\otimes_{\boldsymbol{Z}} b'$ by a result of [7].

COROLLARY (A.2). *The foliated cobordism class (within the category of foliations of class $C^{L.\mathcal{V}_\beta}$ $(\beta\geqq 1)$) of transversely oriented transversely piecewise linear foliations of closed oriented 3-manifolds is characterized by its (discrete) Godbillon-Vey class.*

To prove Theorem (A.1) we need the following proposition which is shown in [20, Theorem (3.2)]. A piecewise linear homeomorphism of $\boldsymbol{R}$ with compact support is said to be elementary if it has at most 3 nondifferentiable points.

PROPOSITION (A.3). *There exist positive real numbers c and C satisfying the following conditions. Let ε be positive real number such that $\varepsilon \leqq c$. Let f be an elementary piecewise linear homeomorphism of $\boldsymbol{R}$ with support in* $[1/8, 7/8]$. *Assume that*

$$|||\log f'|||_1 \leqq \varepsilon^2 .$$

Then f is written as a product (composition) of 3 commutators of piecewise linear homeomorphisms of $\boldsymbol{R}$ as follows.

$$f = [g_1, g_2][g_3, g_4][g_5, g_6] ,$$

where the supports of g_i $(i=1, \cdots, 6)$ are contained in $[0, 1]$ and

$$|||\log g_i'|||_1 \leqq C_\varepsilon .$$

PROPOSITION (A.4). *Let (a_i, b_i) $(i=1, 2, \cdots)$ be disjoint open intervals whose union is bounded. Let f_i be a piecewise linear homeomorphism of $\boldsymbol{R}$ with support in $[(7a_i+b_i)/8, (a_i+7b_i)/8]$ which is a composition of at most k elementary piecewise linear homeomorphisms. Suppose that $\sum |||\log f_i'|||_1^{1/2} < \infty$. Then $f = \prod f_i$ is written as a product (composition) of $3k$ commutators of piecewise linear homeomorphisms of $\boldsymbol{R}$ as follows.*

$$f = \prod_{j=1}^{3k} [g_{2j-1}, g_{2j}] ,$$

where $g_i \in G_c^{L, \mathcal{V}_1}(\boldsymbol{R})$, the supports of g_i $(i=1, \cdots, 6k)$ are contained in $Cl \cup [a_i, b_i]$.

PROOF. By Lemma (4.1) of [**20**], we can take the elementary PL homeomorphisms $h_j^{(i)}$ $(j=1, \cdots, k)$ satisfying the following conditions.

$f_i = h_1^{(i)} \cdots h_k^{(i)}$,

the support of $h_j^{(i)}$ is contained in $[(7a_i+b_i)/8, (a_i+7b_i)/8]$ and

$|||\log (h_j^{(i)})'||| \leqq 2|||\log f_i'|||_1$.

By Proposition (A.3), we write $h_j^{(i)}$ as a product of 3 commutators of piecewise linear homeomorphisms with support in $[a_i, b_i]$ whose norms are estimated by $|||\log f_i'|||_1^{1/2}$. Since $\sum |||\log f_i'|||_1^{1/2} < \infty$, $\prod_i h_j^{(i)}$ is written as a product of 3 commutators of elements of $G_c^{L, \mathcal{V}_1}(\boldsymbol{R})$ with support in $Cl \cup [a_i, b_i]$. Thus the proposition follows.

COROLLARY (A.5). *Let (a_i, b_i), f_i and f be as in Proposition* (A.4). *Let g be an element of $G_c^{L, \mathcal{V}_1}(\boldsymbol{R})$ such that* $\mathrm{Int\,Supp}\, g \cap \bigcup_i (a_i, b_i) = \emptyset$. *Then the 2-cycle $(f, g)-(g, f)$ is homologous to zero.*

PROOF OF THEOREM (A.1). When f and g are commuting PL homeomorphisms of $\boldsymbol{R}$, we write the homology class of the 2-cycle $(f, g)-(g, f)$ by $\{f, g\}$.

