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Local limit theorem and distribution of periodic orbits of Lasota-Yorke transformations with infinite Markov partition

Dedicated to Professor Takesi Watanabe on his sixtieth birthday

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0. Introduction.

A transformation T from the unit interval $[0, 1]$ into itself is called a Lasota-Yorke transformation, simply, an L-Y map, if it is piecewise C^2 and uniformly expanding. A partition $\mathcal{P}=\{I_j\}_j$ of $[0, 1]$ consisting of intervals is called a defining partition for an L-Y map T if $T|\mathrm{Int}I_j$ is of class C^2 and the end points of I_j's are necessarily singularities of T. We consider a class $\mathcal{T}$ of L-Y maps which have an infinite defining partition with Markov property. A typical example of such a map is the so-called the Gauss transformation $T_G x=(1/x)-[1/x]$, $x\in[0, 1]$, where $[x]$ denotes the integral part of x. We regard two maps to be identical if they coincide up to a set of the Lebesgue measure zero. Therefore the maps need not be defined for all $x\in[0, 1]$.

The purpose of this paper is to study the following problems for the map $T\in\mathcal{T}$ by using the spectral properties of the transfer operators acting on the space $BV=BV([0, 1]\to C)$ of functions of bounded variation:

(I) The integral central limit problems and local ones for the sum

$$S_n f = \sum_{k=0}^{n-1} f\circ T^k, \tag{0.1}$$

where f is a real valued function belonging to an appropriate function space.

(II) The problem on the asymptotic distributions of the periodic orbits of T.

For the sake of simplicity we assume in Section 3 the mixing condition (M) which implies that T has a unique absolutely continuous invariant probability measure μ with support $[0, 1]$ and the measure-theoretic dynamical system (T, μ) is mixing. Usually the main goal of the problem (I) is to show the central limit theorem which states that there is a positive number V such that

$$\mu\left\{x \in [0, 1] \mid a \leqq \frac{1}{\sqrt{n}}\left(S_n f(x) - n\int_0^1 f\,d\mu\right) \leqq b\right\} \tag{0.2}$$
$$\longrightarrow \frac{1}{\sqrt{2\pi V}}\int_a^b \exp\left(-\frac{t^2}{2V}\right)dt\,.$$

The number V is called the limiting variance which will be given by

$$V = \lim_{n\to\infty}\frac{1}{n}\int_0^1 (S_n f(x) - \int_0^1 f\,d\mu)^2 d\mu\,. \tag{0.3}$$

If the self-correlation

$$R_n(f) = \int_0^1\left(f\circ T^n - \int_0^1 f\,d\mu\right)\left(f - \int_0^1 f\,d\mu\right)d\mu \tag{0.4}$$

decays rapidly, then the general theory for the stationary sequences can be applied to $\{f\circ T^n\}_{n=0}^{\infty}$. In fact many authors proved the central limit theorem (0.2) by showing that $R_n(f)$ decays exponentially fast as n goes to ∞ if f belongs to a nice class of functions (see [7], [8], [10], [22] etc.). But most results are incomplete because they make assumption that the limiting variance does not vanish, although the convergence of the limit in (0.3) is guaranteed by the exponential decay of $R_n(f)$.

One of the remarkable facts we shall prove later in Section 4 is that the limiting variance turns out to be positive whenever f is a non-constant function in $\mathcal{F}(T)$. The space $\mathcal{F}(T)$ will be defined to contain $\log|T'|$ as well as all the real functions of bounded variation (see Section 1). Moreover, the transfer operator approach allows us to prove in Section 5 not only the central limit theorem (Theorem 5.1) but also the so-called local limit theorem in the following general form:

THEOREM 0.1 (Theorem 5.2). *Let f be a function in $\mathcal{F}(T)$ which is not identically zero with $\int_0^1 f\,d\mu=0$. Let V be the limiting variance as in* (0.3). *Then for any rapidly decreasing function u on $\boldsymbol{R}$ and any function g of bounded variation on* [0, 1], *the following asymptotic formula holds*:

$$\limsup_{n\to\infty}\sup_{\alpha\in\boldsymbol{R}}\left|\sqrt{n}\int_0^1 u(S_n f(x)+\alpha)g(x)m(dx) - \int_{-\infty}^{\infty} u(t)\Phi_{n,\alpha}(dt)\int_0^1 g(x)m(dx)\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{\alpha^2}{2nV}\right)\right| = 0 \tag{0.5}$$

where m denotes the Lebesgue measure on [0, 1] *and $\{\Phi_{n,\alpha}\}_{n,\alpha}$ is a bounded family of Radon measures on $\boldsymbol{R}$ which are expressed by using the Fourier transform $\hat{u}(t)=\int_{-\infty}^{\infty} e^{-\sqrt{-1}ty}u(y)dy$ as*

(0.6) $$\int_{-\infty}^{\infty} u(t)\Phi_{n,\alpha}(dt) = \sum_{k=-\infty}^{\infty} \hat{u}(ka)\exp(\sqrt{-1}(k(nb+a\alpha))).$$

In the above $a \in (0, \infty]$ *and* $0 \leqq b < 2\pi$ *are quantities determined by* f *(see Theorem 5.2). We adopt also the convention that* $u(\infty)=0$ *and* $0 \cdot \infty = 0$.

Similar assertions have been obtained in [**15**] for a general L-Y map T and for a $f \in BV([0, 1] \to \boldsymbol{R})$. The present space $\mathfrak{F}(T)$ coincides with $BV([0, 1] \to \boldsymbol{R})$ if T has a finite defining partition. However, if T has an infinite defining partition, $\mathfrak{F}(T)$ contains many unbounded functions. For example, if we consider the Gauss transformation T_G, the functions $\log x$ and $\sum_{k=1}^{\infty} \chi_{(1/(k+1), 1/k]}(x) \times \log k$ belong to $\mathfrak{F}(T_G)$. It is well-known that these functions play important roles in the study of the metrical theory of continued fractions. This is one of the reasons why we restrict ourselves to the maps in the class $\mathfrak{T}$ and extend the results in [**15**] to the functions in $\mathfrak{F}(T)$.

Concerning the problem (II), we owe a great deal to the results in Parry [**16**] and Baladi and Keller [**1**]. We consider the η-function

(0.9) $$\eta(f, s) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} (S_n f)(x) \exp(-s(S_n f_0)(x))$$

for $f \in \mathfrak{F}(T)$, where $f_0 = \log|T'|$. We can show that $\eta(f, s)$ is analytic in the domain $\mathrm{Re}\, s > 1$ and it can be extended meromorphically beyond the axis $\mathrm{Re}\, s = 1$. The analytic properties of $\eta(f, s)$ are closely related to the local limit theorem in the above, and therefore the problem (II) is linked to the problem (I). More precisely, if the function $f_0 = \log|T'|$ satisfies a certain condition which ensures $a = a(f_0) = \infty$ in the local limit theorem (0.6), then $s=1$ is a unique pole of $\eta(f, s)$ on the axis $\mathrm{Re}\, s = 1$, and it is a simple pole with residue

$$\int_0^1 f d\mu \Big/ \int_0^1 \log|T'| d\mu = \int_0^1 f d\mu / h_\mu(T),$$

where $h_\mu(T)$ denotes the metrical entropy of T with respect to μ. In other words $\eta(f, s)$ can be expressed as

(0.10) $$\eta(f, s) = \int_0^1 f d\mu h_\mu(T)^{-1}(s-1)^{-1} + \phi(s)$$

in a neighborhood of $\mathrm{Re}\, s = 1$, where ϕ is an analytic function in the neighborhood. It is meaningful to give a sufficient condition for $a(f_0) = \infty$ in Theorem 0.1. We shall prove in Section 4 that if we can label the member of the defining partition $\mathcal{P}$ so that

(P) $$0 < \lim_{j \to \infty} (\sup_{\mathrm{Int} I_j} |T'| j^{-p}) < \infty$$

for some $p > 1$, then we have $a(f_0) = \infty$. The condition (P) will be called the polynomial growth condition (for T'). It is clear that the Gauss transformation satisfies the condition (P). An answer to the problem (II) is the following:

THEOREM 0.2 (Theorem 7.2). *Assume that* $T\in\mathfrak{T}$ *satisfies the mixing conditions* (*M*) *and the condition* (*P*). *Let* γ *denote the prime periodic orbit of* T *and* $P(\gamma)$ *denote its period. For* $g\in\mathfrak{F}(T)$, *let*

$$N(g, \gamma)=\sum_{k=0}^{P(\gamma)-1} g(T^k x) \quad for\ x\in\gamma .$$

Then we have

$$\sum_{N(f_0, \gamma)\leq t} \frac{N(f, \gamma)}{N(f_0, \gamma)} \sim \frac{e^t}{t h_\mu(T)}\int_0^1 f\, d\mu \tag{0.11}$$

and

$$\sum_{N(f_0, \gamma)\leq t} N(f, \gamma) \sim \frac{e^t}{h_\mu(T)}\int_0^1 f\, d\mu , \tag{0.12}$$

as $t\to\infty$, *where* $f_0=\log|T'|$, *and* $h_\mu(T)$ *denotes the metrical entropy of* T *with respect to* μ. *Here we write* $a(t)\sim b(t)$ *if* $\lim_{t\to\infty} b(t)/a(t)=1$.

The proofs of Theorem 0.1 and Theorem 0.2 are carried out in the same way as the proofs of Theorem 4.1 in [**15**] and Theorem 4 in [**16**] respectively. To this end we introduce the transfer operators $L(s, t)$ for T defined by

$$L(s, t)g(x)=\sum_{Ty=x} |T'y|^{-s}A(y)^{\sqrt{-1}t}g(y) \tag{0.13}$$

with $A(x)=\exp(f(x))$ for $f\in\mathfrak{F}(T)$.

By using the method in [**7**] (see also [**1**] and [**24**]), we prove that there is a neighborhood U of the set $\{(s, t)|\mathrm{Re}\, s\geq 1, t\in\boldsymbol{R}\}$ in $\boldsymbol{C}\times\boldsymbol{C}$ such that the family $\{L(s, t)\}_{(s,t)\in U}$ becomes an analytic family of quasi-compact operators on $BV=BV([0, 1]\to\boldsymbol{C})$ which have the essential spectral radii uniformly less than 1. This fact enables us to apply the results in Baladi and Keller [**1**], Morita [**15**], and Parry [**16**] to our problems (I) and (II). We would like to note that any Markov map f_K constructed in Bowen and Series [**3**, Section 2] associated with the Fuchsian group with parabolic elements behaves similarly to the member of $\mathfrak{T}$ satisfying the conditions (M) and (P). Therefore we expect that our results work well in the study of dynamical properties of the geodesic flows on the corresponding Riemann surfaces. For example, in the last section, we try to explain the relation between our results and the results obtained by Pollicott in [**20**] which are concerned with the metrical theorems for the closed orbits of the geodesic flow on the modular surface. We must note that Pollicott uses the results in Mayer [**13**] on the zeta functions for the Gauss transformation T_G and Mayer makes a further investigation for the zeta functions for T_G in terms of the thermodynamic formalism in his recent paper [**14**].

In the first section we prepare some basic facts. The next three sections are devoted to the study of the spectral properties of transfer operators $L(s, t)$. In Section 5, we give a proof of the local limit theorem. In Section 6 and Section 7, we consider the second problem and prove Theorem 0.2 (Theorem 7.2). In the last section we give new metrical theorems for continued fraction expansions and make some comments on the Pollicott's paper [**20**]

The author would like to express his gratitude to Professor H. Nakada who introduces the paper [**20**] to him.

1. Preliminaries.

In this section we prepare some definitions and notations which will be used throughout the paper. We denote by m the Lebesgue measure on $[0, 1]$. Unless otherwise stated, we ignore the difference occuring on an m-null set. This causes us no trouble because all the phenomena appearing in this paper are observed by the Lebesgue measure. A transformation T from the unit interval into itself is called a *Lasota-Yorke transformation* (an L-Y map in abbreviation) if it satisfies the following conditions (see [**12**]):

(L-Y.1) There exists a partition $\mathcal{P}=\{I_j\}_j$ such that (a) $T|\mathrm{Int}\, I_j$ is of class C^2 and can be extended to $\bar{I}_j$ as a C^2-function, and (b) the set of intervals $\{T(\mathrm{Int}\, I_j)\}_j$ consists of a finite number of distinct intervals.

(L-Y.2) (Lasota-Yorke condition). There is a positive number $c<1$ such that $\operatorname{ess\,inf}_{x\in[0,1]}|(T^N)'(x)| > 1/c$ for some positive integer N, where T^N denotes the N times iteration $\overbrace{T\circ T\circ\cdots\circ T}^{N}$ of T.

For any L-Y map, the partition $\mathcal{P}$ in the above can be chosen to be minimal in the sense that if $Q=\{J_k\}_k$ is another partition satisfying (L-Y.1), then for any k, there is $j=j(k)$ with $\mathrm{Int}\, I_j\supset\mathrm{Int}\, J_k$. We call such a minimal partition a *defining partition* for T. We note that the defining partition is unique up to the difference of the endpoints. It is easy to see that if T is an L-Y map, so is T^n for each $n\geqq 1$.

Our main concern is a class $\mathcal{T}$ of L-Ymaps defined as follows. An L-Y map T is an element of the class $\mathcal{T}$ if it satisfies the next three conditions:

(T.1) (Markov property). The defining partition $\mathcal{P}=\{I_j\}_j$ is a Markov partition for T in the sense:

$$T(\mathrm{Int}\, I_j)\cap\mathrm{Int}\, I_k \neq \emptyset \quad \text{implies} \quad T(\mathrm{Int}\, I_j) \supset \mathrm{Int}\, I_k\,.$$

(T.2) (Strong Rényi condition). There is a positive number $\delta=\delta(T)$ such that

$$\sup_j \frac{\operatorname{ess\,sup}_{x\in I_j}|T''x|}{\operatorname{ess\,inf}_{x\in I_j}|T'x|^{2-\delta}} < \infty,$$

and

$$\sum_j m(I_j)^{1-\delta} < \infty .$$

(T.3) $\qquad T(\text{Int } I_j) = (0, 1)$ for infinitely many j.

REMARK 1.1. If T is an L-Y map satisfying the condition (T.2), then there is a positive number C_T such that

$$C_T^{-1}(\operatorname*{ess\,inf}_J |T'|)^{-1} \leqq m(J) \leqq C_T(\operatorname*{ess\,inf}_J |T'|)^{-1}$$

for any element J of the defining partition $\mathcal{P}$ for T. In fact, if $x,\ y \in \text{Int} J$ for $J \in \mathcal{P}$, we have

$$(1.1) \quad \left|\frac{T'x}{T'y} - 1\right| = \frac{1}{|T'y|}|T'x - T'y| \leqq \frac{\text{ess sup}_J |T''|}{|T'y|} |x-y|$$
$$\leqq \frac{\text{ess sup}_J |T''|}{|T'y| \text{ess inf}_J |T'|} \operatorname*{ess\,inf}_J |T'| |x-y| .$$

Since

$$(1.2) \qquad \operatorname*{ess\,inf}_J (m(J)|T'|) \leqq \int_J |T'| dm \leqq 1$$

and

$$(1.3) \qquad \operatorname*{ess\,sup}_J (m(J)|T'|) \geqq \int_J |T'| dm \geqq \min_{J\in\mathcal{P}} \int_J |T'| dm > 0$$

by the condition (L-Y.1) (b), we obtain the desired inequality in virtue of the Rényi condition.

REMARK 1.2. It is clear that the strong Rényi condition (T.2) implies the original Rényi condition $\text{ess sup}_{[0,1]} |T''|/|T'|^2 < \infty$. If T has a finite defining partition, these are equivalent to each other.

For any Borel measure ν on $[0, 1]$, and $1 \leqq p \leqq \infty$, $L^p(\nu)$ denotes the usual L^p-space with norm $\|g\|_{p,\nu} = \left(\int_0^1 |g|^p d\nu\right)^{1/p}$ if $p < \infty$, and $\|g\|_{\infty,\nu} = \nu\text{-ess sup}|g|$. $BV = BV([0, 1] \to \boldsymbol{C})$ is the totality of elements in $L^1(m)$ with version of bounded variation. $BV([0, 1] \to \boldsymbol{R})$ denotes the subspace of BV consisting of real valued elements. For $g \in BV$, we define $\mathbf{V}g$ and $\mathbf{V}_J g$ as the infimum of the total variations taken over all the versions of g on $[0, 1]$ and J, respectively. It is easy to see that $\|g\|_{BV,p} = \|g\|_{p,m} + \mathbf{V}g$ becomes a Banach norm on BV for each p. Since we can show $\|g\|_{BV,1} \leqq \|g\|_{BV,\infty} \leqq 2\|g\|_{BV,1}$ for $g \in BV$, the norms $\|g\|_{BV,p}$ are all equivalent. Thus we always regard BV as the Banach space with norm $\|g\|_{BV} = \|g\|_{BV,\infty}$, unless otherwise stated.

Now we introduce the function space $\mathcal{F}(T)$ for an L-Y map in $\mathcal{T}$ in the following way: A real valued measurable function f is an element in $\mathcal{F}(T)$ if it satisfies the next three conditions:

(F.1) For each $J \in \mathcal{P}$, $f|J$ has a version of bounded variation.

(F.2) There is a positive number $C=C(f)$ such that $|f|\leqq\log|T'|+C$ holds m-almost everywhere.

(F.3) $\sup_{J\in\mathcal{P}}\mathbf{V}_J f<\infty$ and $\lim_{n\to\infty}\mathbf{V}_{J_n}f=0$ for some sequence of intervals $\{J_n\}_n\subset\mathcal{P}$ with $T(\mathrm{Int}\,J_n)=(0, 1)$ for $n\geqq1$. Here $\mathcal{P}\in\{J\}$ is the defining partition for T.

If T has a finite defining partition, $\mathcal{F}(T)$ coincides with the space $BV([0, 1]\to\boldsymbol{R})$ but if T has an infinite defining partition, unbounded functions like $\log|T'|$ belong to $\mathcal{F}(T)$.

REMARK 1.3. If we put $A=\exp(f)$, then the conditions (F.1), (F.2), and (F.3) are equivalent to the following conditions (1), (2), and (3), respectively:

(1) For each $J\in\mathcal{P}$, $A|J$ has a version of bounded variation.

(2) There is a number $C'=C'(f)\geqq1$ such that $A|T'|\leqq C'$ and $A|T'|^{-1}\geqq C'^{-1}$ hold m-almost everywhere.

(3) $\sup_{J\in\mathcal{P}}(\mathbf{V}_J A/\mathrm{ess\,inf}_J A)\leqq\infty$ and $\lim_{n\to\infty}(\mathbf{V}_{J_n}A/\mathrm{ess\,inf}_{J_n}A)=0$ for some sequence of intervals $\{J_n\}_n\subset\mathcal{P}$ with $T(\mathrm{Int}\,J_n)=(0, 1)$ for $n\geqq1$. The expressions of (1), (2), and (3) are sometimes more convenient than those of (F.1), (F.2), and (F.3).

In the rest of this section we give typical examples of the elements of $\mathcal{F}$.

EXAMPLE 1.1. The Gauss transformation $T_G x=(1/x)-[1/x]$ has the defining partition

$$\left\{\left(\frac{1}{k+1}, \frac{1}{k}\right]\right\}_{k=1}^{\infty}.$$

EXAMPLE 1.2. For $s>1$, we define a transformation T_s: $[0, 1]\to[0, 1]$ by $T_s x=-\zeta_R(s)k^s(x-1+\zeta_R(s)^{-1}\sum_{n=1}^{k}n^{-s})$ for $x\in(1-\zeta_R(s)^{-1}\sum_{n=1}^{k+1}n^{-s}, 1-\zeta_R(s)^{-1}\sum_{n=1}^{k}n^{-s}]$, $k=1, 2, \cdots$, where $\zeta_R(s)$ is the Riemann's zeta function.

EXAMPLE 1.3. $\tilde{T}x=2^n(x-\sum_{k=1}^{n-1}2^{-k})$ for $x\in[\sum_{k=1}^{n-1}2^{-k}, \sum_{k=1}^{n}2^{-k})$, $n=1, 2, \cdots$, where we regard $\sum_{k=1}^{0}2^{-k}$ as 0.

These transformations are well-defined on $[0, 1]$ except for countably many points.

2. Analytic family of transfer operators.

In this section we consider the transfer operator which has the form:

$$L_G g(x)=\sum_{T^n y=x}G(y)g(y). \tag{2.1}$$

Here G is an m-measurable function and g will be chosen from BV. First of all we recall the results in Baladi and Keller [1, pp. 463-466] and apply them to our situation. Let T be an L-Y map with defining partition $\mathcal{P}=\{J\}$. If G satisfies

(2.2) $$\theta = \lim_{n\to\infty} \|G_n\|_{\infty,m}^{1/n} < 1,$$

and

(2.3) $$M = \sum_{J\in\mathcal{P}} \operatorname*{ess\,sup}_J |G| < \infty,$$

then L_G can be realized as a quasi-compact operator acting on the space BV with

(2.4) $$\|L_G\|_{BV} \leqq 3(\mathbf{V}G+M),$$

where $G_n(x)=G(x)G(Tx)\cdots G(T^{n-1}x)$. More precisely, let $\mathcal{P}_n=\{J\}$ be the defining partition for T^n. We choose a point x_J from $\operatorname{Int} J$ for each member J of $\mathcal{P}_n$ and define an operator Π_n by

(2.5) $$\Pi_n g(x) = \sum_{J\in\mathcal{P}} \chi_J(x) g(x_J),$$

where χ_J is the indicator function of the interval J. Then for any $\tilde{\theta}$ with $\theta<\tilde{\theta}<1$, we can choose a positive number C_0 depending only on $\mathbf{V}G$, M, and the minimal positive integer n_0 such that $\|G_n\|_{\infty,m}<\tilde{\theta}^n$ for any $n\geqq n_0$, and

(2.6) $$\|L_G^n - L_G^n \Pi_n\|_{BV} \leqq C_0 \tilde{\theta}^n$$

holds. Since $L_G^n \Pi_n$ can be easily seen to be a nuclear operator in the sense of Grothendieck, and consequently, a compact operator on BV, we conclude that L_G is quasi-compact. For details, one can consult the paper [**1**]. Keeping these facts in mind we consider the operator with the form:

(2.7) $$L(s, t)g(x) = L_{G(s,t)}g(x) = \sum_{Ty=x} G(s, t)(y)g(y),$$

where T is an L-Y map in $\mathcal{T}$, $G(s, t)=|T'|^{-s}A^{\sqrt{-1}t}$, and $A=\exp(f)$ with $f\in\mathcal{F}(T)$. The main purpose of the present section is to prove:

LEMMA 2.1. *Let T belong to $\mathcal{T}$ and let $L(s, t)$ be as in* (2.7). *Then there exists a neighborhood U of the set $\{(s, t)|\operatorname{Re} s\geqq 1, t\in\boldsymbol{R}\}$ in $\boldsymbol{C}\times\boldsymbol{C}$ such that the family $\{L(s, t)\}_{(s,t)\in U}$ becomes an analytic family of quasi-compact operators and* $\lim_{n\to\infty}\|G(s, t)_n\|_{\infty,m}^{1/n}<\theta$ *holds for any $(s, t)\in U$ and some $\theta<1$. In particular, the essential spectral radii of $L(s, t)$'s are uniformly smaller than θ.*

PROOF. For $s=\sigma+\sqrt{-1}\tau$ with $\sigma\geqq 1$, $\tau\in\boldsymbol{R}$ we put

(2.8) $$G(s, t) = |T'|^{-s}A^{\sqrt{-1}t},$$

and

(2.9) $G(s, t, k, l) = |T'|^{-s}A^{\sqrt{-1}t}(\log|T'|)^k(\log A)^l$ for $k, l = 0, 1, 2, 3, \cdots$.

Let us write formally,

$$L(s+p, t+q) = L_{G(s+p, t+q)} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(\sqrt{-1})^{2k+l}}{k!\,l!}p^k q^l L_{G(s,t,k,l)} \tag{2.10}$$

for $(p, q)\in \boldsymbol{C}\times\boldsymbol{C}$. If we show that there are positive numbers C_1, C_2, a, and b which are independent of k, l, and (s, t) with $\mathrm{Re}\, s\geqq 1$ and $t\in\boldsymbol{R}$ such that

$$\mathbf{V}G(s, t, k, l) \leqq C_1(|s|+|t|)k!\,l!\,a^k b^l, \tag{2.11}$$

and

$$\sum_{J\in\mathcal{P}}\operatorname*{ess\,sup}_J |G(s, t, k, l)| \leqq C_2 k!\,l!\,a^k b^l, \tag{2.12}$$

then $L_{G(s,t,k,l)}$'s are realized as operators on BV with norm not greater than $3(C_1(|s|+|t|)+C_2)k!\,l!\,a^k b^l$ in virtue of the inequality (2.4). Therefore the right hand side of the equation (2.10) is absolutely convergent with respect to the uniform operator topology provided that $|p|<a^{-1}$ and $|q|<b^{-1}$. Combining this fact with the estimate (2.6), the lemma is easily verified.

Before we prove the estimates (2.11) and (2.12), we note that there are positive numbers C_3 and C_4 so that

$$\sup_{J\in\mathcal{P}}\ \sup_{x,y\in \mathrm{Int}J}\frac{|T'x|}{|T'y|}\leqq C_3 \tag{2.13}$$

and

$$|\log A| \leqq |\log|T'||+C_4. \tag{2.14}$$

Indeed, (2.13) follows directly from Remark 1.1 and (2.14) is an easy consequence of the condition (F.2).

Now we choose a small number ε so that $2\varepsilon<\delta$, where δ is the number which appeared in the strong Rényi condition (T.2) in Definition 1.1. For $s=\sigma+\sqrt{-1}\tau$ with $\sigma\geqq 1$, $\tau\in\boldsymbol{R}$, we have:

$$\begin{aligned}
|G(s, t, k, l)(x)| &= ||T'x|^{-s}A(x)^{\sqrt{-1}t}(\log|T'x|)^k(\log A(x))^l| \\
&\leqq |T'x|^{-\sigma}|\log|T'x||^k|\log A(x)|^l \\
&\leqq |T'x|^{-\sigma}|\log|T'x||^k\sum_{i=0}^{l}\binom{l}{i}|\log|T'x||^i C_4^{l-i} \quad \text{(by (2.14))} \\
&\leqq |T'x|^{-\sigma+2\varepsilon}|T'x|^{-\varepsilon}|\log|T'x||^k\sum_{i=0}^{l}\binom{l}{i}|T'x|^{-\varepsilon}|\log|T'x||^i C_4^{l-i}.
\end{aligned}$$

Applying the inequality $x^{-\alpha}(\log x)^n\leqq(\alpha e)^{-n}n^n$ for $x\geqq 1$, and $\alpha>0$, to $|T'x|^{-\varepsilon}|\log|T'x||^n$, we obtain

(2.15) $|T'x|^{-\varepsilon}|\log|T'x||^{n}\leqq\max\{(\text{ess inf}|T'x|^{-\varepsilon})|\log\text{ess inf}|T'x|^{-\varepsilon}|^{n},\ (\alpha e)^{-n}n^{n}\}$
$$\leqq C_5^n n^n$$

for some positive constant C_5 depending only on T. Therefore we have

(2.16) $$\operatorname*{ess\,sup}_J|G(s, t, k, l)|\leqq\operatorname*{ess\,sup}_J|T'|^{-\sigma+2\varepsilon}C_5^k k^k\sum_{i=0}^{l}\binom{l}{i}C_5^i i^i C_4^{l-1}$$
$$\leqq\operatorname*{ess\,sup}_J|T'|^{-1+\delta}C_5^k(C_4+C_5)^l k^k l^l.$$

On the other hand, we know $0<\inf_{J\in\mathcal{P}}\text{ess sup}_J|T'|m(J)$ from the inequality (1.3). Combining this with (2.13), we have $(\text{ess inf}_J|T'|)^{-1+\delta}\leqq(C_3m(J))^{1-\delta}$. The sum of the right hand side taken over all $J\in\mathcal{P}$ is convergent due to the second inequality of the strong Rényi condition (T.2). Thus we obtain

(2.17) $$\sum_{J\in\mathcal{P}}\operatorname*{ess\,sup}_J|G(s, t, k, l)|\leqq\sum_{J\in\mathcal{P}}\operatorname*{ess\,sup}_J|T'|^{-1+\delta}C_5^k(C_4+C_5)^l k^k l^l$$
$$=C_6C_5^k(C_4+C_5)^l k^k l^l.$$

Applying the Stirling's formula $\lim_{n\to\infty}(n!/n^{n+1/2}e^{-n}\sqrt{2\pi})=1$ to the inequality (2.17), we obtain the inequality (2.12) with $a=C_5$, and $b=C_4+C_5$.