Let h be a piecewise linear homeomorphism with support in $[-2, 2]$ such that $h(x)=(x+2)/2$ for $x\in[-1, 2]$. Put $U=(-2/3, 2/3)$. Then $h^j(U)$ are disjoint.

For a real number u such that $|u|\leqq 1$, let f_u be an elementary PL homeomorphism of $\boldsymbol{R}$ with support in $[-1/4, 0]$ such that $\log f_u'(x)=u$ for $x\in[-1/2^4, 0)$ and $|||\log f_u'|||_1\leqq 4|u|$. In the same way, for a real number v such that $|v|\leqq 1$, let g_v be an elementary PL homeomorphism of $\boldsymbol{R}$ with support in $[0, 1/4]$ such that $\log g_v'(x)=v$ for $x\in(0, 1/2^4]$ and $|||\log g_v'|||_1\leqq 4|v|$. We may assume that f_a, $f_{a'}$, g_b and $g_{b'}$ in Theorem (A.1) are those defined above.

Note that it is sufficient to prove the proposition when $0<a<1$, $0<b<1$ and $a'=1$. Let

$$a=\sum_{i=1}^{\infty}\frac{a_i}{2^i} \qquad (a_i\in\{0, 1\})$$

be the dyadic expansion of a. Put

$$A_0=\sum_{i=1}^{\infty}\frac{a_{2i}}{2^{2i}},$$

$$A_1=\sum_{i=1}^{\infty}\frac{a_{2i-1}}{2^{2i}}.$$

Then $a=A_0+2A_1$. Since

$$\{f_a, g_b\}=\{f_{A_0}, g_b\}+2\{f_{A_1}, g_b\},$$

it is sufficient to show the proposition for a such that the dyadic expansion of a is

$$a=\sum_{i=1}^{\infty}\frac{a_{2i}}{2^{2i}} \qquad (a_{2i}\in\{0, 1\}).$$

For a nonpositive integer j, put

$$c_j=2^j\sum_{i=j+1}^{\infty}\frac{a_{2i}}{2^{2i}}.$$

Then

$$c_j\leqq 2^j\sum_{i=j+1}^{\infty}\frac{1}{2^{2i}}<2^{-1-j}.$$

Let F_1 and G_1 be the homeomorphisms defined by

$$F_1=\prod_{j=0}^{\infty}h^j f_{c_j}h^{-j} \quad \text{and} \quad G_1=\prod_{j=0}^{\infty}h^j g_{b/2^j}h^{-j},$$

respectively. Since

$$|||\log(h^j f_{c_j}h^{-j})'|||_1=|||\log(f_{c_j})'|||_1\leqq 4c_j$$

and $\sum c_j<\infty$, F_1 is an element of $G^{L,\mathcal{V}_1}$. Similarly, G_1 is an element of $G^{L,\mathcal{V}_1}$.

Moreover, since $\sum c_j^{1/2}<\infty$ and $\sum(b/2^j)^{1/2}<\infty$, by Proposition (A.4) we have the following lemma.

LEMMA (A.6). *Let Θ be a subset of $\boldsymbol{Z}_+$. Then*

$$\prod_{j\in\Theta} h^j f_{c_j} h^{-j} \quad \textit{and} \quad \prod_{j\in\Theta} h^j g_{b/2^j} h^{-j}$$

are written as products of 3 *commutators of elements of $G_c^{L\cdot cv_1}(\boldsymbol{R})$ with support in $\bigcup_{j\in\Theta} h^j(U)$.*

Consider the homology class $\{F_1, G_1\}$ of the 2-cycle $(F_1, G_1)-(G_1, F_1)$. Since the conjugation acts as the identity of the homology of the group,

$$\{F_1, G_1\} = \{hF_1h^{-1}, hG_1h^{-1}\}.$$

LEMMA (A.7). $\{hF_1h^{-1}, hG_1h^{-1}\}=2\{hF_1h^{-1}, G_1|[2/3, 2]\}$.