Next we choose any $x,\ y\in\text{Int}\,J$ and consider

$G(s, t, k, l)(x)-G(s, t, k, l)(y)=D_1+D_2+D_3+D_4$, where

$$D_1=(|T'x|^{-s}-|T'y|^{-s})A(x)^{\sqrt{-1}t}(\log|T'x|)^k(\log A(x))^l,$$
$$D_2=|T'y|^{-s}(A(x)^{\sqrt{-1}t}-A(y)^{\sqrt{-1}t})(\log|T'x|)^k(\log A(x))^l,$$
$$D_3=|T'y|^{-s}A(y)^{\sqrt{-1}t}((\log|T'x|)^k-(\log|T'y|)^k)(\log A(x))^l$$

and

$$D_4=|T'y|^{-s}A(y)^{\sqrt{-1}t}(\log|T'y|)^k((\log A(x))^l-(\log A(y))^l).$$

Then we have

$$|D_1|\leqq|s|\operatorname*{ess\,sup}_J(|T'|^{-\sigma-1}|T''|)|x-y||\log|T'x||^k|\log A(x)|^l$$
$$\leqq|s|\operatorname*{ess\,sup}_J(|T'|^{-\sigma-1+2\varepsilon}|T''|)\operatorname*{ess\,sup}_J(|T'|^{-2\varepsilon})|\log|T'x||^k|\log A(x)|^l|x-y|.$$

By using the strong Rényi condition (T.2), (2.13), (2.14), and (2.15) we obtain

(2.18) $$|D_1|\leqq|s||x-y|C_7C_5^k(C_4+C_5)^l k^k l^l$$

in the same way as (2.16). For D_2,

$$|D_2|\leqq 2|t|\operatorname*{ess\,sup}_J(|T'|^{-\sigma})\frac{|A(x)-A(y)|}{\text{ess inf}_J A}|\log|T'x||^k|\log A(x)|^l.$$

Therefore we have

(2.19) $$|D_2|\leqq 2|t|\operatorname*{ess\,sup}_J(|T'|^{-\sigma+2\varepsilon})\frac{|A(x)-A(y)|}{\text{ess inf}_J A}C_8C_5^k(C_4+C_5)^l k^k l^l,$$

in virtue of (2.13), (2.14), and (2.15). Next,

$$(2.20)\qquad |D_3| \leqq \operatorname*{ess\,sup}_J (|T'|^{-\sigma})k \operatorname*{ess\,sup}_J (|T'|^{-1}|\log|T'||^{k-1})|\log A(x)|^l|x-y|$$
$$\leqq \operatorname*{ess\,sup}_J(|T'|^{-\sigma+2\varepsilon})|x-y|C_9C_5^k(C_4+C_5)^lk^kl^l$$

in the same way as the above. Finally we have

$$(2.21)\qquad |D_4| \leqq \operatorname*{ess\,sup}_J (|T'|^{-\sigma})|\log|T'||^kl \operatorname*{ess\,sup}_J|\log A(x)|^{l-1}\frac{|A(x)-A(y)|}{\operatorname{ess\,inf}_J A}$$
$$\leqq \operatorname*{ess\,sup}_J(|T'|^{-\sigma+2\varepsilon})\frac{|A(x)-A(y)|}{\operatorname{ess\,inf}_J A}C_{10}C_5^k(C_4+C_5)^lk^kl^l.$$

(2.18), (2.19), (2.20), (2.21) and the condition (F.3) imply

$$\sum_{J\in\mathcal{P}} \underset{J}{\mathbf{V}} G(s, t, k, l) \leqq (|t|+|s|)C_{11}C_5^k(C_4+C_5)^lk^kl^l.$$

Thus we conclude that

$$(2.22)\qquad \mathbf{V}G(s, t, k, l) \leqq \sum_{J\in\mathcal{P}} \underset{J}{\mathbf{V}} G(s, t, k, l)+2\sum_{J\in\mathcal{P}} \operatorname*{ess\,sup}_J|G(s, t, k, l)|$$
$$\leqq (|t|+|s|)C_{12}C_5^k(C_4+C_5)^lk^kl^l.$$

Here we have used the inequality (2.17). Applying the Stirling's formula again to the inequality (2.22), we obtain the inequality (2.11). This completes the proof of the lemma. //

REMARK. We do not need the Markov property of T for the validity of Lemma 2.1. The conditions (L-Y.1), (L-Y.2), and (T.2) on T are sufficient.

3. Spectral properties of $L(s, t)$.

As a consequence of Lemma 2.1 in the previous section and the general perturbation theory for linear operators in [**5**], and [**11**] (see also [**1**]), we obtain the spectral decomposition of $L(s, t)$.

PROPOSITION 3.1. *Let $T\in\mathfrak{T}$. Consider the neighborhood U of the set $\{(s, t)|\operatorname{Re}s\geqq 1,\ t\in\boldsymbol{R}\}$ in $\boldsymbol{C}\times\boldsymbol{C}$ and the number $\theta<1$ which appeared in Lemma 2.1. For any $(s_0, t_0)\in U$, choose any $\tilde\theta>\theta$ so that $L(s_0, t_0)$ has no eigenvalues with modulus $\tilde\theta$ as an operator on BV. Then there is an open subset $U(s_0, t_0)$ of U and there are analytic families $\{M(s, t)\}_{(s,t)\in U(s_0,t_0)}$ and $\{R(s, t)\}_{(s,t)\in U(s_0,t_0)}$ of operators on BV such that the following spectral decomposition holds*:

$$(3.1)\qquad L(s, t) = M(s, t)+R(s, t),$$

$$(3.2)\qquad M(s, t) = \sum_{j=1}^{n(s,t,\tilde\theta)} \lambda_j(s, t)E_j(s, t)(E_j(s, t)+N_j(s, t))$$

and

$$R(s, t)=E(s, t)L(s, t). \tag{3.3}$$

In this decomposition, $\lambda_j(s, t)$ are the eigenvalues of $L(s, t)$ on BV with modulus greater than $\tilde{\theta}$, and $E_j(s, t)$ are the projection operators onto the finite dimensional eigenspaces corresponding to $\lambda_j(s, t)$. $n(s, t, \tilde{\theta})$ is the number of the Jordan blocks of the operator $L(s, t)$ restricted to the finite dimensional space spanned by these eigenspaces, and the operators $N_j(s, t)=(L(s, t)-\lambda_j(s, t))E_j(s, t)$ are nilpotent. The operator $E(s, t)$ is defined as the projection onto the complementary subspace of $\sum_{j=1}^{n(s,t,\tilde{\theta})}E_j(s, t)BV$ and therefore $R(s, t)=E(s, t)L(s, t)$ has the spectral radius not greater than $\tilde{\theta}$. In particular, we have

$$\sum_{j=1}^{n(s,t,\tilde{\theta})} E_j(s, t)+E(s, t)=id_{BV}, \tag{3.4}$$

$$E_j(s, t)L(s, t)=L(s, t)E_j(s, t), \tag{3.5}$$

$$E_i(s, t)E_j(s, t)=E_j(s, t)E_i(s, t)=\delta_{i,j}E_j(s, t), \tag{3.6}$$

$$E_j(s, t)E(s, t)=E(s, t)E_j(s, t)=O, \tag{3.7}$$

and

$$E_j(s, t)N_j(s, t)=N_j(s, t)E_j(s, t)=N_j(s, t). \tag{3.8}$$

PROOF. We only give an outline of the proof because this proposition is proved in [**1**, Section 2] in the case that (s_0, t_0) is fixed. Since $\{L(s, t)\}$ is an analytic family in $(s, t)\in U$ in virtue of Lemma 2.1, we can choose an open neighborhood $U(s_0, t_0)\subset U$ of (s_0, t_0) and numbers θ_1, and θ_2 with $\theta_1<\tilde{\theta}<\theta_2$ such that the set $\{z\in \boldsymbol{C} \mid \theta_1\leqq|z|\leqq\theta_2\}$ is contained in the resolvent set for $L(s, t)$ for all $(s, t)\in U(s_0, t_0)$ and $\rho=\sup_{(s,t)\in U(s_0,t_0)}\|L(s, t)\|_{BV}<\infty$. Then we have the following analytic families of projections defined by the Dunford integrals:

$$P_1(s, t)=\frac{1}{2\pi\sqrt{-1}}\left(\int_{|z|=\rho+1}-\int_{|z|=\theta_2}\right)(z-L(s, t))^{-1}dz \tag{3.9}$$

and

$$P_2(s, t)=\frac{1}{2\pi\sqrt{-1}}\int_{|z|=\theta_1}(z-L(s, t))^{-1}dz. \tag{3.10}$$

Clearly, $E(s, t)$ in (3.3) must be $P_2(s, t)$, $M(s, t)$ in (3.1) must be $L(s, t)P_1(s, t)$ and the Jordan decomposition for $M(s, t)$ on $P_1(s, t)BV$ gives the decomposition (3.2). //

From now on we impose the following mixing condition (M) on the map T in $\mathfrak{T}$.

(M) T has a unique m-absolutely continuous invariant probability measure

μ with support [0, 1], and the measure-theoretic dynamical system (T, μ) is mixing, i.e., $\lim_{n\to\infty}\int_0^1 g_1\circ T^n g_2 d\mu = \int_0^1 g_1 d\mu \int_0^1 g_2 d\mu$ for any $g_1, g_2 \in L^2(\mu)$.

The Radon-Nikodym derivative $d\mu/dm$ will be denoted by h_0. It is well-known that it has a version of bounded variation (see [**12**]).

REMARK 3.1. As noted in [**15**], the condition (M) is not essential for our arguments. In fact for $T\in\mathcal{T}$, any absolutely continuous invariant measure has a density in BV and can be decomposed into a finite number of ergodic components. Each ergodic component can be decomposed into finitely many mixing components which are mapped cyclically by T. For details one can consult [7] and [**24**].

REMARK 3.2. If $T\in\mathcal{T}$ satisfies the condition (M), the density h_0 satisfies $\operatorname{ess\,inf} h_0>0$. In particular, $1/h_0$ is also in BV. Although this is a well-known fact, we give the proof for completeness.

Suppose this is not true, there would be a point $x_0\in[0, 1]$ such that $h_0(x_0-)=0$ or $h_0(x_0+)=0$ holds for any version of h_0. From now on, we fix a version of h_0 and write it as h_0 again. It is well-known that h_0 satisfies the Perron-Frobenius equation

$$L_T h_0 = \sum_{Ty=x} \frac{1}{|T'y|} h_0(y) = h_0(x), \qquad m\text{-a.e.}$$

where $L_T\,(=L(0, 1)$ in our notation) is the so-called Perron-Frobenius operator for T with respect to the Lebesgue measure m. We prove that there is an open interval on which $h_0(x)=0$. This contradicts the condition (M). We assume $x_0\in(0, 1)$. In the case $x_0=0$, or 1, we can show a contradiction in the same way. Take any open interval I. Then for any $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)$ such that $m(T^n I\cap(x_0-\varepsilon, x_0+\varepsilon))>0$ for $n\geqq n_0$ since the dynamical system (T, μ) is mixing. This implies that there is an element J_ε^n of the defining partition $\mathcal{P}_n$ for T^n such that $\operatorname{Int} I\cap \operatorname{Int} J_\varepsilon^n\neq\emptyset$ and $m(T^n J_\varepsilon^n\cap(x_0-\varepsilon, x_0+\varepsilon))>0$ for all $n\geqq n_0$. If there are infinitely many n with $T^n(\operatorname{Int} J_\varepsilon^n)\supset(x_0-\varepsilon, x_0+\varepsilon)$, there is a point y in the $m(J_\varepsilon^n)$-neighborhood of I such that $h_0(y-)=0$ or $h_0(y+)=0$ by the Perron-Frobenius equation. On the contrary, suppose that we can choose a sequence $\varepsilon_k\downarrow 0$ so that $x_0\notin T^n \operatorname{Int}(J_{\varepsilon_k}^n)$ for infinitely many n. Then we can choose a sequence $n_k<n_{k+1}$ such that either $\sup T^{n_k}\operatorname{Int}(J_{\varepsilon_k}^{n_k})<x_0+\varepsilon_k$ and $\sup T^{n_k}\operatorname{Int}(J_{\varepsilon_k}^{n_k})>\sup T^{n_k}\operatorname{Int}(J_{\varepsilon_{k+1}}^{n_{k+1}})$ for all k, or $\inf T^{n_k}\operatorname{Int}(J_{\varepsilon_k}^{n_k})>x_0-\varepsilon_k$ and $\inf T^{n_k}\operatorname{Int}(J_{\varepsilon_k}^{n_k})<\inf T^{n_k}\operatorname{Int}(J_{\varepsilon_{k+1}}^{n_{k+1}})$ for all k. But this is impossible because T is Markov and there are at most finitely many points which can be the end point of $T(\operatorname{Int} J)$ for $J\in\mathcal{P}$ in virtue of the condition (L-Y.1) (b). Since the interval I is arbitrary, we conclude that $D_-\cup D_+$ is dense in [0, 1], where

$D_{-}=\{y \mid h_0(y-)=0\}$ and $D_{+}=\{y \mid h_0(y+)=0\}$. Therefore either the closure of D_{-} or the closure of D_{+} must contain an open interval. On such an interval h_0 must vanishes. We now reach the desired contradiction. //

REMARK 3.3. The maps in Example 1.1, Example 1.2, and Example 1.3 satisfy the condition (M). Some sufficient conditions for T to satisfy the condition (M) are discussed in [**2**].

We recall Proposition 1.1 in [**15**] for the later convenience.

PROPOSITION 3.2. *Let L_T be the Perron-Frobenius operator for T as above. Let L_μ be the operator acting on $L^1(\mu)$ which is defined by*

$$L_\mu g = \frac{d}{d\mu}\int_{T^{-1}(\cdot)} g\,d\mu \quad \textit{for } g \in L^1(\mu).$$

Then for $g \in L^1(\mu)$ and S^1-valued measurable function ϕ, the following are equivalent:

$$(1) \qquad L_T(\phi g h_0) = g h_0 \quad \textit{in } L^1(m),$$

$$(2) \qquad L_\mu(\phi g) = g \quad \textit{in } L^1(\mu), \ \textit{and}$$

$$(3) \qquad g \circ T = \phi g \quad \textit{in } L^1(\mu).$$

Now we are in a position to state the main results in this section.

PROPOSITION 3.3. *Let T be a map in $\mathfrak{T}$ satisfying the mixing condition (M) and $U \subset \boldsymbol{C} \times \boldsymbol{C}$ be the open set which appeared in Lemma 2.1. Then we have the following:*

(1) *For each $t \in \boldsymbol{R}$, $L(1, t)$ can be extended to an operator on $L^1(m)$ with norm not greater than* 1.

(2) *If $t_0 \in \boldsymbol{R}$ and $L(1, t_0)$ has an eigenvalue λ with modulus* 1 *as an operator on $L^1(m)$, then the corresponding eigenfunction must be a constant multiplication of h_0.*

(3) *For each $t_0 \in \boldsymbol{R}$, $L(1, t_0)$ has at most one eigenvalue with modulus* 1 *as an operator on $L^1(m)$.*

(4) *For each $t_0 \in \boldsymbol{R}$, there is an open set $V(1, t_0)$ of $(1, t_0)$ in U with the following properties:*

(4.a) *If $L(1, t_0)$ does not have an eigenvalue with modulus* 1, *then the spectral radius of $L(s, t)$ as an operator on BV is less than* 1 *for any $(s, t) \in V(1, t_0)$;*

(4.b) *If $L(1, t_0)$ has an eigenvalue $\lambda(1, t_0)$ with modulus* 1, *then $L(s, t)$ has the spectral decomposition*

$$(3.11) \qquad L(s, t)^n = \lambda(s, t)^n E_1(s, t) + S(s, t)^n, \qquad \textit{for} \quad n \geqq 1$$

as an operator on BV for $(s, t) \in V(1, t_0)$ with the following properties:

(4.b.1) *$\lambda(s, t)$ is the analytic function in $V(1, t_0)$ and coincides with the simple eigenvalue of $L(s, t)$ with maximal modulus. Moreover we have*

$$\left.\frac{d\lambda(1, t)}{dt}\right|_{t=t_0} = \sqrt{-1}\int_0^1 f d\mu\, \lambda(1, t_0) \tag{3.12}$$

and

$$\begin{aligned}\left.\frac{d^2\lambda(1, t)}{dt^2}\right|_{t=t_0} &= -\lim_{n\to\infty}\frac{1}{n}\int_0^1\left(\sum_{k=0}^{n-1}\left(f\circ T^k-\int_0^1 f d\mu\right)\right)^2 d\mu\, \lambda(1, t_0) \\ &= -\lambda(1, t_0)V(f).\end{aligned} \tag{3.13}$$

(4.b.2) *$E_1(s, t)$ is the projection operator onto the one dimensional eigenspace corresponding to $\lambda(s, t)$ which depends holomorphically on $(s, t)\in V(1, t_0)$ and satisfies*

$$\int_0^1 E_1(1, t_0)g dm = \int_0^1 g dm \quad \text{and} \quad \int_0^1 S(1, t_0) g dm = 0 \quad \text{for any } g\in BV. \tag{3.14}$$

(4.b.3) *$S(s, t)$ is the operator valued holomorphic function in $V(1, t_0)$ with spectral radius less than* 1 *as the operator on BV.*

(4.b.4) *$E_1(1, t_0)$ and $S(1, t_0)$ are extended to bounded operators on $L^1(m)$ and the decomposition* (3.11) *still has a meaning. Moreover, $\|S(1, t_0)^n g\|_{1,m}\to 0$ $(n\to\infty)$ for any $g\in L^1(m)$.*

PROOF. (1) This is a trivial fact.

(2) If $L(1, t_0)h=\lambda h$ for $h\in L^1(m)$ *and* $\lambda\in S^1$, we have $(hh_0^{-1})\circ T=\bar{\lambda}A^{\sqrt{-1}t}hh_0^{-1}$ in virtue of Proposition 3.2, where $\bar{\lambda}$ denotes the complex conjugation of λ and $A=\exp(f)$. For any element J of the defining partition for T with $T(\mathrm{int}\, J)=(0, 1)$, we have

$$\begin{aligned}\mathbf{V}(hh_0^{-1}) &= \mathbf{V}_J(hh_0^{-1}\circ T) = \mathbf{V}_J(\bar{\lambda}A^{\sqrt{-1}t_0}hh_0^{-1}) \\ &\leqq \frac{|t_0|\mathbf{V}_J A}{\text{ess inf}_J A}\|hh_0^{-1}\|_{\infty,m}+\mathbf{V}_J(hh_0^{-1}).\end{aligned}$$

Since hh_0^{-1} is in BV by Remark 3.2, and f satisfies the condition (F.3), we obtain $\mathbf{V}(hh_0^{-1})=0$. This implies hh_0^{-1} is a constant function in $L^1(m)$.

(3) If λ_1 and λ_2 are eigenvalues of $L(1, t_0)$ with modulus 1, $1=\bar{\lambda}_i A^{\sqrt{-1}t_0}$ for $i=1, 2$ holds in virtue of Proposition 3.2 and the assertion (2) above. Therefore we can conclude $\lambda_1=\lambda_2$.

(4) All the assertions except for the equations (3.12), (3.13), and (3.14) follow from Ionescu Tulcea and Marinescu Theorem [**9**] and Proposition 3.1 (see also [**7**], [**10**], [**15**], and [**24**]). To prove (3.12) and (3.13), we first show that $f\in L^k(m)$ for all $k\geqq 1$. Since $L_T=L(0, 1)$ is a positive operator which preserves the value of the integration, we have $\int_0^1 |f|^k dm=\int_0^1 L_T(|f|^k)dm$. On

the other hand $L_T(|f|^k)=\sum_{Ty=x}|T'y|^{-1}|\log A(y)|^k\in BV$ in virtue of the estimate in the proof of (2.11) and (2.12) in Lemma 2.1. Thus $f\in L^k(m)$ for all $k\geqq 1$.

For the sake of simplicity we may assume $\int_0^1 f d\mu=0$. For t so that $t+t_0\in V(1, t_0)$ we have

$$\begin{aligned}\int_0^1\exp(\sqrt{-1}tS_nf)d\mu &= \int_0^1\exp(\sqrt{-1}tS_nf)h_0dm\\ &= \lambda(1, t_0)^{-n}\int_0^1\exp(\sqrt{-1}(t_0+t)S_nf)h_0dm\\ &= \lambda(1, t_0)^{-n}\int_0^1 L_T^n(\exp(\sqrt{-1}(t_0+t)S_nf)h_0)dm\\ &= \lambda(1, t_0)^{-n}\int_0^1 L(1, t_0+t)^n h_0dm\,.\end{aligned}$$

In the above we have used the fact $\exp(\sqrt{-1}t_0S_nf)=\lambda(1, t_0)^n$ which is a consequence of Proposition 3.2. In virtue of the spectral decomposition (3.11) and the above equation, we have

$$\begin{aligned}\int_0^1\exp(\sqrt{-1}tS_nf)d\mu &= \lambda(1, t_0)^{-n}\int_0^1 L(1, t_0+t)^n h_0dm\\ &= \lambda(1, t_0)^{-n}\lambda(1, t_0+t)^n\int_0^1 E_1(1, t_0+t)h_0dm\\ &\quad+\lambda(1, t_0)^{-n}\int_0^1 S(1, t_0+t)^n h_0dm\\ &= p_n(t)+r_n(t)\,.\end{aligned}\tag{3.15}$$

Thus we have

$$\frac{d}{dt}\int_0^1\exp\Big(\sqrt{-1}t\frac{S_nf}{n}\Big)d\mu\Big|_{t=0}=\frac{dp_n(tn^{-1})}{dt}\Big|_{t=0}+\frac{dr_n(tn^{-1})}{dt}\Big|_{t=0}\tag{3.16}$$

and

$$\frac{d^2}{dt^2}\int_0^1\exp\Big(\sqrt{-1}t\frac{S_nf}{\sqrt{n}}\Big)d\mu\Big|_{t=0}=\frac{d^2p_n(tn^{-1/2})}{dt^2}\Big|_{t=0}+\frac{d^2r_n(tn^{-1/2})}{dt^2}\Big|_{t=0}\,.\tag{3.17}$$

We note that the second terms in (3.16) and (3.17) go to 0 exponentially fast since the operator $S(1, t)^n$ can be expressed by the Dunford integral as $S(1, t)^n=(1/2\pi\sqrt{-1})\int_{|z|=r}z^n(z-L(1, t))^{-1}dz$ with $r<1$ for any $n\geqq 1$. The left hand side of (3.16) goes to 0 by the ergodic theorem. Using the Taylor expansion of p_n, we can show that the right hand side goes to $\lambda(1, t_0)^{-1}(d\lambda(1, t)/dt)|_{t=t_0}$ as n goes to ∞. The left hand side of (3.17) equals $-(1/n)\int_0^1(S_nf)^2d\mu$. On the other hand, it is not hard to show that the right hand side goes to $\lambda(1, t_0)^{-1}d^2\lambda(1, t)/dt^2|_{t=t_0}$ as n goes to ∞ in the same manner as (3.16). The proof of

(3.12) and (3.13) is now complete.

Finally, using the fact $\exp(\sqrt{-1}t_0S_nf)=\lambda(1, t_0)^n$ again, we obtain

$$\int_0^1 gdm = \int_0^1 \lambda(1, t_0)^{-n} \exp(\sqrt{-1}t_0S_nf)gdm = \lambda(1, t_0)^{-n}\int_0^1 L(1, t_0)^n gdm$$

$$= \int_0^1 E_1(1, t_0)gdm+\lambda(1, t_0)^{-n}\int_0^1 S(1, t_0)^n gdm \to \int_0^1 E_1(1, t_0)gdm\,(n\to\infty).$$

Thus $\int_0^1 E_1(1, t_0)gdm=\int_0^1 gdm$ and $\int_0^1 S(1, t_0)^n gdm=0$ for $n\geqq 1$, getting (3.14). //

4. Classification of $\mathcal{F}_0(T)$.

In this section we prove the non-degeneracy of the limiting variance given by (0.3) or (3.13) for a non-constant element in $\mathcal{F}(T)$. Furthermore we classify the elements in $\mathcal{F}(T)$ in terms of the spectral properties of the transfer operator $L(1, t)$. As before, we assume T is an L-Y map in $\mathcal{F}$ satisfying the mixing condition (M). Without loss of generality we may restrict ourselves to the subspace $\mathcal{F}_0(T)=\left\{f\in\mathcal{F}(T)\,\middle|\,\int_0^1 fd\mu=0\right\}$. For $f\in\mathcal{F}_0(T)$, we introduce the sets

$$\Lambda_T(f) = \{t \in \boldsymbol{R}\,|\,\text{the transfer operator } L(1, t) \text{ has an eigenvalue with modulus } 1\}. \tag{4.1}$$

and

$$G_T(f) = \{\lambda \in S^1\,|\,L(1, t)h_0 = \lambda h_0 \text{ for some } t \in \Lambda_T(f)\}. \tag{4.2}$$

In other words, $G_T(f)$ is the totality of numbers which are realized as eigenvalues of $L(1, t)$ with modulus 1 for some $t\in\boldsymbol{R}$. Before classifying $\mathcal{F}_0(T)$ we show:

THEOREM 4.1. *For $f\in\mathcal{F}(T)$, the limiting variance $V=V(f)=0$ if and only if f is a constant function.*

PROOF. It suffices to show that if $V(f)=0$ for $f\in\mathcal{F}_0(T)$, then $f=0$ m-a.e.. If $V(f)=0$, then there is a real valued function $g\in L^2(\mu)$ such that $f=g\circ T-g$ by the Leonov's result (see [8, Ch. 18]). Then we have $\exp(\sqrt{-1}tg\circ T)=\exp(\sqrt{-1}tf)\exp(\sqrt{-1}tg)$ for any $t\in\boldsymbol{R}$. In virtue of Proposition 3.2 and the assertion (2) of Proposition 3.3, we conclude that $\exp(\sqrt{-1}tf)=1$ for any $t\in\boldsymbol{R}$. Therefore we have $f=0$ m-a.e.. //

Next we prove

LEMMA 4.1. *For $f\in\mathcal{F}_0(T)$, $\Lambda_T(f)$ and $G_T(f)$ are closed subgroups of $\boldsymbol{R}$ and S^1, respectively.*

PROOF. By using Proposition 3.2, it is easy to see that $\Lambda_T(f)$ and $G_T(f)$

are subgroups of $\boldsymbol{R}$ and S^1, respectively. It remains to show the closedness. Assume $t_n \in \Lambda_T(f)$ converges to t_∞. From Proposition 3.2, we have $\exp(\sqrt{-1}t_n f) = \lambda_n$ m-a.e.. We may assume λ_n converges to $\lambda \in S^1$ as $n \to \infty$. Consequently, we have $\exp(\sqrt{-1}t_\infty f) = \lambda$ m-a.e.. Proposition 3.2 implies that $L(1, t_\infty)h_0 = \lambda h_0$. Therefore $\Lambda_T(f)$ is closed in $\boldsymbol{R}$. $G_T(f)$ is closed in S^1 by the same reason.

//

Before stating the classification theorem we put

(4.3) $\mathcal{F}_1(T) = \{f \in \mathcal{F}_0(T) \mid f = \alpha + \beta K$ for some non-constant integer valued function K and real numbers α and $\beta \neq 0\}$

and

(4.4) $$\mathcal{F}_2(T) = \mathcal{F}_0(T) - (\mathcal{F}_1(T) \cup \{0\}).$$

THEOREM 4.2. *The elements in* $\mathcal{F}_0(T)$ *are classified as follows*:

(1) $f=0$ *in* $L^1(m)$ *if and only if* $\Lambda_T(f) = \boldsymbol{R}$ *and* $G_T(f) = \{1\}$.

(2) $f \in \mathcal{F}_1(T)$ *if and only if* $\Lambda_T(f) = a\boldsymbol{Z}$ *and* $G_T(f) = \{1, \lambda, \cdots, \lambda^{k-1}\}$ *for* $0 < a < \infty$ *and a primitive k-th root* λ *of* 1 *with* $L(1, a)h_0 = \lambda h_0$.

(3) $f \in \mathcal{F}_2(T)$ *if and only if* $\Lambda_T(f) = \{0\}$, *and* $G_T(f) = \{1\}$.