PROOF. First the supports of $(G_1|[2/3, 2])^2$ and hG_1h^{-1} are both the union $\bigcup_{j=1}^{\infty} h^j([-1/4, 0])$ and these two homeomorphisms coincide on the union $\bigcup_{j=1}^{\infty} h^j([-1/2^4, 0])$. Hence the supports of $(G_1|[2/3, 2])^2hG_1^{-1}h^{-1}$ is contained in the union $\bigcup_{j=1}^{\infty} h^j([-1/4, -1/2^4])$. Since

$$|||\log((G_1|h^j([-1/4, -1/2^4])^2(hG_1^{-1}h^{-1})|h^j([-1/4, -1/2^4]))'|||_1{}^{1/2}$$

is estimated by $(4b/2^j+4b/2^j+4b/2^{j-1})^{1/2}=b/2^{j/2-2}$, by Proposition (A.4), this is written as a product of at most 9 commutators of elements of $G_c^{L\cdot cv_1}(\boldsymbol{R})$ with support in the union $\bigcup_{j=1}^{\infty} h^j([-1/2, 0])$. Hence by Corollary (A.5), $\{F_1, (G_1|[2/3, 2])^2hG_1^{-1}h^{-1}\}=0$ in $H_2(G_c^{L\cdot cv_1}(\boldsymbol{R}); \boldsymbol{Z})$. Thus the lemma is proved.

Note here that

$$2c_{j-1}-c_j = 2^j\sum_{i=j}^{\infty}\frac{a_{2i}}{2^{2i}}-2^j\sum_{i=j+1}^{\infty}\frac{a_{2i}}{2^{2i}}=\frac{a_{2j}}{2^j}.$$

Put

$$F_2 = \prod_{j=1}^{\infty} h^j f_{a_{2j}} h^{-j}.$$

The proof of the following lemma is similar to that of Lemma (A.7).

LEMMA (A.8). $\{F_2, G_1|[2/3, 2]\}=\{(F_1|[2/3, 2])^{-1}hF_1^2h^{-1}, G_1|[2/3, 2]\}$.

By the perfectness of $PL_c(\boldsymbol{R})$, we have

$$\{F_1, G_1\} = \{F_1|U, G_1|U\}+\{F_1|[2/3, 2], G_1|[2/3, 2]\}.$$

Hence we have

$$\begin{aligned}\{f_a, g_b\} &= \{F_1|U, G_1|U\}\\ &= \{F_1, G_1\} - \{F_1|[2/3, 2], G_1|[2/3, 2]\}\\ &= 2\{hF_1h^{-1}, G_1|[2/3, 2]\} - \{F_1|[2/3, 2], G_1|[2/3, 2]\}\\ &= \{F_2, G_1|[2/3, 2]\}.\end{aligned}$$

This formula is similar to

$$a\cdot b = \sum_{j=1}^{\infty} \frac{a_{2i}}{2^{2i}} = \frac{a_2}{2}\cdot\frac{b}{2} + \frac{a_4}{2^2}\cdot\frac{b}{2^2} + \frac{a_6}{2^3}\cdot\frac{b}{2^3} + \frac{a_8}{2^4}\cdot\frac{b}{2^4} + \cdots.$$

The next step is to show the formula corresponding to the following.