PROOF. (1) If $f \neq 0$ in $L^1(\mu)$, $V(f) > 0$ by Theorem 4.1. Therefore $\lambda(1, 0) = 1$, $d\lambda(1, t)/dt|_{t=0} = 0$, and $d^2\lambda(1, t)/dt^2|_{t=0} = -V(f) < 0$ from the assertions in Proposition 3.3. Thus $|\lambda(1, t)| < 1$ for small $t \neq 0$. This implies

(4.5) $$a = \inf\{t > 0 \mid t \in \Lambda_T(f)\} > 0.$$

Here we regard a as ∞ if the set above is empty. We have proved that if $f \neq 0$ in $L^1(\mu)$, then $\Lambda_T(f) \neq \boldsymbol{R}$. Hence $\Lambda_T(f) = \boldsymbol{R}$ implies $f=0$ in $L^1(\mu)$ and consequently $G_T(f) = \{1\}$. The converse is trivial.

(2) If $f \in \mathcal{F}_1(T)$, that is, $f = \alpha + \beta K$ as in (4.3), then we have $\exp(\sqrt{-1}2\pi f/\beta) = \exp(\sqrt{-1}2\pi\alpha/\beta)$. From Proposition 3.2, we have $L(1, 2\pi/\beta)h_0 = \exp(\sqrt{-1}2\pi\alpha/\beta)h_0$ m-a.e.. Combining this and (4.5) we have $a < \infty$. Since $\Lambda_T(f)$ is closed in $\boldsymbol{R}$, we conclude that $\Lambda_T(f) = a\boldsymbol{Z}$. Let λ satisfy $L(1, a)h_0 = \lambda h_0$. We have $G_T(f) = \{\lambda^n \mid n \in \boldsymbol{Z}\}$. Since $G_T(f)$ is closed in S^1, $G_T(f) = \{1, \lambda, \cdots, \lambda^{k-1}\}$ and λ must be a primitive k-th root of unity. Conversely, if $\Lambda_T(f) = a\boldsymbol{Z}$, we obtain $L(1, a)h_0 = \lambda h_0$ for some $\lambda \in S^1$. Proposition 3.2 implies that $\exp(\sqrt{-1}af) = \lambda$. Therefore f must be in $\mathcal{F}_1(T)$.

(3) If $f \neq 0$ in $L^1(\mu)$, and $f \notin \mathcal{F}_1(T)$, then we have $a = \infty$. Therefore $\Lambda_T(f) = \{0\}$, and $G_T(f) = \{1\}$. Conversely, a can not be ∞ if $f=0$ in $L^1(\mu)$ or $f \in \mathcal{F}_1(T)$. Now we have completed the proof. //

REMARK 4.1. If we can label the elements of the defining partition for T so that

$$\text{(P)} \qquad 0<\varliminf_{j\to\infty}\frac{\operatorname{ess\,sup}_{I_j}|T'|}{j^p}<\infty \quad \text{for some } p>1,$$

we say that T satisfies the polynomial growth condition (for T'). In this case, any element f in $\mathcal{F}_0(T)$ for which

$$\text{(F.4)} \qquad \lim_{j\to\infty}\operatorname*{ess\,sup}_{I_j}\left|\frac{A}{T'}\right|>0 \text{ exists and } \lim_{j\to\infty}\frac{\mathbf{V}_{I_j}A}{\operatorname{ess\,inf}_{I_j}A}=0$$

is always contained in $\mathcal{F}_2(T)$, where $A=\exp(f)$. In particular, $\log|T'|-\int_0^1\log|T'|\,d\mu\in\mathcal{F}_2(T)$. This is verified as follows: Applying the strong Rényi condition to the inequality (1.1) instead of the Rényi condition, we have

$$\text{(4.6)} \qquad \begin{aligned}\left|\frac{T'x}{T'y}-1\right| &\leqq \frac{\operatorname{ess\,sup}_J|T''|}{|T'y|\operatorname{ess\,inf}_J|T'|^{1-\delta}}\operatorname*{ess\,inf}_J|T'|\,|x-y|\operatorname*{ess\,inf}_J|T'|^{-\delta}\\ &\leqq C\operatorname*{ess\,inf}_J|T'|^{-\delta}|x-y|\end{aligned}$$

for any $x, y\in J\in\mathcal{P}$. (4.6) implies that $\log|T'|$ satisfies (F.4). On the other hand, there are finitely many j such that $\operatorname{ess\,sup}_{I_j}|T'|\leqq K$ for given $K>0$ in virtue of the inequality (1.3). Therefore we can change the labeling so that $\operatorname{ess\,sup}_{I_j}|T'|\leqq\operatorname{ess\,sup} I_{j+1}|T'|$ for $j=1, 2, \cdots$ without breaking the condition (P). It is easy to see that the labeling yields $\lim_{j\to\infty}(\operatorname{ess\,sup}_{I_{j+1}}|T'|/\operatorname{ess\,sup}_{I_j}|T'|)=1$. Combining this fact with (4.6), we obtain

$$\text{(4.7)} \qquad \lim_{j\to\infty}\sup_{x\in I_{j+1},\, y\in I_j}\left|\frac{T'x}{T'y}\right|=1.$$

The first condition in (F.4) and (4.7) implies

$$\text{(4.8)} \qquad \lim_{j\to\infty}\sup_{x\in I_{j+1},\, y\in I_j}|f(x)-f(y)|=\lim_{j\to\infty}\sup_{x\in I_{j+1},\, y\in I_j}\left|\log\frac{A(x)}{A(y)}\right|=0.$$

The second condition in (F.4) yields

$$\text{(4.9)} \qquad \lim_{j\to\infty}\sup_{x,\, y\in I_j}|f(x)-f(y)|\leqq\lim_{j\to\infty}\mathbf{V}_{I_j}f\leqq\lim_{j\to\infty}\frac{\mathbf{V}_{I_j}A}{\operatorname{ess\,inf}_{I_j}A}=0.$$

Therefore if f is in $\mathcal{F}_1(T)$, f and consequently, A must be constant on $\bigcup_{j=j_0}^{\infty}I_j$ for sufficiently large j_0 from (4.8) and (4.9). This contradicts the first condition in (F.4).

Example 1.1, and Example 1.2 satisfy the condition (P) but Example 1.3 does not. In fact, the function $\log|\tilde{T}'|-\int_0^1\log|\tilde{T}'|\,d\mu=(\sum_{n=1}^{\infty}n\chi_{[\sum_{k=1}^{n-1}2^{-k},\, \sum_{k=1}^{n}2^{-k})}-2)\log 2$ belongs to $\mathcal{F}_1(\tilde{T})$.

5. Limit theorems.

In the following three sections we give our answers to the problems (I) and (II) by using the results in Section 3, and 4. The present section is mainly devoted to the proof of Theorem 0.1 in Introduction. For any L-Y map $T\in\mathfrak{T}$ and $f\in\mathfrak{F}_0(T)-\{0\}$, we employ the following convention:

$$(5.1)\qquad \begin{cases} \Lambda_T(f)=\langle a\rangle=\begin{cases} a\boldsymbol{Z} & \text{if } a<\infty, \\ \{0\} & \text{if } a=\infty, \end{cases} \\ G_T(f)=\langle\lambda\rangle=\begin{cases} \{1,\lambda,\cdots,\lambda^{k-1}\} & \text{if } a<\infty, \\ \{1\} & \text{if } a=\infty, \end{cases} \end{cases}$$

where $a=\min(\Lambda_T(f)-\{0\})$ as in the previous section and λ is a primitive k-th root of unity with $L(1, a)h_0=\lambda h_0$ (if $a<\infty$) and $\lambda=1$ (if $a=\infty$). The (integral) central limit theorem for the sum $S_nf=\sum_{k=0}^{n-1}f\circ T^k$ is stated as follows:

THEOREM 5.1. *Let T be a map in $\mathfrak{T}$ satisfying the mixing condition (M). Let f be an element in $\mathfrak{F}_0(T)-\{0\}$. Then there exist positive numbers A_1, A_2, A_3, A_4, and $0<\gamma<1$ depending only on T and f such that*

$$(5.2)\quad \left|\int_0^1\exp\left(\frac{\sqrt{-1}t}{\sqrt{n}}S_nf(x)\right)g(x)m(dx)-\lambda^k\int_0^1 g(x)m(dx)\exp\left(-\frac{V(f)t^2}{2}\right)\right|$$
$$\leqq\left(\exp\left(-\frac{t^2}{4}\right)\left(A_1\frac{|t|^3}{\sqrt{n}}+A_2\frac{|t|}{\sqrt{n}}\right)+A_3\frac{|t|}{\sqrt{n}}\gamma^n\right)\|g\|_{BV}$$

holds for any $g\in BV$ and for any $k\in\boldsymbol{Z}$ whenever $|t|\leqq A_4\sqrt{n}$ (if $a=\infty$, we consider only the case $k=0$). Here $V(f)$ is the limiting variance defined by (0.3). *In particular, for any probability measure m_g with density $g\in BV([0,1]\to\boldsymbol{R})$, we have*

$$(5.3)\qquad \sup_{z\in\boldsymbol{R}}\left|m_g\{x\in[0,1]\,|\,\frac{1}{\sqrt{n}}S_nf(x)\leqq z\}-\frac{1}{\sqrt{2\pi V(f)}}\int_{-\infty}^z\exp\left(-\frac{t^2}{2V(f)}\right)dt\right|$$
$$\leqq\frac{A_5}{\sqrt{n}}\|g\|_{BV}$$

for some positive number A_5 independent of g (c.f. [**10**], [**15**], *and* [**22**]).

PROOF. From Proposition 3.3, we have

$$\int_0^1\exp\left(\frac{\sqrt{-1}t}{\sqrt{n}}S_nf(x)\right)g(x)m(dx)=\int_0^1 L\left(1,\frac{t}{\sqrt{n}}\right)^n g(x)m(dx)$$
$$=\lambda\left(1,\frac{t}{\sqrt{n}}\right)^n\int_0^1 E_1\left(1,\frac{t}{\sqrt{n}}\right)^n g(x)m(dx)+\int_0^1 S\left(1,\frac{t}{\sqrt{n}}\right)^n g(x)m(dx).$$

Comparing the first few terms of the Taylor expansion of the both sides, we obtain the estimate (5.2). The second assertion follows directly from the Berry-Esseen inequality which asserts that

$$\sup_{z\in\boldsymbol{R}}|F(z)-G(z)| \leqq \frac{2}{\pi}\int_0^t \frac{h(u)}{u}du+\frac{24}{\pi t}\sup_{z\in\boldsymbol{R}}|G'(z)|,$$

where $F(z)=\nu_1((-\infty, z])$, and $G(z)=\nu_2((-\infty, z])$ are distribution functions for probability measures ν_1 and ν_2 on $\boldsymbol{R}$ respectively, G is assumed to be differentiable, and $h(u)=\int_{-\infty}^{\infty}\exp(\sqrt{-1}uz)(\nu_1-\nu_2)(dz)$ (see [8]). //

REMARK 5.1. The estimate (5.3) is remarkable in the probability theoretical point of view, because the convergence rate of the central limit theorem is very near to that of the independent and identically distributed random sequence. Philipp [18] obtained more general results for mixing sequences but the convergence rate is not so good in his general setting.

In the sequel, $\mathcal{S}(\boldsymbol{R})$ denotes the space of rapidly decreasing functions on $\boldsymbol{R}$ and $\hat{u}(t)=\int_{-\infty}^{\infty}e^{-\sqrt{-1}ty}u(y)dy$ is the Fourier transform of u.

Now we are ready to prove the main theorem in this section.

THEOREM 5.2 (Local limit theorem). *Let T be a map in $\mathfrak{T}$ an $f\in\mathfrak{F}_0(T)-\{0\}$. $\Lambda_T(f)=\langle a\rangle$ and $G_T(f)=\langle\lambda\rangle$ are in (5.1) and $V=V(f)$ is the limiting variance. Then for any $u\in\mathcal{S}(\boldsymbol{R})$ and any $g\in BV$, we have*

$$\begin{aligned}(5.4)\qquad \lim_{n\to\infty}\sup_{\alpha\in\boldsymbol{R}}\Big|&\sqrt{n}\int_0^1 u(S_nf(x)+\alpha)g(x)m(dx)\\ &-\int_{-\infty}^{\infty}u(t)\Phi_{n,\alpha}(dt)\int_0^1 g(x)m(dx)\frac{1}{\sqrt{2\pi V}}\exp\Big(-\frac{\alpha^2}{2nV}\Big)\Big|=0,\end{aligned}$$

where $\{\Phi_{n,\alpha}\}_{n,\alpha}$ is a bounded family of Radon measures on $\boldsymbol{R}$, represented as

$$(5.5)\qquad \int_{-\infty}^{\infty}u(t)\Phi_{n,\alpha}(dt)=\sum_{k=-\infty}^{\infty}\hat{u}(ka)\exp(\sqrt{-1}(k(nb+a\alpha)))$$

with $\lambda=\exp(\sqrt{-1}b)$. In addition, the Radon measure $\Phi_{n,\alpha}$ has the following descriptions:

If $a=\infty$, $\Phi_{n,\alpha}$ is the Lebesgue measure for each n and α:

$$\int_{-\infty}^{\infty}u(t)\Phi_{n,\alpha}(dt)=\int_{-\infty}^{\infty}u(t)dt.$$

If $a<\infty$, $\Phi_{n,\alpha}$ is the counting measure on the lattice $(2\pi/a)\boldsymbol{Z}+(bn/a)+\alpha$:

$$\int_{-\infty}^{\infty}u(t)\Phi_{n,\alpha}(dt)=\sum_{k=-\infty}^{\infty}u\Big(\frac{2\pi k}{a}+\frac{bn}{a}+\alpha\Big).$$

PROOF. It suffices to prove the theorem for $g\in BV$ with $g\geqq 0$ and $\int_0^1 gdm=1$. Assume first that $\hat{u}\in\mathcal{D}((-r, r))$ for some $r>0$, where $\mathcal{D}(K)$ denotes the totality of smooth functions with support in $K\subset\boldsymbol{R}$. Then we have

$$\sqrt{n}\int_0^1 u(S_nf(x)+\alpha)g(x)m(dx)$$
$$=\frac{\sqrt{n}}{2\pi}\int_{-\infty}^{\infty}\hat{u}(t)\phi_n(t)\exp(\sqrt{-1}\alpha t)dt$$
$$=\frac{\sqrt{n}}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-a/2}^{a/2}\hat{u}(ka+t)\phi_n(ka+t)e^{\sqrt{-1}\alpha(ka+t)}dt,$$

where $\phi_n(t)=\int_{-\infty}^{\infty}\exp(\sqrt{-1}tS_nf)gdm$. Fix $k\in\boldsymbol{Z}$ for a while. Now

$$\frac{\sqrt{n}}{2\pi}\int_{-a/2}^{a/2}\hat{u}(ka+t)\phi_n(ka+t)e^{\sqrt{-1}\alpha(ka+t)}dt-\hat{u}(ka)\lambda^{kn}\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{\alpha^2}{2nV}\right) \tag{5.6}$$
$$=R_1(k,n)+R_2(k,n)+R_3(k,n)+R_4(k,n)\quad\text{with}\quad V=V(f),$$

where

$$R_1(k,n)=\frac{\sqrt{n}}{2\pi}\int_{\varepsilon_n\leq|t|\leq a/2}\hat{u}(ka+t)\phi_n(ka+t)e^{\sqrt{-1}\alpha(ka+t)}dt, \tag{5.7}$$

$$R_2(k,n)=\frac{1}{2\pi}\int_{|t|<\varepsilon_n\sqrt{n}}(\hat{u}(ka+t/\sqrt{n})-\hat{u}(ka))\phi_n(ka+t/\sqrt{n})e^{\sqrt{-1}\alpha(ka+t/\sqrt{n})}dt, \tag{5.8}$$

$$R_3(k,n)=\frac{1}{2\pi}\int_{|t|<\varepsilon_n\sqrt{n}}(\phi_n(ka+t/\sqrt{n})-\lambda^{kn}\exp(-Vt^2/2))e^{\sqrt{-1}\alpha(ka+t/\sqrt{n})}dt\,\hat{u}(ka), \tag{5.9}$$

and

$$R_4(k,n)=-\frac{1}{2\pi}\int_{|t|>\varepsilon_n\sqrt{n}}\exp(-Vt^2/2+\sqrt{-1}\alpha/\sqrt{n})dt\lambda^{kn}\hat{u}(ka)e^{\sqrt{-1}\alpha ka}. \tag{5.10}$$

The number ε_n in the above will be determined later.