$$\begin{aligned}&\frac{a_2}{2}\cdot\frac{b}{2} + \frac{a_4}{2^2}\cdot\frac{b}{2^2} + \frac{a_6}{2^3}\cdot\frac{b}{2^3} + \frac{a_8}{2^4}\cdot\frac{b}{2^4} + \cdots\\ &= a_2\cdot\left(\frac{1}{2}\cdot\frac{b}{2}\right) + a_4\cdot\left(\frac{1}{2^2}\cdot\frac{b}{2^2}\right) + a_6\cdot\left(\frac{1}{2^3}\cdot\frac{b}{2^3}\right) + a_8\cdot\left(\frac{1}{2^4}\cdot\frac{b}{2^4}\right) + \cdots\\ &= \frac{1}{2^1}\cdot\frac{a_2b}{2} + \frac{1}{2^2}\cdot\frac{a_4b}{2^2} + \frac{1}{2^3}\cdot\frac{a_3b}{2^3} + \frac{1}{2^4}\cdot\frac{a_4b}{2^4} + \cdots.\end{aligned}$$

That is, let G_2 be a homeomorphism defined by

$$G_2 = \prod_{j=1}^{\infty} h^j g_{a_{2j}b/2^j} h^{-j}.$$

Since the support of $G_1(G_2)^{-1}$ is contained in the union $\bigcup_{a_{2j}=0} h^j([-1/4, 0])$ and the support of F_2 is contained in the union $\bigcup_{a_{2j}\neq 0} h^j([0, 1/4])$, by Proposition (A.4) and Corollary (A.5),

$$\{F_2, G_1|[2/3, 2]\} = \{F_2, G_2\}.$$

Now let F_3 be the homeomorphism such that

$$F_3 = \prod_{j=0}^{\infty} h^j f_{1/2^j} h^{-j}.$$

Since the support of $(F_3|[2/3, 2])(F_2)^{-1}$ is contained in the union $\bigcup_{a_{2j}=0} h^j([-1/4, 0])$ and the support of G_2 is contained in the union $\bigcup_{a_{2j}\neq 0} h^j([0, 1/4])$, by Lemma (A.6) and Corollary (A.5),

$$\{F_2, G_2\} = \{F_3|[2/3, 2], G_2\}.$$

The final step is to show the formula corresponding to the following.

$$\frac{1}{2}\cdot\frac{a_2b}{2} + \frac{1}{2^2}\cdot\frac{a_4b}{2^2} + \frac{1}{2^3}\cdot\frac{a_6b}{2^3} + \frac{1}{2^4}\cdot\frac{a_8b}{2^4} + \cdots = 1\cdot\sum\frac{a_{2i}b}{2^{2i}} = 1\cdot(ab).$$

Put

$$d_j = 2^j \sum_{i=j+1}^{\infty} \frac{a_{2i}b}{2^{2i}}.$$

Let G_3 be the homeomorphism defined by

$$G_3 = \prod_{j=0}^{\infty} h^j g_{d_j} h^{-j}.$$

Then as in Lemma (A.7),

$$\{F_3, G_3\} = \{hF_3h^{-1}, hG_3h^{-1}\} = 2\{F_3|[2/3, 2], hG_3h^{-1}\}.$$

As in Lemma (A.8), we see that

$$\{F_3|[2/3, 2], G_2\} = \{F_3|[2/3, 2], (G_3|[2/3, 2])^{-1}hG_3^2h^{-1}\}.$$

By the perfectness of $PL_c(\boldsymbol{R})$, we have

$$\{F_3, G_3\} = \{F_3|U, G_3|U\} + \{F_3|[2/3, 2], G_3|[2/3, 2]\}.$$

Thus we have

$$\begin{aligned} \{f_1, g_{ab}\} &= \{F_3|U, G_3|U\} \\ &= \{F_3, G_3\} - \{F_3|[2/3, 2], G_3|[2/3, 2]\} \\ &= 2\{F_3|[2/3, 2], hG_3h^{-1}\} - \{F_3|[2/3, 2], G_3|[2/3, 2]\} \\ &= \{F_3|[2/3, 2], G_2\}. \end{aligned}$$

Thus we obtain $\{f_a, g_b\} = \{f_1, g_{ab}\}$ and we proved Theorem (A.1).

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