From the equations (3.12) and (3.13), we have $d\lambda(1,t)/dt|_{t=ka}=0$, and $d^2\lambda(1,t)/dt^2|_{t=ka}=-\lambda(1,ka)V=-\lambda^kV$. On the other hand, the spectral radius of $L(1,ka+t)$ is less than 1 for $\varepsilon_n\leqq|t|\leqq a/2$, in virtue of the assertion (4.a) in Proposition 3.3. Combining these facts with the spectral decomposition (3.11) we obtain

$$|R_1(k,n)|\leqq\frac{\sqrt{n}}{2\pi}\int_{\varepsilon_n\leq|t|\leq a/2}\|L(1,ka)^n\|_{BV}\|g\|_{BV}dt\sup|\hat{u}| \tag{5.11}$$
$$\leqq C_1\int_{\varepsilon_n\leq|t|\leq a/2}(1-Vt^2/4)^n dt\sqrt{n}\|g\|_{BV}\sup|\hat{u}|,$$

where C_1 is a positive constant independent of g and u. It is easy to see that

$$|R_2(k,n)|\leqq\frac{1}{2\pi}\varepsilon_n^2\sqrt{n}\|g\|_{BV}\sup\left|\frac{d\hat{u}}{dt}\right|. \tag{5.12}$$

In virtue of the central limit theorem (5.2) we obtain

(5.13) $$|R_3(k, n)| \leqq \int_{|t|<\varepsilon_n\sqrt{n}}((A_1|t|^3/\sqrt{n}+A_2|t|/\sqrt{n})e^{-Vt^2/4} + A_3(|t|/\sqrt{n})\gamma^n)\|g\|_{BV}\sup|\hat{u}| \leqq C_2(\varepsilon_n^4 n^{3/2}+\varepsilon_n^2 n^{1/2})\|g\|_{BV}\sup|\hat{u}|,$$

where C_2 is the constant depending only on r. Clearly we have

(5.14) $$|R_4(k, n)| \leqq \frac{1}{2\pi}\int_{|t|>\varepsilon_n\sqrt{n}}\exp(-Vt^2/2)dt\|g\|_{BV}\sup|\hat{u}|.$$

Choosing ε_n so that

(5.15) $$\varepsilon_n \downarrow 0, \quad \varepsilon_n n^{1/2} \uparrow \infty, \quad \text{and} \quad \varepsilon_n^4 n^{3/2} \downarrow 0 \quad (n \to \infty),$$

we have

(5.16) $$\sum_{k=-\infty}^{\infty}|R_1(k, n)+R_2(k, n)+R_3(k, n)+R_4(k, n)| \leqq C_r\gamma_n\left(\sup|\hat{u}|+\sup\left|\frac{d\hat{u}}{dt}\right|\right)\|g\|_{BV},$$

where the number C_r depends only on r and γ_n is a sequence with $\gamma_n \downarrow 0$ $(n \uparrow \infty)$.

Combining (5.16) with the fact $|\sum_{k=-\infty}^{\infty}\hat{u}(ka)\lambda^{kn}e^{\sqrt{-1}\alpha ka}| \leqq 2[r/a]\sup|\hat{u}|$, we obtain

$$\left|\sqrt{n}\int_{-\infty}^{\infty}\hat{u}(t)\phi_n(t)e^{\sqrt{-1}\alpha t}dt\right| \leqq C'_r\left(\sup|\hat{u}|+\sup\left|\frac{d\hat{u}}{dt}\right|\right)\|g\|_{BV}$$

with a positive number C'_r depending only on r. This implies that $\{\sqrt{n}\phi_n(\cdot)e^{\sqrt{-1}(\cdot)\alpha}\}_{n,\alpha}$ is a bounded family in the distribution space $\mathcal{D}((-r, r))'$. Since each $\sqrt{n}\phi_n(\cdot)e^{\sqrt{-1}(\cdot)\alpha}$ is a distribution of positive type, the family $\{\sqrt{n}\phi_n(\cdot)e^{\sqrt{-1}(\cdot)\alpha}\}_{n,\alpha}$ turns out to be a bounded family in the space $\mathcal{B}(\boldsymbol{R})'$ of the bounded distributions (see Schwartz [**25**, p. 276 in Ch. VII]).

Next we take a sequence $\{\rho_j\}_{j=1}^{\infty}$ of probability measures on $\boldsymbol{R}$ which converges to δ_0 (the unit mass at 0) weakly as $j\to\infty$ so that $\hat{\rho}_j\in\mathcal{D}(\boldsymbol{R})$ for every j. Write $\sqrt{n}\int_0^1 u(S_nf+\alpha)gdm=\int_{-\infty}^{\infty}u(t)\mu_{n,\alpha}(dt)$ for convenience. Choose any $u\in\mathcal{S}(\boldsymbol{R})$ and fix it for a while.

(5.17) $$\left|\int_{-\infty}^{\infty}u(t)(\rho_j*\mu_{n,\alpha})(dt)-\int_{-\infty}^{\infty}u(t)\mu_{n,\alpha}(dt)\right| \leqq \int_{|t|<\delta}\rho_j(ds)\left|\int_{-\infty}^{\infty}(u(t+s)-u(t))\mu_{n,\alpha}(dt)\right| + \int_{|t|\geqq\delta}\rho_j(ds)\left|\int_{-\infty}^{\infty}(u(t+s)-u(t))\mu_{n,\alpha}(dt)\right| = I_n+II_n.$$

Since $\{\hat{\mu}_{n,\alpha}(\cdot)=\sqrt{n}\phi_n(\cdot)e^{\sqrt{-1}(\cdot)\alpha}\}_{n,\alpha}$ is a bounded set in $\mathcal{B}(\boldsymbol{R})'$ and $\hat{u}\in\mathcal{S}(\boldsymbol{R})\subset\mathcal{B}(\boldsymbol{R})$, we have

$$(5.18)\qquad \sup_{s\in\boldsymbol{R}}\left|\int_{-\infty}^{\infty}u(t+s)\mu_{n,\alpha}(dt)\right|=\sup_{s\in\boldsymbol{R}}\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(t)\sqrt{n}\phi_n(t)e^{\sqrt{-1}t(\alpha+s)}dt\right|\leqq C_1(u).$$

Since the set $\{\nu_s(\cdot)=s^{-1}(u(\cdot+s)-u(\cdot))\}_{0<|s|\leq 1}$ is bounded in $\mathcal{S}(\boldsymbol{R})$, it is bounded in $\mathcal{B}(\boldsymbol{R})$. Therefore we have

$$(5.19)\qquad \sup_{\alpha\in\boldsymbol{R}}\sup_{|s|\leq 1}\left|\int_{-\infty}^{\infty}\nu_s(t)\mu_{n,\alpha}(dt)\right|\leqq C_2(u).$$

In the above, $C_1(u)$ and $C_2(u)$ are positive numbers which depend on u but do not depend on n and α. Now we obtain

$$(5.20)\qquad |I_n|\leqq\int_{|t|<\delta}\rho_j(ds)|s|\left|\int_{-\infty}^{\infty}\nu_s(t)\mu_{n,\alpha}(dt)\right|\leqq C_2(u)\delta,$$

and

$$(5.21)\qquad |II_n|\leqq\rho_j(|s|\geqq\delta)2C_1(u).$$

Thus we have shown that

$$(5.22)\qquad |I_n+II_n|\leqq C_3(u)\delta$$

if j is large, where $C_3(u)$ is a positive number which does not depend on n, α, and δ. On the other hand, for fixed j, we have

$$(5.23)\qquad \sup_{\alpha\in\boldsymbol{R}}\left|\int_{-\infty}^{\infty}u(t)(\rho_j*\mu_{n,\alpha})(dt)-\sum_{k=-\infty}^{\infty}(\hat{u}\hat{\rho}_j)(ka)e^{\sqrt{-1}ka\alpha}\lambda^{kn}\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{\alpha^2}{2\pi V}\right)\right|$$
$$\longrightarrow 0\quad(n\to\infty)$$

in virtue of the estimate (5.16). In addition we have

$$(5.24)\qquad \left|\sum_{k=-\infty}^{\infty}(\hat{u}(ka)-(\hat{u}\hat{\rho}_j)(ka))e^{\sqrt{-1}ka\alpha}\lambda^{kn}\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{\alpha^2}{2\pi V}\right)\right|$$
$$\leqq\sum_{k=-\infty}^{\infty}|\hat{u}(ka)(\hat{\rho}_j(ka)-1)\leqq\delta$$

if j is large. If we choose j so that (5.22) and (5.24) are satisfied, then we obtain by (5.23) that

$$\limsup_{n\to\infty}\sup_{\alpha\in\boldsymbol{R}}\left|\sqrt{n}\int_0^1 u(S_nf+\alpha)g\,dm-\sum_{k=-\infty}^{\infty}(\hat{u})(ka)e^{\sqrt{-1}ka\alpha}\lambda^{kn}\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{\alpha^2}{2\pi V}\right)\right|$$
$$\leqq(C_3(u)+1)\delta.$$

Since $\delta>0$ is arbitrary, we complete the proof of (5.4). //

6. Zeta functions.

Let T be an L-Y map satisfying the mixing condition (M). For $f\in\mathcal{F}(T)$, we consider the zeta function defined formally as

$$\zeta(s, t)=\exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{T^n x=x}\prod_{k=0}^{n-1}|T'(T^k x)|^{-s}A(T^k x)^{\sqrt{-1}t} \tag{6.1}$$

$$=\exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{T^n x=x}\exp(S_n(-s\log|T'|+\sqrt{-1}tf)(x)).$$

Here the sum $\sum_{T^n x=x}$ is taken over all fixed points of T^n contained in the interior of some elements of the defining partition for T^n. The following assertions are proved by Pollicott [**21**] and Haydn [**6**] with the dynamical system T being replaced by a mixing subshift of finite type and $\log|T'|$ and f being replaced by appropriate Hölder continuous functions on the shift space.

THEOREM 6.1. *For any* (s_0, t_0) *with* $\mathrm{Re}\, s\geqq 1$ *and* $t_0\in\boldsymbol{R}$, *let* θ *and* $U(s_0, t_0)$ *be the same as in Proposition* 3.1. *Choose any* $\tilde{\theta}>0$ *with* $1>\tilde{\theta}>\theta$ *so that* $L(s_0, t_0)$ *has no eigenvalues with modulus* $\tilde{\theta}$. *Then the function* $\phi(s, t)=\prod_{j=1}^{n(s,t,\tilde{\theta})}(1-\lambda_j(s, t))^{\mathrm{rank}\, E_j(s,t)}\zeta(s, t)$ *is realized as a nonvanishing analytic function in an open neighbourhood* $W(s_0, t_0)\subset U(s_0, t_0)$ *of* (s_0, t_0). $\zeta(s, t)$ *can be extended meromorphically to a neighborhood of the set* $\{(s, t)|\mathrm{Re}\, s\geqq 1$ *and* $t\in\boldsymbol{R}\}$ *in* $\boldsymbol{C}\times\boldsymbol{C}$.

REMARK 6.1. Our proof of this theorem is based on the method in [**1**] which is also used to relate the eigenvalues of transfer operators to the poles of Ruelle zeta functions ([**23**]) for piecewise monotonic transformations. The results in [**1**] can be directly applied to an L-Y map with finite defining partition (generating partition in terms of [**1**]). In such a case we do not need the Markov property of T, because we use the Markov extension $\bar{T}$ instead of T. But we have to estimate the difference of the original zeta function for T and the zeta function for $\bar{T}$ by using the number of the intervals in the defining partition for T. This causes a technical difficulty in dealing with an L-Y map with infinite defining partition. We expect that the Markov property can be removed from the assumption in Theorem 6.1.

PROOF OF THEOREM 6.1. First of all we recall Proposition 3.1. In the neighborhood $U(s_0, t_0)$ of (s_0, t_0), we can write

$$L(s, t)=M(s, t)+R(s, t)=L(s, t)P_1(s, t)+L(s, t)P_2(s, t), \tag{6.2}$$

where $P_1(s, t)$ and $P_2(s, t)$ are the projection operators defined by the Dunford integrals in (3.9) and (3.10). Since $P_1(s, t)$ and $P_2(s, t)$ depend analytically on (s, t) in $U(s_0, t_0)$, we have

$$\sup_{(s,t)\in U(s_0,t_0)} \|P_1(s,t)\|_{BV} \leqq C_1 \tag{6.3}$$

and

$$\sup_{(s,t)\in U(s_0,t_0)} \|P_2(s,t)^n\|_{BV} \leqq C_2\tilde{\theta}^n \qquad \text{for any } n\geqq 1, \tag{6.4}$$

where C_1 and C_2 are positive numbers. Put

$$\xi(s,t) = \prod_{j=1}^{n(s,t,\tilde{\theta})} (1-\lambda_j(s,t))^{\operatorname{rank} E_j(s,t)}. \tag{6.5}$$

The decomposition $M(s,t)=L(s,t)P_1(s,t)=\sum_{j=1}^{n(s,t,\tilde{\theta})}\lambda_j(s,t)E_j(s,t)(E_j(s,t)+N_j(s,t))$ in (3.2) is the Jordan decomposition of the operator $M(s,t)$ acting on the finite dimensional space $P_1(s,t)BV$. Therefore $\operatorname{trace}(L(s,t)^nP_1(s,t))=\operatorname{trace}(L(s,t)^nP_1(s,t)|_{P_1(s,t)BV})$ is well-defined and

$$\operatorname{trace}(L(s,t)^nP_1(s,t)) = \sum_{j=1}^{n(s,t,\tilde{\theta})} \lambda_j(s,t)^n \operatorname{rank} E_j(s,t). \tag{6.6}$$

From the general theory in [**11**] and [**21**] we can see that $\xi(s,t)$ and $\operatorname{trace}(L(s,t)^nP_1(s,t))$ are analytic in $U(s_0,t_0)$, although each $\lambda_j(s,t)$ is not necessarily analytic. Put

$$\zeta_n(s,t) = \sum_{T^nx=x} G_n(s,t), \tag{6.7}$$

where $G(s,t)=|T'(\cdot)|^{-s}A(\cdot)^{\sqrt{-1}t}$ and

$$G_n(s,t)(x)=G(s,t)(x)G(s,t)(Tx)\cdots G(s,t)(T^{n-1}x).$$

If we can show that

$$\sup_{(s,t)\in W(s_0,t_0)} |\zeta_n(s,t)-\operatorname{trace}(L(s,t)^nP_1(s,t))| \leqq C_3\tilde{\theta}^n n \tag{6.8}$$

in some neighborhood $W(s_0,t_0)$ of (s_0,t_0) in $U(s_0,t_0)$, then $\exp\{\sum_{n=1}^{\infty}(\zeta_n(s,t)-\operatorname{trace}(L(s,t)^nP_1(s,t)))/n\}$ is analytic and non-vanishing in $W(s_0,t_0)$. This implies $\phi(s,t)=\zeta(s,t)\xi(s,t)$ has an analytic continuation to $W(s_0,t_0)$. Consequently, $\zeta(s,t)$ can be extended meromorphically to $W(s_0,t_0)$.

It remains to prove the estimate (6.8). For the sake of the notational simplicity, we may drop (s,t) if there occurs no confusion. For example, $L(s,t)=L$, $P_j(s,t)=P_j$, $E_j(s,t)=E_j$, and $G(s,t)=G$ and so forth. In addition, we always consider the right continuous version for an element in BV to avoid the unexpected ambiguity of equations. For each n and for each element J of the defining partition $\mathcal{P}_n$ for T^n, we choose a point $x_J\in\operatorname{Int}J$ so that x_J is a fixed point of T^n if $T^nJ\supset\operatorname{Int}J$. Since the condition (L-Y.2) guarantees the uniqueness of a fixed point in J with $T^nJ\supset\operatorname{Int}J$, we have

(6.9) $$\zeta_n = \sum_{T^n J \supset J} G_n(x_J)\chi_J(x_J) = \sum_{J \in \mathcal{P}_n} (L^n \chi_J)(x_J),$$

where χ_J denotes the indicator function of the interval J. We note that we have used the fact $L^n\chi_J(x_J)=0$ if $m(T^nJ\cap J)=0$ in (6.9). For the finite dimensional space P_1BV, we can choose a basis $e_k=e_k(s, t)$ and $\hat{e}_k=\hat{e}_k(s, t)\in BV'$, $k=1, 2, \cdots, d=\dim P_1BV$, with the following properties:

(6.10) $$\hat{e}_k(e_l)=\delta_{kl} \text{ (Kronecker's delta)}, \ \|e_k\|_{BV}=1, \text{ and } \|\hat{e}_k\|_{BV'}\leqq 2^d$$
$$k=1, 2, \cdots, d.$$

Note that $d=\dim P_1BV$ is independent of $(s, t)\in U(s_0, t_0)$. We explain briefly how to choose e_k's and $\hat{e}_k$'s. Since P_1BV is finite dimensional, we can choose a basis e_k with $\|e_k\|_{BV}=1$, and $\min\{\|e_k-x\|_{BV} \mid x\in[e_1, e_2, \cdots, e_{k-1}]\}=1$ in virtue of the finite dimensional Riesz' lemma. Thus the functionals $e_1^0, e_2^0, \cdots, e_d^0$ with $e_k^0(e_l)=\delta_{kl}$ satisfy $\|e_k^0\|_{P_1BV'}\leqq 2^{d-k}$. We can extend each e_k^0 to a functional $\hat{e}_k$ on BV with $\|\hat{e}_k\|_{BV'}=\|e_k^0\|_{P_1BV'}$ in virtue of the Hahn Banach theorem.

Combining (6.2), (6.9) and (6.10), we can write

(6.11) $$\begin{aligned}\zeta_n &= \sum_{J\in\mathcal{P}_n}(P_1L^n\chi_J)(x_J)+\sum_{J\in\mathcal{P}_n}(P_2L^n\chi_J)(x_J)\\ &= \sum_{J\in\mathcal{P}_n}\sum_{k=1}^{d}\hat{e}_k(P_1L^n\chi_J)e_k(x_J)+\sum_{J\in\mathcal{P}_n}(EL^n\chi_J)(x_J),\end{aligned}$$

where $E(s, t)=P_2(s, t)$ in terms of Proposition 3.1. On the other hand, since $P_1L^n=L^nP_1$, we have

(6.12) $$\operatorname{trace}(L^nP_1)=\sum_{k=1}^{d}\hat{e}_k(L^nP_1e_k)=\sum_{J\in\mathcal{P}_n}\sum_{k=1}^{d}\hat{e}_k(P_1L^n(\chi_Je_k)).$$

It follows that

(6.13) $$\begin{aligned}&\zeta_n-\operatorname{trace}(L^nP_1)\\ &=\sum_{k=1}^{d}\sum_{J\in\mathcal{P}_n}(\hat{e}_k(P_1L^n\chi_J)e_k(x_J)-\hat{e}_k(P_1L^n(\chi_Je_k)))+\sum_{J\in\mathcal{P}_n}(EL^n\chi_J)(x_J)\\ &=I+II.\end{aligned}$$

For each k, we obtain

$$\begin{aligned}|\hat{e}_k(P_1L^n\chi_J)e_k(x_J)-\hat{e}_k(P_1L^n(\chi_Je_k))| &= |\hat{e}_k(P_1L^n(\chi_J(e_k(x_J)-e_k)))|\\ &\leqq 2^dC_1\|L^n(\chi_J(e_k(x_J)-e_k))\|_{BV},\end{aligned}$$

where C_1 is the constant in the inequality (6.3). Therefore we have

(6.14) $$|I| \leqq \sum_{k=1}^{d}2^dC_1\|L^n(\chi_J(e_k(x_J)-e_k))\|_{BV}.$$

We can show that

$$\sum_{J\in\mathcal{P}_n} \|L^n(\chi_J(f(x_J)-f))\|_{BV} \leqq \sum_{J\in\mathcal{P}_n} \bigvee_J f\,(4\operatorname{ess\,sup}_J|G_n|+\bigvee_J G_n) \tag{6.15}$$

in the same manner as Lemma 2.6 in [1]. If we choose the neighborhood $W(s_0, t_0)\subset U(s_0, t_0)$ of (s_0, t_0) so that

$$\sup_{(s,t)\in W(s_0,t_0)} \operatorname{ess\,sup}|G_n| \leqq C_4\tilde{\theta}^n \tag{6.16}$$

and

$$\sup_{J\in\mathcal{P}_n} \sup_{(s,t)\in W(s_0,t_0)} \bigvee_J G_n \leqq C_5\tilde{\theta}^n \tag{6.17}$$

hold for some positive numbers C_4 and C_5, we conclude that

$$\sup_{(s,t)\in W(s_0,t_0)} |I| \leqq C_6\tilde{\theta}^n \tag{6.18}$$

in virtue of the estimates (6.14) and (6.15).

It remains to estimate $\sup_{(s,t)\in W(s_0,t_0)} |II|$. For each $J\in\mathcal{P}_k$, choose a point $y_J\in\operatorname{Int} J$ and define

$$Y_J = \begin{cases} L^k\chi_J - G(y_J)L^{k-1}\chi_{TJ} & (k\geqq 2) \\ L\chi_J & (k=1). \end{cases} \tag{6.19}$$

As in (5.2) in Baladi and Keller [1], we can show

$$\|Y_J\|_{BV} \leqq \bigvee_J G\,(4\operatorname{ess\,sup}|G_{k-1}|+\bigvee_{TJ} G_k). \tag{6.20}$$

Thus we have

$$\sum_{J\in\mathcal{P}_k} \|Y_J\|_{BV} \leqq C_7 \bigvee G\,\tilde{\theta}^k \tag{6.21}$$

holds with a positive constant C_7 independent of $(s, t)\in W(s_0, t_0)$ and the choice of y_J in virtue of (6.16) and (6.17). As y_J, we employ x_J which was chosen before. Observe

$$\begin{aligned} L^n\chi_J &= L^n\chi_J - G(x_J)L^{n-1}\chi_{TJ} + G(x_J)L^{n-1}\chi_{TJ} - G(x_J)G(Tx_J)L^{n-2}\chi_{T^2J} + \\ &\qquad \cdots + G(x_J)G(Tx_J)\cdots G(T^{n-1}x_J)L\chi_{T^{n-1}J} \\ &= \sum_{k=0}^{n-1} G_k(x_J)Y_{T^kJ}\,. \end{aligned} \tag{6.22}$$

On the other hand, the Markov property of T implies that $TJ\in\mathcal{P}_k$ if $J\in\mathcal{P}_{k+1}$ $(k\geqq 1)$. Therefore we can estimate II as follows:

$$|II| = \Big|\sum_{J\in\mathcal{P}_n} (EL^n\chi_J)(x_J)\Big| = \Big|\sum_{\substack{J\in\mathcal{P}_n \\ T^nJ\supset J}} \sum_{k=0}^{n-1} G_k(x_J)EY_{T^kJ}(x_J)\Big|$$

$$= \left|\sum_{k=0}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} \sum_{\substack{J\in\mathcal{P}_n \\ T^kJ=J'}} G_k(x_J)EY_{J'}(x_J)\right| = \left|\sum_{k=0}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} (L^kEY_{J'})(T^kx_J)\right|$$

$$\leqq \sum_{k=0}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} \|R^kY_{J'}\|_{BV} \leqq C_8\sum_{k=0}^{n-1}\tilde{\theta}^k \sum_{J'\in\mathcal{P}_{n-k}} \|Y_{J'}\|_{BV} \leqq C_9 n\tilde{\theta}^n ,$$

in virtue of the estimate (6.21), where C_8 and C_9 are positive constants independent of (s, t) in $W(s_0, t_0)$. The absolute convergence of the series in the second line is guaranteed as follows:

$$\sum_{k=0}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} \sum_{\substack{J\in\mathcal{P}_n, \\ T^kJ=J'}} |G_k(x_J)EY_{J'}(x_J)| = \sum_{k=0}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} L_{|G|}{}^k(|EY_{J'}|)(T^kx_J)$$

$$\leqq C_{10}\sum_{k=1}^{n-1} \sum_{J'\in\mathcal{P}_{n-k}} \|Y_{J'}\|_{BV} \leqq C_{11}\frac{\mathbf{V}_G}{1-\tilde{\theta}}$$

by the estimate (6.21), where C_{10} and C_{11} are positive numbers independent of (s, t) in $W(s_0, t_0)$. This completes the proof of the estimate (6.8). The proof of the theorem is completed. //

7. Asymptotic distribution of periodic orbits.

In this section we apply the results of Parry [16] to the zeta function $\zeta(s, t)$ and prove the limit theorems concerning the asymptotic distribution of periodic orbits of $T\in\mathcal{T}$. Let $\gamma=\{x, Tx, \cdots, T^{P(\gamma)-1}x\}$ be a prime periodic orbit of T with period $P(\gamma)$, that is, $x, Tx, \cdots, T^{P(\gamma)-1}x$ are distinct and $T^{P(\gamma)}x=x$. From the Markov property of T, there are at most a finite number of prime periodic orbits which contain a division point of a defining partition. The contribution of such a periodic orbit does not influence the asymptotic distribution of the prime periodic orbits. Therefore we may ignore it. For a prime periodic orbit γ with period $P(\gamma)$ and a function f in $\mathcal{F}(T)$, we define a norm $N(f, \gamma)$ by

(7.1) $$N(f, \gamma) = S_{P(\gamma)}f(x) = f(x)+f(Tx)+\cdots+f(T^{P(\gamma)-1}x),$$

where x is any element in γ. The η-function $\eta(f, s)$ is defined by

(7.2) $$\eta(f, s) = \sum_{n=1}^{\infty}\sum_{\gamma} N(f, \gamma)\exp(-snN(f_0, \gamma)),$$

where $f_0=\log|T'|$. As a consequence of Proposition 3.3 and Theorem 6.1, we obtain

THEOREM 7.1. *Assume that $T\in\mathcal{T}$ satisfies the mixing condition (M) and the polynomial growth condition (P). If f belongs to $\mathcal{F}_2(T)$, then $\eta(f, s)$ is analytic in the domain with* $\mathrm{Re}\, s>1$ *and can be extended meromorphically to the neighborhood of the axis* $\mathrm{Re}\, s=1$. *Moreover,* $s=1$ *is a unique pole of $\eta(f, s)$ on the axis* *and it is a simple pole with residue*

$$\int_0^1 f\,d\mu \Big/ \int_0^1 \log|T'|\,d\mu = \int_0^1 f\,d\mu \Big/ h_\mu(T),$$

where $h_\mu(T)$ denotes the metrical entropy of T with respect to μ.

PROOF. Consider the zeta function $\zeta(s, t)$ given in (6.1). If $\operatorname{Re} s_0>1$ and $t_0\in \boldsymbol{R}$, then $\zeta(s, t)$ is well-defined, non-vanishing and analytic in some neighborhood of (s_0, t_0) in $\boldsymbol{C}\times\boldsymbol{C}$. Since T satisfies the condition (P), the function $f_0-\int_0^1 f_0\,d\mu=f_0-h_\mu(T)$ belongs to $\mathcal{F}_2(T)$ by Remark 4.1 and the classification theorem (Theorem 4.2). Therefore the transfer operator $L(s, t)$ with $s=1+\sqrt{-1}\tau$, $t\in\boldsymbol{R}$, $\tau\in\boldsymbol{R}$ can not have an eigenvalue with modulus 1 except the case $(s_0, t_0)=(1, 0)$. Combining this fact, Theorem 6.1 and Proposition 3.3, we can see the following:

(1) $\zeta(s, t)$ is non-vanishing and analytic in some neighborhood of (s_0, t_0) with $\operatorname{Re} s_0=1$, $\operatorname{Im} s_0\neq 0$, and $t_0\in\boldsymbol{R}$.

(2) $\zeta(s, t)$ can be expressed in some neighborhood D of $(1, 0)$ as

$$\zeta(s, t)=\phi(s, t)(1-\lambda(s, t))^{-1}, \tag{7.3}$$

where $\phi(s, t)$ and $\lambda(s, t)$ are analytic functions in D with the following properties.

(2.a) $\phi(s, t)$ is non-vanishing in D.

(2.b) $\lambda(1, 0)=1$

(2.c) $\lambda(s, t)$ coincides with the simple eigenvalue of $L(s, t)$ with the maximal modulus.

(2.d) $$\left.\frac{\partial\lambda}{\partial t}\right|_{(s,t)=(1,0)}=\sqrt{-1}\int_0^1 f\,d\mu \quad\text{and}\quad \left.\frac{\partial\lambda}{\partial s}\right|_{(s,t)=(1,0)}=\int_0^1 f_0\,d\mu .$$

From the definition (6.1) of $\zeta(s, t)$, we can rewrite

$$\zeta(s, t)=\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\gamma}\exp(-snN(f_0, \gamma)+\sqrt{-1}\,tnN(f, \gamma))\Big).$$

Thus by taking the logarithmic derivative of $\zeta(s, t)$ at $t=0$, we have

$$\frac{\zeta_t(s, 0)}{\zeta(s, 0)}=\sqrt{-1}\sum_{n=1}^{\infty}\sum_{\gamma}N(f, \gamma)\exp(-snN(f_0, \gamma)) = \sqrt{-1}\,\eta(f, s). \tag{7.4}$$

On the other hand, the left hand side of (7.4) can be expressed as

$$\frac{\zeta_t(s, 0)}{\zeta(s, 0)}=\frac{\lambda_t(s, 0)}{1-\lambda(s, 0)}+\frac{\phi_t(s, 0)}{\phi(s, 0)}$$

in virtue of the equation (7.3).

Therefore $\eta(f, s)$ is analytic in $\operatorname{Re} s>1$ and can be extended meromorphically to a neighborhood of $\{s\in \boldsymbol{C} \mid \operatorname{Re} s\geqq 1\}$. In addition, $s=1$ is a unique pole on the axis $\operatorname{Re} s=1$ and it is simple. By using the equalities in (2.d) above, we can conclude that

$$\lim_{s\to 1}\eta(f, s)(s-1)=\lim_{s\to 1}\frac{\sqrt{-1}(s-1)\lambda_t(s, 0)}{1-\lambda(s, 0)}=\frac{-\sqrt{-1}\lambda_t(1, 0)}{\lambda_s(1, 0)}=\int_0^1 f\,d\mu\Big/\int_0^1 f_0\,d\mu.$$

Hence we obtain the theorem. //

We can write

$$\eta(f, s)=\int_1^\infty x^{-s}dF_f(x), \tag{7.5}$$

where $F_f(x)=\sum_{\exp(nN(f_0,\gamma))\leq x}N(f, \gamma)$. Ikehara's Tauberian theorem implies

$$\lim_{s\to 1}\frac{F_f(x)}{x}=\int_0^1 f\,d\mu\Big/\int_0^1 f_0\,d\mu \tag{7.6}$$

in virtue of Theorem 7.1. If we put

$$\eta^1(f, s)=\sum_{\gamma}N(f, \gamma)\exp(-sN(f_0, \gamma)), \tag{7.7}$$

it is not hard to see that Theorem 7.1 is valid for $\eta^1(f, s)$. Thus if we put $F_f^1(x)=\sum_{\exp(N(f_0,\gamma))\leq x}N(f, \gamma)$, we obtain by Ikehara's Tauberian theorem that

$$\lim_{s\to 1}\frac{F_f^1(x)}{x}=\int_0^1 f\,d\mu\Big/\int_0^1 f_0\,d\mu. \tag{7.8}$$

Hence we can show the following theorem in the same manner as in [**16**] and [**17**].

THEOREM 7.2. *Assume $T\in\mathfrak{T}$ satisfies the conditions (M) and (P). If f belongs to $\mathfrak{F}_2(T)$, then we have*

$$\sum_{N(f_0,\gamma)\leq t}\frac{N(f, \gamma)}{N(f_0, \gamma)}\sim\frac{e^t}{th_\mu(T)}\int_0^1 f\,d\mu \tag{7.9}$$

and

$$\sum_{N(f_0,\gamma)\leq t}N(f, \gamma)\sim\frac{e^t}{h_\mu(T)}\int_0^1 f\,d\mu. \tag{7.10}$$

8. Applications.

First we consider the Gauss transformation

$$T_G x=\frac{1}{x}-\left[\frac{1}{x}\right], \tag{8.1}$$

where $[x]$ denotes the integral part of x. Clearly T_G satisfies the mixing condition (M) and the polynomial growth condition (P). Since $T'_G x=-1/x^2$, the function $\log x-\pi^2/(12\log 2)\in\mathcal{F}_2(T_G)$ by the Remark 4.1. Consider the simple continued fraction expansion of an irrational $x\in(0, 1)$:

$$x=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\ddots}}=[a_1(x),\ a_2(x),\ \cdots], \tag{8.2}$$

where $a_n(x)$'s are positive integers with $a_{n+1}(x)=a_n(T_G x)$, $n=1, 2, \cdots$. The n-th order convergent of x is written as

$$\frac{p_n(x)}{q_n(x)}=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\ddots+\cfrac{1}{a_n(x)}}}=[a_1(x),\ a_2(x),\ \cdots,\ a_n(x)], \tag{8.3}$$

where $p_n(x)/q_n(x)$ is the reduced fraction. It is well-known that

$$\left|\log q_n(x)+\sum_{k=0}^{n-1}\log T^k x\right|\leqq 2 \tag{8.4}$$

and

$$(a_1(x)a_2(x)\cdots a_n(x))^{1/n}\longrightarrow\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2+k}\right)^{(\log k)/(\log 2)}\quad m\text{-a.e.}\quad(n\to\infty) \tag{8.5}$$

(see [4, Chapter 7]). Therefore the central limit theorem applied to $\log x$ and $\log a_1(x)=\sum_{k=0}^{\infty}\chi_{(1/(k+1),1/k]}\log k$ reads:

THEOREM 8.1. *There are positive constants V_1, V_2, C_1 and C_2 such that*

$$\sup_{\substack{a,b\in\mathbf{R}\\ a<b}}\Big|m\{x\in(0, 1)\,|\exp(a\sqrt{n}+\pi^2 n/12\log 2)\leqq q_n(x)\leqq\exp(b\sqrt{n}+\pi^2 n/12\log 2)\}-\frac{1}{\sqrt{2\pi V_1}}\int_a^b\exp(-t^2/(2V_1))dt\Big|\leqq C_1\frac{1}{\sqrt{n}} \tag{8.6}$$

and

$$\sup_{\substack{a,b\in\mathbf{R}\\ a<b}}\Big|m\Big\{x\in(0, 1)\,|\exp\Big(a\sqrt{n}+\prod_{k=1}^{\infty}(1+1/(k^2+k))^{(\log k)/(\log 2)}n\Big)\leqq a_1(x)a_2(x)\cdots a_n(x)\leqq\exp\Big(b\sqrt{n}+\prod_{k=1}^{\infty}(1+1/(k^2+k))^{(\log k)/(\log 2)}n\Big)\Big\}-\frac{1}{\sqrt{2\pi V_2}}\int_a^b\exp(-t^2/(2V_2))dt\Big|\leqq C_2\frac{1}{\sqrt{n}}. \tag{8.7}$$

This theorem gives an improvement of the results in [19, p. 37, Theorem 2.1.2]. If we apply the local limit theorem Theorem 5.2 to the same functions, we obtain:

THEOREM 8.2. *For $a<b$, we have*

(8.8)
$$\lim_{n\to\infty}\Big|\sqrt{n}m\{x\in(0,1)|\exp(a\sqrt{n}+\pi^2/(12\log 2)+\alpha)\leqq q_n(x)$$
$$\leqq \exp(b\sqrt{n}+\pi^2/(12\log 2)+\alpha)\}-(b-a)\frac{1}{\sqrt{2\pi V_1}}$$
$$\cdot\exp\Big(-\frac{(\alpha-(\pi^2 n)/(12\log 2))^2}{2V_1 n}\Big)\Big|=0$$

locally uniformly in α and

(8.9)
$$\limsup_{n\to\infty}\sup_{\alpha\in\boldsymbol{R}}\Big|\sqrt{n}m\Big\{x\in(0,1)|\exp\Big(a\sqrt{n}+\prod_{k=1}^{\infty}(1+1/(k^2+k))^{(\log k)/(\log 2)}+\alpha\Big)$$
$$\leqq a_1(x)a_2(x)\cdots a_n(x)\leqq\exp\Big(b\sqrt{n}+\prod_{k=1}^{\infty}(1+1/(k^2+k))^{(\log k)/(\log 2)}+\alpha\Big)\Big\}$$
$$-(b-a)\frac{1}{\sqrt{2\pi V_2}}\exp\Big(-\frac{\alpha^2}{2V_2 n}\Big)\Big|=0\,.$$

If we apply Theorem 7.2 to T_G, we have:

THEOREM 8.3. *For any $f\in\mathfrak{F}(T_G)$, we obtain*

(8.10)
$$\sum_{N(f_0,\gamma)\leq t}\frac{N(f,\gamma)}{N(f_0,\gamma)}\sim\frac{6e^t}{\pi^2 t}\int_0^1\frac{f(x)}{1+x}dm$$

and

(8.11)
$$\sum_{N(f_0,\gamma)\leq t}N(f,\gamma)\sim\frac{6e^t}{\pi^2}\int_0^1\frac{f(x)}{1+x}dm\,,$$

Next we show a limit theorem on the digits $a_1(x)$, $a_2(x)$, $\cdots$ by using the transformation T_s ($s>1$) in Example 1.2. If $x\in(0,1)$ is a periodic point of T_G with period p, we write $x=[a_1(x), a_2(x),\cdots]=[\dot{a}_1(x), a_2(x),\cdots,\dot{a}_p(x)]$. The periodic points of T_G and those of T_s are under the one to one correspondence so that the T_s-periodic point y corresponding to $[\dot{a}_1(x), a_2(x),\cdots,\dot{a}_p(x)]$ satisfies $|(T_s^p)'(y)|=\zeta_R(s)^p(a_1(x)a_2(x)\cdots a_p(x))^s$. Let $\gamma=\{x, T_Gx,\cdots, T_G^{p(\gamma)-1}x\}$ be a prime periodic orbit of T_G. Then γ can be regarded as a $GL(2,\boldsymbol{Z})$-orbit equivalence class of reduced quadratic irrationals. Put $M(\gamma)=a_1(x)a_2(x)\cdots a_{P(\gamma)}(x)$. Then Theorem 7.2 applied to T_s gives:

THEOREM 8.4.
$$\#\{\gamma|\zeta_R(s)^{P(\gamma)/s}M(\gamma)\leqq t\}\sim\frac{s\log t}{t^s}\qquad for\ s>1\,.$$

In the sequel, we explain briefly the relation between Pollicott's paper [20] and our results. Consider the geodesic flow φ_t on the unit tangent bundle T_1M over the modular surface $M=H/PSL(2,\boldsymbol{Z})$, where H denotes the complex upper half plane. Its closed orbits naturally correspond to the periodic orbits of the Gauss transformation T_G. For example, a φ_t-closed orbit which corresponds to a T_G-periodic orbit $\{[\dot{a}_i, a_{i+1}, \cdots, a_{2n}, \cdots, \dot{a}_{i-1}]\}_{i=1}^{2n}$ has the least period $-2\log\sum_{i=1}^{2n}[\dot{a}_i, a_{i+1}, \cdots, a_{2n}, \cdots, \dot{a}_{i-1}]$. For details, one can consult [**20**] and [**26**]. Pollicott uses the Mayer's results in [**13**] on the zeta function which are based on the study of the transfer operator acting not on BV but on a class of holomorphic functions. Any way most results in [**20**] can be reduced to the study of the analytic properties of the η-function defined by

$$\hat{\eta}(s, f) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{a_1, \cdots, a_{2n}} \Big(\sum_{i=1}^{2n} f([\dot{a}_i, a_{i+1}, \cdots, a_{2n}, \cdots, \dot{a}_{i-1}]) \Big) \cdot \prod_{i=1}^{2n} [\dot{a}_i, a_{i+1}, \cdots, a_{2n}, \cdots, \dot{a}_{i-1}]^{2s} . \tag{8.12}$$

Noting that $\log|(T_G^2)'| = (x \cdot T_G x)^{-2}$, we can write

$$\begin{aligned}\hat{\eta}(s, f) &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{T_G^2 x = x} \Big(\sum_{k=0}^{n-1} (f + f \circ T_G)(T_G^{2k} x) \Big) \exp\Big(\sum_{k=1}^{n-1} \log|(T_G^2)'(x)| \Big) \\ &= \eta(f + f \circ T_G, s)\end{aligned}$$

in our notation if we employ T_G^2 as T in the equation (7.2) or (0.9). Thus we can apply the results in Section 7 to $f \in \mathfrak{T}(T_G^2)$ without approximating it by suitable analytic functions because it is easy to see that T_G^2 is also in $\mathfrak{T}$ and satisfies the conditions (M) and (P). For example, take $\log x$ as f we can prove Sarnak-Woo Theorem which asserts $\#\{\gamma \mid \exp(l(\gamma)) \leqq t\} \sim t/(\log t)$ $(t \to \infty)$, where γ denotes a prime closed orbit of φ_t and $l(\gamma)$ denotes its hyperbolic length.

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